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HOMOGENIZATION OF THE LINEAR BOLTZMANN EQUATION IN A DOMAIN WITH A PERIODIC DISTRIBUTION OF HOLES*

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Abstract. Consider a linear Boltzmann equation posed on the Euclidian plane with a periodic system of circular holes and for particles moving at speed 1. Assuming that the holes are absorbing, i.e., that particles falling in a hole remain trapped there forever, we discuss the homogenization limit of that equation in the case where the reciprocal number of holes per unit surface and the length of the circumference of each hole are asymptotically equivalent small quantities. We show that the mass loss rate due to particles falling into the holes is governed by a renewal equation that involves the distribution of free path lengths for the periodic Lorentz gas. In particular, it is proved that the total mass of the particle system at time t decays exponentially quickly as $t \to +\infty$. This is at variance with the collisionless case discussed in [E. Caglioti and F. Golse, *Comm. Math. Phys.*, 236 (2003), pp. 199–221], where the total mass decays as C/t as $t \to +\infty$.

Key words. linear Boltzmann equation, periodic homogenization, periodic Lorentz gas, renewal equation

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1. Introduction.

The homogenization of a transport process describing the motion of particles in a system of fixed obstacles—such as scatterers or holes—leads to very different results according to whether the distribution of obstacles is periodic or random. Before describing the specific problem analyzed in the present work, we recall a few results recently obtained on a more complicated and yet related problem.

An important example of the phenomenon mentioned above is the Boltzmann–Grad limit of the Lorentz gas. The Lorentz gas is the dynamical system corresponding to the free motion of a single point particle in a system of fixed spherical obstacles, assuming that each collision of the particle with any one of the obstacles is purely elastic. Since the particle is not subject to any external force, we assume without loss of generality that its speed is 1. The Boltzmann–Grad limit is the scaling limit where the obstacle radius and the reciprocal number of obstacles per unit volume vanish in such a way that the average free path length of the particle between two consecutive collisions with the obstacles is of the order of unity.

Call $f(t, x, v)$ the particle distribution function in phase space in that scaling limit—in other words, the probability that the particle be located in an infinitesimal volume dx around the position x with direction in an infinitesimal element of solid angle dv around the direction v at time $t \geq 0$ is $f(t, x, v)dxdv$.

In the case of a random system of obstacles—more precisely, assuming that the obstacles' centers are independent and distributed in the three-dimensional Euclidian space under Poisson's law—Gallavotti proved in [15, 16] (see also [17] on pp. 48–55)

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that the average of f over obstacle configurations (i.e., the mathematical expectation of f) is a solution of the linear Boltzmann equation

$$(\partial_t + v \cdot \nabla_x + \sigma) f(t,x,v) = \frac{\sigma}{\pi} \int_{\substack{\omega\cdot v>0\\ |\omega|=1}} f(t,x,v-2(\omega\cdot v)\omega)\omega\cdot v d\omega\,.$$

If, on the contrary, the obstacles are periodically distributed—specifically, if they are centered at the vertices of a cubic lattice—the limiting particle distribution function f cannot be the solution of any linear Boltzmann equation of the form

$$(\partial_t + v \cdot \nabla_x + \sigma) f(t,x,v) = \sigma \int_{|w|=1} p(v|w) f(t,x,w) dw\,,$$

where p is a continuous, symmetric transition probability density on the unit sphere. See [18] for a complete proof of this negative result, based on earlier estimates on the distribution of free path lengths for the periodic Lorentz gas [6, 19].

The correct limiting equation for the Boltzmann–Grad limit of the periodic Lorentz gas was found only very recently; see [8, 25]. In the two-dimensional case, the most striking feature of the theory presented in these references is that the limiting equation is set on an *extended phase space* involving not only the particle position x and direction v, as in all classical kinetic models, but also the (rescaled) distance τ to the next collision point with the obstacles and the impact parameter h at this next collision point.

The particle motion is described in terms of its distribution function in this extended phase space, $F \equiv F(t,x,v,\tau,h)$, which is governed by an equation of the form

$$\begin{aligned}(1)\qquad &(\partial_t + v\cdot\nabla_x - \partial_\tau) F(t,x,v,\tau,h)\\ &\qquad = \int_{-1}^{1} P(\tau,h|h') F(t,x,R[\pi - 2\arcsin(h')]v, 0, h') dh',\end{aligned}$$

where $R[\theta]$ designates the rotation of an angle θ and $P(\tau,h|h')$ is a nonnegative integral kernel whose explicit expression is given in [8] but is of little interest for the present discussion. The particle distribution function in the classical phase space of kinetic theory is recovered in terms of F by the following formula:

$$f(t,x,v) = \int_0^{+\infty} \int_{-1}^{1} F(t,x,v,\tau,h) dh d\tau\,.$$

However, the particle distribution function f itself does not satisfy a linear Boltzmann equation in closed form.

Loosely speaking, in the case of a periodic distribution of obstacles, the particle "feels" the correlations between the obstacles since its trajectory consists of segments of maximal length avoiding the obstacles. This explains the need for an extended phase space in order to describe the Boltzmann–Grad limit of the Lorentz gas in the periodic case. In the random case studied by Gallavotti, the obstacles' centers are assumed to be independent, which reduces the complexity of the limiting dynamics.

In the present work, we shall study a much simpler homogenization problem, which can be formulated as follows.

Problem. Consider a system of point particles whose distribution function is governed by a linear Boltzmann equation. The particles are assumed to move in a periodic system of holes. Describe the asymptotic behavior of the total mass of the particle system in the long time limit, assuming that the radius of the holes and their reciprocal number per unit volume vanish so that the average distance between the holes is of the order of 1.

This problem is the analogue in kinetic theory of the one studied in [23] and [11] for the diffusion equation and in [2] for the Stokes equation.

Although the underlying dynamics in this problem are a lot simpler than those of the Lorentz gas, the homogenized equation is also set on an extended phase space, analogous to the one described above.

As we shall see, the mathematical derivation of the homogenized equation in the extended phase space for the problem above involves only very elementary arguments from functional analysis—at variance with the case of the Boltzmann–Grad limit of the Lorentz gas, which requires a fairly detailed knowledge of particle trajectories.

2. The model. We consider the monokinetic, linear Boltzmann equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon) = 0 \tag{2}$$

in space dimension 2.

The unknown function $f(t,x,v)$ is the density at time $t \in \mathbb{R}_+$ of particles with velocity $v \in \mathbb{S}^1$, located at $x \in \mathbb{R}^2$. For each $\phi \in L^2(\mathbb{S}^1)$, we denote

$$K\phi(v) := \frac{1}{2\pi}\int_{\mathbb{S}^1} k(v,w)\phi(w)dw,$$

where dw is the uniform measure (arc length) on the unit circle $\mathbb{S}^1$. We henceforth assume that

$$\begin{aligned} k \in L^2(\mathbb{S}^1 \times \mathbb{S}^1)\,, \quad k(v,w) = k(w,v) \geq 0 \ \text{ a.e. in } v,w \in \mathbb{S}^1,\\ \text{and } \frac{1}{2\pi}\int_{\mathbb{S}^1} k(v,w)dw = 1 \text{ a.e. in } v \in \mathbb{S}^1. \end{aligned} \tag{3}$$

The case of isotropic scattering, where k is a constant, is a classical model in the context of radiative transfer. Likewise, the case of Thomson scattering in radiative transfer involves the integral kernel

$$k(v,w) = \frac{3}{16}(1 + (v \cdot w)^2);$$

see, for instance, Chapter I, section 16 of [10]. Finally, the collision frequency is a constant $\sigma > 0$.

The linear Boltzmann equation (2) is set on the spatial domain Z_ε, i.e., the space $\mathbb{R}^2$ with a periodic system of holes removed:

$$Z_\varepsilon := \left\{ x \in \mathbb{R}^2 \,|\, \operatorname{dist}(x, \varepsilon\mathbb{Z}^2) > \varepsilon^2 \right\}\,.$$

We assume an absorption boundary condition on ∂Z_ε

$$f_\varepsilon = 0 \text{ for } (t,x,v) \in \mathbb{R}_+^* \times \partial Z_\varepsilon \times \mathbb{S}^1 \text{ whenever } v \cdot n_x > 0\,,$$

where n_x denotes the inward unit normal vector to Z_ε at the point $x \in \partial Z_\varepsilon$. This condition means that a particle falling into any one of the holes remains there forever.

The same problem could, of course, be considered in any space dimension. Notice, however, that in space dimension $N \geq 2$, the appropriate scaling, analogous to the one considered here, would be to consider holes of radius $\varepsilon^{N/(N-1)}$ centered at the points of the cubic lattice $\varepsilon\mathbb{Z}^N$; see, for instance, [6, 19]. Most of the arguments considered in the present paper can be adapted without change to the higher dimensional case, except that the expression of one particular coefficient appearing in the homogenized equation is not yet known explicitly at the time of this writing.

The most natural question related to the dynamics of the system above is the asymptotic behavior of the total mass of the particle system in the small obstacle radius $\varepsilon \ll 1$ and long time limit.

The last two authors have considered in [7] the noncollisional case ($\sigma = 0$) and proved that, in the limit as $\varepsilon \to 0^+$, the solution f_ε converges in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ to a solution f of the following nonautonomous equation:

$$\partial_t f + v \cdot \nabla_x f = \frac{\dot{p}(t)}{p(t)} f \,, \tag{4}$$

where p is a positive decreasing function defined below. In that case, the total mass of the particle system decays like C/t as $t \to +\infty$.

Observe that, starting from the free transport equation, we obtain a nonautonomous (in time) equation in the small ε limit. In particular, the solution of (4) cannot be given by a semigroup in a function space such as $L^p(\mathbb{R}^2_x \times \mathbb{S}^1_v)$. As we shall see, the homogenization of the linear Boltzmann equation in the collisional case ($\sigma > 0$) leads to an even more spectacular change of structure in the equivalent equation obtained in the limit.

The work of the last two authors [7] relies upon an explicit computation of the solution of the free transport equation, where the effect of the system of holes is handled with continued fraction techniques. In the present paper, we investigate the analogous homogenization problem in the collisional case ($\sigma > 0$). As we shall see, there is no explicit representation formula for the solution of the linear Boltzmann equation, other than the one based on the transport process, a particular stochastic process, defined, for example, in [26].

This representation formula was used in a previous work of the first author [3], who established a uniform in ε upper bound for the total mass of the particle system by a quantity of the form $Ce^{-a_\sigma t}$ for some $a_\sigma > 0$. This exponential decay is quite remarkable; indeed, there is a "phase transition" between the collisionless case in which the total mass decays algebraically as $t \to +\infty$ and the collisional case in which the total mass decays at least exponentially quickly in that same limit.

In the present paper, we further investigate this phenomenon and show that the exponential decay estimate found in [3] is sharp by giving an asymptotic equivalent of the total mass of the particle system in the small ε limit as $t \to +\infty$.

Instead of the semiexplicit representation formula by the transport process, our argument is based on the very special structure of the homogenized problem. The key observation in the present work is that this homogenized problem involves a renewal equation, for which exponential decay is a classical result that can be found in classical monographs such as [14].

3. The main results. First we recall the definition of the free path length in the direction v for a particle starting from x in Z_ε:

$$\tau_\varepsilon(x, v) := \inf\{t > 0 \,|\, x - tv \in \partial Z_\varepsilon\} \ . \tag{5}$$

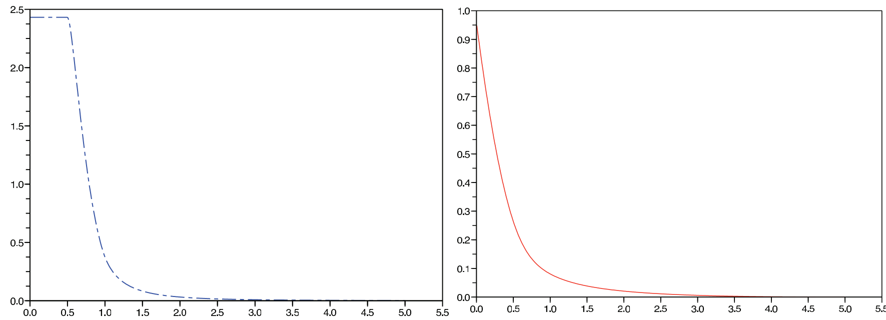

FIG. 1. *The graphs of* Υ *(left) and* p *(right)*

The distribution of the free path length has been studied in [6, 19, 7, 4]. In particular, it is proved that, for each arc $I \subset \mathbb{S}^1$ and each $t \geq 0$, one has

$$\text{meas}(\{(x,v) \in (Z_\varepsilon \cap [0,1]^2) \times I \,|\, \varepsilon\tau_\varepsilon(x,v) > t\}) \to p(t)|I| \tag{6}$$

as $\varepsilon \to 0^+$, where $|I|$ denotes the length of I and the measure considered in the statement above is the uniform measure on $[0,1]^2 \times \mathbb{S}^1$.

The following estimate for p can be found in [6]: there exist $C, C' > 0$ such that for all $t \geq 1$,

$$\frac{C}{t} \leq \text{meas}(\{(x,v) \in (Z_\varepsilon \cap [0,1]^2) \times I \,|\, \varepsilon\tau_\varepsilon(x,v) > t\}) \leq \frac{C'}{t} \tag{7}$$

uniformly as $\varepsilon \to 0^+$ so that

$$\frac{C}{t} \leq p(t) \leq \frac{C'}{t}\,. \tag{8}$$

In [4] Boca and Zaharescu have obtained an explicit formula for p as

$$p(t) = \int_t^{+\infty} (\tau - t)\Upsilon(\tau)d\tau\,, \tag{9}$$

where the function Υ is expressed as follows (see the graphs of Υ and p in Figure 1):
(10)

$$\Upsilon(t) = \frac{24}{\pi^2}\begin{cases} 1 & \text{if } t \in (0, \frac{1}{2}], \\ \dfrac{1}{2t} + 2\left(1 - \dfrac{1}{2t}\right)^2 \ln\left(1 - \dfrac{1}{2t}\right) - \dfrac{1}{2}\left(1 - \dfrac{1}{t}\right)^2 \ln\left|1 - \dfrac{1}{t}\right| & \text{if } t \in \left(\dfrac{1}{2}, +\infty\right). \end{cases}$$

This formula had been conjectured earlier by Dahlqvist in [12] by an argument based on some equidistribution assumption left unverified.

This is precisely at this point that the case of space dimension 2 differs from the higher dimensional case. Indeed, in a space dimension higher than 2, the existence of the limit (6) has been proved in [24], while the uniform estimate analogous to (7) is to be found in [19]. However, no explicit formula analogous to (9) is known in that case, at least at the time of this writing. We have chosen to treat in the present paper only the case of the square lattice in space dimension 2 as it is the only case where the limit (6)–(9) is known completely.

Throughout this paper, we assume that the initial data of (Ξ_ε) satisfies the assumption

$$f^{in} \geq 0 \text{ on } \mathbb{R}^2 \times \mathbb{S}^1 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv + \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v) < +\infty. \tag{11}$$

For each $0 < \varepsilon \ll 1$, let f_ε be the (mild) solution of the initial boundary value problem

$$(\Xi_\varepsilon)\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma(f_\varepsilon - Kf_\varepsilon) = 0, & (x,v)\in Z_\varepsilon\times\mathbb{S}^1, t>0,\\ f_\varepsilon = 0 \text{ if } v\cdot n_x > 0, & (x,v)\in\partial Z_\varepsilon\times\mathbb{S}^1,\\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v)\in Z_\varepsilon\times\mathbb{S}^1. \end{cases}$$

The classical theory of the linear Boltzmann equation guarantees the existence and uniqueness of a mild solution f_ε of the problem (Ξ_ε) satisfying

$$\begin{aligned} &0 \leq f_\varepsilon(t,x,v) \leq \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v) \quad \text{ a.e. on } \mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1\,,\\ &\iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\,. \end{aligned} \tag{12}$$

Consider next $F := F(t,s,x,v)$ the solution of the Cauchy problem

$$(\Sigma)\begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + \dfrac{\dot p}{p}(t\wedge s)F, & t,s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1,\\ F(t,0,x,v) = \sigma\displaystyle\int_0^{+\infty} KF(t,s,x,v)ds, & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1,\\ F(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \end{cases}$$

with the notation $t\wedge s := \min(t,s)$. Notice that F is a density defined on the extended phase space

$$\left\{(s,x,v) | s\geq 0, x\in\mathbb{R}^2, v\in\mathbb{S}^1\right\}$$

involving the extra variable s, whose physical meaning is explained as follows.

Recall that the solution f_ε of the linear Boltzmann equation can be expressed in terms of the transport process (see [26]), a stochastic process involving a jump process in the v variable, perturbed by a drift in the x variable. The variable s is the "age" of the current velocity v in that process, i.e., the time since the last jump in the v variable.

Therefore, between jumps in the v variable, s increases with t, and this accounts for the sign of the additional term $+\partial_s F$ in the system (Σ).

On the contrary, in (1), the extra variable τ (the rescaled distance to the next collision point with one of the scatterers) decreases as t increases between collisions with the scatterers, which accounts for the minus sign in the additional term $-\partial_\tau F$ in that equation.

Henceforth, we shall frequently need to extend functions defined a.e. on Z_ε by 0 inside the holes (that is, in the complement of $\overline{Z_\varepsilon}$). We therefore introduce the following piece of notation.

Notation. For each function $\varphi \equiv \varphi(x)$ defined a.e. on Z_ε, we denote

$$\{\varphi\}(x) = \begin{cases} \varphi(x) & \text{if } x \in Z_\varepsilon, \\ 0 & \text{if } x \notin \overline{Z_\varepsilon}. \end{cases}$$

We use the same notation $\{f_\varepsilon\}$ or $\{F_\varepsilon\}$ to designate the same extension by 0 inside the holes for functions defined on Cartesian products involving Z_ε as one of their factors, such as $\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ in the case of f_ε and $\mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ in the case of F_ε.

Our first main main result is shown in the following theorem.

THEOREM 1. *Under the assumptions above,*

$$\{f_\varepsilon\} \rightharpoonup \int_0^{+\infty} F ds$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$, where F is the unique (mild) solution of (Σ).

Notice that the limit of the (extended) distribution function of the particle system is indeed defined in terms of the solution F of the homogenized integro-differential equation (Σ). However, it does not seem that the limit of $\{f_\varepsilon\}$ itself satisfies any natural equation.

Next we discuss the asymptotic decay as $t \to +\infty$ of the total mass of the particle system in the homogenization limit $\varepsilon \ll 1$. Obviously, the particle system loses mass due to particles falling into the holes.

In order to do so, we introduce the quantity

$$m(t,s) := \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v) dx dv \, .$$

A key observation in our work is that m is the solution of a renewal type partial differential equation (PDE), as explained in the next proposition.

PROPOSITION 1. *Denote*

$$B(t,s) = \sigma - \frac{\dot{p}}{p}(t \wedge s) \, ,$$

and assume that f^{in} satisfies the condition (11).

Then the renewal PDE

$$\begin{cases} \partial_t \mu(t,s) + \partial_s \mu(t,s) + B(t,s)\mu(t,s) = 0, & t, s > 0 \, , \\ \mu(t,0) = \sigma \displaystyle\int_0^{+\infty} \mu(t,s) ds, & t > 0 \, , \\ \mu(0,s) = \sigma e^{-\sigma s}, & s > 0 \end{cases}$$

has a unique mild solution $\mu \in L^\infty([0,T]; L^1(\mathbb{R}_+))$ for all $T > 0$.

Moreover, one has

$$m(t,s) = \frac{\mu(t,s)}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v) dx dv$$

a.e. in $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Renewal equations are frequently met in many different contexts. For instance, they are used as a mathematical model in biology to study the dynamics of structured populations. The interested reader can consult [22] or [27] for more information on this subject.

Consider next the quantity

$$M(t) := \frac{1}{2\pi}\int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{S}^1} F(t,s,x,v)dxdvds = \int_0^{+\infty} m(t,s)ds\,. \tag{13}$$

As explained in the theorem below, $M(t)$ is the total mass at time t of the particle system in the limit as $\varepsilon \to 0^+$; besides, the asymptotic behavior of M as $t \to +\infty$ is a consequence of the renewal PDE satisfied by the function $(t,s) \mapsto m(t,s)$.

THEOREM 2. *Under the same assumptions as in Theorem* 1*,*

(1) *the total mass*

$$\frac{1}{2\pi}\iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \to M(t)$$

in $L^1_{loc}(\mathbb{R}_+)$ *as* $\varepsilon \to 0^+$ *and a.e. in* $t \geq 0$ *after extracting a subsequence of* $\varepsilon \to 0^+$*;*

(2) *the limiting total mass is given by the representation formula*

$$M(t) = \frac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\sum_{n\geq 1}\kappa^{*n}(t),\ \ t>0$$

with

$$\kappa(t) := \sigma e^{-\sigma t}p(t)\mathbf{1}_{t\geq 0}\,, \quad \kappa^{*n} := \underbrace{\kappa * \cdots * \kappa}_{n \text{ factors}}$$

and $*$ *denoting as usual the convolution product on the real line;*

(3) *for each* $\sigma > 0$*, there exists* $\xi_\sigma \in (-\sigma, 0)$ *such that*

$$M(t) \sim C_\sigma e^{\xi_\sigma t} \text{ as } t \to +\infty$$

with

$$C_\sigma := \frac{1}{2\pi\sigma}\frac{\displaystyle\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv}{\displaystyle\int_0^\infty tp(t)e^{-(\sigma+\xi_\sigma)t}dt}\,; \text{ and}$$

(4) *finally the exponential mass loss rate* ξ_σ *satisfies*

$$\xi_\sigma \sim -\sigma \text{ as } \sigma \to 0^+ \text{ and } \xi_\sigma \to -2 \text{ as } \sigma \to +\infty\,.$$

Statement (1) above means that M is the limiting mass of the particle system at time t as $\varepsilon \to 0^+$. Statement (3) gives a precise asymptotic equivalent of $M(t)$ as $t \to +\infty$.

As recalled in the previous section, if $\sigma = 0$ in the linear Boltzmann equation (Ξ_ε), the total mass of the particle system in the vanishing ε limit is asymptotically equivalent to

$$\frac{\displaystyle\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv}{\pi^2 t}$$

as $t \to +\infty$. The reason for this slow, algebraic decay is the existence of channels—infinite open strips included in the spatial domain Z_ε, i.e., avoiding all the holes.

Particles located in one such channel and moving in a direction close to the channel's direction will not fall into a hole before exiting the channel, and this can take an arbitrarily long time as the particles' direction approaches that of the channel. This construction based on channels leads to a sufficiently large fraction of the single-particle phase space and accounts for the algebraic lower bound in (8). The asymptotic equivalent mentioned above in the collisionless case $\sigma = 0$ is a consequence of a more refined analysis based on continued fractions given in [7].

When $\sigma > 0$, particles whose distribution function solves the linear Boltzmann equation in (Ξ_ε) travel on trajectories whose direction is discontinuous in time. More specifically, time discontinuities are distributed under an exponential law of parameter σ. Obviously, this circumstance destroys the channel structure that is responsible for the algebraic decay of the total mass of the particle system in the collisionless case so that one expects that the total mass decay is faster than algebraic as $t \to +\infty$. That this decay is indeed exponential whenever $\sigma > 0$ is by no means obvious; see the argument in [3], leading to an upper bound for the total mass. Statement (3) above leads to an asymptotic equivalent of the total mass, thereby refining the conclusions of [3].

In section 4, we give the proof of Theorem 1; the evolution of the total mass in the vanishing ε limit (governing equation and asymptotic behavior as $t \to +\infty$) is discussed in section 5.

4. The homogenized kinetic equation.

Our argument for the proof of Theorem 1 is split into several steps.

4.1. A new formulation of the transport equation.

Perhaps the most surprising feature in Theorem 1 is the introduction of the extended phase space involving the additional variable s.

As a matter of fact, this additional variable s can be used already at the level of the original linear Boltzmann equation, i.e., in the formulation of the problem (Ξ_ε).

Let us indeed return to the initial boundary value problem (Ξ_ε) for the linear Boltzmann equation.

As recalled above, the last two authors have obtained the homogenized equation corresponding to (Ξ_ε) in the noncollisional case ($\sigma = 0$) by explicitly computing the solution of the linear Boltzmann equation for each $0 < \varepsilon \ll 1$. In the collisionnal case ($\sigma > 0$), as recalled above, there is no such explicit formula giving the solution of the linear Boltzmann equation except the semiexplicit formula involving the transport process defined in [26].

However, not all the information in that semiexplicit formula is needed for the proof of Theorem 1. The additional variable s is precisely the exact amount of information contained in that semiexplicit formula needed in the description of the homogenized process in the limit as $\varepsilon \to 0^+$.

Consider therefore the initial boundary value problem

$$(\Sigma_\varepsilon)\begin{cases} \partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon + \partial_s F_\varepsilon + \sigma F_\varepsilon = 0, & t,s>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1,\\ F_\varepsilon(t,s,x,v) = 0 \text{ if } v\cdot n_x > 0, & t,s>0, (x,v)\in(\partial Z_\varepsilon\times\mathbb{S}^1),\\ F_\varepsilon(t,0,x,v) = \sigma\displaystyle\int_0^\infty KF_\varepsilon(t,s,x,v)ds, & t>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1,\\ F_\varepsilon(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v), & s>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1 \end{cases}$$

with unknown $F_\varepsilon := F_\varepsilon(t,s,x,v)$.

The relation between these two initial boundary value problems, (Ξ_ε) and (Σ_ε), is explained by the following proposition.

PROPOSITION 2. *Assume that f^{in} satisfies the assumption* (11). *Then*

(a) *for each $\varepsilon > 0$, the problem (Σ_ε) has a unique mild solution such that*

$$(t,x,v) \mapsto \int_0^{+\infty} |F_\varepsilon(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T] \times Z_\varepsilon \times \mathbb{S}^1)$$

for each $T > 0$;

(b) *moreover,*

$$0 \le F_\varepsilon(t,s,x,v) \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}\sigma e^{-\sigma s}$$

a.e. in $t, s \ge 0$, $x \in Z_\varepsilon$ and $v \in \mathbb{S}^1$, and

$$\int_0^{+\infty} F_\varepsilon(t,s,x,v)ds = f_\varepsilon(t,x,v)$$

for a.e. $t \ge 0, x \in Z_\varepsilon$, and $v \in \mathbb{S}^1$, where f_ε is the solution of (Ξ_ε).

Proof. Applying the method of characteristics, we see that, should a mild solution F_ε of the problem (Σ_ε) exist, it must satisfy

$$F_\varepsilon(t,s,x,v) = F_{1,\varepsilon}(t,s,x,v) + F_{2,\varepsilon}(t,s,x,v) \tag{14}$$

with

$$\begin{aligned} F_{1,\varepsilon}(t,s,x,v) &= \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{s<t}e^{-\sigma s}F_\varepsilon(t-s,0,x-vs,v) \\ &= \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{s<t}\sigma e^{-\sigma s}\int_0^{+\infty} KF_\varepsilon(t-s,\tau,x-sv,v)d\tau \end{aligned} \tag{15}$$

and

$$\begin{aligned} F_{2,\varepsilon}(t,s,x,v) &= \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{t<s}e^{-\sigma t}F_\varepsilon(0,s-t,x-vt,v) \\ &= \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{t<s}\sigma e^{-\sigma s}f^{in}(x-tv,v) \end{aligned} \tag{16}$$

a.e. in $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$.

First define $\mathcal{X}_T$ to be, for each $T > 0$, the set of measurable functions G defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ such that

$$(t,x,v) \mapsto \int_0^{+\infty} |G(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T] \times Z_\varepsilon \times \mathbb{S}^1)\,,$$

which is a Banach space for the norm

$$\|G\|_{\mathcal{X}_T} = \left\| \int_0^{+\infty} |G(\cdot,s,\cdot,\cdot)|ds \right\|_{L^\infty([0,T]\times Z_\varepsilon\times\mathbb{S}^1)}.$$

Next, for each $G \in \mathcal{X}_T$, we define

$$\mathcal{T}G(t,s,x,v) := \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{s<t}\sigma e^{-\sigma s}\int_0^{+\infty} KG(t-s,\tau,x-sv,v)d\tau\,.$$

Obviously

$$\begin{aligned}\left\|\int_0^{+\infty}|\mathcal{T}^nG(t,s,\cdot,\cdot)|ds\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}\\ \le\sigma\int_0^t\left\|\int_0^{+\infty}|\mathcal{T}^{n-1}G(t_1,\tau,\cdot,\cdot)|d\tau\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}dt_1\\ \le\sigma^n\int_0^t\dots\int_0^{t_{n-1}}\left\|\int_0^{+\infty}|G(t_n,s,\cdot,\cdot)|ds\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}dt_n\dots dt_1\end{aligned}$$

so that

$$\|\mathcal{T}^nG\|_{\mathcal{X}_T}\le\frac{(\sigma T)^n}{n!}\|G\|_{\mathcal{X}_T}\,.$$

Now $F_{1,\varepsilon}=\mathcal{T}F_\varepsilon$ so that (14) can be recast as

$$F_\varepsilon=F_{2,\varepsilon}+\mathcal{T}F_\varepsilon\,.$$

This integral equation has a solution $F_\varepsilon\in\mathcal{X}_T$ for each $T>0$, given by the series

$$F_\varepsilon=\sum_{n\ge0}\mathcal{T}^nF_{2,\varepsilon}$$

which is normally convergent in the Banach space $\mathcal{X}_T$ since

$$\sum_{n\ge0}\|\mathcal{T}^nF_{2,\varepsilon}\|_{\mathcal{X}_T}\le\sum_{n\ge0}\frac{(\sigma T)^n}{n!}\|F_{2,\varepsilon}\|_{\mathcal{X}_T}<+\infty\,.$$

Assuming that the integral equation above has another solution $F'_\varepsilon\in\mathcal{X}_T$ would imply that

$$F_\varepsilon-F'_\varepsilon=\mathcal{T}(F_\varepsilon-F'_\varepsilon)=\dots=\mathcal{T}^n(F_\varepsilon-F'_\varepsilon)$$

so that

$$\|F_\varepsilon-F'_\varepsilon\|_{\mathcal{X}_T}=\|\mathcal{T}^n(F_\varepsilon-F'_\varepsilon)\|_{\mathcal{X}_T}\le\frac{(\sigma T)^n}{n!}\|F_\varepsilon-F'_\varepsilon\|_{\mathcal{X}_T}\to0$$

as $n\to+\infty$; hence $F'_\varepsilon=F_\varepsilon$. Thus we have proved statement (a).

As for statement (b), observe that $\mathcal{T}G\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ if $G\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$. Hence, if $f^{in}\in L^\infty(\mathbb{R}^2\times\mathbb{S}^1)$ satisfies $f^{in}\ge0$ a.e. on $\mathbb{R}^2\times\mathbb{S}^1$, one has $F_{2,\varepsilon}\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ so that $\mathcal{T}^nF_{2,\varepsilon}\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ and the series defining F_ε is a.e. nonnegative on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$.

Next, integrating both sides of (14) with respect to s and setting

$$g_\varepsilon(t,x,v):=\int_0^{+\infty}F_\varepsilon(t,s,x,v)ds\,,$$

we arrive at

$$\begin{aligned} g_\varepsilon(t,x,v) &= \int_0^{+\infty} F_{2,\varepsilon}(t,s,x,v)ds + \int_0^{+\infty} F_{1,\varepsilon}(t,s,x,v)ds \\ &= \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} f^{in}(x-tv,v) \int_0^{+\infty} \mathbf{1}_{t<s}\sigma e^{-\sigma s} ds \\ &\quad + \int_0^{+\infty} \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbf{1}_{s<t}\sigma e^{-\sigma s} \left(\int_0^{+\infty} KF_\varepsilon(t-s,\tau,x-sv,v)d\tau \right) ds \\ &= \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} f^{in}(x-tv,v) e^{-\sigma t} \\ &\quad + \int_0^t e^{-\sigma s} \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \sigma K g_\varepsilon(t-s,x-sv,v) ds \end{aligned}$$

in which we recognize the Duhamel formula giving the unique mild solution f_ε of (Ξ_ε). Hence

$$f_\varepsilon(t,x,v) = \int_0^{+\infty} F_\varepsilon(t,s,x,v)ds \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1 .$$

Finally, since (Ξ_ε) satisfies the maximum principle, one has

$$f_\varepsilon(t,x,v) \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1 .$$

Going back to (14), we recast it in the form

$$\begin{aligned} F_\varepsilon(t,s,x,v) &= \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbf{1}_{s<t}\sigma e^{-\sigma s} K f_\varepsilon(t-s,x-sv,v) \\ &\quad + \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbf{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v) \\ &\le \mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbf{1}_{s<t}\sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \\ &\quad + \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbf{1}_{t<s}\sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \\ &\le \sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \end{aligned}$$

a.e. in $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$, which concludes the proof. ◻

Observe that if

$$F_\varepsilon(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v)$$

is replaced with

$$F_\varepsilon(0,s,x,v) = \Pi(s) f^{in}(x,v),$$

where Π is any probability density on $\mathbb{R}_+$ vanishing at ∞, the conclusion of the proposition above remains valid. In other words, the dependence of the solution F_ε of the problem (Σ) upon the choice of the initial probability density Π disappears after integration in s so that the particle distribution function f_ε is indeed independent of the choice of Π.

The choice $\Pi(s) = \sigma e^{-\sigma s}$ corresponds with the situation where the gas molecules have been evolving under the linear Boltzmann equation for $t < 0$ and the holes are suddenly opened at $t = 0$.

Before giving the proof of Theorem 1, we need to establish a few technical lemmas.

4.2. The distribution of free path lengths. A straightforward consequence of the limit in (6) is the following lemma, which accounts eventually for the coefficient $\dot{p}(t \wedge s)/p(t \wedge s)$ in the limiting equation (Σ).

LEMMA 1. *Let τ_ε be the free path length defined in* (5)*. Then for each $t > 0$,*

$$\{\mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\} \rightharpoonup p(t)$$

in $L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$.
(See the definition before Theorem 1 for the notation $\{\mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\}$.)

Proof. Since the linear span of functions $\phi \equiv \phi(x,v)$ of the form

$$\phi(x,v) = \chi(x)\mathbf{1}_I(v)\,, \quad \chi \in C_0^\infty(\mathbb{R}^2), \text{ and } I \text{ is an arc of } \mathbb{S}^1$$

is dense in $L^1(\mathbb{R}^2 \times \mathbb{S}^1)$, and the family $\mathbf{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t}$ is bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, it is enough to prove that

$$\iint_{Z_\varepsilon\times\mathbb{S}^1} \phi(x,v)\mathbf{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t}dxdv \to p(t)\iint_{\mathbb{R}^2\times\mathbb{S}^1} \phi(x,v)dxdv \text{ as } \varepsilon \to 0\,.$$

Write

$$\begin{aligned}\iint_{Z_\varepsilon\times\mathbb{S}^1} \phi(x,v)\mathbf{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t}dxdv &= \int_{Z_\varepsilon} \chi(x)\left(\int_I \mathbf{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t}dv\right)dx \\ &= \int_{Z_\varepsilon} \chi(x)T_\varepsilon\left(\frac{x}{\varepsilon}\right)dx\end{aligned}$$

with

$$T_\varepsilon(y) := \int_I \mathbf{1}_{\varepsilon\tau_\varepsilon(y,v)>t}dv\,.$$

Obviously T_ε is 1-periodic in y_1 and y_2 and satisfies $0 \le T_\varepsilon \le |I|$. Hence

$$\mathbf{1}_{d(y,\mathbb{Z}^2)>\varepsilon}T_\varepsilon(y) = \sum_{k\in\mathbb{Z}^2} \hat{T}_\varepsilon(k)e^{2i\pi k\cdot y}$$

in $L^2(\mathbb{R}^2/\mathbb{Z}^2)$ with

$$\hat{T}_\varepsilon(k) := \int_{\substack{\max(|z_1|,|z_2|)<1/2\\|z|>\varepsilon}} T_\varepsilon(z)^{-2i\pi k\cdot z}dz$$

for each $k \in \mathbb{Z}^2$.

Then, by Parseval's identity,

$$\begin{aligned}\int_{Z_\varepsilon} \chi(x)T_\varepsilon\left(\frac{x}{\varepsilon}\right)dx &= \int_{\mathbb{R}^2}\chi(x)\left(\sum_{k\in\mathbb{Z}^2}\hat{T}_\varepsilon(k)e^{2i\pi\frac{k\cdot x}{\varepsilon}}\right)dx \\ &= \hat{\chi}(0)\hat{T}_\varepsilon(0) + \sum_{k\in\mathbb{Z}^2\setminus(0,0)}\hat{T}_\varepsilon(k)\hat{\chi}\left(\frac{-2\pi k}{\varepsilon}\right)\end{aligned}$$

with

$$\hat{\chi}(\xi) := \int_{\mathbb{R}^2}\chi(x)e^{-i\xi\cdot x}dx\,.$$

Applying again Parseval's identity,

$$\sum_{k\in\mathbb{Z}^2} |\hat{T}_\varepsilon(k)|^2 = \int_{\substack{\max(|y_1|,|y_2|)<1/2\\ |y|>\varepsilon}} |T_\varepsilon(y)|^2 dy \le |I|$$

while

$$|\hat{\chi}(\xi)| \le \frac{1}{|\xi|^2}\|\nabla^2\chi\|_{L^\infty}$$

so that

$$\left|\hat{\chi}\left(\frac{-2\pi k}{\varepsilon}\right)\right| \le \frac{\varepsilon^2}{4\pi^2|\xi|^2}\|\nabla^2\chi\|_{L^\infty}\,.$$

Hence, by the Cauchy–Schwarz inequality,

$$\left|\sum_{k\in\mathbb{Z}^2\setminus(0,0)} \hat{T}_\varepsilon(k)\hat{\chi}(-2\pi k/\varepsilon)\right|^2 \le \sum_{k\in\mathbb{Z}^2\setminus(0,0)} |\hat{T}_\varepsilon(k)|^2 \sum_{k\in\mathbb{Z}^2\setminus(0,0)} \frac{\varepsilon^4\|\nabla^2\chi\|^2_{L^\infty}}{16\pi^4|k|^4} = O(\varepsilon^4),$$

and therefore

$$\int_{Z_\varepsilon} \chi(x)T_\varepsilon\left(\frac{x}{\varepsilon}\right)dx = \hat{\chi}(0)\hat{T}_\varepsilon(0) + O(\varepsilon^2)$$

as $\varepsilon \to 0^+$.

By (6)

$$\hat{T}_\varepsilon(0) = \int_{\substack{\max(|y_1|,|y_2|)<1/2\\ |y|>\varepsilon}} T_\varepsilon(y)dy \to p(t)|I| \quad \text{as } \varepsilon \to 0^+$$

so that

$$\hat{\chi}(0)\hat{T}_\varepsilon(0) \to p(t)|I|\int_{\mathbb{R}^2}\chi(x)dx = p(t)\iint_{\mathbb{R}^2\times\mathbb{S}^1}\phi(x,v)dxdv$$

as $\varepsilon \to 0^+$, and hence

$$\int_{Z_\varepsilon} \chi(x)T_\varepsilon\left(\frac{x}{\varepsilon}\right)dx = p(t)\iint_{\mathbb{R}^2\times\mathbb{S}^1}\phi(x,v)dxdv + o(1) + O(\varepsilon^2)$$

which entails the announced result. □

4.3. Extending f_ε by 0 in the holes. We begin with the equation satisfied by the (extension by 0 inside the holes of the) distribution function $\{f_\varepsilon\}$.

LEMMA 2. *For each $\varepsilon > 0$, the function $\{f_\varepsilon\}$ satisfies*

$$(\partial_t + v\cdot\nabla_x)\{f_\varepsilon\} + \sigma(\{f_\varepsilon\} - K\{f_\varepsilon\}) = (v\cdot n_x)f_\varepsilon\big|_{\partial Z_\varepsilon\times\mathbb{S}^1}\delta_{\partial Z_\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{S}^1)$, where $\delta_{\partial Z_\varepsilon}$ is the surface measure concentrated on the boundary of Z_ε and n_x is the unit normal vector at $x\in\partial Z_\varepsilon$ pointing toward the interior of Z_ε.

Proof. One has

$$\partial_t\{f_\varepsilon\} = \{\partial_t f_\varepsilon\}$$

and

$$\nabla_x \{f_\varepsilon\} = \{\nabla_x f_\varepsilon\} + f_\varepsilon \,|_{\partial Z_\varepsilon \times \mathbb{S}^1}\, \delta_{\partial Z_\varepsilon} n_x$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{S}^1)$. Hence

$$\begin{aligned} 0 &= \{\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon)\} \\ &= \partial_t \{f_\varepsilon\} + v \cdot \nabla_x \{f_\varepsilon\} + (v \cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon \times \mathbb{S}^1} \delta_{\partial Z_\varepsilon} + \sigma(\{f_\varepsilon\} - K\{f_\varepsilon\}) \end{aligned}$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{S}^1)$. ∎

A straightforward consequence of the scaling considered here is that the family of Radon measures

$$(v \cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon \times \mathbb{S}^1} \delta_{\partial Z_\varepsilon}$$

is controlled uniformly as $\varepsilon \to 0^+$ in the following manner.

LEMMA 3. *For each $R > 0$, the family of Radon measures*

$$(v \cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon \times \mathbb{S}^1} \delta_{\partial Z_\varepsilon}\big|_{[-R,R]^2 \times \mathbb{S}^1}$$

is bounded in[1] $\mathcal{M}([-R,R]^2 \times \mathbb{S}^1)$.

Proof. The total mass of the measure

$$(v \cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon \times \mathbb{S}^1} \delta_{\partial Z_\varepsilon}\big|_{[-R,R]^2 \times \mathbb{S}^1}$$

is less than or equal to

$$2\pi \|f_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1)} \|\delta_{\partial Z_\varepsilon}\,|_{[-R,R]^2}\,\|_{\mathcal{M}([-R,R]^2)}$$

which is itself less than or equal to

$$2\pi \|f^{in}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} \left\|\delta_{\partial Z_\varepsilon}\,|_{[-R,R]^2}\right\|_{\mathcal{M}([-R,R]^2)}.$$

Since $\delta_{\partial Z_\varepsilon}\,|_{[-R,R]^2}$ is the union of $O((\frac{2R}{\varepsilon})^2)$ circles of radius ε^2,

$$\|\delta_{\partial Z_\varepsilon}\,|_{[-R,R]^2}\,\|_{\mathcal{M}([-R,R]^2)} = O\left(\left(\frac{2R}{\varepsilon}\right)^2\right) 2\pi\varepsilon^2 = O(1)R^2$$

as $\varepsilon \to 0^+$, whence, the announced result. ∎

4.4. The velocity averaging lemmas. As is the case of all homogenization results, the proof of Theorem 1 is based on the strong L^1_{loc} convergence of certain quantities defined in terms of F_ε. In the case of kinetic models, strong L^1_{loc} compactness is usually obtained by velocity averaging; see, for instance, [1, 21, 20] for the first results in this direction. Below we recall a classical result in velocity averaging that is a special case of theorem 1.8 in [5].

PROPOSITION 3. *Let $p > 1$, and assume that $f_\varepsilon \equiv f_\varepsilon(t,x,v)$ is a bounded family in $L^p_{loc}(\mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{S}_v^{d-1})$ such that*

$$\sup_\varepsilon \int_0^T \iint_{B(0,R)\times\mathbb{S}^{d-1}} |\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon| dx dv dt < +\infty$$

[1] For each compact subset K of $\mathbb{R}^N$, we denote by $\mathcal{M}(K)$ the space of signed Radon measures on K, i.e., the set of all real-valued continuous linear functionals on $C(K)$ endowed with the topology of uniform convergence on K.

for each $T > 0$ and $R > 0$. Then, for each $\psi \in C(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$, the family $\rho_\psi[f_\varepsilon]$ defined by

$$\rho_\psi[f_\varepsilon](t,x,v) = \int_{\mathbb{S}^{d-1}} f_\varepsilon(t,x,v)\psi(v,w)dw$$

is relatively compact in $L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^d_x \times \mathbb{S}^{d-1}_v)$.

A straightforward consequence of Proposition 3 is the following compactness result in L^1_{loc} strong, which is the key argument in the proof of Theorem 1.

LEMMA 4. *Let $f_\varepsilon \equiv f_\varepsilon(t,x,v)$ be the family of solutions of the initial boundary value problem (Ξ_ε). Then the families*

$$K\{f_\varepsilon\} = \{Kf_\varepsilon\}$$

and

$$\int_{\mathbb{S}^1} \{f_\varepsilon\} dv$$

are relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ strong.

Proof. We recall that, by the maximum principle for (Ξ_ε),

$$|f_\varepsilon(t,x,v)| \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}$$

a.e. in $t \ge 0, x \in Z_\varepsilon$ and $v \in \mathbb{S}^1$, so that

$$\sup_\varepsilon \| \{f_\varepsilon\} \|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}. \tag{17}$$

By Lemma 2, $\{f_\varepsilon\}$ satisfies

$$\partial_t \{f_\varepsilon\} + v \cdot \nabla_x \{f_\varepsilon\} = \sigma(K\{f_\varepsilon\} - \{f_\varepsilon\}) - \delta_{\partial Z_\varepsilon}(v.n_x) f_\varepsilon\,|_{\partial Z_\varepsilon\times\mathbb{S}^1}$$

in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$. Because of (17) and the fact that the scattering kernel k is a.e. nonnegative (see (3)), one has

$$\begin{aligned}\|\sigma(K\{f_\varepsilon\} - \{f_\varepsilon\})\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} &\le \sigma(1 + \|K1\|_{L^\infty(\mathbb{S}^1)})\| \{f_\varepsilon\} \|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \\ &= 2\sigma\| \{f_\varepsilon\} \|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)}\end{aligned}$$

since $K1 = 1$ (see again (3)). Besides, the family of Radon measures

$$\mu_\varepsilon = f_\varepsilon\,|_{\partial Z_\varepsilon\times\mathbb{S}^1}\,(v \cdot n_x)\delta_{\partial Z_\varepsilon}$$

satisfies

$$\sup_\varepsilon \int_{[0,T]\times\overline{B(0,R)}\times\mathbb{S}^1} |\mu_\varepsilon| < +\infty$$

for each $T > 0$ and $R > 0$ according to Lemma 3.

Applying the velocity averaging result recalled above implies that the family

$$\int_{\mathbb{S}^1} g_\varepsilon dv$$

is relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$.

By density of $C(\mathbb{S}^1 \times \mathbb{S}^1)$ in $L^2(\mathbb{S}^1 \times \mathbb{S}^1)$, replacing the integral kernel k with a continuous approximant and applying the velocity averaging Proposition 3 in the same way as above, we conclude that the family Kg_ε is also relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$. ∎

4.5. Uniqueness for the homogenized equation. Consider the Cauchy problem with unknown $G \equiv G(t,s,x,v)$

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G + \dfrac{\dot p(t\wedge s)}{p(t\wedge s)}G, & t,s>0, x\in\mathbb{R}^2, v\in\mathbb{S}^1,\\ G(t,0,x,v) = S(t,x,v), & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1,\\ G(0,s,x,v) = G^{in}(s,x,v), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1. \end{cases}$$

If, for a.e. $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$, the function $\tau\mapsto G(t+\tau,s+\tau,x+\tau v,v)$ is C^1 in $\tau>0$, then, since the function $p\in C^1(\mathbb{R}_+)$ and $p>0$ on $\mathbb{R}_+$, one has

$$\begin{aligned}\left(\frac{d}{d\tau}+\sigma-\frac{\dot p(t\wedge s+\tau)}{p(t\wedge s+\tau)}\right)&G(t+\tau,s+\tau,x+\tau v,v)\\ &= e^{-\sigma\tau}p(t\wedge s+\tau)\frac{d}{d\tau}\left(\frac{e^{\sigma\tau}G(t+\tau,s+\tau,x+\tau v,v)}{p(t\wedge s+\tau)}\right)=0\,.\end{aligned}$$

Hence

$$\Gamma:\,\tau\mapsto\frac{e^{\sigma\tau}G(t+\tau,s+\tau,x+\tau v,v)}{p(t\wedge s+\tau)}$$

is a constant. Therefore

$$\Gamma(0)=\begin{cases}\Gamma(-t) & \text{if } t<s,\\ \Gamma(-s) & \text{if } s<t\end{cases}$$

so that

$$G(t,s,x,v)=\mathbf{1}_{t<s}e^{-\sigma t}p(t)G^{in}(s-t,x-tv,v)+\mathbf{1}_{s<t}e^{-\sigma s}p(s)S(t-s,x-sv,v)\,.$$

PROPOSITION 4. *Assume that $f^{in}\in L^\infty(\mathbb{R}^2\times\mathbb{S}^1)$. Then the problem (Σ) has a unique mild solution F such that*

$$(t,x,v)\mapsto\int_0^{+\infty}|F(t,s,x,v)|ds \textit{ belongs to } L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{S}^1)$$

for each $T>0$. This solution satisfies

$$\begin{aligned}F(t,s,x,v)=\mathbf{1}_{t<s}&\sigma e^{-\sigma t}p(t)f^{in}(x-tv,v)\\ &+\mathbf{1}_{s<t}\sigma e^{-\sigma s}p(s)\int_0^{+\infty}KF(t-s,\tau,x-sv,v)d\tau\end{aligned}$$

for a.e. $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$.

Besides, $F\geq 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$ if $f^{in}\geq 0$ a.e. on $\mathbb{R}^2\times\mathbb{S}^1$.

Proof. That a mild solution of the problem (Σ), should it exist, satisfies the integral equation above follows from the computation presented before the proposition.

As above, let $\mathcal{Y}_T$ be, for each $T>0$, the set of measurable functions G defined a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$ and such that

$$(t,x,v)\mapsto\int_0^{+\infty}|G(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{S}^1)\,,$$

which is a Banach space for the norm

$$\|G\|_{\mathcal{Y}_T} = \left\| \int_0^{+\infty} |G(\cdot, s, \cdot, \cdot)| ds \right\|_{L^\infty([0,T]\times Z_\epsilon \times \mathbb{S}^1)} .$$

Next, for each $G \in \mathcal{Y}_T$, we define

$$\mathcal{Q}G(t, s, x, v) := \mathbf{1}_{s<t} \sigma e^{-\sigma s} p(s) \int_0^{+\infty} KG(t-s, \tau, x - sv, v) d\tau .$$

Since $0 < e^{-\sigma s} p(s) \leq 1$, the integral kernel $k \geq 0$ on $\mathbb{S}^1 \times \mathbb{S}^1$ and $K1 = 1$ by (3), one has

$$\int_0^{+\infty} |\mathcal{Q}G(t, s, x, v)| ds \leq \sigma \int_0^t \left\| \int_0^{+\infty} |G(t-s, \tau, \cdot, \cdot)| d\tau \right\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} ds$$

a.e. in $(t, x, v) \in [0, T] \times \mathbb{R}^2 \times \mathbb{S}^1$, meaning that

$$\begin{aligned} &\left\| \int_0^{+\infty} |\mathcal{Q}^n G(t, s, \cdot, \cdot)| ds \right\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} \\ &\quad \leq \sigma \int_0^t \left\| \int_0^{+\infty} |\mathcal{Q}^{n-1} G(t_1, s, \cdot, \cdot)| ds \right\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} dt_1 \\ &\quad \leq \sigma^n \int_0^t \dots \int_0^{t_{n-1}} \left\| \int_0^{+\infty} |G(t_n, s, \cdot, \cdot)| ds \right\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} dt_n \dots dt_1 . \end{aligned}$$

In particular,

$$\|\mathcal{Q}^n G\|_{\mathcal{Y}_T} \leq \frac{(\sigma T)^n}{n!} \|G\|_{\mathcal{Y}_T} .$$

The integral equation in the statement of the proposition is

$$F = F_2 + \mathcal{Q}F,$$

where

$$F_2(t, s, x, v) = \mathbf{1}_{t<s} \sigma e^{-\sigma t} p(t) f^{in}(x - tv, v) .$$

Therefore, arguing as in the proof of Proposition 2, one obtains a mild solution of (Σ) as the sum of the series

$$F = \sum_{n \geq 0} \mathcal{Q}^n F_2 ,$$

which is normally convergent in the Banach space $\mathcal{Y}_T$ for each $T > 0$.

Should there exist another mild solution, say, F', it would satisfy

$$(F - F') = \mathcal{Q}(F - F') = \dots = \mathcal{Q}^n(F - F')$$

for all $n \geq 0$ so that

$$\|F - F'\|_{\mathcal{Y}_T} = \|\mathcal{Q}^n(F - F')\|_{\mathcal{Y}_T} \leq \frac{(\sigma T)^n}{n!} \|F - F'\|_{\mathcal{Y}_T} \to 0$$

as $n \to +\infty$, which implies that $F = F'$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$.

Finally, $\mathcal{Q}F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ if $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$. Since F is given by the series above, one has $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ whenever $f^{in} \geq 0$ a.e. on $\mathbb{R}^2 \times \mathbb{S}^1$. ∎

4.6. Proof of the homogenization theorem. Start from the decomposition (14) of F_ε. Passing to the limit as $\varepsilon \to 0^+$ in the term $F_{2,\varepsilon}$ is easy. Indeed, by Lemma 1,

$$\{\mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\} \rightharpoonup p(t) \tag{18}$$

in $L^\infty(\mathbb{R}^2_x \times \mathbb{S}^1_v)$ weak-$*$ for each $t > 0$ as $\varepsilon \to 0^+$. Hence

$$\begin{aligned}\{F_{2,\varepsilon}\}(t,s,x,v) =&\mathbf{1}_{t<s}e^{-\sigma s}f^{in}(x-tv,v)\{\mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\}\\ &\rightharpoonup \mathbf{1}_{t<s}e^{-\sigma s}f^{in}(x-tv,v)p(t) =: F_2(t,s,x,v)\end{aligned} \tag{19}$$

in $L^\infty(\mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}^2_x \times \mathbb{S}^1_v)$ weak-$*$ as $\varepsilon \to 0^+$.

Next we analyze the term $F_{1,\varepsilon}$; this is obviously more difficult as this term depends on the (unknown) solution F_ε itself.

We recall the uniform bound

$$\sup_\varepsilon \| \{f_\varepsilon\} \|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}$$

(see Proposition 2(b)) so that by the Banach-Alaoglu theorem

$$\{f_\varepsilon\} \rightharpoonup f \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ weak-}* \tag{20}$$

for some $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$, possibly after extracting a subsequence of $\varepsilon \to 0^+$.

Thus, applying the strong compactness Lemma 4 shows that

$$K\{f_\varepsilon\} \to Kf \text{ in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ strong}$$

as $\varepsilon \to 0^+$.

This and the weak-$*$ convergence in Lemma 1 imply that

$$\begin{aligned}\{F_{1,\varepsilon}\} =&\mathbf{1}_{s<t}\sigma e^{-\sigma s}K\{f_\varepsilon\}(t-s,x-sv,v)\mathbf{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\\ &\rightharpoonup \mathbf{1}_{s<t}\sigma e^{-\sigma s}Kf(t-s,x-sv,v)p(s)\end{aligned} \tag{21}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$. Therefore

$$\begin{aligned}\{F_\varepsilon\}(t,s,x,v) \rightharpoonup&\mathbf{1}_{s<t}\sigma e^{-\sigma s}Kf(t-s,x-sv,v)p(s) + F_2(t,s,x,v)\\ &=: \tilde{F}(t,s,x,v)\end{aligned} \tag{22}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$.

Fix $T > 0$; then, for $t \in [0, T]$, one has

$$\int_0^\infty F_\varepsilon(t,s,x,v)ds = \int_0^T F_{1,\varepsilon}(t,s,x,v)ds + e^{-\sigma t} f^{in}(x - tv, v)\mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}$$

since $F_{1,\varepsilon}$ is supported in $s \le t \le T$ so that

$$\begin{aligned}\int_0^\infty \{F_\varepsilon\}(t,s,x,v)ds \rightharpoonup \int_0^T &\mathbf{1}_{s\le t} Kf(t-s, x-vs, v)\sigma e^{-\sigma s}p(s)ds \\ &+ f^{in}(x-tv,v)e^{-\sigma t}p(t) \\ = \int_0^\infty &\tilde{F}(t,s,x,v)ds\end{aligned} \tag{23}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weakly as $\varepsilon \to 0^+$, where $\tilde{F}$ is defined in (22).

On the other hand,

$$\int_0^\infty \{F_\varepsilon\}(t,s,x,v)ds = \{f_\varepsilon\}(t,x,v) \rightharpoonup f(t,x,v)$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$ and therefore also in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$. By uniqueness of the limit, we conclude that

$$f(t,x,v) = \int_0^\infty \tilde{F}(t,s,x,v)ds \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1 \tag{24}$$

so that $\tilde{F}$ satisfies

$$\begin{aligned}\tilde{F}(t,s,x,v) = \mathbf{1}_{s<t}\sigma e^{-\sigma s} K &\left(\int_0^\infty \tilde{F}(t-s,u,x-sv,\cdot)du\right)(v)p(s) \\ &+ \mathbf{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)p(t)\end{aligned}$$

a.e. in $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$. By Proposition 4, this means that $\tilde{F}$ is a solution of the Cauchy problem (Σ).

By uniqueness of the solution of (Σ), we conclude that $\tilde{F} = F$ and that the whole family

$$F_\varepsilon \rightharpoonup F \text{ in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$$

weakly as $\varepsilon \to 0^+$.

Finally, (20) and (24) imply that

$$\{f_\varepsilon\} \rightharpoonup f = \int_0^\infty F ds$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$, which concludes the proof of Theorem 1. ∎

5. Asymptotic behavior of the total mass in the long time limit. The formulation of the homogenized equation (problem (Σ)) as an integro-differential equation set on the extended phase space involving the additional variable s is of considerable importance in understanding the asymptotic behavior of the total mass of the particle system as the time variable $t \to +\infty$. Indeed, this formulation implies that the total mass of the particle system satisfies a renewal equation, i.e., a class of integral equations for which a lot is known on the asymptotic behavior of the solutions in the long time limit; see, for instance, in [14] the basic results on renewal type integral equations.

5.1. The renewal PDE governing the mass. We begin with a proof of Proposition 1.

Proof. That μ is a mild solution of the renewal PDE means that, for a.e. $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\mu(t,s) = \mathbf{1}_{t<s}\sigma e^{-\sigma(s-t)}e^{-\sigma t}p(t) + \mathbf{1}_{s<t}e^{-\sigma s}p(s)\int_0^{+\infty}\mu(t-s,\tau)d\tau$$
$$= \sigma e^{-\sigma s}p(t\wedge s)\left(\mathbf{1}_{t<s} + \mathbf{1}_{s<t}\int_0^{+\infty}\mu(t-s,\tau)d\tau\right).$$

For each $T>0$, define

$$\mathcal{R}\mu(t,s) = \mathbf{1}_{s<t}\sigma e^{-\sigma s}p(s)\int_0^{+\infty}\mu(t-s,\tau)d\tau$$

a.e. in $(t,s)\in\mathbb{R}_+\times\mathbb{R}_+$. Obviously, for each $\phi \in L^\infty([0,T];L^1(\mathbb{R}_+))$ and a.e. $t\ge 0$,

$$\|\mathcal{R}\phi(t,\cdot)\|_{L^1(\mathbb{R}_+)} \le \int_0^t \sigma e^{-\sigma(t-s)}p(t-s)\|\phi(s,\cdot)\|_{L^1(\mathbb{R}_+)}ds$$
$$\le \sigma\int_0^t \|\phi(s,\cdot)\|_{L^1(\mathbb{R}_+)}ds$$

so that, for each $n\ge 0$, one has

$$\|\mathcal{R}^n\phi(t,\cdot)\|_{L^1(\mathbb{R}_+)} \le \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}}\|\phi(t_n,\cdot)\|_{L^1(\mathbb{R}_+)}dt_n\dots dt_1$$
$$\le \frac{(\sigma t)^n}{n!}\|\phi\|_{L^\infty([0,T];L^1(\mathbb{R}_+))}$$

a.e. in $t\in\mathbb{R}_+$.

Arguing as in the proof of Proposition 2, we see that the renewal PDE has a unique mild solution $\mu \in L^\infty([0,T];L^1(\mathbb{R}_+))$ for all $T>0$, which is given by the series

$$\mu = \sum_{n\ge 0}\mathcal{R}^n(\mu^{in}),$$

where

$$\mu^{in}(s) := \sigma e^{-\sigma s}.$$

Obviously $\mathcal{R}\phi \ge 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+$ if $\phi\ge 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+$ so that $\mu\ge 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+$. Besides, for each $T>0$,

$$\|\mu\|_{L^\infty([0,T];L^1(\mathbb{R}_+))} \le \sum_{n\ge 0}\frac{(\sigma T)^n}{n!}\|\mu^{in}\|_{L^1(\mathbb{R}_+)} = e^{\sigma T},$$

which implies, in turn, that

$$0 \le \mu(t,s) \le \sigma e^{-\sigma s}p(t\wedge s)\left(\mathbf{1}_{t<s} + \mathbf{1}_{s<t}e^{\sigma T}\right) \le \sigma e^{\sigma T}e^{-\sigma s}$$

a.e. in $(t,s)\in[0,T]\times\mathbb{R}_+$.

Finally, let F be the mild solution of the problem (Σ) obtained in Proposition 2. Since $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ is measurable, one can apply Fubini's theorem to show that

$$\begin{aligned}
m(t,s) :&= \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} F(t,s,x,v)dxdv \\
&= \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(t) \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x-tv,v)dxdv \\
&\quad + \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(s) \int_0^\infty \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} KF(t-s,\tau,x-sv,v)dxdvd\tau \\
&= \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(t) \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv \\
&\quad + \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(s) \int_0^\infty \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} KF(t-s,\tau,y,v)dydvd\tau \\
&= \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(t) \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv \\
&\quad + \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(s) \int_0^\infty \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} F(t-s,\tau,y,w)dydwd\tau \\
&= \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(t) \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x-tv,v)dxdv \\
&\quad + \mathbf{1}_{t<s}\sigma e^{-\sigma t} p(s) \int_0^\infty m(t-s,\tau)d\tau\,,
\end{aligned}$$

where the second equality follows from the substitution $y = x - tv$ that leaves the Lebesgue measure invariant, while the third equality follows from the identity

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} k(v,w)dv = 1\,,$$

which implies that

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} KF(t-s,\tau,y,v)dv = \frac{1}{2\pi} \int_{\mathbb{S}^1} F(t-s,\tau,y,w)dw\,.$$

In other words, $m(t,s)$ satisfies the same integral equation as

$$\frac{\mu(t,s)}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv.$$

Now the solution f_ε of (Ξ_ε) satisfies

$$f_\varepsilon \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon(t,y,v)dydv \leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv\,,$$

which implies by Theorem 1 that

$$\int_{|y|\leq R} \int_{\mathbb{S}^1} f_\varepsilon(t,y,v)dvdy \rightharpoonup \int_0^{+\infty} \int_{|y|\leq R} \int_{\mathbb{S}^1} F(t,s,y,v)dvdyds\,.$$

Hence, by Fatou's lemma,

$$\begin{aligned}
\int_0^{+\infty} \int_{|y|\leq R} \int_{\mathbb{S}^1} F(t,s,y,v)dvdyds &\leq \underline{\lim}_{\varepsilon\to 0^+} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \\
&\leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv
\end{aligned}$$

a.e. in $t \geq 0$.

Letting $R \to +\infty$ in the inequality above, we see that $m \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+))$, and we have proved that the difference

$$\Lambda(t,s) = m(t,s) - \frac{\mu(t,s)}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv$$

satisfies

$$\Lambda \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+)) \quad \text{ and } \quad \Lambda = \mathcal{R}\Lambda\,.$$

By the same uniqueness argument as in the proof of Proposition 4, we conclude that $\Lambda = 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+$. ◻

5.2. The total mass in the vanishing ε limit. By Theorem 1, the solution f_ε of (Ξ_ε) satisfies

$$\{f_\varepsilon\} \rightharpoonup \int_0^{+\infty} F ds \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ weak-}*;$$

therefore, checking that

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1} \{f_\varepsilon\}dxdv \rightharpoonup \int_0^{+\infty} \iint_{\mathbb{R}^2\times\mathbb{S}^1} F dxdvds =: 2\pi M(t)$$

reduces to proving that there is no mass loss at infinity in the x variable.

Lemma 5. *Under the same assumptions as in Theorem* 1*,*

$$\frac{1}{2\pi}\iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv = \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} \{f_\varepsilon\}(t,x,v)dxdv \to M(t)$$

strongly in $L^1_{loc}(\mathbb{R}_+)$ *as* $\varepsilon \to 0^+$.

Proof. Going back to the proof of Proposition 2 (whose notations are kept in the present discussion), we have seen that

$$F_\varepsilon = \sum_{n\geq 0} \mathcal{T}^n F_{2,\varepsilon} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$$

with the notation

$$F_{2,\varepsilon}(t,s,x,v) = \mathbf{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbf{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Since $\mathcal{T}\Phi \geq 0$ a.e. whenever $\Phi \geq 0$ a.e., the formula above implies that

$$F_\varepsilon \leq G := \sum_{n\geq 0}\mathcal{T}^n G_2 \text{ a.e. in } (t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1\,,$$

where

$$G_2(t,s,x,v) := \mathbf{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Thus, G satisfies the integral equation

$$G = G_2 + \mathcal{T}G,$$

meaning that G is the mild solution of

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G\,, & t,s>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,,\\ G(t,0,x,v) = \sigma\displaystyle\int_0^{+\infty} KG(t,s,x,v)ds\,, & t>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,,\\ G(0,s,x,v) = f^{in}(x,v)\sigma e^{-\sigma s}\,, & s>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1. \end{cases}$$

Reasoning as in Proposition 2 shows that

$$g(t,x,v) := \int_0^{+\infty} G(t,s,x,v)ds$$

is the solution of the linear Boltzmann equation

$$\begin{cases} (\partial_t + v\cdot\nabla_x)g + \sigma(g-Kg) = 0\,, & t>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,,\\ g(0,x,v) = f^{in}(x,v)\,, & x\in\mathbb{R}^2\,,\ |v|=1\,. \end{cases}$$

In view of the assumption (11) bearing on f^{in}, we know that

$$G \geq 0 \text{ a.e. on } \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$$

and

$$\begin{aligned}\int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{S}^1} G(t,s,x,v)dxdvds &= \iint_{\mathbb{R}^2\times\mathbb{S}^1} g(t,x,v)dxdv\\ &= \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\end{aligned}$$

for each $t\geq 0$.

Summarizing, we have

$$0 \leq \{F_\varepsilon\} \leq G$$

and

$$\iiint_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1} G(t,s,x,v)dsdxdv = \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv < +\infty\,.$$

Then we conclude as follows: for each $R>0$, one has

$$\begin{aligned}&\iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv - \int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{S}^1} F(t,s,x,v)dxdvds\\ &\quad= \int_0^{+\infty}\int_{|x|>R}\int_{\mathbb{S}^1}\{F_\varepsilon\}(t,s,x,v)dvdxds\\ &\qquad+ \int_0^{+\infty}\int_{|x|\leq R}\int_{\mathbb{S}^1}\left(\{F_\varepsilon\}-F\right)(t,s,x,v)dvdxds\\ &\qquad- \int_0^{+\infty}\int_{|x|>R}\int_{\mathbb{S}^1}\{F\}(t,s,x,v)dvdxds = I_{R,\varepsilon}(t) + II_{R,\varepsilon}(t) + III_R(t)\,.\end{aligned}$$

First, for a.e. $t > 0$, the term $I_{R,\varepsilon}(t) \to 0$ as $R \to +\infty$ uniformly in $\varepsilon > 0$ since $0 \leq \{F_\varepsilon\} \leq G$ and $G \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1))$.

Next, the term $II_{R,\varepsilon}(t) \to 0$ strongly in $L^1_{loc}(\mathbb{R}_+)$ as $\varepsilon \to 0^+$ for each $R > 0$ by Lemma 4.

Finally, since $\{F_\varepsilon\} \rightharpoonup F$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$, one has $0 \leq \{F\} \leq G$ so that $F \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1))$. Hence the term $III_R(t) \to 0$ as $R \to +\infty$ for a.e. $t \geq 0$.

Thus we have proved that

$$\iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \to \int_0^{+\infty} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v)dxdvds$$

in $L^1_{loc}(\mathbb{R}_+)$ and therefore for a.e. $t \geq 0$, possibly after extraction of a subsequence of $\varepsilon \to 0^+$. ∎

5.3. An integral equation for *M*. Given a function ψ defined (a.e.) on the half-line $\mathbb{R}_+$, we abuse the notation $\psi \mathbf{1}_{\mathbb{R}_+}$ to designate its extension by 0 on $\mathbb{R}^*_-$.

Henceforth we also denote

$$\kappa(t) := p(t)\sigma e^{-\sigma t}\mathbf{1}_{t \geq 0}.$$

LEMMA 6. *The function M defined in* (13) *satisfies the integral equation*

$$M(t) = \kappa * (M\mathbf{1}_{\mathbb{R}_+})(t) + \frac{1}{2\pi\sigma}\kappa(t)\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv, \ t \geq 0,$$

where $*$ denotes the convolution on the real line.

Proof. We apply the same method as for deriving the explicit representation formula for F starting from the equation in Corollary 1 in order to find an exact formula for m. Indeed, by the method of characteristics,

$$\begin{aligned} m(t,s) &= \mathbf{1}_{s<t}p(s)e^{-\sigma s}m(t-s,0) + \mathbf{1}_{t<s}p(t)e^{-\sigma t}m(0,s-t) \\ &= \mathbf{1}_{s<t}p(s)\sigma e^{-\sigma s}\int_0^\infty m(t-s,u)du \\ &\quad + \mathbf{1}_{t<s}p(t)\sigma e^{-\sigma s}\frac{1}{2\pi}\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv\,. \end{aligned}$$

The function m therefore satisfies

$$\begin{aligned} m(t,s) &= \mathbf{1}_{s<t}p(s)\sigma e^{-\sigma s}M(t-s) \\ &\quad + \mathbf{1}_{t<s}p(t)\sigma e^{-\sigma s}\frac{1}{2\pi}\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv\,. \end{aligned} \tag{25}$$

We next integrate both sides of (25) in $s \in \mathbb{R}_+$. By the definition (13) of M, we obtain

$$M(t) = \int_0^t \sigma p(s)e^{-\sigma s}M(t-s)ds + p(t)e^{-\sigma t}\frac{1}{2\pi}\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv$$

a.e. in $t \geq 0$, which is precisely the desired integral equation for M:

$$M(t) = \int_0^t \kappa(s)M(t-s)ds + \frac{1}{2\pi\sigma}\kappa(t)\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv\,. \tag{26}$$ ∎

5.4. An explicit representation formula for M. We first establish the following elementary representation formula for M.

LEMMA 7. *Let M be the function defined in* (13). *Then*

$$M = \frac{1}{2\pi\sigma} \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \sum_{n\geq 1} \kappa^{*n}$$

with the notation

$$\kappa^{*n} = \underbrace{\kappa * \cdots * \kappa}_{n \text{ factors}} .$$

Proof. Observe that

$$\begin{aligned}\int_0^{+\infty} \kappa(t)dt &= \sigma \int_0^{+\infty} e^{-\sigma t} p(t)dt \\ &= 1 + \int_0^{+\infty} \dot{p}(t)e^{-\sigma t}dt < 1 \,,\end{aligned} \tag{27}$$

where the second equality results from integrating by parts the integral defining κ and the final inequality is implied by the fact that p is a C^1 decreasing function.

By Lemma 5, $M \in L^1_{loc}(\mathbb{R}_+)$ and $M \geq 0$ a.e. on $\mathbb{R}_+$ since $f_\varepsilon \geq 0$ a.e. on $\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ because $f^{in} \geq 0$ a.e. on $\mathbb{R}^2 \times \mathbb{S}^1$; see the positivity assumption in (11). Applying Fubini's theorem shows that

$$\begin{aligned}\int_0^{+\infty} M(t)dt &= \int_0^{+\infty}\int_0^t \kappa(t-s)M(s)dsdt + \frac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \int_0^{+\infty}\kappa(t)dt \\ &= \int_0^{+\infty} M(s)\left(\int_s^{+\infty}\kappa(t-s)dt\right)ds + \frac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \int_0^{+\infty}\kappa(t)dt.\end{aligned}$$

In other words,

$$\|M\|_{L^1(\mathbb{R}_+)} \leq \|M\|_{L^1(\mathbb{R}_+)}\|\kappa\|_{L^1(\mathbb{R}_+)} + \frac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv$$

so that $M \in L^1(\mathbb{R}_+)$ since $\|\kappa\|_{L^1(\mathbb{R}_+)} < 1$, and

$$\|M\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{2\pi\sigma(1-\|\kappa\|_{L^1(\mathbb{R}_+)})}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \,.$$

In particular, if

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv = 0,$$

then $M = 0$ a.e. on $\mathbb{R}_+$ so that the representation formula to be established obviously holds in this case.

Otherwise,

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv > 0 \,;$$

define then

$$\psi(t) := 2\pi\sigma \left(\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \right)^{-1} M(t),\ t \geq 0\,.$$

According to Lemma 6, the function ψ verifies the integral equation

$$\psi(t) = (\kappa * (\psi \mathbf{1}_{\mathbb{R}_+}))(t) + \kappa(t) \qquad \text{a.e. in } t \geq 0\,. \tag{28}$$

Applying Fubini's theorem as above shows that the linear operator

$$\mathcal{A}:\ L^1(\mathbb{R}_+) \ni f \mapsto \kappa * (f\mathbf{1}_{\mathbb{R}_+}) \in L^1(\mathbb{R}_+)$$

satisfies

$$\|\mathcal{A}f\|_{L^1(\mathbb{R}_+)} \leq \|\mathcal{A}\|\|f\|_{L^1(\mathbb{R}_+)} \qquad \text{with } \|\mathcal{A}\| = \int_0^{+\infty} \kappa(t)dt < 1\,.$$

Therefore $(1-\mathcal{A})$ is invertible in the class of bounded operators on $L^1(\mathbb{R}_+)$ with inverse

$$(1-\mathcal{A})^{-1} = \sum_{n\geq 0} \mathcal{A}^n\,.$$

In particular,

$$\psi = (I-\mathcal{A})^{-1}\kappa = \sum_{n\geq 1} \kappa^{\star n}$$

is the unique solution of the integral equation (28) in $L^1(\mathbb{R}_+)$, which establishes the representation formula in the lemma. ∎

5.5. Asymptotic behavior of M in the long time limit.

5.5.1. The characteristic exponent ξ_σ. The characteristic exponent governing the long time limit of the total mass is defined as follows.

LEMMA 8. *For each $\sigma > 0$,*

$$\int_0^\infty \sigma e^{-(\sigma+\xi)t} p(t)dt = 1$$

with unknown ξ has a unique real solution ξ_σ. This solution ξ_σ satisfies

$$-\sigma < \xi_\sigma < 0.$$

Proof. Consider the Laplace transform of the function κ defined above:

$$\mathcal{L}[\kappa](\xi) := \int_0^\infty \sigma e^{-(\sigma+\xi)t} p(t)dt.$$

As $0 < p \leq 1$, $\mathcal{L}[\kappa]$ is of class C^1 on $]-\sigma,+\infty[$, and

$$\dot{\mathcal{L}}[\kappa](\xi) = -\int_0^\infty \sigma e^{-(\sigma+\xi)t} tp(t)dt < 0$$

as $p(t) > 0$ for each $t \geq 0$. The function $\mathcal{L}[\kappa]$ is therefore decreasing on $]-\sigma,+\infty[$.

For each $t > 0$,

$$\kappa(t)e^{-\xi t} \to 0^+ \quad \text{as } \xi \to +\infty\,,$$

while

$$\kappa(t)e^{-\xi t} \leq \sigma e^{-\sigma t} \quad \text{for each } t \geq 0$$

since $0 < p \leq 1$. By dominated convergence, one concludes that

$$\mathcal{L}[\kappa](\xi) \to 0^+ \text{ as } \xi \to +\infty.$$

Besides, for each $t > 0$,

$$\sigma p(t)e^{-(\sigma+\xi)t} \uparrow \sigma p(t) \quad \text{as } \xi \downarrow -\sigma^+\,.$$

By monotone convergence,

$$\mathcal{L}[\kappa](\xi) \to \sigma \int_0^{+\infty} p(t)dt = +\infty \quad \text{as } \xi \to -\sigma^+\,.$$

(Notice that

$$\int_0^{+\infty} p(t)dt = +\infty$$

follows from the lower bound in (8).)

By the intermediate value theorem, there exists a unique $\xi_\sigma > -\sigma$ such that

$$\mathcal{L}[\kappa](\xi_\sigma) = 1.$$

Besides, $\xi_\sigma < 0$ as $\mathcal{L}[\kappa]$ is decreasing, and

$$\mathcal{L}[\kappa](0) = \int_0^{\infty} \kappa(t)dt < \int_0^{+\infty} \sigma e^{-\sigma t}dt = 1 = \mathcal{L}[\kappa](\xi_\sigma)\,,$$

which concludes the proof. ∎

In particular,

$$t \mapsto \kappa(t)e^{-\xi_\sigma t}$$

is a decreasing probability density on $\mathbb{R}_+$.

5.5.2. The renewal equation. It remains to prove statement (3) in Theorem 2.

First, for each $\lambda \in \mathbb{R}$ and for each locally bounded measurable function $f : \mathbb{R} \mapsto \mathbb{R}$ supported in $\mathbb{R}_+$, denote

$$f_\lambda(t) := e^{\lambda t} f(t) \text{ for each } t \in \mathbb{R}\,.$$

Notice that for each such f, g, we have

$$e^{\lambda t}(f * g)(t) = (f_\lambda * g_\lambda)(t) \text{ for each } t \in \mathbb{R}.$$

Hence, if ψ is a solution of the integral equation (28), the function $\psi_{-\xi_\sigma}$ satisfies

$$\psi_{-\xi_\sigma}(t) = (\kappa_{-\xi_\sigma} * \psi_{-\xi_\sigma})(t) + \kappa_{-\xi_\sigma}\,, \tag{29}$$

which is a renewal integral equation in the sense of [14].

Moreover, as noticed above, $\kappa_{-\xi_\sigma}$ is a decreasing probability density on $\mathbb{R}_+$, so that, in particular, $\kappa_{-\xi_\sigma}$ is directly Riemann integrable (see [14], pp. 348–349). Thus, applying Theorem 2 on p. 349 in [14] shows that

$$\psi(t)e^{-\xi_\sigma t} \to \frac{1}{\displaystyle\int_0^\infty t\kappa(t)e^{-\xi_\sigma t}dt} \quad \text{as } t \to +\infty. \tag{30}$$

By definition of ψ, this is precisely the asymptotic behavior of M in Theorem 2 (3).

5.6. Two important limiting cases for ξ_σ. We conclude our proof of Theorem 2 with a discussion of the asymptotic behavior of ξ_σ (statement (4) of Theorem 2) in the two following regimes:

1. the collisionless regime $\sigma \to 0^+$, and
2. the highly collisional regime $\sigma \to +\infty$.

End of the proof of Theorem 2. Denote, for the sake of simplicity, $\lambda_\sigma := \sigma + \xi_\sigma$. Establishing that $\xi_\sigma \sim -\sigma$ as $\sigma \to 0^+$ amounts to proving that $\lambda_\sigma = o(\sigma)$. First notice that, since $-\sigma < \xi_\sigma$,

$$0 < \lambda_\sigma < \sigma$$

so $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$. Keeping this in mind, we have

$$\int_0^{+\infty} e^{-\lambda_\sigma t} p(t)dt = \frac{1}{\sigma} \tag{31}$$

by definition of ξ_σ. Substituting $z = \lambda_\sigma t$ in the integral above, we obtain

$$0 < \frac{\lambda_\sigma}{\sigma} = \int_0^{+\infty} e^{-z} p\left(\frac{z}{\lambda_\sigma}\right) dz.$$

Since $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$ and $p(t) \to 0^+$ as $t \to +\infty$, one has $p(z/\lambda_\sigma) \to 0^+$ as $\sigma \to 0^+$. Besides, $0 \le e^{-z}p(z/\lambda_\sigma) \le e^{-z}$ so that, by dominated convergence,

$$\frac{\lambda_\sigma}{\sigma} \to 0 \text{ as } \sigma \to 0^+.$$

This establishes the asymptotic behavior of ξ_σ in the collisionless regime.

As for the highly collisional regime, we return to (31) defining ξ_σ (written in terms of λ_σ):

$$\begin{aligned} 1 &= \sigma \int_0^{+\infty} e^{-\lambda_\sigma t} p(t)dt \\ &= \lambda_\sigma \int_0^\infty e^{-\lambda_\sigma t} p(t)dt - \xi_\sigma \int_0^\infty e^{-\lambda_\sigma t} p(t)dt \\ &= 1 + \int_0^\infty e^{-\lambda_\sigma t} \dot{p}(t)dt - \xi_\sigma \int_0^\infty e^{-\lambda_\sigma t} p(t)dt, \end{aligned}$$

where the last equality follows from integrating by parts the first integral on the left-hand side. Therefore,

$$\xi_\sigma = \frac{\displaystyle\int_0^\infty e^{-\lambda_\sigma t}\dot{p}(t)dt}{\displaystyle\int_0^\infty e^{-\lambda_\sigma t}p(t)dt}$$

or, after substituting $t' = \lambda_\sigma t$,

$$\xi_\sigma = \frac{\displaystyle\int_0^\infty e^{-t}\dot{p}\left(\frac{t}{\lambda_\sigma}\right)dt}{\displaystyle\int_0^\infty e^{-t}p\left(\frac{t}{\lambda_\sigma}\right)dt}. \tag{32}$$

(31) shows that $\lambda_\sigma \to +\infty$ as $\sigma \to +\infty$. Passing to the limit in the right-hand side of (32), we find, by dominated convergence,

$$\xi_\sigma \to \frac{\displaystyle\int_0^\infty e^{-t}\dot{p}(0)dt}{\displaystyle\int_0^\infty e^{-t}p(0)dt} = \dot{p}(0) \quad \text{ as } \sigma \to +\infty\,.$$

Indeed, p is decreasing and convex, as can be verified, for instance, on the Boca–Zaharescu [4] explicit formula[2] (9)–(10) for p so that

$$0 \le -\dot{p}(t) \le -\dot{p}(0) \quad \text{ for each } t \ge 0\,.$$

We conclude by observing that the same explicit formulas of Boca–Zaharescu [4] imply that

$$\dot{p(0)} = -2\,. \qquad \square$$

6. Final remarks and open problems. The present work provides a complete description of the homogenization of the linear Boltzmann equation for monokinetic particles in the periodic system of holes of radius ε^2 centered at the vertices of the square lattice $\varepsilon\mathbb{Z}^2$ (Theorem 1.) In particular, we have given an asymptotic equivalent of exponential type of the total mass of the particle system in the long time limit (Theorem 2.)

Since the discussion in the present paper is restricted to the two-dimensional setting, it would be useful to extend the results above to the case of higher space dimensions and to lattices other than the square or cubic lattice. Most of the arguments considered here can be adapted to these more general cases; however, the analogue of the distribution of free path lengths (the function $p(t)$) is not known explicitly so far.

Otherwise, it would also be interesting to investigate other scalings than the Boltzmann–Grad type of scaling considered here—holes of radius ε^2 centered at the vertices of a square lattice whose fundamental domain is a square of size ε in the case of space dimension 2. Typically, one would like to mix the homogenization procedure

[2]In space dimension higher than 2, one can show that the analogue of p is also nonincreasing and convex, by using a variant of a formula due to L.A. Santalò established in [13], for want of an explicit formula giving the limiting distribution of free path lengths.

considered in the present work with the assumption of a highly collisional regime $\sigma \gg 1$ so that the size of the holes and the distance between neighboring holes are scaled in a way that differs from the one considered here. We hope to return to this problem in a forthcoming publication.

Another problem of potential interest is the case where the periodically distributed holes considered in the present paper are replaced with scatterers, assuming that particles are specularly reflected on the surface of each scatterer. In other words, the problem (Ξ_ε) is replaced with

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^1, t > 0, \\ f_\varepsilon(t,x,v) = f_\varepsilon(t,x,v - 2(v \cdot n_x)n_x, v), & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^1, t > 0, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^1. \end{cases}$$

Assume for simplicity that f_ε is periodic with period 1 in x_1, x_2, while ε designates the sequence of $1/n$ for each integer $n \geq 1$.

Most likely, the homogenized equation governing the vanishing ε limit of f_ε should involve an extended phase space, as in the case of the Boltzmann–Grad limit of the periodic Lorentz gas [8, 25]. The structure of this homogenized equation should be such that its solution converges to a constant state exponentially quickly in the long time limit for each $\sigma > 0$. However, while the limiting constant state is fully determined by conservation of mass and is therefore independent of $\sigma > 0$, the exponential decay to that constant state is not expected to hold uniformly as $\sigma \to 0$. Indeed, the case $\sigma = 0$ is precisely the Boltzmann–Grad limit of the periodic Lorentz gas governed by (1), and according to Theorem 3.5 in [9], (1) does not have the spectral gap property.

Finally, the homogenization result considered in the present paper raises an interesting question of quite general bearing. Usually, homogenization is a limiting process leading to a macroscopic description of some material that is known at the microscopic scale. In the problem considered here, it has been necessary to use a more detailed description of the particle system than that provided by the linear Boltzmann equation (problem (Ξ_ε) set in the extended phase space that involves the additional variable s).

In other words, the formulation of the macroscopic homogenization limit for the linear Boltzmann equation considered here involves remnants of an *even more microscopic description* of the system than the linear Boltzmann equation itself, namely, the extended phase space and the additional variable s.

We do not know whether this phenomenon (i.e., the need for a more microscopic description of a system to arrive at the formulation of a homogenized equation for that system) can be observed in homogenization problems other than the one considered here—for instance, in the case of equations other than those found in the context of kinetic theory.

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