Fluid Dynamic Limits of Kinetic Equations II
Convergence Proofs for the Boltzmann Equation

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To the memory of R. J. DiPerna.

Abstract

Using relative entropy estimates about an absolute Maxwellian, it is shown that any properly scaled sequence of DiPerna-Lions renormalized solutions of some classical Boltzmann equations has fluctuations that converge to an infinitesimal Maxwellian with fluid variables that satisfy the incompressibility and Boussinesq relations. Moreover, if the initial fluctuations entropically converge to an infinitesimal Maxwellian then the limiting fluid variables satisfy a version of the Leray energy inequality. If the sequence satisfies a local momentum conservation assumption, the momentum densities globaly converge to a solution of the Stokes equation. A similar discrete time version of this result holds for the Navier-Stokes limit with an additional mild weak compactness assumption. The continuous time Navier-Stokes limit is also discussed. ©1993 John Wiley & Sons., Inc.

1. Preliminaries

Introduction

The incompressible Navier-Stokes equations describe the evolution of the velocity field $u = u(t,x)$ of an idealized fluid over a given spatial domain in $\mathbb{R}^D$:

$$\begin{aligned} \nabla_x \cdot u &= 0 \,, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p &= \nu \Delta_x u \,, \\ u(0,x) &= u^{in}(x) \,, \end{aligned} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity of the fluid. In a seminal paper of 1934, J. Leray (see [16]) proved the existence of a temporally global weak solution to these equations over the whole space $\mathbb{R}^3$ for any initial data with finite energy. We use the modification of this result for the case when the fluid is contained in a D-dimensional periodic box $\mathbb{T}^D$; this will be stated more precisely below.

If the fluid consists of similar particles then at the kinetic level of description the state of the fluid is given by a density $F = F(t,x,v)$ of particle mass with

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position x and velocity v in the single particle phase space at instant t. If the particles interact only through a conservative interparticle force with a finite range then at low densities all but binary collisions can be neglected and the evolution of the phase space density F is governed by the classical Boltzmann equation:

$$\partial_t F + v\cdot\nabla_x F = B(F,F)\,, \tag{1.2a}$$

$$F(0,x,v) = F^{in}(x,v) \geqq 0\,, \tag{1.2b}$$

where the collision operator $B(F,F)$ is given by

$$B(F,F) = \iint (F_1'F' - F_1F)\, b(v_1 - v,\omega)\, d\omega\, dv_1\,.$$

The Boltzmann kernel $b(v_1 - v,\omega)$ is a non-negative measurable function. The variable ω lies on the unit sphere $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The F, F_1, F', and F_1' appearing in the integrand are understood to mean $F(t,x,\cdot)$ evaluated at the velocities v, v_1, v', and v_1' respectively, where the primed velocities are defined by

$$v' = v + \omega\omega\cdot(v_1 - v)\,, \qquad v_1' = v_1 - \omega\omega\cdot(v_1 - v)\,, \tag{1.3}$$

for any given $(v, v_1, \omega) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1}$.

The unprimed and primed velocities denote possible velocities for a pair of particles either before and after, or after and before they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v_1'\,, \qquad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2\,. \tag{1.4}$$

Equation (1.3) represents the general solution of these D + 1 equations for the 4D unknowns v, v_1, v', and v_1' in terms of the 3D − 1 parameters (v, v_1, ω). Each solution is counted twice by this representation since (1.3) is invariant under the transformation $\omega \mapsto -\omega$. Geometrically, the binary collisions associated with (v, v_1, ω) leave $v + v_1$ unchanged while reflecting $v_1 - v$ through the plane perpendicular to ω.

That the Boltzmann kernel b depends on $v_1 - v$ follows from the Galilean invariance of the collisional physics. Since ω and $-\omega$ represent the same collisions, b is taken to be an even function of ω. The rotational invariance of the collisional physics then implies that b has the classical form

$$b(v_1 - v,\omega) = |v_1 - v|\,\Sigma(|v_1 - v|, |\mu_c|)\,, \qquad \mu_c = \frac{\omega\cdot(v_1 - v)}{|v_1 - v|}\,, \tag{1.5}$$

where $\Sigma \geqq 0$ is the specific differential cross-section. Moreover, it will be assumed that b satisfies the bounds

$$0 \leqq b(v_1 - v,\omega) \leqq C\left(1 + |v_1 - v|^2\right)\,, \tag{1.6}$$

for some constant $C < \infty$ independent of ω. This condition is met by classical Boltzmann kernels with a small deflection cut-off; see [7]. Additional technical requirements on b will be imposed later.

In a recent paper R. J. DiPerna and P.-L. Lions (see [10]) proved the existence of a temporally global weak solution to the Boltzmann equation over the spatial domain $\mathbb{R}^D$ for any initial data satisfying natural physical bounds. With slight modifications, their theory can be extended to the case of a spatial domain which is a periodic box; this result will be stated more precisely below. In many respects this theory is analogous to the Leray global existence theory for the Navier-Stokes equations (1.1). This paper shows how the Leray solutions can be understood as an appropriate fluid dynamic limit of a sequence of DiPerna-Lions solutions. While the results given here are for a periodic box, the relations drawn here should be valid over a much wider context.

Dimensional Analysis

The dimensional scales of the Boltzmann initial-value problem (1.2) can be identified as follows. First, the volume of the periodic box determines a length scale λ_* by setting

$$\int dx = \lambda_*^D \,, \tag{1.7}$$

where here, as with all integrals, the integration is understood to be over the whole domain associated with its measure unless otherwise stated. The sides of the box $\mathbb{T}^D$ need not be the same length; all these length scales, however, are assumed to be of the same order.

Next, after a Galilean transformation to ensure that

$$\iint v F^{in} \, dv \, dx = 0 \,, \tag{1.8a}$$

the initial data F^{in} determines a density scale ρ_* and a velocity scale $\theta_*^{1/2}$ by the relations

$$\iint F^{in} \, dv \, dx = \rho_* \lambda_*^D \,, \qquad \iint \frac{1}{2}|v|^2 F^{in} \, dv \, dx = \frac{D}{2} \rho_* \theta_* \lambda_*^D \,. \tag{1.8b}$$

The parameters ρ_* and θ_* have been chosen so that the equilibrium associated with the initial data F^{in} is given by the absolute (constant in space and time) Maxwellian

$$M = \frac{\rho_*}{(2\pi\theta_*)^{D/2}} \exp\left(-\frac{1}{2\theta_*} |v|^2 \right) \,. \tag{1.9}$$

Here θ_* is related to the physical temperature T_* of this equilibrium by $\theta_* = kT_*/m$, where m is the single particle mass and k is the Boltzmann constant.

Finally, since the Boltzmann kernel b has units of reciprocal density times time, it determines a timescale τ_* by

$$\iiint M_1 M\, b(v_1 - v, \omega)\, d\omega\, dv_1\, dv = \frac{\rho_*}{\tau_*} . \tag{1.10}$$

The finiteness of the above integral is ensured by the assumed bound on b (1.6), so that $0 < \tau_* < \infty$. This is the scale of the average time interval that particles in the equilibrium density M spend traveling freely between collisions, the so-called mean free time. It is related to the length scale of the mean free path ($= \theta_*^{1/2}\tau_*$).

The initial-value problem (1.2) can then be reformulated in terms of dimensionless variables; these are introduced below adorned with hats. Dimensionless time, space, and velocity are defined by

$$t = \frac{\lambda_*}{\theta_*^{1/2}}\hat{t}\,, \quad x = \lambda_* \hat{x}\,, \quad v = \theta_*^{1/2}\hat{v}\,; \tag{1.11}$$

while a dimensionless phase space density is given by

$$F(t,x,v) = \frac{\rho_*}{\theta_*^{D/2}} \hat{F}(\hat{t},\hat{x},\hat{v}) . \tag{1.12}$$

Define the dimensionless Boltzmann kernel $\hat{b}(\hat{v}_1 - \hat{v}, \omega)$ by the relation

$$b(v_1 - v, \omega) = \frac{1}{\rho_* \tau_*} \hat{b}(\hat{v}_1 - \hat{v}, \omega)\ , \tag{1.13}$$

and set the corresponding dimensionless collision operator to be

$$\hat{B}\left(\hat{F},\hat{F}\right) = \iint \left(\hat{F}_1'\hat{F}' - \hat{F}_1\hat{F}\right) \hat{b}(\hat{v}_1 - \hat{v}, \omega)\, d\omega\, d\hat{v}_1 . \tag{1.14}$$

Substituting (1.11)–(1.14) into the Boltzmann equation (1.2) and henceforth dropping all hats yields the dimensionless initial-value problem

$$\partial_t F + v\cdot\nabla_x F = \frac{1}{\varepsilon} B(F,F)\,, \tag{1.15a}$$

$$F(0,x,v) = F^{in}(x,v) \geqq 0\,, \tag{1.15b}$$

where $\varepsilon = \theta_*^{1/2}\tau_*/\lambda_*$ is the dimensionless mean free path or Knudsen number.

The incompressible Navier-Stokes equations are obtained with a scaling in which F is considered close to M in a sense that will be made more precise later. It is natural to introduce the relative density, $G = G(t,x,v)$, defined by $F = MG$, where the dimensionless equilibrium Maxwellian is now

$$M = \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{1}{2}|v|^2\right) . \tag{1.16}$$

Recasting the initial-value problem (1.15) for G yields

$$\partial_t G + v\cdot\nabla_x G = \frac{1}{\varepsilon}Q(G,G)\,, \tag{1.17a}$$

$$G(0,x,v) = G^{in}(x,v) \geqq 0\,, \tag{1.17b}$$

where the collision operator is now given by

$$Q(G,G) = \iint (G_1'G' - G_1G)\, b(v_1 - v,\omega)\, d\omega M_1\, dv_1\,. \tag{1.18}$$

This nondimensionalization has the following normalizations:

$$\int d\omega = 1\,, \quad \int M dv = 1\,, \quad \int dx = 1\,, \tag{1.19}$$

associated with the domains $\mathbb{S}^{D-1}$, $\mathbb{R}^D$, and $\mathbb{T}^D$ respectively;

$$\begin{gathered} \iint G^{in} M dv\, dx = 1\,, \quad \iint v\, G^{in} M dv\, dx = 0\,, \\ \iint \frac{1}{2}|v|^2 G^{in} M dv\, dx = \frac{D}{2}\,, \end{gathered} \tag{1.20}$$

associated with the initial data; and

$$\iiint b(v_1 - v,\omega)\, d\omega M_1 dv_1\, M dv = 1\,, \tag{1.21}$$

associated with the Boltzmann kernel.

Formal Structure

Since $M dv$ is a positive unit measure on $\mathbb{R}^D$, we denote by $\langle\xi\rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle\xi\rangle = \int \xi\, dv\,. \tag{1.22}$$

Since $d\mu \equiv b(v_1 - v,\omega)\, d\omega M_1 dv_1\, M dv$ is a non-negative unit measure on $\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{S}^{D-1}$, we denote by $\langle\!\langle\Xi\rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(v, v_1, \omega)$,

$$\langle\!\langle\,\Xi\,\rangle\!\rangle = \int \Xi\, d\mu\,. \tag{1.23}$$

We now present the basic formal structure of the Boltzmann equation in the setting of this notation for later reference. All of these results are standard and their proofs can be essentially found in [7].

The formal structure of the Boltzmann equation follows from two fundamental properties of the measure $d\mu$. First, that it is invariant under the coordinate transformations

$$(1.24)\qquad \begin{gathered}(v, v_1, \omega) \mapsto (v_1, v, \omega)\,, \quad (v, v_1, \omega) \mapsto (v', v_1', \omega)\,, \\ (v, v_1, \omega) \mapsto (v_1', v', \omega)\,.\end{gathered}$$

These transformations will be referred to as the $d\mu$-symmetries. Second, that it characterizes microscopic conserved quantities in the sense that for any measurable $\xi = \xi(v)$ the following statements are equivalent:

(1.25)

(i) $\xi + \xi_1 - \xi' - \xi_1' = 0$
for $d\mu$-almost every $(v, v_1, \omega) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1}$;

(ii) $\xi = \alpha + \beta \cdot v + \gamma \frac{1}{2}|v|^2$
for some $(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}$.

This property will be referred to as the $d\mu$-characterization.

Repeated application of the $d\mu$-symmetries (1.24) yields the following important identity regarding the collision operator (1.18):

$$(1.26)\qquad \begin{aligned}-\langle \xi Q(G,G)\rangle &= \langle\!\langle \xi(G_1 G - G_1' G')\rangle\!\rangle \\ &= \frac{1}{4}\langle\!\langle (\xi + \xi_1 - \xi' - \xi_1')(G_1 G - G_1' G')\rangle\!\rangle\,,\end{aligned}$$

for every $\xi = \xi(v)$ and $G = G(v)$ for which the integrals make sense.

Successively setting $\xi = 1, v, \frac{1}{2}|v|^2$ in (1.26) and using the microscopic conservation laws (1.4) gives the conservation laws

$$(1.27)\qquad \langle Q(G,G)\rangle = 0\,, \qquad \langle v\,Q(G,G)\rangle = 0\,, \qquad \left\langle \frac{1}{2}|v|^2 Q(G,G)\right\rangle = 0\,,$$

for every $G = G(v)$ for which the integrals make sense. It can be shown that these are essentially all the quantities conserved by $Q(G,G)$ by using the $d\mu$-characterization (1.25). More precisely, the following statements are equivalent:

(1.28)

(i) $\langle \xi Q(G,G)\rangle = 0$
for every $G = G(v)$ for which the integral makes sense;

(ii) $\xi = \alpha + \beta \cdot v + \gamma \frac{1}{2}|v|^2$
for some $(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}$.

If G solves the Boltzmann equation (1.17) then (1.27) implies that it satisfies local conservation laws of mass, momentum, and energy:

$$
\begin{aligned}
&\partial_t\langle G\rangle + \nabla_x\cdot\langle vG\rangle = 0\,, \\
&\partial_t\langle vG\rangle + \nabla_x\cdot\langle v\otimes vG\rangle = 0\,, \qquad (1.29)\\
&\partial_t\left\langle \frac{1}{2}|v|^2G\right\rangle + \nabla_x\cdot\left\langle v\frac{1}{2}|v|^2G\right\rangle = 0\,.
\end{aligned}
$$

Integrating these over space and time yields the global conservation laws of mass, momentum, and energy:

$$
(1.30)\qquad \int\langle G(t)\rangle\,dx = 1\,, \quad \int\langle v\,G(t)\rangle\,dx = 0\,, \quad \int\left\langle \frac{1}{2}|v|^2G(t)\right\rangle\,dx = \frac{\mathrm{D}}{2}\,.
$$

Upon setting $\xi = \log G$ in the collision identity (1.26), Boltzmann observed that the resulting integrand is non-negative and hence obtained the dissipation law

$$
(1.31)\qquad -\langle\log G\,Q(G,G)\rangle = \frac{1}{4}\left\langle\!\!\left\langle \log\!\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle \geqq 0\,,
$$

for every $G = G(v)$ for which the integrals make sense. He then characterized the equilibria of the collision operator by using the $d\mu$-characterization (1.25); he found that for any $G = G(v)$ for which the integrals make sense, the following statements are equivalent:

(1.32)

(i) $Q(G,G) = 0$;

(ii) $\langle\log G\,Q(G,G)\rangle = 0$;

(iii) $G = \exp\!\left(\alpha + \beta\cdot v + \gamma\frac{1}{2}|v|^2\right)$
for some $(\alpha,\beta,\gamma) \in \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}$.

The equilibria characterized in (iii) above will have finite mass, momentum, and energy density when $\gamma < 1$. In that case they can be written as $G = M(\rho,u,\theta)/M$, where $M(\rho,u,\theta)$ are the classical Maxwellians defined by

$$
(1.33)\qquad M(\rho,u,\theta) \equiv \frac{\rho}{(2\pi\theta)^{\mathrm{D}/2}}\exp\!\left(-\frac{1}{2\theta}|v-u|^2\right)\,,
$$

and where the density $\rho \geqq 0$, the velocity $u \in \mathbb{R}^{\mathrm{D}}$, and the temperature $\theta > 0$ are determined by the relations

$$
(1.34)\qquad \rho = \langle G\rangle\,, \quad \rho u = \langle v\,G\rangle\,, \quad \frac{1}{2}\rho|u|^2 + \frac{\mathrm{D}}{2}\rho\,\theta = \left\langle\frac{1}{2}|v|^2G\right\rangle\,.
$$

Now, if G solves the Boltzmann equation (1.17) then the dissipation law (1.31) implies that G satisfies the local entropy dissipation law

$$\begin{aligned}(1.35)\qquad &\partial_t\langle G\log G - G + 1\rangle + \nabla_x\cdot\langle v(G\log G - G + 1)\rangle \\ &\qquad = -\frac{1}{\varepsilon}\frac{1}{4}\left\langle\!\!\left\langle \log\!\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle \leqq 0 .\end{aligned}$$

Integrating this over space and time gives the global entropy equality

$$(1.36)\qquad H(G(t)) + \frac{1}{\varepsilon}\int_0^t R(G(s))\,ds = H(G^{in}) ,$$

where $H(G)$ is the entropy functional

$$(1.37)\qquad H(G) = \int \langle G\log G - G + 1\rangle\,dx ,$$

and $R(G)$ is the entropy dissipation rate functional

$$(1.38)\qquad R(G) = \int \frac{1}{4}\left\langle\!\!\left\langle \log\!\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle\,dx .$$

This choice of H as the entropy functional (1.37) is based on the fact that its integrand is a non-negative strictly convex function of G with a minimum value of zero at $G = 1$. Thus for any G,

$$(1.39)\qquad H(G) \geqq 0 , \quad \text{and} \quad H(G) = 0 \quad \text{iff } G = 1 .$$

This is the so-called relative entropy of G with respect to the absolute equilibrium $G = 1$; it provides a natural measure of the proximity of G to that equilibrium.

Incompressible Navier-Stokes Scalings

Fluid dynamics is obtained in limits where the mean free path becomes small compared with the macroscopic length scales, those with vanishing Knudsen number ($\varepsilon \to 0$). If this is done while the Reynolds number is held fixed, the Mach number must also vanish (cf. [2] or [3]). In order to realize densities corresponding to a small Mach number, it is natural to consider them as perturbations about the equilibrium Maxwellian M. The flow will be incompressible if its kinetic energy in the acoustic modes is smaller than that in the rotational modes. Since the acoustic modes vary on a faster timescale than rotational modes, they may be suppressed by assuming that the solution is consistent with motion on a slow timescale; this scale separation will also be measured with the Knudsen number.

This idea is quantified by rescaling time to the order of ε^{-1} while setting the distance to the absolute Maxwellian M to be of order ε^m for some $m \geqq 1$. Thus, we consider a sequence of solutions G_ε to the scaled Boltzmann equation

$$\varepsilon\,\partial_t G_\varepsilon + v\cdot\nabla_x G_\varepsilon = \frac{1}{\varepsilon}Q(G_\varepsilon, G_\varepsilon)\,, \tag{1.40}$$

in the form

$$G_\varepsilon = 1 + \varepsilon^m g_\varepsilon\,. \tag{1.41}$$

As ε tends to zero, the leading behavior of the fluctuations g_ε is formally consistent with the incompressible Navier-Stokes equations (1.1) when $m = 1$, and with the Stokes equations (the linearization of 1.1) when $m > 1$. We make this more precise below.

Setting (1.41) into (1.40) and Taylor expanding the collision operator gives

$$\varepsilon\,\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon}Lg_\varepsilon = \varepsilon^{m-1}Q(g_\varepsilon, g_\varepsilon)\,, \tag{1.42}$$

where L, the linearized collision operator, is given by

$$Lg \equiv -2\,Q(1,g) = \iint (g + g_1 - g' - g_1')\,b\,d\omega M_1 dv_1\,. \tag{1.43}$$

Repeated application of the $d\mu$-symmetries (1.24) yields the identity

$$\begin{aligned}\langle \xi L g\rangle &= \langle\!\langle \xi(g + g_1 - g' - g_1')\rangle\!\rangle \\ &= \frac{1}{4}\langle\!\langle (\xi + \xi_1 - \xi' - \xi_1')(g + g_1 - g' - g_1')\rangle\!\rangle\,,\end{aligned} \tag{1.44}$$

for every $\xi = \xi(v)$ and $g = g(v)$ for which the integral makes sense. This shows that L is formally self-adjoint and has a non-negative Hermitian form. These properties ensure that L has a self-adjoint extension to the Hilbert space $L^2(M dv)$ with the inner product $\langle fg\rangle$. Furthermore, using the $d\mu$-characterization (1.25), it can be shown that for any $g = g(v)$ in the form domain of L, the following statements are equivalent:

$$\begin{aligned}&\text{(i)}\quad Lg = 0\,;\\ &\text{(ii)}\quad g = \alpha + \beta\cdot v + \gamma\frac{1}{2}|v|^2 \quad \text{for some } (\alpha,\beta,\gamma)\in\mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}\,.\end{aligned} \tag{1.45}$$

This characterizes N(L), the null space of L, as the set obtained by linearizing (iii) of (1.32) about $(\alpha,\beta,\gamma) = (0,0,0)$, the so-called infinitesimal Maxwellians.

In studying the formal incompressible Navier-Stokes limit of the Boltzmann equation, one finds a special role is played by the functions $\phi(v) \in \mathbb{R}^{\mathrm{D}\times\mathrm{D}}$ and $\psi(v) \in \mathbb{R}^{\mathrm{D}}$ that are the unique solutions to the equations

$$L\phi(v) = v\otimes v - \frac{1}{\mathrm{D}}|v|^2 I\,, \qquad L\psi(v) = \frac{1}{2}|v|^2 v - \frac{\mathrm{D}+2}{2}v\,, \tag{1.46}$$

which are orthogonal to N(L); henceforth ϕ and ψ will always refer to these functions. The main formal result of [3] is the following.

THEOREM 1.1. *Let $G_\varepsilon(t,x,v)$ be a sequence of non-negative solutions to the scaled Boltzmann equation* (1.40) *such that, when it is written according to formula* (1.41), *the sequence g_ε converges in the sense of distributions and almost everywhere to a function g as ε tends to zero. Furthermore, assume that the moments*

$$\langle g_\varepsilon\rangle\,,\ \langle v g_\varepsilon\rangle\,,\ \langle v\otimes v g_\varepsilon\rangle\,,\ \langle v|v|^2 g_\varepsilon\rangle\,,$$

$$\langle \phi\otimes v g_\varepsilon\rangle\,,\ \langle \phi Q(g_\varepsilon,g_\varepsilon)\rangle\,,\ \langle \psi\otimes v g_\varepsilon\rangle\,,\ \langle \psi Q(g_\varepsilon,g_\varepsilon)\rangle\,,$$

converge in the sense of distributions to the corresponding moments

$$\langle g\rangle\,,\ \langle v g\rangle\,,\ \langle v\otimes v g\rangle\,,\ \langle v|v|^2 g\rangle\,,$$

$$\langle \phi\otimes v g\rangle\,,\ \langle \phi Q(g,g)\rangle\,,\ \langle \psi\otimes v g\rangle\,,\ \langle \psi Q(g,g)\rangle\,,$$

and that all formally small terms in ε vanish. Then the limiting form of g is that of an infinitesimal Maxwellian,

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,, \tag{1.47}$$

where the velocity u satisfies the incompressibility relation, while the density and temperature fluctuations, ρ and θ, satisfy the Boussinesq relation:

$$\nabla_x\cdot u = 0\,,\qquad \nabla_x(\rho+\theta) = 0\,. \tag{1.48}$$

Moreover, the functions ρ, u, and θ are weak solutions of the equations

$$\partial_t u + \nabla_x p = \nu\Delta_x u\,,\qquad \partial_t\theta = \kappa\Delta_x\theta\,,\quad \text{if } m>1\,; \tag{1.49}$$

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = \nu\Delta_x u\,,\quad \partial_t\theta + u\cdot\nabla_x\theta = \kappa\Delta_x\theta\,,\quad \text{if } m=1\,. \tag{1.50}$$

In these equations the coefficients ν and κ are given by

$$\nu = \frac{1}{(\mathrm{D}-1)(\mathrm{D}+2)}\langle \phi : L\phi\rangle\,,\qquad \kappa = \frac{2}{\mathrm{D}(\mathrm{D}+2)}\langle \psi\cdot L\psi\rangle\,. \tag{1.51}$$

We shall refer to (1.49) as the Stokes system and to (1.50) as the Navier-Stokes system. The momentum equations in these systems will be referred to as the Stokes equation and the Navier-Stokes equation respectively.

Global Solutions

The theory of R. J. DiPerna and P.-L. Lions in [10] (modified slightly for the periodic box) gives the existence of a global weak solution to a whole class of formally equivalent initial-value problems. These are obtained by dividing the Boltzmann equation (1.17a) by normalizing functions $N(G) > 0$:

$$(\partial_t + v\cdot\nabla_x)\Gamma(G) = \frac{1}{\varepsilon}\frac{1}{N(G)}Q(G,G)\,, \tag{1.52a}$$

$$G(0,x,v) = G^{in}(x,v)\,, \tag{1.52b}$$

where the normalization $N(G)$ satisfies $(1+Z)/N(Z) \leqq C$ over $Z > 0$ for some constant $C < \infty$ and where $\Gamma'(z) = 1/N(z)$. They showed that if G is a weak solution of (1.52) for one such $N(G)$ then it is a weak solution for all such $N(G)$. Such solutions they called renormalized solutions of the Boltzmann initial-value problem (1.17).

More specifically, given any initial data in the entropy class $\{G^{in} \geqq 0 : H(G^{in}) < +\infty\}$ that satisfies the initial normalizations (1.20), there exists at least one non-negative weak solution of (1.52) in $C([0,\infty); w\text{-}L^1(Mdv\,x))$ (see Appendix A for the notation regarding spaces) with

$$\begin{aligned} &\frac{1}{N(G)}\,Q^-(G,G) \in L^\infty(dt;L^1(Mdv\,x))\,,\\ &\frac{1}{N(G)}\,Q^+(G,G) \in L^1_{\mathrm{loc}}(dt;L^1(Mdv\,x))\,, \end{aligned} \tag{1.53}$$

where Q^- and Q^+ are the source and sink components of the collision operator (1.18):

$$\begin{aligned} Q^+(G,G) &= \iint G_1'G'\,b(v_1-v,\omega)\,d\omega M_1\,dv_1\,,\\ Q^-(G,G) &= \iint G_1G\,b(v_1-v,\omega)\,d\omega M_1\,dv_1\,. \end{aligned} \tag{1.54}$$

Here, to say G is a weak solution of (1.52) means that it is initially equal to G^{in} and that it satisfies the normalized Boltzmann equation (1.52a) in the sense that for every $\chi \in L^\infty(Mdv; C^1(\mathbb{T}^{\mathrm{D}}))$ and every $0 \leqq t_1 < t_2 < \infty$ it satisfies

$$\begin{aligned} \int\langle\Gamma(G(t_2))\,\chi\rangle\,dx - \int\langle\Gamma(G(t_1))\,\chi\rangle\,dx - \int_{t_1}^{t_2}\!\!\int\langle\Gamma(G)\,v\cdot\nabla_x\chi\rangle\,dx\,dt&\\ = \frac{1}{\varepsilon}\int_{t_1}^{t_2}\!\!\int\left\langle\frac{1}{N(G)}Q(G,G)\chi\right\rangle\,dx\,dt&\,. \end{aligned} \tag{1.55}$$

It also satisfies the global entropy inequality

$$H(G(t)) + \frac{1}{\varepsilon}\int_0^t R(G(s))\,ds \leqq H(G^{in}) , \tag{1.56}$$

the local conservation law of mass

$$\partial_t\langle G\rangle + \nabla_x\cdot\langle vG\rangle = 0 , \tag{1.57}$$

the global conservation law of momentum

$$\int \langle vG(t)\rangle\,dx = 0 , \tag{1.58}$$

and the global energy inequality

$$\int \left\langle \frac{1}{2}|v|^2 G(t)\right\rangle\,dx \leqq \frac{\mathrm{D}}{2} , \tag{1.59}$$

for every $t > 0$.

The finiteness of the entropy is enough to insure the integrability of the conserved densities. The DiPerna-Lions theory, however, does not assert the local conservation of momentum (see (1.29)), the global conservation of energy (see (1.30)), or the global entropy equality (see (1.36)); nor does it assert the uniqueness of the solution.

The DiPerna-Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for Navier-Stokes type systems. For the Navier-Stokes system (1.50) with mean zero initial data, we set the Leray theory in the following Hilbert spaces of vector and scalar valued functions:

$$\begin{aligned}
\mathscr{H}_v &= \left\{ w \in L^2(dx;\mathbb{R}^{\mathrm{D}}) \ :\ \nabla_x\cdot w = 0\,,\ \int w\,dx = 0 \right\} ,\\
\mathscr{H}_s &= \left\{ \chi \in L^2(dx;\mathbb{R}) \ :\ \int \chi\,dx = 0 \right\} ,\\
\mathscr{V}_v &= \left\{ w \in \mathscr{H}_v \ :\ \int |\nabla_x w|^2 dx < \infty \right\} ,\\
\mathscr{V}_s &= \left\{ \chi \in \mathscr{H}_s \ :\ \int |\nabla_x \chi|^2 dx < \infty \right\} .
\end{aligned} \tag{1.60}$$

Let $\mathscr{H} = \mathscr{H}_v \oplus \mathscr{H}_s$ and $\mathscr{V} = \mathscr{V}_v \oplus \mathscr{V}_s$. Given any $(u^{in}, \theta^{in}) \in \mathscr{H}$, there exists a (u, θ) in $C([0,\infty); w\text{-}\mathscr{H}) \cap L^2_{\mathrm{loc}}(dt;\mathscr{V})$ which is initially (u^{in}, θ^{in}) and satisfies the Navier-Stokes system (1.50) in the sense that for every $(w,\chi) \in \mathscr{H} \cap C^1(\mathbb{T}^{\mathrm{D}})$

$$\begin{aligned}
&\int w\cdot u(t_2)\,dx - \int w\cdot u(t_1)\,dx - \int_{t_1}^{t_2}\int \nabla_x w : (u\otimes u)\,dx\,dt \\
&\qquad = -\nu \int_{t_1}^{t_2}\int \nabla_x w : \nabla_x u\,dx\,dt ,
\end{aligned} \tag{1.61a}$$

$$\int \chi\theta(t_2)\,dx - \int \chi\theta(t_1)\,dx - \int_{t_1}^{t_2}\int \nabla_x\chi\cdot(u\,\theta)\,dx\,dt \tag{1.61b}$$
$$= -\kappa\int_{t_1}^{t_2}\int \nabla_x\chi\cdot\nabla_x\theta\,dx\,dt\,,$$

for every $0 \leqq t_1 < t_2$. Moreover, (u, θ) satisfies the dissipation inequalities

$$\int \frac{1}{2}|u(t)|^2\,dx + \int_0^t\int \nu|\nabla_x u|^2\,dx\,dt' \leqq \int \frac{1}{2}|u^{in}|^2\,dx\,, \tag{1.62a}$$

$$\int \frac{1}{2}|\theta(t)|^2\,dx + \int_0^t\int \kappa|\nabla_x \theta|^2\,dx\,dt' \leqq \int \frac{1}{2}|\theta^{in}|^2\,dx\,, \tag{1.62b}$$

for every $t > 0$. Arguing formally from the Navier-Stokes system (1.50) one would expect these inequalities to be equalities, but that is not asserted by the Leray theory. Also, as was the case for the DiPerna-Lions theory, the Leray theory does not assert uniqueness of the solution.

The Program

Let $G_\varepsilon \geqq 0$ be a sequence of DiPerna-Lions renormalized solutions to the scaled Boltzmann initial-value problem

$$\varepsilon\,\partial_t G_\varepsilon + v\cdot\nabla_x G_\varepsilon = \frac{1}{\varepsilon}Q(G_\varepsilon, G_\varepsilon)\,, \tag{1.63a}$$

$$G_\varepsilon(0,x,v) = G_\varepsilon^{in}(x,v) \geqq 0\,. \tag{1.63b}$$

For any given DiPerna-Lions normalization $N(Z)$, the associated normalized Boltzmann equation is

$$(\varepsilon\,\partial_t + v\cdot\nabla_x)\Gamma(G_\varepsilon) = \frac{1}{\varepsilon}\,\frac{1}{N(G_\varepsilon)}\,Q(G_\varepsilon, G_\varepsilon)\,, \tag{1.64}$$

with $\Gamma(Z)$ is related to $N(Z)$ by $\Gamma'(Z) = 1/N(Z)$. The associated DiPerna-Lions entropy inequality is

$$H(G_\varepsilon(t)) + \frac{1}{\varepsilon^2}\int_0^t R(G_\varepsilon(s))\,ds \leqq H(G_\varepsilon^{in})\,. \tag{1.65}$$

Assume that the initial data G_ε^{in} satisfies the normalizations (1.20) and the entropy bound

$$H(G_\varepsilon^{in}) \leqq C^{in}\varepsilon^{2m}, \tag{1.66}$$

for some fixed $C^{in} > 0$ and $m \geqq 1$. Moreover, assume that the initial data has the form $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in}$ where

$$g_\varepsilon^{in} \to \rho^{in} + u^{in}\cdot v + \theta^{in}\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right), \tag{1.67}$$

in $L^1(M dv\, dx)$ as ε tends to zero, where $\rho^{in} + \theta^{in} = 0$ and $(u^{in}, \theta^{in}) \in \mathscr{H}$.

Consider the sequence g_ε as defined by the relation $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon$ as ε tends to zero. The DiPerna-Lions entropy inequality (1.65) and the entropy bound (1.66) are consistent with this order of fluctuation about the equilibrium $G = 1$. Given the formal result contained in Theorem 1.1, it is natural to ask whether, and in what sense, one has the limits

$$g_\varepsilon \to \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right), \tag{1.68a}$$

$$\langle v\, g_\varepsilon\rangle \to u, \quad \left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right) g_\varepsilon\right\rangle \to \theta, \tag{1.68b}$$

where $\rho + \theta = 0$ and $(u, \theta) \in C([0, \infty); w\text{-}\mathscr{H}) \cap L^2_{\mathrm{loc}}(dt; \mathscr{V})$ is a solution of the Stokes system (1.49) when $m > 1$, or else a Leray solution of the Navier-Stokes system (1.50) when $m = 1$.

While this program is not yet complete, we present significant partial results in this paper. The next section gives precise statements of all the main results contained in the remainder of this paper; in particular, it clearly sets out the additional assumptions necessary for their proofs. In doing so, we also provide an outline of the body of the article and give an impressionistic overview of the strategies employed.

The program above deals with globally defined, weak solutions of the Boltzmann equation or the Stokes and Navier-Stokes systems. It is possible to analyze fluid dynamic limits when dealing with sufficiently smooth solutions. De Masi, Esposito, and Lebowitz (see [9]) have used asymptotic expansions "à la" Hilbert or Chapman-Enskog to construct solutions of the Boltzmann equation having a prescribed hydrodynamic limit. It is still unknown, however, whether the existence of smooth solutions of the Boltzmann equation or of the Navier-Stokes system is a generic fact. This makes the derivation of hydrodynamic limits for weak solutions a problem of definite interest, although more complicated than for smooth solutions.

It is clear that completion of the program may require a better knowlege of properties of the DiPerna-Lions solutions. For example, in order to obtain the dynamical equation for u, we shall assume that the local momentum conservation law is satified. In any event, it is certainly a major success for the DiPerna-Lions theory that this program can be carried so far.

2. Main Results

The Normalized Boltzmann Equation

Throughout this article, the Boltzmann initial-value problem will be taken in its scaled form (1.63). Based on the formal arguments in the last section, we expect the renormalized solutions of (1.63) to have the form $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon$. With this in mind, we choose to work with a DiPerna-Lions normalization in the form

$$N_\varepsilon = N(G_\varepsilon) = \frac{2}{3} + \frac{1}{3}G_\varepsilon = 1 + \frac{1}{3}\varepsilon^m g_\varepsilon \,. \tag{2.1}$$

One reason for this choice is such that formally $N_\varepsilon \to 1$ as ε tends to zero; thus, the normalizing factor will conveniently disappear from all algebraic expressions considered in this limit. Another reason lies in simplification of the specifics encountered during some subsequent estimates. Of course, our main results are independent of this particular choice of normalization.

Given this choice, we then choose to write the normalized Boltzmann equation (1.64) as

$$\varepsilon\, \partial_t \gamma_\varepsilon + v \cdot \nabla_x \gamma_\varepsilon = \frac{1}{\varepsilon^{m+1}} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} \,, \tag{2.2}$$

where we have introduced γ_ε by

$$\gamma_\varepsilon = \frac{1}{\varepsilon^m}\Gamma(G_\varepsilon) = \frac{3}{\varepsilon^m} \log\left(1 + \frac{1}{3}\varepsilon^m g_\varepsilon\right) \,. \tag{2.3}$$

Since γ_ε formally behaves like g_ε for small ε, it should be thought of as the normalized form of the fluctuations g_ε.

Implications of the Entropy Inequality

The first objective of the paper is to characterize the limiting form of the fluctuations g_ε; the formal argument indicated that this should have the form of an infinitesimal Maxwellian (1.47). Since the quantities of interest are indeed fluctuations, they do not have a definite sign; therefore the conservation laws do not provide any a priori estimate on the family g_ε. The a priori estimates needed are to be sought in the combination of the entropy inequality (1.65) and the entropy bound (1.66) assumed for the initial data:

$$H(G_\varepsilon(t)) + \frac{1}{\varepsilon^2}\int_0^t R(G_\varepsilon(s))\,ds \leqq H(G_\varepsilon^{in}) \leqq C^{in}\varepsilon^{2m} \,. \tag{2.4}$$

As can be seen from (1.37), the terms involving the entropy H measure the proximity of G_ε and G_ε^{in} to the absolute equilibrium value of 1. On the other hand, the terms involving the dissipation rate R, defined in (1.38), can be understood to measure the proximity of G_ε to any Maxwellian through their characterization (1.32). More precisely, the object of interest regarding the dissipation rate R is the scaled collision integrand given by

$$q_\varepsilon = \frac{1}{\varepsilon^{m+1}} \left(G'_{\varepsilon 1} G'_\varepsilon - G_{\varepsilon 1} G_\varepsilon \right) . \tag{2.5}$$

One observes that the entropy and dissipation rate can be recast as

$$\begin{aligned} H(G_\varepsilon) &= \int \langle h(\varepsilon^m g_\varepsilon) \rangle \, dx , \\ R(G_\varepsilon) &= \int \frac{1}{4} \left\langle\!\!\left\langle \frac{1}{\varepsilon^2} r \left(\frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon} \right) G_{\varepsilon 1} G_\varepsilon \right\rangle\!\!\right\rangle dx , \end{aligned} \tag{2.6}$$

where the integrands are written in terms of the convex functions

$$h(z) = (1+z)\log(1+z) - z , \qquad r(z) = z \log(1+z) .$$

Since $h(z) = O(z^2)$ and $r(z) = O(z^2)$ as $z \to 0$, one easily sees that $H(G_\varepsilon)$ and $R(G_\varepsilon)$ asymptotically behave almost like L^2 norms of g_ε and q_ε respectively as ε tends to zero. Using this observation, the bound (2.4) results in the following statement.

PROPOSITION 2.1. (THE INFINITESIMAL MAXWELLIAN FORM) *Let the family $G_\varepsilon = G_\varepsilon(t,x,v)$ satisfy the entropy inequality and bound* (2.4). *Let g_ε and q_ε be the corresponding families of fluctuations* (1.41) *and scaled collision integrands* (2.5). *Then*

(1) *The family $(1 + |v|^2) g_\varepsilon$ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx))$;*
(2) *The family $(1 + |v|^2) q_\varepsilon / N_\varepsilon$ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\, dx))$;*
(3) *Any convergent subsequence of g_ε as $\varepsilon \to 0$ has a limit g of the form of an infinitesimal Maxwellian for some $(\rho, u, \theta) \in L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}))$,*

$$g = \rho + u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2} \right) . \tag{2.7}$$

It is remarkable that the statement above does not involve the fact that g_ε will eventually represent fluctuations of the number density in the Boltzmann equation; the only features of the Boltzmann equation used in these result are the entropy and entropy dissipation bounds resulting from the entropy inequality and bound (2.4). More precisely, the entropy and entropy dissipation bounds provide the weak compactness statements regarding g_ε and q_ε respectively. The limiting local

Maxwellian form (2.7) is a consequence of the weak compactness property of q_ε. The statement above is proved in Section 3 (see Propositions 3.1, 3.4, and 3.8). Section 3 also contains other consequences of the entropy inequality that are used throughout this article.

Implications of the Normalized Boltzmann Equation

Let G_ε be a family of renormalized solution of the Boltzmann initial-value problem (1.63) with initial data satisfying the entropy bound (1.66). Let g_ε and q_ε be the corresponding families of fluctuations (1.41) and scaled collision integrands (2.5). As a consequence of the above proposition, we may assume g_ε converges to g in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1((1+|v|^2)M dv\,dx))$, $q_\varepsilon/N_\varepsilon$ converges to q in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\,dx))$, and g has the form of an infinitesimal Maxwellian (2.7). Passing to the limit in the normalized Boltzmann equation (2.2) leads to the following result.

PROPOSITION 2.2. (THE LIMITING BOLTZMANN EQUATION) *Given g and q as described above, then q inherits the symmetries of q_ε under the $d\mu$-symmetries* (1.24), *is in $L^2_{\mathrm{loc}}(dt; L^2(d\mu\,dx))$, and satisfies*

$$v\cdot\nabla_x g = \iint q\, b(v_1 - v, \omega)\, d\omega M_1\, dv_1 \,. \tag{2.8}$$

Combining this result with that of the last subsection and using the microscopic conservation laws (1.4) and the $d\mu$-symmetries (1.24) yields the following relations.

PROPOSITION 2.3. (INCOMPRESSIBILITY AND BOUSSINESQ RELATIONS) *Given g as described above, it has the form of an infinitesimal Maxwellian* (2.7) *where (ρ, u, θ) satisfy*

$$\nabla_x\cdot u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,. \tag{2.9}$$

The proof makes critical use of compactness results from Section 3.

Before going further in this direction, we introduce the notion of "entropic convergence" that will be of repeated use later. A sequence of fluctuations g_ε is said to converge entropically of order ε^m to g if and only if

$$g_\varepsilon \to g \text{ in } w\text{-}L^1(M dv\,dx)\,,$$

$$\text{and} \tag{2.10}$$

$$\lim_{\varepsilon\to 0}\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon)\right\rangle dx = \int \frac{1}{2}\langle g^2\rangle\, dx\,.$$

We shall show that this notion of convergence is in fact stronger than that of $L^1((1+|v|^2)M dv\,dx)$.

One of the most remarkable features of the incompressible Navier-Stokes scalings is that the DiPerna-Lions entropy inequality (2.4) transforms into a global form of the Leray energy inequalities (1.62) as ε tends to zero. More precisely, we shall consider the entropy inequality (2.4) multiplied by ε^{-2m} and pass to the limit in the resulting inequality as ε tends to zero, to obtain the following result.

PROPOSITION 2.4. (THE LERAY ENERGY INEQUALITY) *Let*

$$(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$$

and define the infinitesimal Maxwellian g^{in} in $L^2(M dv\, dx)$ by the formula

$$g^{in} = \rho^{in} + u^{in} \cdot v + \theta^{in} \left(\frac{1}{2}|v|^2 - \frac{D}{2} \right) . \tag{2.11}$$

Suppose that $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$ such that $g_\varepsilon^{in} \to g^{in}$ entropically of order ε^m for some $m \geqq 1$. Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem (1.63) and let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g and q be limits of the sequences g_ε and $q_\varepsilon / N_\varepsilon$ in w-$L^1_{\text{loc}}(dt;$ w-$L^1(M dv\, dx))$ and w-$L^1_{\text{loc}}(dt;$ w-$L^1(d\mu\, dx))$ respectively. Then g has the form of an infinitesimal Maxwellian (2.7), where $\rho \in L^2(dt; \mathcal{V}_s)$, $u \in L^2(dt; \mathcal{V}_v)$, and $\nabla_x \theta \in L^2(dt; L^2(dx))$ satisfy the inequality

$$\begin{aligned} &\int \frac{1}{2} \left(\rho(t)^2 + |u(t)|^2 + \frac{D}{2}\, \theta(t)^2 \right) dx \\ &+ \int_0^t \int \frac{1}{2} \nu \left| \nabla_x u + (\nabla_x u)^T \right|^2 + \frac{D+2}{2} \kappa \left| \nabla_x \theta \right|^2 dx\, ds \\ &\leqq \int \frac{1}{2} \left(\rho^{in2} + |u^{in}|^2 + \frac{D}{2} \theta^{in2} \right) dx . \end{aligned} \tag{2.12}$$

The proof is based essentially on the convexity of the integrands of both the entropy H and the entropy dissipation rate R, and on the $d\mu$-symmetries (1.24). Various implications of the normalized Boltzmann equation are used in the proof, in particular, the limiting Boltzmann equation (2.8) and the incompressibility and Boussinesq relations (2.9).

The implications of the normalized Boltzmann equation are the subject matter of Section 4. In particular, the above results are contained in Propositions 4.1, 4.2, and 4.9.

The Stokes Limit

So far, the local conservation laws associated to the Boltzmann equation have not been used. The only local conservation law known to be satisfied by all renormalized solutions of the Boltzmann equation, however, is that of mass (1.57). In order to formulate the hydrodynamic limits (which are obviously based on the fundamental principle of dynamics), we are consistently led to restrict our attention to sequences of renormalized solutions G_ε of the scaled Boltzmann initial-value problem (1.63) such that the following assumption holds:

(H0). The solutions G_ε satisfy the local momentum conservation law:

$$\partial_t\langle vG_\varepsilon\rangle + \frac{1}{\varepsilon}\nabla_x\cdot\langle v \otimes vG_\varepsilon\rangle = 0 . \tag{2.13}$$

Whether renormalized solutions of (1.63) generally satisfy (H0) is still an open problem.

The Stokes equation will be obtained as the limiting form of the above local momentum conservation law as ε tends to zero. But, in order to take the small ε limit in the local momentum conservation law, it is essential to control the high velocity tails of the quantities involved. High velocities are obviously generated by the collision operator. It is therefore little wonder that controlling the high velocity tails can be achieved by some assumptions bearing on the Boltzmann kernel b. To achieve the Stokes and the time-discretized Navier-Stokes limits, we shall make the following assumption:

(H1). The Boltzmann kernel b is that of a cut-off hard potential (see [7]) such that the two following inequalities hold

$$\big(|\phi(v)| + |\phi(v_1)|\big)\, b(v_1 - v, \omega) \leqq C\left(1 + |v|^2 + |v_1|^2\right) , \tag{2.14a}$$

$$(1 + |v|^2) \leqq C\left(1 + |\phi(v)|\right)^2 , \tag{2.14b}$$

where $\phi = \phi(v)$ is the matrix valued function defined by (1.46).

Assumption (H1) is certainly satisfied by Maxwell potentials. In that case the key observation is that the entries of the matrix ϕ are eigenfunctions of the linearized collision operator L (see [7]); both inequalities in (H1) then follow from (1.46). The properties of the linearized collision operator L related to assumption (H1) will be discussed in Appendix C.

Our main result concerning the Stokes limit is the following.

THEOREM 2.5. (THE STRONG STOKES LIMIT) *Assume* (H1). *Let* $u^{in} \in \mathcal{H}_v$ *and define the infinitesimal Maxwellian* g^{in} *by*

$$g^{in} = u^{in}\cdot v . \tag{2.15}$$

Let $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$ *be any sequence such that* $g_\varepsilon^{in} \to g^{in}$ *entropically of order* ε^m *for some* $m > 1$. *Let* $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon \geqq 0$ *be any corresponding sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *that satisfies* (H0). *Then*

$$g_\varepsilon(t) \to u(t)\cdot v \qquad \textit{entropically of order } \varepsilon^m \textit{ for almost every } t > 0 , \tag{2.16}$$

where $u(t)$ is the unique solution of the Stokes initial-value problem

(2.17a) $$\partial_t u + \nabla_x p = \nu \Delta_x u \,, \qquad \nabla_x \cdot u = 0 \,,$$

(2.17b) $$u(0) = u^{in} \,,$$

with the viscosity ν given by formula (1.51). *Moreover, the normalized scaled collision integrands converge strongly to q:*

(2.18) $$\frac{q_\varepsilon}{N_\varepsilon} \to q = \left(\nabla_x u + (\nabla_x u)^T\right) : \Phi \quad \text{in } L^1_{\mathrm{loc}}\left(dt; L^1((1+|v|^2)d\mu\, dx)\right) ,$$

where $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$ and ϕ is given by (1.46).

The strong Stokes limit theorem is proved in Section 6 (see Theorem 6.2).

A key step in its proof is the following compactness result: that any consistently scaled sequence of DiPerna-Lions solutions has a subsequence whose velocity moments converge weakly to a solution of the Stokes equation (see Theorem 6.1). This result is based on the following line of arguments. Recast the local momentum conservation law as

(2.19) $$\partial_t \langle v g_\varepsilon \rangle + \nabla_x \cdot \frac{1}{\varepsilon} \langle (L\phi)\, g_\varepsilon \rangle = -\nabla_x \frac{1}{\varepsilon} \left\langle \frac{1}{\mathrm{D}} |v|^2 g_\varepsilon \right\rangle .$$

Then, using the self-adjointness of L, the quadratic character of Q, and the definition of the scaled collision integrand q_ε, one obtains

(2.20) $$\begin{aligned} &\frac{1}{\varepsilon} \langle (L\phi)\, g_\varepsilon \rangle \\ &= \frac{1}{\varepsilon} \langle \phi\, L g_\varepsilon \rangle \\ &= \frac{1}{\varepsilon} \left\langle \phi \left(1 - \frac{1}{N_\varepsilon}\right) L g_\varepsilon \right\rangle + \varepsilon^{m-1} \left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle - \frac{1}{\varepsilon^{m+1}} \left\langle \phi \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} \right\rangle \\ &= \frac{1}{\varepsilon} \left\langle \phi \left(1 - \frac{1}{N_\varepsilon}\right) L g_\varepsilon \right\rangle + \varepsilon^{m-1} \left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle - \left\langle\!\!\left\langle \phi \frac{q_\varepsilon}{N_\varepsilon} \right\rangle\!\!\right\rangle . \end{aligned}$$

Using various implications of the entropy bounds (2.4), one can show that the first two terms in the right side of (2.20) vanish as ε tends to zero. The last term in the right side of (2.20) is first computed with the help of the limiting Boltzmann equation (2.8) and the infinitesimal Maxwellian form (2.7):

$$\left\langle\!\!\left\langle \phi \frac{q_\varepsilon}{N_\varepsilon} \right\rangle\!\!\right\rangle \to \langle\!\langle \phi q \rangle\!\rangle = \langle \phi v \cdot \nabla_x g \rangle = \nu \left(\nabla_x u + (\nabla_x u)^T\right) .$$

One sees from the above relation that this last term in the right side of (2.20) corresponds exactly to the viscosity term in the Stokes equation.

The local conservation of momentum is integrated against a test vector field which is divergence free (with respect to x); only then does one take the limits of the various terms involved as ε tends to zero. This procedure eliminates the pressure term on right side of (2.19). The pressure in the Stokes equation is therefore nothing but a Lagrange multiplier associated with the constraint of the incompressibility relation of (2.9).

The Time-Discretized Navier-Stokes Limit

The scaling leading to the nonlinear Navier-Stokes equation corresponds to the case $m = 1$ in the entropy bound (2.4) on the initial data. For various reasons discussed below, we have not been able to prove the exact analog of the Stokes limit theorem in the case where $m = 1$. The main simplification we have to concede is to study time-discretized analogs of the evolution equations above. The scaled time-discretized Boltzmann problem is

$$\varepsilon \frac{G_\varepsilon - G_\varepsilon^{in}}{\Delta t} + v \cdot \nabla_x G_\varepsilon = \frac{1}{\varepsilon} Q(G_\varepsilon, G_\varepsilon) \; ; \tag{2.21}$$

it is an implicit time discretization of the scaled Boltzmann equation (1.63a). Throughout this article, we shall always set the time step $\Delta t = 1$. With the same definitions as in (2.1) and (2.3), the normalized Boltzmann equation reads:

$$\varepsilon \frac{g_\varepsilon - g_\varepsilon^{in}}{N_\varepsilon} + v \cdot \nabla_x \gamma_\varepsilon = \frac{1}{\varepsilon^2} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} \, . \tag{2.22}$$

The DiPerna-Lions theory can be transposed to this new problem without significant change. The form of the entropy inequality is, however, somewhat different:

$$H(G_\varepsilon) + J(G_\varepsilon^{in}, G_\varepsilon) + \frac{1}{\varepsilon^2} R(G_\varepsilon) \leqq H(G_\varepsilon^{in}) \, , \tag{2.23}$$

where $J(G_\varepsilon^{in}, G_\varepsilon)$ is the relative entropy of G_ε^{in} with respect to G_ε which is given by

$$J(G_\varepsilon^{in}, G_\varepsilon) = \int \left\langle G_\varepsilon^{in} \log\left(\frac{G_\varepsilon^{in}}{G_\varepsilon} \right) - G_\varepsilon^{in} + G_\varepsilon \right\rangle dx \, . \tag{2.24}$$

The corresponding time-discretized Navier-Stokes equation reads

$$u + \nabla_x \cdot (u \otimes u) + \nabla_x p = \nu \Delta_x u + u^{in} \, , \qquad \nabla_x \cdot u = 0 \, . \tag{2.25}$$

In any dimension, for every u^{in} in $\mathscr{H}_v$, this equation has a solution in $\mathscr{V}_v$ that satisfies the Leray energy inequality:

$$\int |u|^2 \, dx + \int \nu |\nabla_x u|^2 \, dx \leqq \int u^{in} \cdot u \, dx \, . \tag{2.26}$$

In dimension D = 2, 3, 4, any solution of the time-discretized Navier-Stokes equation in $\mathcal{V}_\nu$ satisfies the equality in (2.26).

For a sequence of initial data for (2.21) chosen to satisfy the entropy bound

$$H(G_\varepsilon^{in}) \leqq C^{in}\varepsilon^2 ,$$

analogs of the results implied by the evolution entropy inequality and the evolution Boltzmann equation hold and are proved in the corresponding sections.

For the same reason as in the previous subsection, in order to derive the Navier-Stokes limit, it has been necessary to assume the local momentum conservation law for the renormalized solutions G_ε of (2.21) considered:

(H0′). G_ε satisfies the time-discretized local momentum conservation law

$$\langle vG_\varepsilon\rangle + \frac{1}{\varepsilon}\nabla_x\cdot\langle v\otimes vG_\varepsilon\rangle = \langle vG_\varepsilon^{in}\rangle \ .$$

Most of the proof of the Stokes limit theorem can be reproduced in the case where $m = 1$. The second term on the right side of (2.20), however, does not tend to zero, but is formally expected to converge to the (quadratic) convection tensor in the Navier-Stokes equation (see Section 1 and reference [3]). It becomes, therefore, essential to control this nonlinear term at high velocities. To this end, we have been led to introduce the supplementary assumption

(H2). The family $(1 + |v|^2)g_\varepsilon^2/N_\varepsilon$ is relatively compact in $w\text{-}L^1(Mdv\,dx)$.

The term $g_\varepsilon^2/N_\varepsilon$ somehow measures the difference between the entropy bound (2.4) and an L^2 bound on g_ε. Section 3 contains a proof of the following partial result in this direction (see Proposition 3.3):

(1) $\dfrac{g_\varepsilon^2}{N_\varepsilon}$ is bounded in $L^\infty(dt;L^1(Mdv\,dx))$,

(2) $|v|^2\dfrac{g_\varepsilon^2}{N_\varepsilon} = O\left(\log\left(\dfrac{1}{\varepsilon|\log(\varepsilon)|}\right)\right)$ in $L^\infty(dt;L^1(Mdv\,dx))$.

This result is enough to take the limit of the term

$$\varepsilon^{m-1}\left\langle \phi\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\right\rangle$$

in the case where $m > 1$; assumption (H2) is needed, however, to achieve the same in the case where $m = 1$.

THEOREM 2.6. (THE STRONG NAVIER-STOKES LIMIT) *Assume* (H1) *and* D $\leqq$ 4. *Let* $u^{in} \in \mathcal{H}_\nu$ *and define the infinitesimal Maxwellian by*

$$g^{in} = u^{in}\cdot v \ .$$

Let $G_\varepsilon^{in} = 1 + \varepsilon g_\varepsilon^{in} \geqq 0$ *be any sequence of initial data such that* g_ε^{in} *converges to* g^{in} *entropically of order* ε. *Let* $G_\varepsilon = 1 + \varepsilon g_\varepsilon \geqq 0$ *be any corresponding family of renormalized solutions of the time-discretized Boltzmann equation* (2.22) *that satisfies* (H0′) *and* (H2). *Then the family* g_ε *is relatively compact in* $w\text{-}L^1((1 + |v|^2)M\,dv\,dx)$ *and for any convergent subsequence (again denoted* g_ε*)*

$$g_\varepsilon \to g = u\cdot v \qquad \textit{entropically of order } \varepsilon\ ,$$

where $u \in \mathcal{V}_v$ *is a weak solution of the time-discretized Navier-Stokes equation*

$$u + \nabla_x\cdot(u \otimes u) + \nabla_x p = \nu\Delta_x u + u^{in}\ , \qquad \nabla_x\cdot u = 0\ ,$$

with the viscosity ν *given by the formula* (1.51). *Moreover, the normalized scaled collision integrands converge strongly to* q*:*

$$\frac{q_\varepsilon}{N_\varepsilon} \to q = \left(\nabla_x u + (\nabla_x u)^T\right) : \Phi\ , \quad \textit{in } L^1((1 + |v|^2)d\mu\,dx)\ ,$$

where $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$.

The main difference with the Stokes limit theorem lies in the treatment of the nonlinearity in

$$\left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle . \tag{2.27}$$

The strategy used in this article is based on earlier results (see [4] and [15]). It is based on a two-step procedure.

Step 1. Write

$$g_\varepsilon = w_\varepsilon + z_\varepsilon$$

with $z_\varepsilon(x, \cdot) \in \mathrm{N}(L)$ for almost every x and $\langle w_\varepsilon \chi\rangle = 0$ for all $\chi \in N(L)$. A consequence of the entropy dissipation bound (2.4) is the fact that $w_\varepsilon \to 0$ for almost every (x, v) as ε tends to zero. In other words, the entropy dissipation bound measures the distance between the fluctuation of the number density and the linear space of infinitesimal Maxwellians.

Step 2. Observe that

$$z_\varepsilon = \langle g_\varepsilon\rangle + \langle v g_\varepsilon\rangle\cdot v + \frac{2}{\mathrm{D}}\left\langle\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) g_\varepsilon\right\rangle \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) . \tag{2.28}$$

Therefore, in view of Step 1, proving the pointwise convergence of g_ε amounts to proving the pointwise convergence of its velocity averages. The tool best adapted to investigating those properties is the velocity averaging theorem given in Golse, Lions, Perthame, and Sentis; see [13] and [14].

THEOREM 2.7. (VELOCITY AVERAGING) *Assume that:*

(i) *The family* $f_\varepsilon = f_\varepsilon(t,x,v)$ *is bounded in* $L^2_{\mathrm{loc}}(dt; L^2((1+|v|^2)M dv\, dx))$,
(ii) *The family* $T_\varepsilon \equiv \varepsilon\,\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon$ *is bounded in* $L^2_{\mathrm{loc}}(dt; L^2((1+|v|^2)M dv\, dx))$.

Then, for any measurable function $\chi = \chi(v)$ *subquadratic as* $|v| \to \infty$, *the family* $\langle \chi f_\varepsilon \rangle$ *is bounded in* $L^2_{\mathrm{loc}}(dt; H^{1/2}(dx))$ *and for every* $t < t'$ *there exists some constant* C *such that the following inequality holds:*

$$\|\langle \chi f_\varepsilon\rangle\|_{L^2([t,t'];H^{1/2}(dx))} \leqq C \|f_\varepsilon\|^{1/2}_{L^2([t,t'];L^2((1+|v|^2)M dv\, dx))} \|T_\varepsilon\|^{1/2}_{L^2([t,t'];L^2((1+|v|^2)M dv\, dx))} \,.$$

Classical interpolation arguments allow one to state an analog of the velocity averaging theorem in L^1 *spaces.*

With the argument sketched above, it can be proved that the nonlinear term (2.27) converges to $u \otimes u$ modulo a matrix proportional to the identity, which can therefore be absorbed in the pressure term. The proof of the strong Navier-Stokes limit theorem is carried through in Section 7 (see Theorem 7.4). As in the case of the Stokes limit, a key step in order to prove the strong Navier-Stokes limit theorem is the following statement, which holds in any dimension: any consistently scaled sequence of DiPerna-Lions solutions has a subsequence whose velocity moments converge to a solution of the time-discretized Navier-Stokes equation (see Theorem 7.3).

Observe that the velocity averaging theorem stated above does not provide relative compactness with respect to the variable t. This is the very reason for which we have proved the Navier-Stokes limit theorem for the time-discretized problem. In other words, the velocity averaging theorem is used in the particular case of functions constant in time.

3. Implications of the Entropy Inequality

Convexity

As stated in Section 1, DiPerna and Lions (see [10]) have proven the global existence of G_ε, a renormalized solution to the scaled Boltzmann initial-value problem (1.63), satisfying the entropy inequality (1.65),

$$\begin{aligned}
&\int \langle G_\varepsilon(t)\log G_\varepsilon(t) - G_\varepsilon(t) + 1\rangle \, dx \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle (G'_{\varepsilon 1}G'_\varepsilon - G_{\varepsilon 1}G_\varepsilon)\log\left(\frac{G'_{\varepsilon 1}G'_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right)\right\rangle\!\!\right\rangle dx\, ds \qquad (3.1)\\
&\leqq \int \langle G^{in}_\varepsilon \log G^{in}_\varepsilon - G^{in}_\varepsilon + 1\rangle \, dx \,.
\end{aligned}$$

Consider a sequence of such solutions G_ε indexed by a vanishing positive sequence ε such that for some constant $C^{in} > 0$, the initial data G_ε^{in} satisfies the entropy bound (1.66):

$$\int \langle G_\varepsilon^{in} \log G_\varepsilon^{in} - G_\varepsilon^{in} + 1 \rangle \, dx \leqq C^{in} \varepsilon^{2m} . \tag{3.2}$$

This then implies bounds on the sequence G_ε through the entropy inequality (3.1). This section contains results that follow directly from the convexity of the integrands in the entropy inequality (3.1) and the entropy bound (3.2).

Since the entropy integrand, $G \log G - G + 1$, is a strictly convex function of G with a quadratic minimum of zero at $G = 1$, the integral approximately measures the square of the deviations from this minimum. This suggests introducing the fluctuation g_ε defined by $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon$ and the convex function $h = h(z)$ defined over $z > -1$ by

$$h(z) = (1 + z) \log(1 + z) - z . \tag{3.3}$$

The entropy inequality (3.1) and entropy bound (3.2) then give

$$\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t)) \right\rangle dx \leqq \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in}) \right\rangle dx \leqq C^{in} . \tag{3.4}$$

Proposition 3.1 will state that these bounds imply that the families g_ε and g_ε^{in} are relatively compact (therefore bounded) sets in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx))$ and $w\text{-}L^1(M dv\, dx)$ respectively, a so-called "entropy control."

The second integral on the left side of the entropy inequality is the entropy dissipation. The convexity here is a bit subtle; its form suggests the introduction of the scaled collision integrand $q_\varepsilon = \varepsilon^{-(m+1)}(G'_{\varepsilon 1} G'_\varepsilon - G_{\varepsilon 1} G_\varepsilon)$ and the convex function $r = r(z)$ defined over $z > -1$ by

$$r(z) = z \log(1 + z) . \tag{3.5}$$

The entropy inequality (3.1) and the entropy bound (3.2) can then be recast in the form

$$\begin{aligned} &\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t)) \right\rangle dx + \int_0^t \int \frac{1}{4} \left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon} \right) G_{\varepsilon 1} G_\varepsilon \right\rangle\!\!\right\rangle dx\, ds \\ &\qquad \leqq \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in}) \right\rangle dx \leqq C^{in} . \end{aligned} \tag{3.6}$$

Proposition 3.5 will state that the family of scaled collision integrands q_ε divided by a normalization N_ε is a relatively compact set in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\, dx))$, a so-called "dissipation control."

The scaled Boltzmann time-discretized problem (2.21) brings a new element to the convexity story. Its renormalized solutions satisfy the entropy inequality (2.23),

$$
\begin{aligned}
&\int \langle G_\varepsilon \log G_\varepsilon - G_\varepsilon + 1\rangle\, dx \\
&+ \int \left\langle G_\varepsilon^{in} \log\left(\frac{G_\varepsilon^{in}}{G_\varepsilon}\right) - G_\varepsilon^{in} + G_\varepsilon \right\rangle dx \\
&+ \frac{1}{\varepsilon^2} \int \frac{1}{4} \left\langle\!\!\left\langle (G'_{\varepsilon 1} G'_\varepsilon - G_{\varepsilon 1} G_\varepsilon) \log\left(\frac{G'_{\varepsilon 1} G'_\varepsilon}{G_{\varepsilon 1} G_\varepsilon}\right) \right\rangle\!\!\right\rangle dx \\
&\leqq \int \langle G_\varepsilon^{in} \log G_\varepsilon^{in} - G_\varepsilon^{in} + 1\rangle\, dx\,.
\end{aligned}
\tag{3.7}
$$

The convexity for the integrands of all but the middle term on the left side of inequality (3.7) is as it was for the continuous time problem. The middle term is just the entropy of G_ε^{in} relative to G_ε; as such, its integrand is easily understood to be a non-negative convex function of G_ε^{in}. It is one of the beautiful properties of the classical entropy that this integrand is a jointly convex function of both of its arguments, G_ε^{in} and G_ε.

Since only the Navier-Stokes limit will be considered for this problem, sequences of such solutions G_ε indexed by a vanishing positive sequence ε are taken with initial data G_ε^{in} satisfying the entropy bound (3.2) with $m = 1$ for some constant $C^{in} > 0$. Once again using the fluctuation g_ε defined by $G_\varepsilon = 1 + \varepsilon g_\varepsilon$, and introduce the convex function $j = j(z, y)$ defined over $z, y > -1$ by

$$
j(z,y) = (1+z)\log\left(\frac{1+z}{1+y}\right) - z + y = h(z) - h(y) - h'(y)(z-y)\,.
\tag{3.8}
$$

The entropy inequality (3.7) and the entropy bound (3.2) (with $m = 1$) can then be recast in the form

$$
\begin{aligned}
&\int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon) \right\rangle dx + \int \left\langle \frac{1}{\varepsilon^2} j(\varepsilon g_\varepsilon^{in}, \varepsilon g_\varepsilon) \right\rangle dx \\
&\qquad + \int \frac{1}{4} \left\langle\!\!\left\langle \frac{1}{\varepsilon^4} r\left(\frac{\varepsilon^2 q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon}\right) G_{\varepsilon 1} G_\varepsilon \right\rangle\!\!\right\rangle dx \\
&\qquad \leqq \int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon^{in}) \right\rangle dx \leqq C^{in}\,.
\end{aligned}
\tag{3.9}
$$

Proposition 3.5 will state that the family of scaled collision integrands q_ε divided by a normalization N_ε is a relatively compact set in $w\text{-}L^1(d\mu\, dx)$), another so-called "dissipation control."

All the results in this section will be obtained from properties of the functions h and r defined over the interval $z > -1$. The foremost of these properties is convexity; this will often be used through the Young inequality. Generally stated, if f and f^* are strictly convex functions defined over the convex domains D and D^* in the dual linear spaces E and E^* respectively that are dual under the Legendre transformation (see [1]) then they satisfy the inequality

$$\langle y; z\rangle_{E^*E} \leqq f^*(y) + f(z)\,, \tag{3.10}$$

for every $z \in D$ and $y \in D^*$. The Legendre transform of h (3.3) is explicitly given by

$$h^*(y) = \exp(y) - 1 - y\,, \tag{3.11}$$

while that of r (3.5) is implicitly determined by

$$r^*(y) = \frac{z^2}{1+z}\,, \qquad y = \log(1+z) + \frac{z}{1+z}\,; \tag{3.12}$$

both are defined for all $y \in \mathbb{R}$. Notice that j is not strictly convex; its Legendre transform is singular and plays no role in the sequel.

Two other properties of these functions play a role in what follows. First, the functions h and r satisfy the elementary reflection inequalities

$$h(|z|) \leqq h(z)\,, \qquad r(|z|) \leqq r(z)\,, \tag{3.13}$$

over the interval $z > -1$. Second, the functions h^* and r^* have superquadratic homogeneity for $y > 0$; this means that

$$h^*(\lambda y) \leqq \lambda^2 h^*(y)\,, \qquad r^*(\lambda y) \leqq \lambda^2 r^*(y)\,, \tag{3.14}$$

for any $y > 0$ and $0 \leqq \lambda \leqq 1$.

Entropy Controls

The first step is the following compactness result which shows that the entropy bound (3.2) provides the necessary control to justify the scaling of the fluctuations used in the formal arguments.

PROPOSITION 3.1. *Let $\varepsilon = \{\varepsilon_n\}$ be any positive sequence converging to zero. Let g_ε be any sequence of functions in $L^\infty(dt; L^1(Mdv\,dx))$ with $1 + \varepsilon^m g_\varepsilon \geqq 0$. Assume there exists a positive constant C^{in} such that g_ε satisfies the entropy bound*

$$\int \left\langle \frac{1}{\varepsilon^{2m}} h\left(\varepsilon^m g_\varepsilon(t)\right)\right\rangle dx \leqq C^{in} . \tag{3.15}$$

Then the following assertions hold:

(1) *The sequence $(1 + |v|^2)g_\varepsilon$ is bounded in $L^\infty(dt; L^1(Mdv\,dx))$ and relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(Mdv\,dx))$;*
(2) *If g is the $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(Mdv\,dx))$ limit of any converging subsequence of g_ε then $g \in L^\infty(dt; L^2(Mdv\,dx))$ and for almost every $t \in [0, \infty)$ it satisfies*

$$\int \frac{1}{2} \langle g^2(t)\rangle\, dx \leqq \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t))\right\rangle dx \leqq C^{in} . \tag{3.16}$$

Remark. As it is stated, Proposition 3.1 may be applied to the sequence g_ε^{in} by considering the g_ε to be independent of time; the temporal component of the above spaces then trivializes. This observation is also used in later sections when applying this, as well as subsequent propositions, to the treatment of the time-discretized Navier-Stokes limit.

Proof: Let $\alpha > 0$ (to be chosen). First apply the Young inequality (3.10) to h and h^* with $z = \varepsilon^m |g_\varepsilon|$ and $y = \varepsilon^m \frac{1}{4}(1 + |v|^2)/\alpha$ and invoke the reflection inequality for h (3.12). For all ε such that $\varepsilon^m \leqq \alpha$ use the superquadratic homogeneity of h^* (3.13) with $\lambda = \varepsilon^m/\alpha$ to obtain the bound

$$\begin{aligned} \frac{1}{4}(1 + |v|^2)\,|g_\varepsilon| &\leqq \frac{\alpha}{\varepsilon^{2m}} h^*\left(\frac{\varepsilon^m}{\alpha}\frac{1}{4}(1 + |v|^2)\right) + \frac{\alpha}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) \\ &\leqq \frac{1}{\alpha} h^*\left(\frac{1}{4}(1 + |v|^2)\right) + \frac{\alpha}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) . \end{aligned} \tag{3.17}$$

From the explicit form for h^* (3.11), it is manifest that the first term on the right side above is in $L^1(Mdv)$, while the second term is is uniformly bounded in $L^1(Mdv\,dx)$ by (3.15). Thus, integrating (3.17) over $\mathbb{T}^D \times \mathbb{R}^D$ after choosing α larger than any value of ε^m shows that the left side of (3.17) is a bounded sequence in $L^1_{\rm loc}(dt; L^1(Mdv\,dx))$.

The equi-integrability of the sequence must be demonstrated in order to verify its relative compactness; see [11]. Integrating (3.17) over any measurable $\Omega \subset [0,T] \times \mathbb{T}^D \times \mathbb{R}^D$ and using the entropy bound (3.15) gives

$$\iiint_\Omega \frac{1}{4}(1+|v|^2)\,|g_\varepsilon|\,dv\,dx\,dt$$

(3.18)

$$\leqq \frac{1}{\alpha}\iiint_\Omega h^*\left(\frac{1}{4}(1+|v|^2)\right) dv\,dx\,dt + \alpha T C^{in}\,.$$

Take $\eta > 0$ arbitrarily small. First choose $\alpha = \eta/(2TC^{in})$, then pick $\delta > 0$ such that

$$\text{meas}(\Omega) < \delta \quad \text{implies} \quad \iiint_\Omega h^*\left(\frac{1}{4}(1+|v|^2)\right) dv\,dx\,dt \leqq \frac{1}{2}\alpha\eta\,.$$

For all ε with $\varepsilon^m \leqq \alpha$ this choice of δ will ensure that the left side of (3.18) will be smaller than η for any Ω with meas$(\Omega) < \delta$. The finite set of members of ε with $\varepsilon^m > \alpha$ can be accommodated by picking δ as small as necessary. This proves assertion (1).

Now let g be the w-$L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\,dx))$ limit of any convergent subsequence of g_ε. The convexity of h gives the inequality

$$\frac{1}{\varepsilon^{2m}}h(\varepsilon^m g) + \frac{1}{\varepsilon^m}h'(\varepsilon^m g)(g_\varepsilon - g) \leqq \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon)\,.$$

Fix $\lambda > 0$ and multiply this inequality by the indicator (characteristic) function $\mathbf{1}_{|g|<\lambda}$; the non-negativity of h then implies

$$\frac{1}{\varepsilon^{2m}}h(\varepsilon^m g)\mathbf{1}_{|g|<\lambda} + \frac{1}{\varepsilon^m}h'(\varepsilon^m g)\mathbf{1}_{|g|<\lambda}(g_\varepsilon - g) \leqq \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon)\,.$$

Average this over $[t_1,t_2] \times \mathbb{T}^D \times \mathbb{R}^D$ for an arbitrary time interval $[t_1,t_2]$ and then consider its limit as ε tends to zero. Use the strong L^∞ limits

$$\frac{1}{\varepsilon^{2m}}h(\varepsilon^m g)\mathbf{1}_{|g|<\lambda} \to \frac{1}{2}g^2\mathbf{1}_{|g|<\lambda}\,, \qquad \frac{1}{\varepsilon^m}h'(\varepsilon^m g)\mathbf{1}_{|g|<\lambda} \to g\mathbf{1}_{|g|<\lambda}\,,$$

and the w-L^1 limit $(g_\varepsilon - g) \to 0$ to show that

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\int \frac{1}{2}\left\langle g^2\mathbf{1}_{|g|<\lambda}\right\rangle dx\,dt$$

$$\leqq \liminf_{\varepsilon\to 0}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\int \left\langle \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon)\right\rangle dx\,dt \leqq C^{in}\,.$$

Taking $\lambda \to +\infty$ and using the arbitrariness of the interval $[t_1, t_2]$ completes the proof of assertion (2).

The following corollary considers a $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$ convergent subsequence of g_ε (still denoted g_ε) and its limit g in $L^\infty(dt; L^2(M dv\, dx))$. It concerns some technical results regarding certain functions of g_ε that will subsequently be used to approximate g_ε and g. The first is γ_ε, the nonlinear function of g_ε that appears in the normalized Boltzmann equation (2.2). Note that $\gamma_\varepsilon = \varepsilon^{-m} t(\varepsilon^m g_\varepsilon)$ where $t(z) = 3\log(1 + \frac{1}{3}z)$. The second is the decomposition

$$g_\varepsilon = \frac{g_\varepsilon}{N_\varepsilon} + \frac{1}{3}\varepsilon^m \frac{g_\varepsilon^2}{N_\varepsilon}\,.$$

This will be used frequently for various technical reasons, in particular to control terms in the collision operator. These approximations make explicit the fact that g_ε converges in $w\text{-}L^1$ to a limit g that is in L^2, as we shall see from the following corollary.

COROLLARY 3.2. *Given g_ε as above:*

(1) *The sequences $g_\varepsilon/N_\varepsilon$ and γ_ε converge to g in $w\text{-}L^2_{\text{loc}}(dt; w\text{-}L^2(M dv\, dx))$;*
(2) *As ε tends to zero, $g_\varepsilon - \gamma_\varepsilon = O(\varepsilon^m)$ in $L^\infty(dt; L^1(M dv\, dx))$;*
(3) *The sequence $g_\varepsilon^2/N_\varepsilon$ is bounded in $L^\infty(dt; L^1(M dv\, dx))$.*

Proof: Assertion (1) will follow from assertions (2) and (3) once the sequences are shown to be bounded in $L^2_{\text{loc}}(dt; L^2(M dv\, dx))$. But such bounds follow immediately from the entropy bound (3.15) by setting $z = \varepsilon^m g_\varepsilon$ into the following elementary inequalities (for $z \geqq -1$)

$$\left(\frac{z}{1 + \frac{1}{3}z}\right)^2 \leqq 3\,h(z)\,, \qquad t(z)^2 \leqq 2\,h(z)\,.$$

Assertion (2) follows from assertion (3) by setting $z = \varepsilon^m g_\varepsilon$ into the elementary inequality

$$\frac{z}{1 + \frac{1}{3}z} \leqq t(z) \leqq z\,;$$

thus obtaining

$$0 \leqq g_\varepsilon - \gamma_\varepsilon \leqq g_\varepsilon - \frac{g_\varepsilon}{1 + \frac{1}{3}\varepsilon^m g_\varepsilon} = \frac{1}{3}\varepsilon^m \frac{g_\varepsilon^2}{N_\varepsilon}\,.$$

The right side above is then $O(\varepsilon^m)$ in $L^\infty(dt; L^1(M dv\, dx))$ by assertion (3).

In order to prove assertion (3) we introduce the function

$$s(z) = \frac{1}{2}\frac{z^2}{1 + \frac{1}{3}z}\,. \tag{3.19}$$

Observe that $s(z) \leqq h(z)$ for $z \geqq -1$; indeed $s''(z) = (1+\frac{1}{3}z)^{-3} \leqq (1+z)^{-1} = h''(z)$ and $h(z) - s(z) = O(z^4)$ as $z \to 0$. Therefore

$$\frac{1}{2}\frac{g_\varepsilon^2}{N_\varepsilon} = \frac{1}{\varepsilon^{2m}} s(\varepsilon^m g_\varepsilon) \leqq \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) \,,$$

whence assertion (3) follows from the entropy bound.

The most technical point in this section consists of controlling the behavior of $g_\varepsilon^2/N_\varepsilon$ for the large values of v. We obtain only a partial success in this direction in the sense that the following control is not sufficient to prove convergence to the Navier-Stokes equation; it will be strong enough, however, to prove convergence to the Stokes equation. This result should be compared with assertion (3) of Corollary 3.2 above and with assumption (H2).

PROPOSITION 3.3. *As ε tends to zero*

$$|v|^2 \frac{g_\varepsilon^2}{N_\varepsilon} = O\left(\log\left(\frac{1}{\varepsilon|\log(\varepsilon)|}\right)\right) \,, \qquad \text{in } L^\infty(dt; L^1(M dv\, dx)) \,. \tag{3.20}$$

Proof: Start with the observation that the function $w \mapsto h'(w)/s'(w)$ is increasing over $w \in (0,\infty)$. This follows from the fact that for every $w \in (0,\infty)$

$$s'(w)h''(w) - h'(w)s''(w) = \frac{w(1+\frac{1}{6}w)}{(1+\frac{1}{3}w)^2}\frac{1}{1+w} - \log(1+w)\frac{1}{(1+\frac{1}{3}w)^3} > 0 \,.$$

Moreover, the values of $h'(w)/s'(w)$ over $(0,\infty)$ range from 1 to ∞. This monotonicity then implies that every positive w and z satisfy

$$\frac{h'(w)}{s'(w)} s(z) \leqq h(z) + \left(s(w)\frac{h'(w)}{s'(w)} - h(w)\right) \,. \tag{3.21}$$

This inequality is nothing but the Young inequality applied to the function $f = h \circ s^{-1}$, but can be inferred more directly by noticing that equality holds when $z = w$ and using the monotonicity to compare the z derivatives of each of its sides.

Setting $z = \varepsilon^m g_\varepsilon$ into (3.21) and dividing by ε^{2m} gives

$$\frac{h'(w)}{s'(w)}\frac{1}{\varepsilon^{2m}} s(\varepsilon^m g_\varepsilon) \leqq \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) + \frac{1}{\varepsilon^{2m}}\left(s(w)\frac{h'(w)}{s'(w)} - h(w)\right) \,, \tag{3.22}$$

for every positive w. In this proof we shall choose $w = w_\varepsilon(v)$ to be the solution of the equation

$$\frac{h'(w)}{s'(w)} = 1 + \log\left(1 + \varepsilon^m \exp\left(\frac{1}{8}|v|^2\right)\right) \,. \tag{3.23}$$

Notice that the values of the right side of this equation span range of values of the function on the left.

The proof now contains three technical steps which we state here in order of usage, deferring their verification.

Step 1. The function w_ε determined by (3.23) satisfies the bound

$$\zeta(\varepsilon)\frac{1}{8}|v|^2 \leqq \frac{h'(w_\varepsilon)}{s'(w_\varepsilon)}, \tag{3.24}$$

where $\zeta = \zeta(\varepsilon)$ is the solution of

$$1 - \zeta\log(\zeta) - (1-\zeta)\log(1-\zeta) + \zeta\log(\varepsilon^m) = 0\,. \tag{3.25}$$

Step 2. There exists a constant C, independent of ε, such that w_ε determined by (3.23) satisfies the bound

$$\frac{1}{\varepsilon^{2m}}\left\langle\left(s(w_\varepsilon)\frac{h'(w_\varepsilon)}{s'(w_\varepsilon)} - h(w_\varepsilon)\right)\right\rangle < C\,. \tag{3.26}$$

Step 3. Given that $\zeta = \zeta(\varepsilon)$ is the solution of (3.25), then $\zeta(\varepsilon)$ satisfies the asymptotic bound

$$\frac{1}{\zeta(\varepsilon)} = O\left(\log\left(\frac{1}{\varepsilon|\log(\varepsilon)|}\right)\right)\,. \tag{3.27}$$

Given these three steps, the remainder of the proof follows directly. First, combining inequality (3.24) of Step 1 with inequality (3.22) gives

$$\zeta(\varepsilon)\frac{1}{8}|v|^2\frac{1}{\varepsilon^{2m}}s(\varepsilon^m g_\varepsilon) \leqq \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon) + \frac{1}{\varepsilon^{2m}}\left(s(w_\varepsilon)\frac{h'(w_\varepsilon)}{s'(w_\varepsilon)} - h(w_\varepsilon)\right)\,.$$

Integrating this over $\mathbb{T}^D \times \mathbb{R}^D$ while using the entropy bound (3.4) and the bound (3.26) of Step 2 yields the bound

$$\zeta(\varepsilon)\int\left\langle\frac{1}{16}|v|^2\frac{g_\varepsilon^2}{N_\varepsilon}\right\rangle dx \leqq C^{in} + C\,.$$

Dividing this by $\zeta(\varepsilon)$ and using (3.27) of Step 3 yields (3.20), the desired result. All that remains to complete the proof of Proposition 3.3 is the verification of Steps 1–3.

In order to verify Step 1, for every $\zeta \in (0,1)$ and $\varepsilon > 0$ define a function over $k \in \mathbb{R}$ by $k \mapsto 1 + \log(1 + \varepsilon^m \exp\{k\}) - k\zeta$. This function takes on its minimum value of

$$1 + \log\left(\frac{1}{1-\zeta}\right) - \zeta \log\left(\frac{\zeta}{\varepsilon^m(1-\zeta)}\right),$$

at the point k where

$$\exp(k) = \frac{\zeta}{\varepsilon^m(1-\zeta)}.$$

This minimum value will be zero when $\zeta = \zeta(\varepsilon)$ solves (3.25); this solution exists and is unique provided $\log(\varepsilon^m) < -1$. Setting $k = \frac{1}{8}|v|^2$ and $\zeta = \zeta(\varepsilon)$ into this positive function and recalling the definition of w_ε (3.23) then yields (3.24).

To verify Step 2, first notice that (3.23) gives the estimate

$$\left(\frac{h'(w_\varepsilon)}{s'(w_\varepsilon)} - 1\right)^2 \leqq \left(\log\left(1 + \varepsilon^m \exp\left(\frac{1}{8}|v|^2\right)\right)\right)^2 \leqq \varepsilon^{2m} \exp\left(\frac{1}{4}|v|^2\right). \tag{3.28}$$

Next notice that for some $C > 0$ one can obtain the bound

$$\frac{s(w)\frac{h'(w)}{s'(w)} - h(w)}{\left(\frac{h'(w)}{s'(w)} - 1\right)^2} \leqq C(1 + w), \tag{3.29}$$

for every positive w. This follows from direct asymptotic analysis of the defining formulas (3.3) and (3.19) of h and s. Finally, notice that when $\log(\varepsilon^m) < -1$, one has the bound

$$\begin{aligned} 1 + \log\left(1 + \varepsilon^m \exp\left(\frac{1}{8}|v|^2\right)\right) &\leqq 1 + \log\left((1 + \varepsilon^m)\exp\left(\frac{1}{8}|v|^2\right)\right) \\ &\leqq \log(e + 1) + \frac{1}{8}|v|^2. \end{aligned}$$

Thus, the function w_ε defined by (3.23) satisfies

$$\frac{h'(w_\varepsilon)}{s'(w_\varepsilon)} \leqq \log(e + 1) + \frac{1}{8}|v|^2,$$

which is uniform in ε. After some more asymptotic analysis, this leads to

$$w_\varepsilon = O\left(\exp\left(\frac{1}{12}|v|^2\right)\right). \tag{3.30}$$

Combining (3.28), (3.29), and (3.30) shows that

$$\frac{1}{\varepsilon^{2m}}\left(s(w_\varepsilon)\frac{h'(w_\varepsilon)}{s'(w_\varepsilon)} - h(w_\varepsilon)\right) = O\left(\exp\left(\frac{1}{3}|v|^2\right)\right),$$

where the estimate is uniform in ε. Integrating this over $M dv$ then gives (3.26).

The verification of Step 3 follows from a straightforward asymptotic analysis of (3.25), the defining equation for $\zeta(\varepsilon)$. This completes the proof of Proposition 3.3.

Dissipation Controls

Applying exactly the same techniques to the entropy dissipation as were used in the proof of Proposition 3.1 to control the sequence g_ε by the entropy produces a corresponding result for the sequence of scaled collision integrands q_ε. In this case, while similar, the results for the continuous and discrete time problems have differences worthy of distinction.

PROPOSITION 3.4. *Let g_ε and g_ε^{in} be sequences of functions in*

$$L^\infty(dt; L^1(M dv\, dx)) \quad \text{and} \quad L^1(M dv\, dx)$$

respectively such that $1 + \varepsilon^m g_\varepsilon \geqq 0$ and $1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$. Let q_ε be the sequence of scaled collision integrands corresponding to g_ε. If g_ε, q_ε, and g_ε^{in} satisfy the entropy inequality and bound (3.6) *then*

(1) *The sequence $(1 + |v|^2)q_\varepsilon/N_\varepsilon$ is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\, dx))$;*
(2) *If g and q are the $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$ and $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\, dx))$ limits of any converging subsequence of g_ε and $q_\varepsilon/N_\varepsilon$ respectively then q inherits the symmetries of q_ε under the $d\mu$-symmetries* (1.24); *moreover, $q \in L^2(dt; L^2(d\mu\, dx))$ and for almost every $t \in [0, \infty)$ it satisfies*

$$\int \frac{1}{2} \langle g^2(t) \rangle \, dx + \int_0^t \int \frac{1}{4} \langle\langle q^2 \rangle\rangle \, dx\, ds \leqq \liminf_{\varepsilon \to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in}) \right\rangle dx \leqq C^{in} . \tag{3.31}$$

Proof: Let $\alpha > 0$ (to be chosen). First apply the Young inequality (3.10) to r and r^* with $z = \varepsilon^{m+1}|q_\varepsilon|/(G_{\varepsilon 1} G_\varepsilon)$ and $y = \varepsilon^{m+1} 1/4 (1 + |v|^2)/(\alpha N_\varepsilon)$ and invoke the reflection inequality for r (3.13) to obtain the bound

$$\begin{aligned} \frac{1}{4}\left(1 + |v|^2\right) \frac{|q_\varepsilon|}{N_\varepsilon} \leqq{}& \frac{\alpha}{\varepsilon^{2m+2}} r^*\!\left(\frac{\varepsilon^{m+1} \frac{1}{4}(1 + |v|^2)}{\alpha N_\varepsilon} \right) G_{\varepsilon 1} G_\varepsilon \\ &+ \frac{\alpha}{\varepsilon^{2m+2}} r\!\left(\frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon} \right) G_{\varepsilon 1} G_\varepsilon . \end{aligned}$$

Since $N_\varepsilon = 1 + 1/3\,\varepsilon^m g_\varepsilon \geqq 2/3$, the superquadratic homogeneity of r^* (3.14) can be used with $\lambda = \varepsilon^{m+1}/(\alpha N_\varepsilon)$ provided ε satisfies $\varepsilon^{m+1} \leqq 2/3\,\alpha$. This leads to

$$\begin{aligned}\frac{1}{4}\,(1+|v|^2)\,\frac{|q_\varepsilon|}{N_\varepsilon} &\leqq \frac{1}{\alpha}\,r^*\!\left(\frac{1}{4}\,(1+|v|^2)\right)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^2} \\ &\quad + \frac{\alpha}{\varepsilon^{2m+2}}\,r\!\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right)G_{\varepsilon 1}G_\varepsilon\,. \end{aligned} \tag{3.32}$$

Now consider the first term of the right side above. From the implicit formula for r^* (3.12) it can be easily shown that $r^*(y) = O(\exp\{y\})$ as $y \to +\infty$; thus, the function $r^*(1/4(1+|v|^2))$ is $O(\exp\{1/4\,|v|^2\})$ for large $|v|$. The factor $G_\varepsilon/N_\varepsilon^2$ is uniformly bounded by the value 9/8 and converges to 1 almost everywhere while the factor $(1+|v_1|^2)G_{\varepsilon 1}$ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M\,dv_1\,dx))$. These facts along with the bound (1.6) on b imply that the first term on the right side of (3.32) is relatively compact (and therefore equi-integrable) in

$$w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\,dx))\,.$$

Integrating (3.32) over any measurable $\Omega \subset [0,T]\times\mathbb{T}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1}$ and using the entropy bound (3.4) with the entropy inequality (3.6) gives

$$\begin{aligned}&\iiint_\Omega \frac{1}{4}(1+|v|^2)\frac{|q_\varepsilon|}{N_\varepsilon}\,d\mu\,dx \\ &\qquad\leqq \frac{1}{\alpha}\iiint_\Omega r^*\!\left(\frac{1}{4}\,(1+|v|^2)\right)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^2}\,d\mu\,dx\,dt + \alpha C^{in}\,. \end{aligned} \tag{3.33}$$

Taking $\Omega = [0,T]\times\mathbb{T}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1}$ and choosing α larger than any value of $3/2\,\varepsilon^{m+1}$ shows that the left side of (3.32) is a bounded sequence in $L^1_{\mathrm{loc}}(dt; L^1(d\mu\,dx))$.

To prove equi-integrability, take $\eta > 0$ arbitrarily small. Choose $\alpha = \eta/2C^{in}$. Invoking the equi-integrability of the first term on the right side of (3.32), pick $\delta > 0$ such that

$$\mathrm{meas}(\Omega) < \delta \quad \text{implies} \quad \iiint_\Omega r^*\!\left(\frac{1}{4}\,(1+|v|^2)\right)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^2}\,d\mu\,dx\,dt \leqq \frac{1}{2}\alpha\eta\,.$$

For every ε with $\varepsilon^{m+1} \leqq 2/3\,\alpha$ this choice will ensure that the left side of (3.33) will be smaller than η for any Ω with $\mathrm{meas}(\Omega) < \delta$. The finite set of members of ε such that $\varepsilon^{m+1} > 2/3\,\alpha$ can then be accommodated by picking δ as small as necessary. This proves assertion (1).

Now let q be the $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\,dx))$ limit of any convergent subsequence of $q_\varepsilon/N_\varepsilon$. The product limit theorem (see Appendix B) implies that

$$\frac{q_\varepsilon}{N_\varepsilon N_{\varepsilon 1} N'_\varepsilon N'_{\varepsilon 1}} \to q$$

in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(d\mu\, dx))$ as ε tends to zero. Since each of these approximating functions has the symmetries of q_ε under the $d\mu$-symmetries (1.24), q inherits these symmetries.

In order to obtain the inequality (3.31), take the limit inferior on both sides of the entropy inequality (3.6) and apply assertion (2) of Proposition 3.1 to obtain

$$\begin{aligned}\int \frac{1}{2}\left\langle g^2(t)\right\rangle dx + \liminf_{\varepsilon\to 0}\int_0^t\int\left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\right\rangle\!\!\right\rangle dx\, ds \\ \leqq \liminf_{\varepsilon\to 0}\int\left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in})\right\rangle dx \leqq C^{in}\,.\end{aligned} \tag{3.34}$$

It remains to show that the second term on the left side above is an upper bound for the corresponding term in (3.31).

The convexity of r gives the inequality

$$\frac{1}{\varepsilon^{2m+2}} r\left(\varepsilon^{m+1}q\right) + \frac{1}{\varepsilon^{m+1}} r'\left(\varepsilon^{m+1}q\right)\left(\frac{q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon} - q\right) \leqq \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right).$$

Fix $\lambda > 0$ and multiply this inequality by the indicator function $\mathbf{1}_{|q|<\lambda}$ times $G_{\varepsilon 1}G_\varepsilon$ over the normalization $N_\varepsilon^{abs} \equiv 1 + \frac{1}{3}\varepsilon^m |g_\varepsilon|$; the non-negativity of r then implies

$$\begin{aligned}\frac{1}{\varepsilon^{2m+2}} r(\varepsilon^{m+1}q)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^{abs}}\mathbf{1}_{|q|<\lambda} + \frac{1}{\varepsilon^{m+1}} r'(\varepsilon^{m+1}q)\left(\frac{q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon} - q\right)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^{abs}}\mathbf{1}_{|q|<\lambda} \\ \leqq \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\,.\end{aligned} \tag{3.35}$$

Integrate this over $(0,t)\times \mathbb{T}^D \times (\mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1})$ then consider its limit as ε tends to zero. The normalization N_ε^{abs} is chosen so that $G_{\varepsilon 1}G_\varepsilon / N_\varepsilon^{abs} \to 1$ in $L^1_{loc}(dt; L^1(d\mu\, dx))$ (by the estimates of Corollary 3.2), and that

$$\frac{q_\varepsilon}{N_\varepsilon^{abs}} = \frac{q_\varepsilon}{N_\varepsilon}\frac{N_\varepsilon}{N_\varepsilon^{abs}} \to q$$

in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(d\mu\, dx))$ (by hypothesis and the product limit theorem). The $w\text{-}L^1$ limit of the left side of inequality (3.35) can be evaluated using the strong L^∞ limits

$$\frac{1}{\varepsilon^{2m+2}} r(\varepsilon^{m+1}q)\,\mathbf{1}_{|q|<\lambda} \to q^2\,\mathbf{1}_{|g|<\lambda}\,, \qquad \frac{1}{\varepsilon^{m+1}} r'(\varepsilon^{m+1}q)\,\mathbf{1}_{|q|<\lambda} \to 2q\,\mathbf{1}_{|q|<\lambda}\,,$$

and the $w\text{-}L^1$ limit

$$\left(\frac{q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon} - q\right)\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^{abs}}\mathbf{1}_{|q|<\lambda} = \left(\frac{q_\varepsilon}{N_\varepsilon^{abs}} - q\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^{abs}}\right)\mathbf{1}_{|q|<\lambda} \to 0\,;$$

this leads to the bound

$$\int_0^t \int \langle\langle q^2 \mathbf{1}_{|q|<\lambda} \rangle\rangle \, dx \, ds \leqq \liminf_{\varepsilon \to 0} \int_0^t \int \left\langle\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon}\right) G_{\varepsilon 1} G_\varepsilon \right\rangle\right\rangle \, dx \, ds \, .$$

Taking $\lambda \to +\infty$ then provides the estimate needed in (3.34) to complete the proof of assertion (2).

Remark. Control of the first term on the right side of (3.32) required only the boundedness of the sequence $G_\varepsilon/N_\varepsilon^2$. This can be achieved by normalizations $N(G)$ that grow sublinearly as $G \to \infty$. For example, $N(G) = (2/3 + 1/3G)^{1/2}$ works fine. In fact, the whole DiPerna-Lions theory can be recast with such normalizations. So far, however, we have been unable to gain significant new results from this observation.

The time-discretized version of Propositon 3.4 mainly differs in the limiting form of the entropy inequality obtained (compare (3.31) above with (3.36) below). This difference plays a critical role in obtaining the strong Navier-Stokes limit in Section 7.

PROPOSITION 3.5. *Let g_ε and g_ε^{in} be sequences of functions in $L^1(M dv\, dx)$ such that $1 + \varepsilon g_\varepsilon \geqq 0$ and $1 + \varepsilon g_\varepsilon^{in} \geqq 0$. Let q_ε be the sequence of scaled collision integrands corresponding to g_ε. If g_ε, q_ε, and g_ε^{in} satisfy the entropy inequality and bound* (3.9) *then*

(1) *The sequence $(1 + |v|^2) q_\varepsilon / N_\varepsilon$ is relatively compact in w-$L^1(d\mu\, dx)$;*
(2) *If g, g^{in}, and q are the w-$L^1(M dv\, dx)$, w-$L^1(M dv\, dx)$, and w-$L^1(d\mu\, dx)$ limits of any converging subsequence of g_ε, g_ε^{in}, and $q_\varepsilon/N_\varepsilon$ respectively then q inherits the symmetries of q_ε under the $d\mu$-symmetries* (1.24); *moreover, $q \in L^2(d\mu\, dx))$ and it satisfies*

$$\int \frac{1}{2}\langle g^2 \rangle dx + \int \frac{1}{2} \left\langle |g - g^{in}|^2 \right\rangle dx + \int \frac{1}{4} \langle\langle q^2 \rangle\rangle \, dx$$

(3.36)

$$\leqq \liminf_{\varepsilon \to 0} \int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon^{in}) \right\rangle dx \leqq C^{in} \, .$$

Proof: The verification of assertion (1) and the first parts of assertion (2) proceed like the corresponding parts of Proposition 3.4; all that remains is to verify (3.36). The only term in (3.9) that is unlike those already analyzed in the proof of Proposition 3.4 is the middle term on its left side. The convexity of j gives the inequality

$$\frac{1}{\varepsilon^2} j(\varepsilon g^{in}, \varepsilon g) + \frac{1}{\varepsilon} \partial_z j(\varepsilon g^{in}, \varepsilon g)(g_\varepsilon^{in} - g^{in}) + \frac{1}{\varepsilon} \partial_y j(\varepsilon g^{in}, \varepsilon g)(g_\varepsilon - g)$$

$$\leqq \frac{1}{\varepsilon^2} j(\varepsilon g_\varepsilon^{in}, \varepsilon g_\varepsilon) \, .$$

Fix $\lambda > 0$ and multiply this inequality by $\mathbf{1}_\lambda$, the indicator (characteristic) function for the set $\{(x, v) \in \mathbb{T}^D \times \mathbb{R}^D : |g|^{in}, |g| < \lambda\}$ and passing to limits as before leads to the bound

$$\int \frac{1}{2} \left\langle |g - g^{in}|^2 \mathbf{1}_\lambda \right\rangle dx \leqq \liminf_{\varepsilon \to 0} \int \left\langle \frac{1}{\varepsilon^2} j(\varepsilon g_\varepsilon^{in}, \varepsilon g_\varepsilon) \right\rangle dx .$$

Taking $\lambda \to \infty$ then provides the estimate needed to complete the proof of Proposition 3.5.

An immediate corollary of Proposition 3.4 (or 3.5) that will be used frequently in the sequel is the relative compactness of the family of normalized collision operators that appears on the right side of the normalized scaled Boltzmann equation (2.2) (or (2.22)).

COROLLARY 3.6. *Let $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon > 0$ where g_ε is as in Proposition 3.4 (or with $m = 1$, as in Proposition 3.5). Then the sequence*

$$(1 + |v|^2) \frac{1}{\varepsilon^{m+1}} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon}$$

is relatively compact in

$$w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx)) \quad (\text{or with } m = 1, w\text{-}L^1(M dv\, dx)) .$$

Proof: Just observe that for any χ in $L^\infty_{\mathrm{loc}}(dt; L^\infty(M dv\, dx))$ and $[t_1, t_2] \subset \mathbb{R}^+$ one has

$$\int_{t_1}^{t_2}\!\!\int \left\langle \chi \, (1 + |v|^2) \frac{1}{\varepsilon^{m+1}} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} \right\rangle dx\, dt = \int_{t_1}^{t_2}\!\!\int \left\langle\!\!\left\langle \chi \, (1 + |v|^2) \frac{q_\varepsilon}{N_\varepsilon} \right\rangle\!\!\right\rangle dx\, dt ,$$

and apply Proposition 3.4 to the right side. The proof for the time-discretized case is similar.

Another consequence of the entropy inequalities is a bound on the difference between Q_ε^+ and Q_ε^-, the gain and loss components of the collision operator. These are defined by

$$Q_\varepsilon^+ = Q^+(G_\varepsilon, G_\varepsilon) = \iint G'_{\varepsilon 1} G'_\varepsilon \, b(v_1 - v, \omega) \, d\omega \, M_1 dv_1 ,$$

$$Q_\varepsilon^- = Q^-(G_\varepsilon, G_\varepsilon) = \iint G_{\varepsilon 1} G_\varepsilon \, b(v_1 - v, \omega) \, d\omega \, M_1 dv_1 .$$

PROPOSITION 3.7. *The bound*

$$(3.37)\quad 0 \leqq (Q_\varepsilon^+ - Q_\varepsilon^-)\log\left(\frac{Q_\varepsilon^+}{Q_\varepsilon^-}\right) \leqq \iint r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\ ,$$

holds for almost every $(t, x, v) \in [0, \infty) \times \mathbb{T}^D \times \mathbb{R}^D$.

Remark. The significance of this result lies in the fact that the right side of the inequality divided by ε^{2m+2} is in a bounded set of $L^1_{loc}(dt; L^1(M dv\, dx))$ by the dissipation bound (3.6). This inequality is a slightly sharper version of the one exploited by DiPerna and Lions (see [10]) to obtain some of their compactness results. While those results now follow from Proposition 3.4, we feel that inequality (3.37) has intrinsic interest.

Proof: For almost every (t, x, v) the values of Q_ε^+ and Q_ε^- are finite. If $G_\varepsilon(t, x, \cdot) = 0$ almost everywhere then all members of the inequality (3.37) vanish, giving the result. Consider the quantity $\ell(G_\varepsilon)$ defined by

$$\ell(G_\varepsilon) \equiv \iint G_{\varepsilon 1}\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1 \geqq 0\ .$$

Since $Q_\varepsilon^- = G_\varepsilon \ell(G_\varepsilon)$, it is clear that $\ell(G_\varepsilon)$ also has a finite value for almost every (t, x, v). If $G_\varepsilon(t, x, \cdot) > 0$ on a set of positive measure then it follows that $\ell(G_\varepsilon(t, x, \cdot)) > 0$ provided the Boltzmann kernel b is sufficiently positive. Wherever this is the case, define the positive unit measure $d\mu_\varepsilon^\ell$ on $\mathbb{R}^D \times \mathbb{S}^{D-1}$ by

$$d\mu_\varepsilon^\ell = \frac{1}{\ell(G_\varepsilon)} G_{\varepsilon 1}\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\ .$$

The mean of the function

$$Z_\varepsilon = \frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon} = \frac{G'_{\varepsilon 1}G'_\varepsilon}{G_{\varepsilon 1}G_\varepsilon} - 1$$

over this measure is simply

$$\int Z_\varepsilon\, d\mu_\varepsilon^\ell = \frac{Q_\varepsilon^+}{Q_\varepsilon^-} - 1\ .$$

Application of the Jensen inequality to the non-negative convex function r gives

$$0 \leqq r\left(\int Z_\varepsilon\, d\mu_\varepsilon^\ell\right) \leqq \int r(Z_\varepsilon)\, d\mu_\varepsilon^\ell\ ,$$

which yields inequality (3.37) upon multiplication by Q_ε^- $(= G_\varepsilon \ell(G_\varepsilon))$.

The Infinitesimal Maxwellian

One of the main consequences of the entropy and dissipation controls is the determination of the limiting form for the fluctuations g_ε. The following proposition relies on none of the assumptions (H0), (H1), or (H2); the only assumption used is the bound (1.6) on the Boltzmann kernel.

PROPOSITION 3.8. (THE INFINITESIMAL MAXWELLIAN FORM) *Let g be the limit of a converging sequence of g_ε in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$ that satisfies the entropy inequality and bound (3.6). Then, for almost every (t,x), $g(t,x,\cdot) \in \mathrm{N}(L)$, which means that g is of the form*

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right), \tag{3.38}$$

where $(\rho, u, \theta) \in L^\infty(dt; L^2(dx; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$

Proof: Consider the identity

$$Lg_\varepsilon = \varepsilon^m Q(g_\varepsilon, g_\varepsilon) - \frac{1}{\varepsilon^m} Q(G_\varepsilon, G_\varepsilon)\,. \tag{3.39}$$

The bound (1.6) on the Boltzmann kernel implies that L is a continuous linear map from $L^1((1+|v|^2)M dv)$ to $L^1(M dv)$. Since $g_\varepsilon \to g$ in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1((1+|v|^2)M dv\, dx))$ as ε tends to zero, this continuity implies that

$$Lg_\varepsilon \to Lg \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))\,. \tag{3.40}$$

Thus, the left side of (3.39) is $w\text{-}L^1$ convergent. Evaluating the limit of the right side, however, is complicated by its nonlinear nature.

There is some information about terms that appear on the right side of (3.39). For example, Corollary 3.6 gives that

$$\frac{1}{\varepsilon^m}\frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} = O(\varepsilon) \quad \text{in } L^1_{\text{loc}}(dt; L^1(M dv\, dx))\,, \tag{3.41}$$

while, by an estimate given below, Proposition 3.3 will imply that

$$\varepsilon^m \frac{Q(g_\varepsilon, g_\varepsilon)}{\langle (1+|v|^2) N^{abs}_\varepsilon \rangle} = O\left(\varepsilon^m \log\left(\frac{1}{\varepsilon|\log(\varepsilon)|}\right)\right) \quad \text{in } L^\infty(dt; L^1(M dv\, dx))\,, \tag{3.42}$$

where $N^{abs}_\varepsilon = 1 + \frac{1}{3}\varepsilon^m |g_\varepsilon|$.

It is convenient to define $\xi = \xi(v) \equiv \frac{1}{D+1}(1 + |v|^2)$ (so that $\langle \xi \rangle = 1$). The preceding information suggests dividing (3.39) by $N_\varepsilon \langle \xi N_\varepsilon^{abs} \rangle$ to obtain

$$\frac{1}{N_\varepsilon \langle \xi N_\varepsilon^{abs} \rangle} L g_\varepsilon = \frac{1}{N_\varepsilon} \left(\varepsilon^m \frac{Q(g_\varepsilon, g_\varepsilon)}{\langle \xi N_\varepsilon^{abs} \rangle} \right) - \frac{1}{\langle \xi N_\varepsilon^{abs} \rangle} \left(\frac{1}{\varepsilon^m} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} \right) . \tag{3.43}$$

Observing the uniform bounds

$$\frac{1}{N_\varepsilon} \leqq \frac{3}{2} \quad \text{and} \quad \frac{1}{\langle \xi N_\varepsilon^{abs} \rangle} \leqq 1 , \tag{3.44}$$

it is clear that (3.41) and (3.42) show that the right side of (3.43) vanishes in L^1 as ε tends to zero. Proposition 3.1 implies

$$\frac{1}{N_\varepsilon} \to 1 \quad \text{and} \quad \frac{1}{\langle \xi N_\varepsilon^{abs} \rangle} \to 1 \quad \text{almost everywhere,} \tag{3.45}$$

so using (3.40), (3.44), and (3.45) in the product limit theorem (see Appendix B) shows that the left side of (3.43) converges to Lg in $w\text{-}L^1$ as ε tends to zero. Therefore, the limiting form of (3.43) is just

$$Lg = 0 ,$$

and formula (3.38) follows from the characterization of the null space of L, (1.45). Assertion (2) of Proposition 3.1 states that g is in $L^\infty(dt; L^2(M dv\, dx))$, so that (ρ, u, θ) is in $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ then follows.

What remains is to verify (3.42). This is done for the gain and loss components of the collision operator separately. The bound (1.6) on the Boltzmann kernel gives the estimate

$$\left| \langle Q^\pm (g_\varepsilon, g_\varepsilon) \rangle \right| \leqq \langle Q^\pm (|g_\varepsilon|, |g_\varepsilon|) \rangle \leqq C \, \langle \xi \, |g_\varepsilon| \rangle^2$$

for some $C < \infty$; hence

$$\frac{|\langle Q^\pm (g_\varepsilon, g_\varepsilon) \rangle|}{\langle \xi N_\varepsilon^{abs} \rangle} \leqq C \frac{\langle \xi \, |g_\varepsilon| \rangle^2}{\langle \xi N_\varepsilon^{abs} \rangle} \leqq C \frac{\langle \xi \, |g_\varepsilon| \rangle^2}{1 + \frac{1}{3} \varepsilon^m \langle \xi \, |g_\varepsilon| \rangle} .$$

This last term is a convex function of $\langle \xi \, |g_\varepsilon| \rangle$, so an application of the Jensen inequality gives

$$\frac{|\langle Q^\pm (g_\varepsilon, g_\varepsilon) \rangle|}{\langle \xi N_\varepsilon^{abs} \rangle} \leqq C \left\langle \xi \frac{|g_\varepsilon|^2}{N_\varepsilon^{abs}} \right\rangle \leqq C \left\langle \xi \frac{g_\varepsilon^2}{N_\varepsilon} \right\rangle .$$

Applying Proposition 3.3 to estimate the last term above and multiplying the result by ε^m verifies (3.42), whence Proposition 3.8 holds.

4. Implications of the Normalized Boltzmann Equation

The Limiting Boltzmann Equation

Consider the normalized Boltzmann equation (2.2) written in the form

$$\varepsilon\, \partial_t \gamma_\varepsilon + v\cdot\nabla_x \gamma_\varepsilon = \frac{1}{\varepsilon^{m+1}} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} = \iint \frac{q_\varepsilon}{N_\varepsilon}\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1 \,. \tag{4.1}$$

This means (see 1.55) that for every χ in $L^\infty(Mdv; C^1(\mathbb{T}^D))$ and every $0 \leqq t_1 < t_2 < \infty$

$$\begin{aligned} &\varepsilon \int \langle \gamma_\varepsilon(t_2)\, \chi\rangle\, dx - \varepsilon \int \langle \gamma_\varepsilon(t_1)\, \chi\rangle\, dx - \int_{t_1}^{t_2} \int \langle \gamma_\varepsilon\, v\cdot\nabla_x \chi\rangle\, dx\, dt \\ &\qquad = \int_{t_1}^{t_2} \int \left\langle\!\!\left\langle \frac{q_\varepsilon}{N_\varepsilon}\, \chi \right\rangle\!\!\right\rangle dx\, dt \,. \end{aligned} \tag{4.2}$$

Taking the limits in (4.2) as ε tends to zero while using Proposition 3.2 and Proposition 3.4 to establish the limits of the terms involving γ_ε and q_ε respectively yields

$$-\int_{t_1}^{t_2} \int \langle g\, v\cdot\nabla_x \chi\rangle\, dx\, dt = \int_{t_1}^{t_2} \int \langle\langle q\, \chi\rangle\rangle\, dx\, dt \,;$$

hence, the limiting form of the normalized Boltzmann equation is

$$v\cdot\nabla_x g = \iint q\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1 \,. \tag{4.3}$$

More precisely, we have proved the following.

PROPOSITION 4.1. (THE LIMITING BOLTZMANN EQUATION) *Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *satisfying the entropy bound* (3.2) *for some $m \geqq 1$. Let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g and q be limits of the sequences g_ε and $q_\varepsilon/N_\varepsilon$ in w-$L^1_{\rm loc}(dt; w\text{-}L^1(Mdv\, dx))$ and w-$L^1_{\rm loc}(dt; w\text{-}L^1(d\mu\, dx))$ respectively. Then g and q satisfy the limiting Boltzmann equation* (4.3).

Remark. The limiting Boltzmann equation for the time-discretized case is also (4.3); both the formulation of this theorem and its proof are analogous to those of Proposition 4.1. For this reason we omit both here, but rather shall also refer to this corresponding time-discretized result as Proposition 4.1.

The principal implications of (4.3) are twofold. First, it provides the starting point for the derivation of the incompressiblity and Boussinesq relations satisfied by the fluid variables (ρ, u, θ) that arise in the infinitesimal Maxwellian form for g (3.38). Second, it provides the key relations between q and the dissipation tensors involving gradients of these fluid variables. These formulas yield spatial regularity for the limiting fluctuations and are used in this section to derive a version of the Leray energy inequality (2.12) from the limiting form of the DiPerna-Lions entropy inequality (3.27), and later to obtain the fluid equations.

The Incompressibility and Boussinesq Relations

In the formal argument leading to incompressible fluid dynamics (see [3] and Theorem 1.1) the incompressibility and Boussinesq relations arise as the first manifestations of the local conservation laws. This route to these relations is not available here since, except for that of mass, these local conservation laws are not known to hold for DiPerna-Lions renormalized solutions of the Boltzmann equation. With the aid of the $d\mu$-symmetries (1.24), however, the local conservation laws can be established for the limiting Boltzmann equation (4.3) and the incompressibility and Boussinesq relations can be derived as follows.

Given that q is in $L^2(d\mu\, dx))$ then for every $\xi = \xi(v)$ in $L^2(d\mu)$, an application of the Cauchy-Schwarz inequality shows that $\langle\!\langle \xi q\rangle\!\rangle$ is in $L^2(dx))$. By a repeated application of the $d\mu$-symmetries (1.24) together with the symmetries that q inherited from the sequence q_ε (see Propositions 3.4 and 3.5), one has that for any ξ in $L^2(d\mu)$,

$$\langle\!\langle \xi q\rangle\!\rangle = \frac{1}{4}\langle\!\langle (\xi_1 + \xi - \xi_1' - \xi')q\rangle\!\rangle \,. \tag{4.4}$$

As was done in the formal derivation of the local conservation laws for the Boltzmann equation (1.29), successively apply this relation for $\xi = 1, v, \frac{1}{2}|v|^2$ and use the microscopic conservation laws (1.4) to obtain

$$\langle\!\langle q\rangle\!\rangle = 0\,, \quad \langle\!\langle vq\rangle\!\rangle = 0\,, \quad \left\langle\!\!\left\langle \frac{1}{2}|v|^2 q\right\rangle\!\!\right\rangle = 0\,,$$

The fact that these ξ are in $L^2(d\mu)$ follows from the bound on the Boltzmann kernel (1.6). Since these ξ are also in $L^2(Mdv)$, it then follows from the limiting Boltzmann equation (4.3) that g satisfies the local conservation laws of mass, momentum, and energy:

$$\nabla_x\cdot\langle vg\rangle = 0\,, \quad \nabla_x\cdot\langle v\otimes vg\rangle = 0\,, \quad \nabla_x\cdot\left\langle v\frac{1}{2}|v|^2 g\right\rangle = 0\,. \tag{4.5}$$

Proposition 3.7 states that g has the form of the infinitesimal Maxwellian (3.38),

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,. \tag{4.6}$$

Substituting this into (4.5), the local mass and energy conservation laws yield the incompressibility relation for the velocity field u while that of momentum yields the Boussinesq relation between ρ and θ:

$$\nabla_x \cdot u = 0 \,, \qquad \nabla_x(\rho + \theta) = 0 \,. \tag{4.7}$$

More precisely, we have proved the following.

PROPOSITION 4.2. (INCOMPRESSIBILITY AND BOUSSINESQ RELATIONS) *Assume that $g \in L^2(M dv\, dx)$ and $q \in L^2(d\mu\, dx)$ satisfy the limiting Boltzmann equation and let g have the form of an infinitesimal Maxwellian* (4.6). *Then (ρ, u, θ) $\in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ satisfy the incompressibility and Boussinesq relations* (4.7) *in the sense that for every test function $\chi \in C^1(\mathbb{T}^D)$*

$$\int u \cdot \nabla_x \chi \, dx = 0 \,, \qquad \int (\rho + \theta) \nabla_x \chi \, dx = 0 \,. \tag{4.8}$$

Remark. Notice that the above proposition does not require that the limiting initial data g^{in} be an infinitesimal Maxwellian nor that its fluid variables $(\rho^{in}, u^{in}, \theta^{in})$ satisfy either the incompressibility or the Boussinesq relation. For the continuous time problem, (4.8) is only asserted almost everywhere in time, not pointwise. Physically, this result implies that the weak limit of the acoustic modes of the flow must vanish, but does not mean they are not there.

The Dissipation Tensor Relations

The limiting Boltzmann equation (4.3) leads to relations between q and the gradients of the fluid variables u and θ; these relations are stated in the following proposition.

PROPOSITION 4.3. *Let $g \in L^2(M dv\, dx)$ and $q \in L^2(d\mu\, dx)$ satisfy the limiting Boltzmann equation and let g have the form of an infinitesimal Maxwellian* (4.6). *Then for almost every x*

$$\langle\!\langle \phi q \rangle\!\rangle = \nu \left(\nabla_x u + (\nabla_x u)^T \right) \,, \qquad \langle\!\langle \psi q \rangle\!\rangle = \frac{D+2}{2} \kappa \nabla_x \theta \,, \tag{4.9}$$

where ν and κ are given by

$$\nu = \frac{1}{(D-1)(D+2)} \langle \phi : L\phi \rangle \,, \qquad \kappa = \frac{2}{D(D+2)} \langle \psi \cdot L\psi \rangle \,, \tag{4.10}$$

and where ϕ and ψ are defined in (1.46).

Proof: Starting from the limiting Boltzmann equation (4.3) satisfied by g and q, set g to its infinitesimal Maxwellian form (4.6) and use the incompressibility and Boussinesq relations (4.7) to obtain

$$\begin{aligned}
&\iint q\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1 \\
&= v\cdot\nabla_x \rho + (v \otimes v):\nabla_x u + \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) v\cdot\nabla_x\theta \\
&= \left(v \otimes v - \frac{1}{\mathrm{D}}|v|^2 I\right):\nabla_x u + \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}+2}{2}\right) v\cdot\nabla_x\theta \\
&= L\phi:\nabla_x u + L\psi\cdot\nabla_x\theta \,.
\end{aligned} \tag{4.11}$$

Multiplying this limiting relation by ϕ and ψ then integrating over v gives

$$\langle\!\langle \phi q \rangle\!\rangle = \langle \phi \otimes L\phi \rangle:\nabla_x u \,, \qquad \langle\!\langle \psi q \rangle\!\rangle = \langle \psi \otimes L\psi \rangle\cdot\nabla_x\theta \,. \tag{4.12}$$

The result now follows from the following classical lemma that is also used in the proofs of Lemmas 4.6 and 4.8.

LEMMA 4.4. *The components of* $\langle \phi \otimes L\phi \rangle$ *and* $\langle \psi \otimes L\psi \rangle$ *satisfy the following identities:*

$$\langle \phi_{ij} \otimes L\phi_{kl} \rangle = \nu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{\mathrm{D}}\delta_{ij}\delta_{kl} \right) , \tag{4.13a}$$

$$\langle \psi_i \otimes L\psi_j \rangle = \frac{\mathrm{D}+2}{2}\kappa\, \delta_{ij} \,, \tag{4.13b}$$

where ν *and* κ *are given by* (4.10).

Proof: Set $T_{ijkl} = \langle \phi_{ij} \otimes L\phi_{kl} \rangle$. Since L is self-adjoint, this tensor has the symmetries

$$T_{ijkl} = T_{jikl} = T_{jilk} = T_{ijlk} = T_{klij} = T_{lkij} = T_{lkji} = T_{klji} \,.$$

Since $B(v) \equiv v \otimes v - \frac{1}{\mathrm{D}}|v|^2 I$ is proportional to the second spherical harmonic, the rotation invariance of L implies that the same holds for $\phi(v)$, the unique solution of $L\phi(v) = B(v)$ that is orthogonal to the null space of L. The symmetries of the defining integral imply that $T_{ijkl} \neq 0$ only if all of its indices are paired (the integrand is odd in the variable corresponding to an unpaired index). The symmetries listed above mean T_{ijkl} has the form

$$T_{ijkl} = \nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu'\delta_{ij}\delta_{kl}$$

for some scalars ν and ν'. But since $B_{kk} = 0$ (summing over repeated indices), one also has $T_{ijkk} = T_{kkij} = 0$, whence $\nu' = -\frac{2}{\mathrm{D}}\nu$. Finally, the most symmetric formula for ν is obtained by taking the double trace of T: $T_{ijij} = \nu(\mathrm{D}+2)(\mathrm{D}-1)$. When rendered in the original notation this is the formula for ν given by (4.10). This completes the proof of formula (4.13a). Formula (4.13b) follows from a similar argument.

The Limiting Global Conservation Laws

Another property of renormalized solutions of the scaled Boltzmann initial-value problem (1.63) is that they satisfy global conservation laws for mass and momentum; see (1.57) and (1.58). In terms of the fluctuations g_ε, the initial normalizations (1.20) imply that these conservation laws take the form

$$\int \langle g_\varepsilon(t)\rangle \, dx = 0 \, , \qquad \int \langle v \, g_\varepsilon(t)\rangle \, dx = 0 \, ,$$

for every $t > 0$. Let ε tend to zero in these expressions and use the form of the infinitesimal Maxwellian (4.6) to obtain

$$\int \rho(t) \, dx = \int \langle g(t)\rangle \, dx = 0 \, , \qquad \int u(t) \, dx = \int \langle v \, g(t)\rangle \, dx = 0 \, ,$$

for almost every $t > 0$. Thus ρ and u have mean zero; this fact, along with the incompressibility relation of (4.7), proves the following result.

PROPOSITION 4.5. *Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *satisfying the entropy bound* (3.2). *Let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g and q be limits of the sequences g_ε and $q_\varepsilon/N_\varepsilon$ in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx))$ and $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\, dx))$ respectively. Then g has the form of an infinitesimal Maxwellian* (4.6) *such that*

$$\rho \in L^\infty(dt; \mathcal{H}_s) \, , \qquad u \in L^\infty(dt; \mathcal{H}_v) \, .$$

Remark. The obstruction to proving both that θ is in $L^\infty(dt; \mathcal{H}_s)$ and that $\rho + \theta = 0$ is the lack of a global energy conservation law for the DiPerna-Lions renormalized solutions. Spatially integrating the Boussinesq relation of (4.7) gives

$$\rho(t,x) + \theta(t,x) = \epsilon(t) \, ,$$

where ϵ depends only on t. If the global energy conservation law were established, even if only in the limit, then it would follow that

$$\frac{\mathrm{D}}{2}\epsilon(t) = \int \frac{\mathrm{D}}{2} \left(\rho(t) + \theta(t)\right) \, dx = \int \left\langle \frac{1}{2}|v|^2 g(t) \right\rangle \, dx = 0 \, .$$

The usual Boussinesq relation, $\rho + \theta = 0$, would then be inferred and the fact that $\theta(t)$ has mean zero (and thus is in $\mathcal{H}_s$) would then follow from Proposition 4.5.

The Limiting Dissipation Inequalities

A limiting form of the entropy inequality for the initial-value problem (1.63) was established in Proposition 3.4. With the additional information contained in the infinitesimal Maxwellian form and the limiting Boltzmann equation, this can be refined.

PROPOSITION 4.6. *Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *with initial data G_ε^{in} satisfying the entropy bound* (3.2). *Let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g and q be limits of the sequences g_ε and $q_\varepsilon/N_\varepsilon$ in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$ and $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\, dx))$ respectively. Then g has the form of an infinitesimal Maxwellian*

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right), \tag{4.14}$$

where $\rho \in L^2(dt; \mathcal{V}_s)$, $u \in L^2(dt; \mathcal{V}_v)$, and $\nabla_x\theta \in L^2(dt; L^2(dx))$ satisfy the inequality

$$\begin{aligned} &\int \frac{1}{2}\left(\rho(t)^2 + |u(t)|^2 + \frac{D}{2}\theta(t)^2\right) dx \\ &+ \int_0^t\int \frac{1}{2}\nu\left|\nabla_x u + (\nabla_x u)^T\right|^2 + \frac{D+2}{2}\kappa\left|\nabla_x\theta\right|^2 dx\, ds \\ &\leqq \liminf_{\varepsilon\to 0}\int\left\langle \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon^{in})\right\rangle dx \leqq C^{in}. \end{aligned} \tag{4.15}$$

Proof: Inequality (3.31) of Proposition 3.4 states that

$$\int \frac{1}{2}\langle g^2(t)\rangle\, dx + \int_0^t\int \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle\, dx\, ds \leqq \liminf_{\varepsilon\to 0}\int\left\langle \frac{1}{\varepsilon^{2m}}h(\varepsilon^m g_\varepsilon^{in})\right\rangle dx \leqq C^{in}. \tag{4.16}$$

Proposition 3.8 states that g has the form of an infinitesimal Maxwellian (4.14); a direct calculation shows that

$$\int \frac{1}{2}\langle g^2(t)\rangle\, dx = \int \frac{1}{2}\left(\rho(t)^2 + |u(t)|^2 + \frac{D}{2}\theta(t)^2\right) dx. \tag{4.17}$$

The result then follows from the limiting entropy inequality (4.11) upon showing that

$$\frac{1}{2}\nu \left|\nabla_x u + (\nabla_x u)^T\right|^2 + \frac{D+2}{2}\kappa\,|\nabla_x \theta|^2 \leqq \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle , \tag{4.18}$$

almost everywhere. But this inequality will follow directly from Proposition 4.3 above and Lemma 4.7 below.

LEMMA 4.7. *Any $q \in L^2(d\mu)$ satisfies the inequality*

$$\frac{1}{2}\frac{1}{\nu}\langle\!\langle \phi q\rangle\!\rangle : \langle\!\langle \phi q\rangle\!\rangle + \frac{2}{D+2}\frac{1}{\kappa}\langle\!\langle \psi q\rangle\!\rangle \cdot \langle\!\langle \psi q\rangle\!\rangle \leqq \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle , \tag{4.19}$$

where ν and κ are given by (4.10).

Proof: Introduce $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$ and $\Psi = \frac{1}{4}(\psi_1 + \psi - \psi_1' - \psi')$. First observe that the symmetries of q under the $d\mu$-symmetries (see (2) of Proposition 3.4) imply

$$\langle\!\langle \phi q\rangle\!\rangle = \langle\!\langle \Phi q\rangle\!\rangle , \qquad \langle\!\langle \psi q\rangle\!\rangle = \langle\!\langle \Psi q\rangle\!\rangle . \tag{4.20}$$

Next, repeated application of the $d\mu$-symmetries (1.24) shows

$$\langle \phi \otimes L\phi\rangle = 4\langle\!\langle \Phi \otimes \Phi\rangle\!\rangle , \qquad \langle \psi \otimes L\psi\rangle = 4\langle\!\langle \Psi \otimes \Psi\rangle\!\rangle . \tag{4.21}$$

Lemma 4.4 then implies that any vector $a \in \mathbb{R}^D$ and any traceless symmetric matrix $A \in \mathbb{R}^{D\times D}$ satisfy the identities

$$\langle\!\langle \Phi \otimes \Phi\rangle\!\rangle : A = \frac{1}{2}\nu A , \qquad \langle\!\langle \Psi \otimes \Psi\rangle\!\rangle \cdot a = \frac{D+2}{8}\kappa a . \tag{4.22}$$

When viewed this way, Lemma 4.4 is seen to give orthogonality relations for the functions Φ and Ψ with respect to the $d\mu$ measure while (4.20) gives the coefficients of the orthogonal expansion of q in terms of Φ and Ψ. All that remains is to check that (4.19) is the Bessel inequality associated with that expansion.

First applying the Cauchy-Schwarz inequality and then the identities (4.22) shows that

$$\begin{aligned}
\left(\langle\!\langle \Phi q\rangle\!\rangle : A + \langle\!\langle \Psi q\rangle\!\rangle \cdot a\right)^2 &= \langle\!\langle (\Phi : A + \Psi \cdot a) q\rangle\!\rangle^2 \\
&\leqq \langle\!\langle (\Phi : A + \Psi \cdot a)^2\rangle\!\rangle \langle\!\langle q^2\rangle\!\rangle \\
&= (A : \langle\!\langle \Phi \otimes \Phi\rangle\!\rangle : A + a \cdot \langle\!\langle \Psi \otimes \Psi\rangle\!\rangle \cdot a)\,\langle\!\langle q^2\rangle\!\rangle \\
&= \left(\frac{1}{2}\nu A : A + \frac{D+2}{8}\kappa a \cdot a\right)\langle\!\langle q^2\rangle\!\rangle .
\end{aligned} \tag{4.23}$$

The result now follows by using (4.20) and setting

$$A = \frac{1}{2}\frac{1}{\nu}\langle\!\langle \Phi q \rangle\!\rangle = \frac{1}{2}\frac{1}{\nu}\langle\!\langle \phi q \rangle\!\rangle\,, \qquad a = \frac{2}{\mathrm{D}+2}\frac{1}{\kappa}\langle\!\langle \Psi q \rangle\!\rangle = \frac{2}{\mathrm{D}+2}\frac{1}{\kappa}\langle\!\langle \psi q \rangle\!\rangle\,,$$

in the inequality (4.23).

Remark. It is clear from the application of the Cauchy-Schwarz inequality in (4.23) that inequality (4.19) is an equality if and only if

$$q = 2\frac{1}{\nu}\langle\!\langle \phi q \rangle\!\rangle : \Phi + \frac{8}{\mathrm{D}+2}\frac{1}{\kappa}\langle\!\langle \psi q \rangle\!\rangle \cdot \Psi\,.$$

This observation plays a key role in establishing the limiting form of the normalized scaled collision integrands.

The time-discretized analog of Proposition 4.6 uses the infinitesimal Maxwellian form and the limiting Boltzmann equation to refine the limiting form of the entropy inequality found in Proposition 3.5.

PROPOSITION 4.8. *Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled time-discretized Boltzmann problem* (2.21) *with initial data G_ε^{in} satisfying the entropy bound* (3.9). *Let g_ε^{in}, g_ε, and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g^{in}, g, and q be limits of the sequences g_ε^{in}, g_ε, and $q_\varepsilon/N_\varepsilon$ in w-$L^1(M dv\, dx)$, w-$L^1(M dv\, dx)$, and w-$L^1(d\mu\, dx)$ respectively. Define the fluid variables $(\rho^{in}, u^{in}, \theta^{in})$ in $L^2(dx; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R})$ corresponding to g^{in} by the formulas*

$$\rho^{in} = \langle g^{in}\rangle\,, \quad u^{in} = \langle v g^{in}\rangle\,, \quad \theta^{in} = \frac{2}{\mathrm{D}}\left\langle \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) g^{in}\right\rangle\,. \tag{4.24}$$

Then g has the form of an infinitesimal Maxwellian

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,, \tag{4.25}$$

where $\rho \in \mathscr{V}_s$, $u \in \mathscr{V}_v$, and $\nabla_x\theta \in L^2(dx)$ satisfy the inequality

$$\begin{aligned}\int \rho^2 + |u|^2 + \frac{\mathrm{D}}{2}\theta^2\, dx - \int \rho^{in}\rho + u^{in}\cdot u + \frac{\mathrm{D}}{2}\theta^{in}\theta\, dx + \int \frac{1}{2}\langle g^{in\,2}\rangle\, dx \\ + \int \frac{1}{2}\nu\, |\nabla_x u + (\nabla_x u)^T|^2 + \frac{\mathrm{D}+2}{2}\kappa\, |\nabla_x\theta|^2\, dx \\ \leqq \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon^{in})\right\rangle dx \leqq C^{in}\,.\end{aligned} \tag{4.26}$$

Proof: Inequality (3.36) of Proposition 3.5 states that

$$\int \frac{1}{2}\langle g^2\rangle\,dx + \int \frac{1}{2}\left\langle |g - g^{in}|^2\right\rangle\,dx + \int \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle\,dx$$

(4.27)

$$\leqq \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon\, g_\varepsilon^{in})\right\rangle\,dx \leqq C^{in}\,.$$

Expanding the middle term on the left side of this inequality gives

$$(4.28)\qquad \int \frac{1}{2}\langle |g - g^{in}|^2\rangle\,dx = \int \frac{1}{2}\langle g^2\rangle\,dx - \int \langle g^{in} g\rangle\,dx + \int \frac{1}{2}\langle g^{in\,2}\rangle\,dx\,.$$

Proposition 3.8 states that g has the form of an infinitesimal Maxwellian (4.25); a direct calculation shows that

$$(4.29)\qquad \int \langle g^2\rangle\,dx = \int \rho^2 + |u|^2 + \frac{\mathrm{D}}{2}\theta^2\,dx\,,$$

while using the definition (4.24) of $(\rho^{in}, u^{in}, \theta^{in})$ gives

$$(4.30)\qquad \int \langle g^{in} g\rangle\,dx = \int \rho^{in}\rho + u^{in}\cdot u + \frac{\mathrm{D}}{2}\theta^{in}\theta\,dx\,.$$

As before, Proposition 4.3 and Lemma 4.7 yield the inequality

$$(4.31)\qquad \frac{1}{2}\nu\,|\nabla_x u + (\nabla_x u)^T|^2 + \frac{\mathrm{D}+2}{2}\kappa\,|\nabla_x\theta|^2 \leqq \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle\,,$$

almost everywhere. The result then follows upon setting (4.28)–(4.31) into the limiting entropy inequality (4.27).

The Leray Energy Inequalities

It was quite clear from the outset that the DiPerna-Lions global existence result for the Boltzmann equation was analogous to Leray's global existence result for the incompressible Navier-Stokes equation. More specifically, the DiPerna-Lions entropy inequalities and the Leray energy inequalities play parallel roles in their respective theories; both inequalities are instrumental in establishing compactness results and serve as criteria to select relevant solutions from among all weak solutions. It is therefore quite satisfying that the dissipation inequalities of the preceding subsection can be refined to make this analogy more explicit; the Leray energy inequalities derive from the DiPerna-Lions entropy inequalities.

This refinement requires the introduction of a stronger notion of convergence than we have used heretofore. We say that $g_\varepsilon \to g$ entropically of order ε^m provided that

$$g_\varepsilon \to g \quad \text{in } w\text{-}L^1(M dv\, dx)$$

$$\text{and} \tag{4.32}$$

$$\lim_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) \right\rangle dx = \int \frac{1}{2}\langle g^2\rangle\, dx\,.$$

The notion of entropic convergence is a strong one since, as will be shown in Proposition 4.11 below, it implies that sequence g_ε converges to g strongly in $L^1((1+|v|^2)M dv\, dx)$.

If the initial data $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$ for the Boltzmann initial-value problem (1.63) is now chosen so that $g_\varepsilon^{in} \to g^{in}$ entropically of order ε^m then the entropy bound (3.2) is automatically satisfied. Proposition 4.6 then has the following refinement.

PROPOSITION 4.9. (THE LERAY ENERGY INEQUALITY) *Let* $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R})$ *and define the infinitesimal Maxwellian* g^{in} *in* $L^2(M dv\, dx)$ *by the formula*

$$g^{in} = \rho^{in} + u^{in}\cdot v + \theta^{in}\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,. \tag{4.33}$$

Suppose that $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$ *such that* $g_\varepsilon^{in} \to g^{in}$ *entropically of order* ε^m *for some* $m \geqq 1$. *Let* $G_\varepsilon \geqq 0$ *be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *and let* g_ε *and* q_ε *be the corresponding sequences of fluctuations and scaled collision integrands. Let* g *and* q *be limits of the sequences* g_ε *and* $q_\varepsilon/N_\varepsilon$ *in* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx))$ *and* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\, dx))$ *respectively. Then* g *has the form of an infinitesimal Maxwellian*

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,, \tag{4.34}$$

where $\rho \in L^2(dt; \mathcal{V}_s)$, $u \in L^2(dt; \mathcal{V}_v)$, *and* $\nabla_x\theta \in L^2(dt; L^2(dx))$ *satisfy the inequality*

$$\begin{aligned} &\int \frac{1}{2}\left(\rho(t)^2 + |u(t)|^2 + \frac{\mathrm{D}}{2}\theta(t)^2\right) dx \\ &+ \int_0^t \int \frac{1}{2}\nu\left|\nabla_x u + (\nabla_x u)^T\right|^2 + \frac{\mathrm{D}+2}{2}\kappa\left|\nabla_x\theta\right|^2 dx\, ds \\ &\leqq \int \frac{1}{2}\left(\rho^{in\,2} + |u^{in}|^2 + \frac{\mathrm{D}}{2}\theta^{in\,2}\right) dx\,. \end{aligned} \tag{4.35}$$

Remark. Note that this is a single dissipation inequality for the fluid variables rather than the two which are given in (1.62) that might be expected from the formal argument. Of course, this reflects the fact there is only one DiPerna-Lions entropy inequality from which to begin.

Proof: Invoking entropic convergence (4.32) and using the initial infinitesimal Maxwellian form (4.33), a direct calculation shows that

$$\lim_{\varepsilon\to 0}\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in})\right\rangle dx = \int \frac{1}{2}\langle g^{in\,2}\rangle\, dx = \int \frac{1}{2}\left(\rho^{in\,2} + |u^{in}|^2 + \frac{\mathrm{D}}{2}\theta^{in\,2}\right) dx\,. \tag{4.36}$$

The result then follows from setting (4.36) into the limiting entropy inequality (4.15) of Proposition 4.6.

The analogous result for the time-discretized Boltzmann problem (2.21) is even easier; it follows directly from Proposition 4.8 upon invoking entropic convergence (4.32) in the limiting entropy inequality (4.26).

Proposition 4.10. *Let $g^{in} \in L^2(M dv\, dx)$ and define the corresponding fluid variables $(\rho^{in}, u^{in}, \theta^{in})$ in $L^2(dx; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R})$ by the formulas*

$$\rho^{in} = \langle g^{in}\rangle\,, \quad u^{in} = \langle v g^{in}\rangle\,, \quad \theta^{in} = \frac{2}{\mathrm{D}}\left\langle \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) g^{in}\right\rangle\,. \tag{4.37}$$

Suppose that $G_\varepsilon^{in} = 1 + \varepsilon g_\varepsilon^{in} \geqq 0$ such that g_ε^{in} converges to g^{in} entropically of order ε. Let $G_\varepsilon \geqq 0$ be a sequence of renormalized solutions of the scaled Boltzmann time-discretized problem (2.22) *and let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands. Let g and q be limits of the sequences g_ε and $q_\varepsilon/N_\varepsilon$ in w-$L^1(M dv\, dx)$ and w-$L^1(d\mu\, dx)$ respectively. Then g has the form of an infinitesimal Maxwellian*

$$g = \rho + u\cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,, \tag{4.38}$$

where $\rho \in \mathcal{V}_s$, $u \in \mathcal{V}_v$, and $\nabla_x\theta \in L^2(dx)$ satisfy the inequality

$$\int \rho^2 + |u|^2 + \frac{\mathrm{D}}{2}\theta^2\, dx + \int \frac{1}{2}\nu\, \left|\nabla_x u + (\nabla_x u)^T\right|^2 + \frac{\mathrm{D}+2}{2}\kappa\, |\nabla_x\theta|^2\, dx \leqq \int \rho^{in}\rho + u^{in}\cdot u + \frac{\mathrm{D}}{2}\theta^{in}\theta\, dx\,. \tag{4.39}$$

Remark. Notice that here, in contrast to Proposition 4.9 for the continuous-time case, g^{in} is not assumed to have the form of an infinitesimal Maxwellian.

Entropic Convergence

The notion of entropic convergence introduced in the last subsection will play a central role in sharpening our convergence results. Here we show that it implies convergence in $L^1((1+|v|^2)M dv\,dx)$.

PROPOSITION 4.11. *If g_ε converges to g entropically of order ε^m then g_ε converges to g in $L^1((1+|v|^2)M dv\,dx)$.*

Proof: For any $z_o > -1$ consider the convex function defined over $z > -1$ by

$$z \mapsto h(z) - h(z_o) - h'(z_o)(z - z_o)\,.$$

When viewed as a function of $z - z_o$ this function satisfies the reflection property

$$h(z_o + |z - z_o|) - h(z_o) - h'(z_o)|z - z_o| \leqq h(z) - h(z_o) - h'(z_o)(z - z_o)\,,$$

so the Young inequality gives

$$\begin{aligned} y|z - z_o| \leqq\ & h^*(h'(z_o) + y) - h^*(h'(z_o)) - z_o y \\ & + h(z) - h(z_o) - h'(z_o)(z - z_o)\,. \end{aligned} \tag{4.40}$$

Moreover, the Legendre dual function

$$y \mapsto h^*(h'(z_o) + y) - h^*(h'(z_o)) - z_o y\,,$$

is a superquadratic function of y in the sense that

$$\begin{aligned} & h^*(h'(z_o) + \lambda y) - h^*(h'(z_o)) - z_o \lambda y \\ & \qquad \leqq \lambda^2 \left(h^*(h'(z_o) + y) - h^*(h'(z_o)) - z_o y\right)\,, \end{aligned} \tag{4.41}$$

for every $\lambda \leqq 1$.

Let α be any number such that $\alpha \geqq \varepsilon^m$ for every value of ε. Set $z_o = \varepsilon^m g$, $z = \varepsilon^m g_\varepsilon$, and $y = \varepsilon^m \frac{1}{4}(1 + |v|^2)/\alpha$ into (4.40) and make use of the superquadratic property (4.41) with $\lambda = \varepsilon^m/\alpha$ to obtain

$$\begin{aligned} \frac{1}{4}(1 + |v|^2)|g_\varepsilon - g| \leqq\ & \frac{1}{\alpha}\Big(h^*\Big(h'(\varepsilon^m g) + \frac{1}{4}(1 + |v|^2)\Big) \\ & - h^*\left(h'(\varepsilon^m g)\right) - \varepsilon^m g \frac{1}{4}(1 + |v|^2)\Big) \\ & + \frac{\alpha}{\varepsilon^{2m}}\left(h(\varepsilon^m g_\varepsilon) - h(\varepsilon^m g) - \varepsilon^m h'(\varepsilon^m g)(g_\varepsilon - g)\right)\,. \end{aligned} \tag{4.42}$$

Fix $\lambda > 0$ and multiply this inequality by the indicator (characteristic) function $\mathbf{1}_{|g|<\lambda}$; integrate the result over $M dv\, dx$ and let ε tend to zero to obtain

$$\limsup_{\varepsilon\to 0} \int \frac{1}{4} \left\langle (1+|v|^2)\, |g_\varepsilon - g| \mathbf{1}_{|g|<\lambda} \right\rangle dx \leqq \frac{1}{\alpha} \left\langle h^*\!\left(\frac{1}{4}(1+|v|^2)\right)\right\rangle .$$

Arguing as in the proof of assertion (3) of Proposition 3.1, the limit of the last term in (4.42) vanishes by the entropic convergence of g_ε to g. By the arbitrariness of α, the limit must be zero. Taking λ to infinity while using the equi-integrability of the sequence g_ε then completes the proof.

5. Weak Compactness of the Convection-Diffusion Tensor

The Convection-Diffusion Tensor

Let G_ε be a family of renormalized solutions of the scaled Boltzmann initial-value problem (1.63) satisfying assumption (H0). This means that the normalized Boltzmann equation satisfied by g_ε is

$$\varepsilon\, \partial_t \gamma_\varepsilon + v\cdot\nabla_x \gamma_\varepsilon = \frac{1}{\varepsilon^{m+1}} \frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon} , \tag{5.1}$$

and that, as assumed in (H0), g_ε satisfies the local momentum conservation law

$$\partial_t \langle v g_\varepsilon \rangle + \frac{1}{\varepsilon} \nabla_x \cdot \langle v\otimes v g_\varepsilon\rangle = 0 . \tag{5.2}$$

The momentum flux is a symmetric $\mathrm{D}\times\mathrm{D}$ tensor that can be split into the sum of its traceless and diagonal part as

$$\begin{aligned} \langle v\otimes v g_\varepsilon\rangle &= \left\langle \left(v\otimes v - \frac{1}{\mathrm{D}}|v|^2 I\right) g_\varepsilon \right\rangle + \left\langle \frac{1}{\mathrm{D}} |v|^2 g_\varepsilon \right\rangle I \\ &= \langle (L\phi) g_\varepsilon \rangle + \left\langle \frac{1}{\mathrm{D}} |v|^2 g_\varepsilon \right\rangle I , \end{aligned}$$

where ϕ was defined in (1.46). The local momentum conservation law can then be recast as

$$\partial_t \langle v g_\varepsilon \rangle + \frac{1}{\varepsilon} \nabla_x \cdot \langle (L\phi) g_\varepsilon\rangle = -\frac{1}{\varepsilon} \nabla_x \left\langle \frac{1}{\mathrm{D}} |v|^2 g_\varepsilon \right\rangle . \tag{5.3}$$

The quantity under the gradient on the right side of (5.3) is to become the pressure in the limiting process and will be consistently referred to as the pressure term; in the same way, the divergence of the tensor field on the left side of (5.3) will eventually converge to the convection and diffusion terms of the Navier-Stokes equation and this tensor field will be referred to as the convection-diffusion tensor.

It will be unessential to control the pressure term in any asymptotics leading to incompressible hydrodynamics since the limit of (5.3) will be considered after integrating against a divergence-free test function; the pressure will then be recovered as a Lagrange multiplier. This is the reason for the above splitting of the momentum flux.

The convection-diffusion tensor, on the other hand, must be controlled with particular care since it remains the only source of nonlinearities in the limiting hydrodynamic equation. Its control is based on the decomposition

$$\begin{aligned}\frac{1}{\varepsilon}\langle (L\phi)g_\varepsilon\rangle &= \frac{1}{\varepsilon}\langle \phi L g_\varepsilon\rangle \\ &= \left\langle \phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon\right\rangle + \left\langle \phi\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle \\ &\quad - \left\langle \phi\frac{1}{\varepsilon^{m+1}}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon}\right\rangle .\end{aligned} \tag{5.4}$$

This decomposition is physically meaningful: the divergence of the last term in (5.4) will converge to the diffusion term in the Navier-Stokes equation, whereas that of the second term in (5.4) will converge to the convection term (when $m = 1$). The first term is only a remainder due to the normalization procedure and will converge to zero.

The limit of the last term in (5.4) is easily computed since, as was seen in the proof of Corollary 3.5, it can be written in terms of the sequence $q_\varepsilon/N_\varepsilon$, which is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(d\mu\, dx))$. Let q be the limit point of any converging subsequence of $q_\varepsilon/N_\varepsilon$. Thus, as ε tends to zero, it is seen that

$$\left\langle \phi\frac{1}{\varepsilon^{m+1}}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon}\right\rangle = \left\langle\!\!\left\langle \phi\frac{q_\varepsilon}{N_\varepsilon}\right\rangle\!\!\right\rangle \to \langle\!\langle \phi q\rangle\!\rangle .$$

But this limiting quantity was already evaluated in equation (4.19) of Lemma 4.5 and found to be given by

$$\langle\!\langle \phi q\rangle\!\rangle = \nu\left(\nabla_x u + (\nabla_x u)^T\right) .$$

This proves the following result.

PROPOSITION 5.1. *Given G_ε as described above, then*

$$\left\langle \phi\frac{1}{\varepsilon^{m+1}}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon}\right\rangle \to \nu\left(\nabla_x u + (\nabla_x u)^T\right)$$

in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$ as ε tends to zero.

This proposition means that the control of the convection-diffusion tensor has been reduced to that of the second term in (5.4); upon doing so, the remainder term will then easily be shown to vanish. The nonlinear character of this term can be seen by considering its formal limit when $m = 1$,

$$\left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle \to \langle \phi Q(g, g) \rangle .$$

This section is essentially aimed at the goal of controlling the convection-diffusion tensor, or equivalently, the convection tensor.

We should make it clear at this point what is meant by "control" here. When $m > 1$ the convection tensor formally vanishes due to the ε^{m-1} sitting in front of it. In that case we shall show that Proposition 3.3 is enough to justify the formal limit. When $m = 1$ then more is needed. Section 7, which treats that case, essentially provides pointwise convergence properties of g_ε. In order eventually to obtain convergence properties in strong L^1, the relative compactness in w-L^1 of all the quantities of interest should be first established. This is exactly the main task of the present section.

Weak Compactness of the Convection Tensor

For technical reasons, it is convenient to decompose the family g_ε into a part that is bounded in $L^\infty(dt; L^2(M dv\, dx))$ and a part that is small in

$$L^\infty(dt; L^1(M dv\, dx)) .$$

Based on parts (1) and (3) of Proposition 3.2, we choose to do this in the form

$$g_\varepsilon = \widehat{g}_\varepsilon + \varepsilon^m \widetilde{g}_\varepsilon , \quad \text{where } \widehat{g}_\varepsilon = \frac{g_\varepsilon}{N_\varepsilon} \quad \text{and} \quad \widetilde{g}_\varepsilon = \frac{1}{3} \frac{g_\varepsilon^2}{N_\varepsilon} . \tag{5.5}$$

This decomposition will be instrumental in controlling the nonlinearities.

Some properties of the Boltzmann kernel will be used in controlling some high velocity tails. These properties are a direct consequence of assumption (H1) which holds throughout the present section and is discussed in Appendix C.

For the case when $m = 1$, we also adopt assumption (H2):

$$(1 + |v|^2)\widetilde{g}_\varepsilon \quad \text{is relatively compact in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\, dx)) .$$

For the case when $m > 1$, the result proved in Proposition 3.3 is sufficient for the purposes of this article.

The main result of the present section is

PROPOSITION 5.2. *Given $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon$ as described above, then the following statements hold.*

(i) *The sequences*

$$(1+|\phi|)\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon} \quad \text{and} \quad (1+|\phi|)\frac{1}{\varepsilon}Lg_\varepsilon$$

are relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt;w\text{-}L^1(M\,dv\,dx))$.

(ii) *The convergence*

$$(1+|\phi|)\varepsilon^{m-1}\left(\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}-Q(\hat{g}_\varepsilon,\hat{g}_\varepsilon)\right)\to 0$$

holds in $L^1_{\mathrm{loc}}(dt;L^1(M\,dv\,dx))$ *as* ε *tends to zero.*

(iii) *The families*

$$(1+|\phi|)\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon \quad \text{and} \quad (1+|\phi|)\frac{1}{\varepsilon}\left(1-\frac{\langle(1+|v|^2)\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)Lg_\varepsilon$$

converge to zero in $L^1_{\mathrm{loc}}(dt;L^1(M\,dv\,dx))$ *as* ε *tends to zero.*

The proof of assertion (i) *being quite involved, it is preferable to begin with a short outline of its main steps, before giving the proof itself. Those of assertions* (ii) *and* (iii) *will be easy corollaries of* (i). *The steps are best understood in the context of the following formula that will be the key to the proof:*

$$\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{\mathcal{N}_\varepsilon}=\frac{1}{\varepsilon}\frac{Lg_\varepsilon}{\mathcal{N}_\varepsilon}+\frac{1}{\varepsilon^{m+1}}\frac{Q(G_\varepsilon,G_\varepsilon)}{\mathcal{N}_\varepsilon}\,. \tag{5.6}$$

Here, $\mathcal{N}_\varepsilon$ *denotes various renormalizations used in the sequel;* $\mathcal{N}_\varepsilon$ *will stand for either* N_ε *or* $\langle(1+|v|^2)N_\varepsilon\rangle/\langle(1+|v|^2)\rangle$ *or even* $N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle/\langle(1+|v|^2)\rangle$.

Step 1. The families

$$(1+|\phi|)\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon \quad \text{and} \quad (1+|\phi|)\frac{1}{\varepsilon}\left(1-\frac{\langle(1+|v|^2)\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)Lg_\varepsilon$$

are relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt;w\text{-}L^1(M\,dv\,dx))$.

Step 2. The sequence

$$(1+|\phi|)\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{\langle(1+|v|^2)N_\varepsilon\rangle}$$

is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt;w\text{-}L^1(M\,dv\,dx))$.

Step 3. The sequence

$$(1 + |\phi|)\frac{1}{\varepsilon}\frac{1}{N_\varepsilon\langle(1 + |v|^2)N_\varepsilon\rangle}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M dv\, dx))$.

Step 4. The sequence

$$(1 + |\phi|)\frac{1}{\varepsilon}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M dv\, dx))$.

Step 5. The sequence

$$(1 + |\phi|)\varepsilon^{m-1}\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}$$

is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M dv\, dx))$.

Proof:

Step 1. Here, $\mathcal{N}_\varepsilon$ stands for either N_ε or $\langle(1 + |v|^2)N_\varepsilon\rangle / \langle(1 + |v|^2)\rangle$. The decomposition in (5.5) yields

$$\frac{1}{\varepsilon}\left(1 - \frac{1}{\mathcal{N}_\varepsilon}\right)Lg_\varepsilon = \frac{1}{\varepsilon}\left(1 - \frac{1}{\mathcal{N}_\varepsilon}\right)L\hat{g}_\varepsilon + \left(1 - \frac{1}{\mathcal{N}_\varepsilon}\right)\varepsilon^{m-1}L\tilde{g}_\varepsilon\,. \tag{5.7}$$

The sequence $(1 + |\phi|)\varepsilon^{m-1}L\tilde{g}_\varepsilon$ is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M dv\, dx))$ because of Proposition 3.3 or (H2); the sequence

$$\left(1 - \frac{1}{\mathcal{N}_\varepsilon}\right)$$

is uniformly bounded by 3/2 and converges almost everywhere to 1 as ε tends to zero. According to the product limit theorem (see Appendix B), the sequence

$$(1 + |\phi|)\left(1 - \frac{1}{\mathcal{N}_\varepsilon}\right)\varepsilon^{m-1}L\tilde{g}_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M dv\, dx))$.

Now, let $\Omega \subset [0,T] \times \mathbb{T}^D \times \mathbb{R}^D$ be measurable. Using the Cauchy-Schwarz inequality twice with Proposition C (see Appendix C) provides the estimate

$$
\begin{aligned}
&\left(\int_\Omega (1 + |\phi|) \left| \frac{1}{\varepsilon} \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right) L\hat{g}_\varepsilon \right| M dv\, dx\, dt \right)^2 \\
&\leqq \frac{1}{\varepsilon^2} \left(\int_\Omega \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right)^2 (1 + |v|^2) M dv\, dx\, dt \right) \\
&\quad \times \left(\int_\Omega \frac{(1 + |\phi|)^2}{(1 + |v|^2)} |L\hat{g}_\varepsilon|^2 M dv\, dx\, dt \right) \\
&\leqq \frac{1}{\varepsilon^2} \left(\int_\Omega \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right)^2 (1 + |v|^2) M dv\, dx\, dt \right) \\
&\quad \times \left(\int_\Omega \frac{(1 + |\phi|)^2}{(1 + |v|^2)} |L|(1)\, |L|(|\hat{g}_\varepsilon|^2) M dv\, dx\, dt \right) \qquad (5.8) \\
&\leqq C \frac{1}{\varepsilon^2} \left(\int_\Omega \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right)^2 (1 + |v|^2) M dv\, dx\, dt \right) \\
&\quad \times \left(\int_\Omega \frac{(1 + |\phi|)^2}{(1 + |v|^2)} |L|(1)\, |L|(\tilde{g}_\varepsilon) M dv\, dx\, dt \right) \\
&\leqq C \left(\int_\Omega \frac{1}{\varepsilon^{m+1}} \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right)^2 (1 + |v|^2) M dv\, dx\, dt \right) \\
&\quad \times \left(\int_\Omega (1 + |\phi|)\, \varepsilon^{m-1} |L|(\tilde{g}_\varepsilon) M dv\, dx\, dt \right) .
\end{aligned}
$$

In the case where $\mathcal{N}_\varepsilon = N_\varepsilon$ one has

$$
\frac{1}{\varepsilon^{m+1}} \left(1 - \frac{1}{\mathcal{N}_\varepsilon} \right)^2 = \frac{1}{9} \varepsilon^{m-1} \hat{g}_\varepsilon^2 \leqq \frac{1}{6} \varepsilon^{m-1} \tilde{g}_\varepsilon .
$$

In the case where $\mathcal{N}_\varepsilon = \langle(1+|v|^2)N_\varepsilon\rangle/\langle(1+|v|^2)\rangle$, the following inequality holds:

$$\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon^{m+1}}\left(1-\frac{1}{\mathcal{N}_\varepsilon}\right)^2 (1+|v|^2)M\,dv\,dx\,dt \\
&\leqq \int_\Omega \frac{1}{9}\varepsilon^{m-1}\left(\frac{\langle(1+|v|^2)g_\varepsilon\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)^2 (1+|v|^2)M\,dv\,dx\,dt \\
&\leqq \frac{1}{6}\int_0^T\int \varepsilon^{m-1}\frac{\langle(1+|v|^2)g_\varepsilon\rangle^2}{\langle(1+|v|^2)N_\varepsilon\rangle}\,dx\,dt \\
&\leqq C\int_0^T\int \varepsilon^{m-1}(1+|v|^2)\,\tilde{g}_\varepsilon M\,dv\,dx\,dt\,,
\end{aligned}$$

and follows from the Jensen inequality applied to the convex map

$$s(z) = \frac{1}{2}z^2\left(1+\frac{1}{3}z\right)^{-1}$$

and the measure $(1+|v|^2)M\,dv$. In both cases therefore,

$$\int_\Omega \frac{1}{\varepsilon^{m+1}}\left(1-\frac{1}{\mathcal{N}_\varepsilon}\right)^2 (1+|v|^2)\,M\,dv\,dx\,dt \tag{5.9}$$
$$\leqq C\int_0^T\iint \varepsilon^{m-1}\tilde{g}_\varepsilon\,(1+|v|^2)\,M\,dv\,dx\,dt\,.$$

Finally, according to (5.8), (5.9), and Proposition 3.3 or (H2):

$$\left(\int_\Omega (1+|\phi|)\left|\frac{1}{\varepsilon}\left(1-\frac{1}{\mathcal{N}_\varepsilon}\right)L\hat{g}_\varepsilon\right| M\,dv\,dx\,dt\right)^2 \tag{5.10}$$
$$\leqq C\int_\Omega (1+|\phi|)\varepsilon^{m-1}|L|(\tilde{g}_\varepsilon)M\,dv\,dx\,dt\,,$$

where C is a generic positive constant. Replacing Ω with the whole $[0,T]\times\mathbb{T}^D\times\mathbb{R}^D$ space in (5.10) and using Proposition 3.3 and assumption (H2) in connection with Proposition C of Appendix C shows that the sequence $(1+|\phi|)|\hat{g}_\varepsilon L\hat{g}_\varepsilon|$ is bounded in $L^1(M\,dv\,dx)$. Taking meas(Ω) going to zero in (5.8) shows by the same argument that this sequence is equi-integrable. Going back to (5.7) gives the announced result announced in Step 1.

Step 2. The statement announced above as Step 2 will be an easy consequence of the following

LEMMA 5.3. *The following assertions hold true:*

(1) *The family*

$$\varepsilon^{m-1}|\hat{g}_\varepsilon||\hat{g}_{\varepsilon 1}|(1+|v|^2)(1+|v_1|^2)$$

is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$;

(2) *The family*

$$\varepsilon^{m-1}|\hat{g}_\varepsilon|\varepsilon^m \tilde{g}_{\varepsilon 1}(1+|v|^2)(1+|v_1|^2) \to 0$$

in $L^1_{\mathrm{loc}}(dt; L^1(M_1 dv_1 M dv\, dx))$ *as* $\varepsilon \to 0$;

(3) *The inequalities*

$$\varepsilon^{2m}\frac{\tilde{g}_\varepsilon}{N_\varepsilon} \leqq 3 \quad \textit{and} \quad \varepsilon^{2m}\frac{\langle (1+|v|^2)\tilde{g}_\varepsilon\rangle}{\langle (1+|v|^2)N_\varepsilon\rangle} \leqq 3$$

hold for almost every (t,x,v) *and* (t,x) *respectively*;

(4) *The family*

$$\varepsilon^{m-1}\frac{\varepsilon^m \tilde{g}_\varepsilon \varepsilon^m \tilde{g}_{\varepsilon 1}}{\langle (1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2) \to 0$$

in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$ *as* $\varepsilon \to 0$;

(5) *The family*

$$\varepsilon^{m-1}\frac{|g_\varepsilon||g_{\varepsilon 1}|}{\langle (1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2)$$

is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$.

Proof: Notice that assertion (5) is an easy consequence of assertions (1), (2), and (4), according to the decomposition (5.5).

To prove assertion (1), it suffices to remark that

$$\begin{aligned}\varepsilon^{m-1}|\hat{g}_\varepsilon||\hat{g}_{\varepsilon 1}|(1+|v|^2)(1+|v_1|^2) &\leqq \varepsilon^{m-1}\frac{1}{2}\left(\hat{g}_\varepsilon^2+\hat{g}_{\varepsilon 1}^2\right)(1+|v|^2)(1+|v_1|^2)\\ &\leqq \varepsilon^{m-1}\frac{3}{4}(\tilde{g}_\varepsilon+\tilde{g}_{\varepsilon 1})(1+|v|^2)(1+|v_1|^2)\,.\end{aligned}$$

Indeed, it follows from Proposition 3.3 or (H2) that $\varepsilon^{m-1}\tilde{g}_\varepsilon(1+|v|^2)(1+|v_1|^2)$ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$, whence assertion (1) easily follows.

The proof of assertion (2) directly follows from the product limit theorem of Appendix B. Indeed, one has $\varepsilon^m|\widehat{g}_\varepsilon| \leqq 3$ and $\varepsilon^m|\widehat{g}_\varepsilon| \to 0$ almost everywhere; moreover the family $\varepsilon^{m-1}\widetilde{g}_{\varepsilon 1}(1+|v|^2)(1+|v_1|^2)$ is relatively compact in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(M_1dv_1Mdv\,dx))$ according to assumption (H2) or Proposition 3.3; their product converges therefore to zero in $L^1_{\rm loc}(dt; L^1(M_1dv_1Mdv\,dx))$.

To prove assertion (3), notice first that

$$1+\frac{1}{3}\varepsilon^m\widehat{g}_\varepsilon \geqq 1-\frac{1}{3}\varepsilon^m\widehat{g}_\varepsilon^- \geqq 1-\frac{1}{2}\varepsilon^m g_\varepsilon^- \geqq \frac{1}{2}\,.$$

Indeed,

$$\varepsilon^m\widehat{g}_\varepsilon = \frac{\varepsilon^m g_\varepsilon}{1+\frac{1}{3}\varepsilon^m g_\varepsilon} \leqq \frac{3}{2}\varepsilon^m g_\varepsilon^- \leqq \frac{3}{2}\,,$$

since the renormalized solutions of the Boltzmann equation $G_\varepsilon = 1+\varepsilon^m g_\varepsilon$ are non-negative. Therefore:

$$\frac{\varepsilon^{2m}\widetilde{g}_\varepsilon}{N_\varepsilon} \leqq \frac{\varepsilon^{2m}\widetilde{g}_\varepsilon}{\frac{2}{3}+\frac{1}{3}\varepsilon^{2m}\widetilde{g}_\varepsilon} \leqq 3\,.$$

To prove assertion (4), proceed as follows. First,

$$\int_0^T\iiint \varepsilon^{m-1}\frac{\varepsilon^m\widetilde{g}_\varepsilon\varepsilon^m\widetilde{g}_{\varepsilon 1}}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2)M_1dv_1\,Mdv\,dx\,dt$$

$$\leqq \int_0^T\iint \varepsilon^{m-1}\widetilde{g}_\varepsilon(1+|v|^2)\left(\int\frac{\varepsilon^{2m}\widetilde{g}_{\varepsilon 1}}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v_1|^2)M_1dv_1\right)Mdv\,dx\,dt\,.$$

Now, in view of Proposition 3.3 or assumption (H2) and assertion (3) above,

$$\int\frac{\varepsilon^{2m}\widetilde{g}_{\varepsilon 1}}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v_1|^2)M_1dv_1 \to 0$$

for almost every (t,x) and is bounded by 3. Applying the product limit theorem shows that

$$\varepsilon^{m-1}\widetilde{g}_\varepsilon(1+|v|^2)\left(\int\frac{\varepsilon^{2m}\widetilde{g}_{\varepsilon 1}}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v_1|^2)M_1dv_1\right) \to 0$$

in $w\text{-}L^1_{\rm loc}(dt; w\text{-}L^1(Mdv\,dx))$; assertion (4) then follows upon integration with respect to (t,x,v).

Continuation of the Proof of Proposition 5.2: The proof of Step 2 now follows easily from Lemma 5.3. Using assumption (H1) with assertion (5) of Lemma 5.3 shows that the family

$$\varepsilon^{m-1}\frac{|g_\varepsilon||g_{\varepsilon 1}|}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|\phi|)\,b(v_1-v;\omega)$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$. Therefore, the renormalized sink term (see Section 1)

$$(1 + |\phi|)\varepsilon^{m-1}\frac{Q^-(g_\varepsilon, g_\varepsilon)}{\langle(1 + |v|^2)N_\varepsilon\rangle}$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$.

To control the source term, observe first that, according to (H1)

$$|\phi(v')|\, b(v_1' - v'; \omega) \leqq C(1 + |v'|^2 + |v_1'|^2) = C(1 + |v|^2 + |v_1|^2)\,.$$

Therefore, assertion (5) of Lemma 5.3 shows that the family

$$\varepsilon^{m-1}\frac{|g_\varepsilon||g_{\varepsilon 1}|}{\langle(1 + |v|^2)N_\varepsilon\rangle}(1 + |\phi(v')|)\, b(v_1' - v'; \omega)$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$. Upon the change of variables $(v, v_1) \mapsto (v', v_1')$ which leaves invariant the measure $M_1 dv_1 M dv$, it follows that the family

$$\varepsilon^{m-1}\frac{|g_\varepsilon'||g_{\varepsilon 1}'|}{\langle(1 + |v|^2)N_\varepsilon\rangle}(1 + |\phi(v)|)\, b(v_1 - v; \omega)$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$. Therefore, the renormalized source term

$$(1 + |\phi|)\varepsilon^{m-1}\frac{Q^-(g_\varepsilon, g_\varepsilon)}{\langle(1 + |v|^2)N_\varepsilon\rangle}$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M_1 dv_1 M dv\, dx))$. This completes Step 2.

Step 3. The essential formula in this step is (5.6) with $\mathcal{N}_\varepsilon = N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle$. Both N_ε and $\langle(1 + |v|^2)N_\varepsilon\rangle$ are uniformly bounded below by 2/3 and converge almost everywhere as ε tends to zero. The product limit theorem shows:

—according to Step 2 above, that

$$(1 + |\phi|)\varepsilon^{m-1}\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon\langle(1 + |v|^2)N_\varepsilon\rangle}$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M dv\, dx))$,

—according to Proposition C of Appendix C, that

$$(1 + |\phi|)\frac{1}{\varepsilon^{m+1}}\frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon\langle(1 + |v|^2)N_\varepsilon\rangle}$$

is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M dv\, dx))$.

Then (5.6) with $\mathcal{N}_\varepsilon = N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle$ shows that

$$(1+|\phi|)\frac{1}{\varepsilon}\frac{1}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$.

Step 4. Start with the identity

$$(5.11)\quad \begin{aligned}&(1+|\phi|)\left(1-\frac{\langle\langle(1+|v|^2)\rangle\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\right)\frac{1}{\varepsilon}Lg_\varepsilon\\ &= (1+|\phi|)\left(1-\frac{1}{N_\varepsilon}\right)\frac{1}{\varepsilon}Lg_\varepsilon + (1+|\phi|)\frac{1}{N_\varepsilon}\left(1-\frac{\langle\langle(1+|v|^2)\rangle\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)\frac{1}{\varepsilon}Lg_\varepsilon .\end{aligned}$$

The first term on the right side of identity (5.11) is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$ in view of Step 1; the second term on the right side of (5.11) is also relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$ because of Step 1. Indeed,

$$(1+|\phi|)\left(1-\frac{\langle\langle(1+|v|^2)\rangle\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)\frac{1}{\varepsilon}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$; apply the product limit theorem with multiplier the sequence $1/N_\varepsilon$, which is uniformly bounded by 3/2 and converges to 1 almost everywhere as ε tends to zero. Therefore,

$$(1+|\phi|)\left(1-\frac{\langle\langle(1+|v|^2)\rangle\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\right)\frac{1}{\varepsilon}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$. Combining this property with the result of Step 2 finally proves that

$$(1+|\phi|)\frac{1}{\varepsilon}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$.

Step 5. Write (5.6) with $\mathcal{N}_\varepsilon = N_\varepsilon$. Using the results of Step 1 and of Step 3 in this proof shows that

$$(1+|\phi|)\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(Mdv\,dx))$.

This completes the proof of assertion (i) in Proposition 5.2. The proof of assertion (ii) follows from assertions (2) and (4) in Lemma 5.3. Indeed, one has:

$$\varepsilon^{m-1}\left(\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} - Q(\hat{g}_\varepsilon, \hat{g}_\varepsilon)\right) = \varepsilon^{m-1}\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\left(1 - \frac{\langle 1+|v|^2\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)$$
$$+ \varepsilon^{m-1}\frac{\langle 1+|v|^2\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\left(Q(g_\varepsilon, g_\varepsilon) - Q(\hat{g}_\varepsilon, \hat{g}_\varepsilon)\right)$$
$$- \varepsilon^{m-1}Q(\hat{g}_\varepsilon, \hat{g}_\varepsilon)\left(1 - \frac{\langle 1+|v|^2\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\right).$$

Observe that the families

$$\varepsilon^{m-1}\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\left(1 - \frac{\langle 1+|v|^2\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right)$$

and

$$\varepsilon^{m-1}Q(\hat{g}_\varepsilon, \hat{g}_\varepsilon)\left(1 - \frac{\langle 1+|v|^2\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\right)$$

converge to zero in $L^1_{loc}(dt; L^1(M\,dv\,dx))$ as $\varepsilon \to 0$ by the product limit theorem—one uses assertion (i) of Proposition 5.2, assertion (1) of Lemma 5.3, and the fact that

$$\left(1 - \frac{\langle 1+|v|^2\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\right) \to 0, \qquad \left(1 - \frac{\langle 1+|v|^2\rangle}{N_\varepsilon\langle(1+|v|^2)N_\varepsilon\rangle}\right) \to 0,$$

almost everywhere as ε tends to zero.

Observe now that

$$\left|\varepsilon^{m-1}\frac{|g_\varepsilon||g_{\varepsilon 1}|}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2)\right.$$
$$\left. -\varepsilon^{m-1}\frac{|\hat{g}_\varepsilon||\hat{g}_\varepsilon|}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2)\right|$$
$$\leqq \varepsilon^{m-1}\frac{|\hat{g}_\varepsilon|\varepsilon^m\tilde{g}_{\varepsilon 1} + |\hat{g}_{\varepsilon 1}|\varepsilon^m\tilde{g}_\varepsilon + \varepsilon^{2m}\tilde{g}_\varepsilon\tilde{g}_{\varepsilon 1}}{\langle(1+|v|^2)N_\varepsilon\rangle}(1+|v|^2)(1+|v_1|^2)$$

which converges to zero in $L^1_{loc}(dt; L^1(M_1dv_1M\,dv\,dx))$ as $\varepsilon \to 0$. Therefore, as in the proof of assertion (i), Step 2, it follows that

$$\varepsilon^{m-1}\frac{\langle 1+|v|^2\rangle}{\langle(1+|v|^2)N_\varepsilon\rangle}\left(Q^-(g_\varepsilon, g_\varepsilon) - Q^-(\hat{g}_\varepsilon, \hat{g}_\varepsilon)\right) \to 0$$

in $L^1_{loc}(dt; L^1(M\,dv\,dx))$ as $\varepsilon \to 0$. The same analysis applies verbatim for the source terms. Therefore, assertion (ii) holds true.

Finally, assertion (iii) easily follows from assertion (i), Step 4, to be more precise and the product limit theorem. Indeed, the families

$$\left(1 - \frac{1}{N_\varepsilon}\right) \quad \text{and} \quad \left(1 - \frac{\langle\!\langle(1+|v|^2)\rangle\!\rangle}{\langle\!\langle(1+|v|^2)N_\varepsilon\rangle\!\rangle}\right)$$

are both bounded by 3/2 and converge pointwise to zero, whereas the family

$$(1 + |\phi|)\frac{1}{\varepsilon}Lg_\varepsilon$$

is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$. Hence, assertion (iii) holds.

6. Convergence to a Solution of the Stokes Equation

Let $G_\varepsilon \geqq 0$ be a sequence of DiPerna-Lions renormalized solutions of the scaled Boltzmann initial-value problem (1.63) with initial data $G_\varepsilon^{in} \geqq 0$ satisfying the entropy bound (1.66) for some $m > 1$; this is the scaling that corresponds to the formal Stokes limit (see (1.49)) and will be assumed throughout this section.

Let g_ε and q_ε be the corresponding sequences of fluctuations and scaled collision integrands defined by

$$G_\varepsilon = 1 + \varepsilon^m g_\varepsilon\,, \qquad q_\varepsilon = \frac{1}{\varepsilon^{m+1}}\left(G'_{\varepsilon 1}G'_\varepsilon - G_{\varepsilon 1}G_\varepsilon\right)\,, \tag{6.1}$$

respectively. Propositions (3.1) and (3.4) imply that the sequences g_ε and $q_\varepsilon/N_\varepsilon$ are relatively compact in

$$w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1((1+|v|^2)M dv\, dx)) \quad \text{and} \quad w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1((1+|v|^2)d\mu\, dx))$$

respectively. Let g be the $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv\, dx))$ limit point for any converging subsequence of g_ε. Extracting a further subsequence if need be, let q be a $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\, dx))$ limit for the corresponding sequence of $q_\varepsilon/N_\varepsilon$.

Assume hypothesis (H0); specifically, that these solutions satisfy the local momentum conservation law, here cast in the form (5.3):

$$\partial_t\langle v g_\varepsilon\rangle + \frac{1}{\varepsilon}\nabla_x\cdot\langle\phi L g_\varepsilon\rangle = -\frac{1}{\varepsilon}\nabla_x\left\langle\frac{1}{\mathrm{D}}|v|^2 g_\varepsilon\right\rangle\,. \tag{6.2}$$

This conservation law is understood to hold in the weak sense. As indicated previously, we shall consider passing to the limit of ε tending to zero in this equation only when it is integrated against a divergence free test vector field w in $\mathcal{H}_v \cap C^1(\mathbb{T}^{\mathrm{D}})$, thus eliminating the pressure term on its right side and yielding

$$\int w\cdot\langle v g_\varepsilon(t_2)\rangle\, dx - \int w\cdot\langle v g_\varepsilon(t_1)\rangle\, dx = \int_{t_1}^{t_2}\!\!\int \nabla_x w : \frac{1}{\varepsilon}\langle\phi L g_\varepsilon\rangle\, dx\, dt\,, \tag{6.3}$$

for every $0 \leqq t_1 < t_2 < \infty$.

Before passing to the limit, first observe that the sequence of time dependent functions appearing on the left side of (6.3),

$$t \mapsto \int w\cdot\langle v g_\varepsilon(t)\rangle\, dx\,, \tag{6.4}$$

when considered as a sequence in $C([0,\infty);\mathbb{R})$, is both pointwise equibounded and equicontinuous (and therefore relatively compact). The equiboundedness follows directly from the first part of assertion (1) of Proposition 3.1. By Proposition 5.2 (i), the quantity

$$\int \nabla_x w : \left\langle \phi \frac{1}{\varepsilon} L g_\varepsilon \right\rangle dx\,,$$

is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt;\mathbb{R})$, so the equicontinuity of (6.4) then follows from (6.3). Since the sequence (6.4) has a $w\text{-}L^1_{\mathrm{loc}}(dt;\mathbb{R})$ limit, however, it has the same $C([0,\infty);\mathbb{R})$ limit. Hence, passing to the limit in (6.4) yields

$$\lim_{\varepsilon\to 0}\int w\cdot\langle v g_\varepsilon\rangle\, dx = \int w\cdot\langle v g\rangle\, dx = \int w\cdot u\, dx \quad \text{in } C([0,\infty);\mathbb{R})\,. \tag{6.5}$$

Note that since $u \in L^\infty(dt;\mathcal{H}_v)$, a simple density argument then shows that u is in $C([0,\infty); w\text{-}\mathcal{H}_v)$; this result will be improved once the limiting dynamics is established and the classical regularity results for the Stokes equation [8] are applied.

In order to pass to the limit on the right side of (6.3), we utilize Proposition 5.2. This requires the additional assumption of hypothesis (H0) regarding the Boltzmann kernel b (see Appendix C). Consider the decomposition (5.4) of the convection-diffusion tensor:

$$\begin{aligned}\left\langle \phi\frac{1}{\varepsilon}Lg_\varepsilon\right\rangle &= \left\langle \phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon\right\rangle \\ &\quad + \left\langle \phi\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle - \left\langle \phi\frac{Q(G_\varepsilon,G_\varepsilon)}{\varepsilon^{m+1}N_\varepsilon}\right\rangle \\ &= \left\langle \phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon\right\rangle + \left\langle \phi\varepsilon^{m-1}\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle - \left\langle\!\left\langle \phi\frac{q_\varepsilon}{N_\varepsilon}\right\rangle\!\right\rangle .\end{aligned} \tag{6.6}$$

By assertion (iii) of Proposition 5.2, the first term on the left side of (6.6) vanishes,

$$\left\langle \phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)Lg_\varepsilon\right\rangle \to 0 \quad \text{in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx;\mathbb{R}^{D\times D}))\,. \tag{6.7}$$

By assertion (ii) of Proposition 5.2, showing the second term on the left side of (6.6) vanishes would follow upon showing that

$$\varepsilon^{m-1}Q(\hat g_\varepsilon,\hat g_\varepsilon) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt; L^1(M\,dv\,dx))\,.$$

By Lemma 5.3 (1),

$$\varepsilon^{m-1}|\hat{g}_\varepsilon||\hat{g}_{\varepsilon 1}|(1+|v|^2)(1+|v_1|^2) \to 0\,,$$

in $L^1_{\text{loc}}(dt; L^1(M_1 dv_1 M dv\, dx))$. Therefore, as in Step 1 of Proposition 5.2,

$$\varepsilon^{m-1}Q^{\pm}(|\hat{g}_\varepsilon|, |\hat{g}_\varepsilon|) \to 0 \quad \text{in } L^1_{\text{loc}}(dt; L^1((1+|\phi|)M dv\, dx))\,.$$

from which it follows that

$$\left\langle \phi\varepsilon^{m-1}\frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\right\rangle \to 0 \quad \text{in } L^1_{\text{loc}}(dt; L^1(dx; \mathbb{R}^{D\times D}))\,. \tag{6.8}$$

Finally, according to Proposition 5.1, the convergence of the last term on the left side of (6.6) is given by

$$\left\langle\!\!\left\langle \phi\frac{q_\varepsilon}{N_\varepsilon}\right\rangle\!\!\right\rangle \to \langle\!\langle \phi q\rangle\!\rangle = \nu(\nabla_x u + (\nabla_x u)^T) \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(dx; \mathbb{R}^{D\times D}))\,. \tag{6.9}$$

Combining the results (6.7), (6.8), and (6.9) into the decompositon (6.6) gives that

$$\begin{aligned}\int \nabla_x w : \left\langle \phi \frac{1}{\varepsilon} L g_\varepsilon\right\rangle dx &\to -\int \nu \nabla_x w : (\nabla_x u + (\nabla_x u)^T)\, dx \\ &= -\int \nu\nabla_x w : \nabla_x u\, dx \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; \mathbb{R})\,.\end{aligned} \tag{6.10}$$

Now passing to the limit in (6.3) by using (6.5) on the left and (6.10) on the right shows that

$$\int w\cdot u(t_2)\, dx - \int w\cdot u(t_1)\, dx = -\int_{t_1}^{t_2}\int \nu\nabla_x w : \nabla_x u\, dx\,, \tag{6.11}$$

which is just the weak form of the Stokes equation (compare with (1.61a)). The initial data for this equation can be recovered by considering (6.5) evaluated at $t = 0$. Since w is an arbitrary test vector field in $\mathcal{H} \cap C^1(\mathbb{T}^D)$, it is seen that the initial condition for (6.11) can be written in terms of the orthogonal projection P_v of $L^2(\mathbb{T}^D; \mathbb{R}^D)$ onto $\mathcal{H}_v$ as

$$u(0) = \lim_{\varepsilon\to 0} P_v \langle v g_\varepsilon^{in}\rangle\,.$$

This result is stated below as a proposition.

THEOREM 6.1. *Assume* (H1). *Let $G_\varepsilon \geqq 0$ be any sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *with initial data* *satisfying the entropy bound* (1.66) *for some* $m > 1$ *and satisfying* (H0). *If the corresponding sequence of fluctuations* g_ε *has a* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1((1+|v|^2)M\,dv\,dx))$ *limit* g *then*

$$u = \langle vg \rangle \in C([0,\infty); w\text{-}\mathcal{H}_v) \cap L^2(dt; \mathcal{V}_v)\,, \tag{6.12}$$

is approximated by $\langle vg_\varepsilon \rangle$ *in the sense that*

$$\lim_{\varepsilon\to 0} \int w\cdot\langle vg_\varepsilon\rangle\,dx = \int w\cdot u\,dx \tag{6.13}$$

in $C([0,\infty);\mathbb{R})$ *for every* $w = w(x)$ *in* $\mathcal{H}_v \cap C^1(\mathbb{T}^D)$. *Moreover,* u *is the unique solution of the Stokes initial-value problem* (2.17) :

$$\partial_t u + \nabla_x p = \nu\Delta_x u\,, \qquad \nabla_x\cdot u = 0\,, \tag{6.14a}$$

$$u(0) = \lim_{\varepsilon\to 0} P_v\langle v g_\varepsilon^{in}\rangle\,, \tag{6.14b}$$

with the viscosity ν *given by formula* (1.51).

Remark. Notice that the existence of the initial data in (6.14b) was inferred through the existence of the weak limit (6.13). Since any sequence of renormalized solutions of the scaled Boltzmann initial-value problem (1.63) with initial data satisfying the entropy bound (1.66) is relatively compact in $w\text{-}L^1$, this theorem shows that the momentum densities associated with each of its convergent subsequences converge to a solution of the Stokes equation (6.14a) which is uniquely determined by the initial condition (6.14b).

With a careful choice of initial data G_ε^{in}, any solution of the Stokes initial-value problem can be attained and be used to uniquely characterize the limiting fluctuation g.

THEOREM 6.2. (THE STRONG STOKES LIMIT) *Assume* (H1). *Let* $u^{in} \in \mathcal{H}_v$ *and define the infinitesimal Maxwellian* g^{in} *by*

$$g^{in} = u^{in}\cdot v\,. \tag{6.15}$$

Let $G_\varepsilon^{in} = 1 + \varepsilon^m g_\varepsilon^{in} \geqq 0$ *be any sequence such that* $g_\varepsilon^{in} \to g^{in}$ *entropically of order* ε^m *for some* $m > 1$. *Let* $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon \geqq 0$ *be any corresponding sequence of renormalized solutions of the scaled Boltzmann initial-value problem* (1.63) *satisfying* (H0). *Then*

$$g_\varepsilon(t) \to u(t)\cdot v \quad \textit{entropically of order } \varepsilon^m \textit{ for almost every } t > 0\,, \tag{6.16}$$

where $u(t)$ *is the unique solution of the Stokes initial-value problem*

$$\partial_t u + \nabla_x p = \nu\Delta_x u\,, \qquad \nabla_x\cdot u = 0\,, \tag{6.17a}$$

$$u(0) = u^{in}\,, \tag{6.17b}$$

with the viscosity ν *given by formula* (1.51). *Moreover, the normalized scaled collision integrands converge strongly to* q*:*

$$\frac{q_\varepsilon}{N_\varepsilon} \to q = \left(\nabla_x u + (\nabla_x u)^T\right) : \Phi \quad \textit{in } L^1_{\text{loc}}(dt; L^1((1 + |v|^2)d\mu\, dx)) \,, \tag{6.18}$$

where $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$.

Remark. In fact we shall obtain a stronger result for the convergence of $q_\varepsilon/N_\varepsilon$, that it converges to $(\nabla_x u + (\nabla_x u)^T) : \Phi$ in a sense analogous to entropically of order ε^{m+1}, but based on the dissipation rate integrand r rather than the relative entropy integrand h.

Proof: The idea is to show that all convergent subsequences of g_ε have the same limit, namely, that which is of the form (6.16) with $u(t)$ prescribed by (6.17). The entropic convergence will follow from a squeezing argument. First, extract any convergent subsequence of g_ε in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1((1 + |v|^2)M dv\, dx))$ and apply Proposition 6.1 to it. Since $g_\varepsilon^{in} \to g^{in}$ entropically, it does so in $w\text{-}L^1((1 + |v|^2)M dv\, dx))$, hence the right side of the initial condition (6.14b) reduces to that of (6.17b). Thus the limiting u for each convergent subsequence of g_ε is determined by (6.17).

Since $g_\varepsilon^{in} \to u^{in} \cdot v$ entropically, Proposition 4.9 states that g has the form of an infinitesimal Maxwellian,

$$g = \rho + u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2} \right) ,$$

such that the fluid variables (ρ, u, θ) satisfies the Leray energy inequality,

$$\begin{aligned} &\int \frac{1}{2} \left(\rho(t)^2 + |u(t)|^2 + \frac{\mathrm{D}}{2}\theta(t)^2 \right) dx \\ &\quad + \int_0^t \int \frac{1}{2}\nu \left| \nabla_x u + (\nabla_x u)^T \right|^2 + \frac{\mathrm{D}+2}{2}\kappa \left| \nabla_x \theta \right|^2 dx\, ds \\ &\leq \int \frac{1}{2} \left| u^{in} \right|^2 dx \,. \end{aligned} \tag{6.19}$$

Solutions of the Stokes initial-value problem (6.17), however, satisfy the energy equality

$$\int \frac{1}{2}|u(t)|^2 dx + \int_0^t \int \frac{1}{2}\nu \left| \nabla_x u + (\nabla_x u)^T \right|^2 dx\, ds = \int \frac{1}{2} \left| u^{in} \right|^2 dx \,. \tag{6.20}$$

Subtracting this equality from the Leray energy inequality (6.19) yields $\rho = \theta = 0$ for all time, whence the only possible form for the limit of an extracted subsequence of g_ε is necessarily the one defined by (6.16).

The squeezing argument to obtain the entropic convergence starts with the following observations. The proof of Proposition 3.1 showed that for almost every $t > 0$

$$\int \frac{1}{2}|u(t)|^2\,dx = \int \frac{1}{2}\langle g^2(t)\rangle\,dx \leqq \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t))\right\rangle\,dx\,, \tag{6.21}$$

while that of Proposition 3.4 showed that

$$\begin{aligned}\int_0^t \int \frac{1}{2}\nu|\nabla_x u + (\nabla_x u)^T|^2\,dx\,ds \\ = \int_0^t \int \frac{1}{4}\langle\!\langle q^2\rangle\!\rangle\,dx\,ds \\ \leqq \liminf_{\varepsilon\to 0} \int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\right\rangle\!\!\right\rangle\,dx\,ds\,.\end{aligned} \tag{6.22}$$

Since $g_\varepsilon^{in} \to u^{in}\cdot v$ entropically, (4.30) gives

$$\lim_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in})\right\rangle\,dx = \int \frac{1}{2}\,\langle g^{in\,2}\rangle\,dx = \int \frac{1}{2}|u^{in}|^2\,dx\,. \tag{6.23}$$

Combining (6.20)–(6.23) with the entropy inequality part of (3.6),

$$\begin{aligned}\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t))\right\rangle\,dx + \int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\right\rangle\!\!\right\rangle\,dx\,ds \\ \leqq \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{in})\right\rangle\,dx\,,\end{aligned} \tag{6.24}$$

implies the existence of the limits

$$\begin{aligned}\lim_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t))\right\rangle\,dx = \int \frac{1}{2}\,|u(t)|^2\,dx\,, \\ \lim_{\varepsilon\to 0} \int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon\right\rangle\!\!\right\rangle dx\,ds = \int_0^t \int \frac{1}{2}\nu\,|\nabla_x u + (\nabla_x u)^T|^2\,dx\,ds\,.\end{aligned} \tag{6.25}$$

The first limit above implies the entropic convergence of g_ε (6.16) while the second will yield (6.18) through an argument given below that is similar to the proof of Proposition 4.11.

For any $z_o > -1$ consider the convex function defined over $z > -1$ by

$$z \mapsto r(z) - r(z_o) - r'(z_o)(z - z_o) .$$

When viewed as a function of $z - z_o$ this function satisfies the usual reflection property so that the Young inequality gives

$$y|z - z_o| \leqq r^*(r'(z_o) + y) - r^*(r'(z_o)) - z_o y + r(z) - r(z_o) - r'(z_o)(z - z_o) . \tag{6.26}$$

Moreover, the Legendre dual function

$$y \mapsto r^*(r'(z_o) + y) - r^*(r'(z_o)) - z_o y ,$$

is a superquadratic function of y in the sense that

$$\begin{aligned} r^* & \left(r'(z_o) + \lambda y\right) - r^*\left(r'(z_o)\right) - z_o \lambda y \\ & \leqq \lambda^2 \left(r^* \left(r'(z_o) + y\right) - r^*\left(r'(z_o)\right) - z_o y\right) , \end{aligned} \tag{6.27}$$

for every $\lambda \leqq 1$.

Let α be any number such that $\alpha \geqq \varepsilon^{m+1}$ for every value of ε. Set

$$z_o = \varepsilon^{m+1} q , \quad z = \frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon} , \quad y = \frac{\varepsilon^{m+1}}{\alpha} \frac{1}{4} \left(1 + |v|^2\right) ,$$

into (6.26) and make use of the superquadratic property (6.27) with $\lambda = \varepsilon^{m+1}/\alpha$ to obtain

$$\begin{aligned} & \frac{1}{4}\left(1 + |v|^2\right) \left| \frac{q_\varepsilon}{N_\varepsilon^{abs}} - q \frac{G_{\varepsilon 1} G_\varepsilon}{N_\varepsilon^{abs}} \right| \\ & \leqq \frac{1}{\alpha} \left(r^*\left(r'(\varepsilon^{m+1} q) + \frac{1}{4}\left(1 + |v|^2\right)\right) - r^*\left(r'(\varepsilon^{m+1} q)\right) \right. \\ & \qquad \left. - \varepsilon^{m+1} q \frac{1}{4}\left(1 + |v|^2\right)\right) \frac{G_{\varepsilon 1} G_\varepsilon}{N_\varepsilon^{abs}} \\ & \quad + \frac{\alpha}{\varepsilon^{2m+2}} \left(r\left(\frac{\varepsilon^{m+1} q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon}\right) - r\left(\varepsilon^{m+1} q\right) \right. \\ & \qquad \left. - \varepsilon^{m+1} r'\left(\varepsilon^{m+1} q\right) \left(\frac{q_\varepsilon}{G_{\varepsilon 1} G_\varepsilon} - q\right)\right) \frac{G_{\varepsilon 1} G_\varepsilon}{N_\varepsilon^{abs}} . \end{aligned} \tag{6.28}$$

Fix $\lambda > 0$ and multiply this inequality by the indicator (characteristic) function $\mathbf{1}_{|q|<\lambda}$; integate the result over $d\mu\, dx$ and let ε tend to zero to obtain

$$\begin{aligned} \limsup_{\varepsilon \to 0} \int_0^t \int \frac{1}{4} & \left\langle\!\left\langle \left(1 + |v|^2\right) \left| \frac{q_\varepsilon}{N_\varepsilon} - q \frac{G_{\varepsilon 1} G_\varepsilon}{N_\varepsilon^{abs}} \right| \mathbf{1}_{|q|<\lambda} \right\rangle\!\right\rangle dx\, ds \\ & \leqq \frac{1}{\alpha} \left\langle r^*\left(\frac{1}{4}\left(1 + |v|^2\right)\right)\right\rangle . \end{aligned} \tag{6.29}$$

The limit of the last term in (6.28) vanishes by (6.25) and an argument as in the proof of assertion (2) of Proposition 3.4 (see after (3.35)). By the arbitrariness of α, the limit in (6.29) must be zero. But, since $G_{\varepsilon 1}G_\varepsilon/N_\varepsilon^{abs} \to 1$ in $L^1_{loc}(dt; L^1)$,

$$\lim_{\varepsilon\to 0}\int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle (1+|v|^2)\left| q\frac{G_{\varepsilon 1}G_\varepsilon}{N_\varepsilon^{abs}} - q\right| \mathbf{1}_{|q|<\lambda}\right\rangle\!\!\right\rangle dx\,ds = 0\,,$$

while an application of the product limit theorem shows that

$$\lim_{\varepsilon\to 0}\int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle (1+|v|^2)\left| \frac{q_\varepsilon}{N_\varepsilon} - \frac{q_\varepsilon}{N_\varepsilon^{abs}}\right| \right\rangle\!\!\right\rangle dx\,ds = 0\,,$$

so the vanishing of the limit in (6.29) then implies

$$\lim_{\varepsilon\to 0}\int_0^t \int \frac{1}{4}\left\langle\!\!\left\langle (1+|v|^2)\left| \frac{q_\varepsilon}{N_\varepsilon} - q\right| \mathbf{1}_{|q|<\lambda}\right\rangle\!\!\right\rangle dx\,ds = 0\,.$$

Taking λ to infinity while using the equi-integrability of the sequence $q_\varepsilon/N_\varepsilon$ then completes the proof.

Remark. One very simple choice for G_ε^{in} that can be used for any $u^{in} \in \mathcal{H}_v$ is

$$G_\varepsilon^{in} = \frac{M(1, \varepsilon^m u^{in}, 1)}{M}\,.$$

It is easy to verify that its fluctuations converge entropically of order ε^m to $g^{in} = u^{in}\cdot v$ (see (4.32)) since a direct calculation shows that the scaled relative entropies are even independent of ε:

$$\int \left\langle \frac{1}{\varepsilon^{2m}} h\left(\varepsilon^m g_\varepsilon^{in}\right)\right\rangle dx = \int \frac{1}{2}\left\langle g^{in\,2}\right\rangle dx = \int \frac{1}{2}|u^{in}|^2\,dx\,.$$

This shows the existence of one entropically convergent sequence of initial fluctuations; the point of Theorem 6.2 is that all such sequences have a unique limiting dynamics.

7. Convergence to a Weak Solution of the Time-Discretized Navier-Stokes Equation

In the present section, the Navier-Stokes and Boltzmann equations will always be taken in the time discretized form. All the results from Sections 3 and 4 hold verbatim by dropping the time dependence in all the statements and making $m = 1$. (This is due to the fact that the entropy inequality, which is really all that is needed for these results, is of the same form no matter that the problem be time dependent or time discretized).

Therefore, $G_\varepsilon = G_\varepsilon(x,v) = 1 + \varepsilon g_\varepsilon$ will denote a family of renormalized solutions of the time-discretized Boltzmann problem (2.21) with initial data

$$G_\varepsilon^{in} = 1 + \varepsilon g_\varepsilon^{in}\,, \qquad H(G_\varepsilon^{in}) \leqq C^{in}\varepsilon^2\,. \tag{7.1}$$

Throughout the present section, (H0′), (H1), and (H2) are assumed.

The main objective of the present section is to prove that, (modulo extraction of subsequences), $\langle v g_\varepsilon\rangle$ converges to a weak solution of the Navier-Stokes equation.

Moment Equations

We shall closely follow the arguments of Section 6. The local conservation law for momentum (see (H0′)) holds and is written in the form (5.3):

$$\langle v g_\varepsilon\rangle + \nabla_x\cdot\frac{1}{\varepsilon}\,\langle (L\phi) g_\varepsilon\rangle = \langle v g_\varepsilon^{in}\rangle - \nabla_x\frac{1}{\varepsilon}\left\langle \frac{1}{D}|v|^2 g_\varepsilon\right\rangle\,. \tag{7.2}$$

Using again decomposition (5.4) leads to:

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle (L\phi) g_\varepsilon\right\rangle &= \left\langle \phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right) L g_\varepsilon\right\rangle\\ &\quad + \left\langle \phi\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle - \left\langle \phi\frac{1}{\varepsilon^2}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon}\right\rangle\,.\end{aligned} \tag{7.3}$$

Therefore, the local conservation of momentum can be recast in the form

$$\begin{aligned}&\langle v g_\varepsilon\rangle + \nabla_x\cdot\left\langle \phi\frac{Q(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle\\ &= -\nabla_x\frac{1}{\varepsilon}\left\langle\frac{1}{D}|v|^2 g_\varepsilon\right\rangle + \nabla_x\cdot\left\langle\phi\frac{1}{\varepsilon^2}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon}\right\rangle\\ &\quad + \langle v g_\varepsilon^{in}\rangle - \nabla_x\cdot\left\langle\phi\frac{1}{\varepsilon}\left(1-\frac{1}{N_\varepsilon}\right)L g_\varepsilon\right\rangle\,.\end{aligned} \tag{7.4}$$

The first term in the left side of (7.4) is to converge to the velocity field, the second term to the convection part of the Navier-Stokes equation. The second term in the right side of (7.4) is to converge to the diffusion part of the Navier-Stokes equation, the third one to the source velocity field. The last term in the right side of (7.4) is a remainder eventually converging to zero. Finally, the first term in the right side of (7.4) contributes to the pressure term and will vanish upon integration of this conservation law against any divergence free test vector field.

Because of the quadratic nature of the convection part in the Navier-Stokes equation, convergence in weak topologies will not be enough for this purpose. It should, however, be observed that

(i) The control of the entropy dissipation rate shows that the distance between the sequence of fluctuations and the manifold of infinitesimal Maxwellian states tends to zero with ε.
(ii) Fluctuations of local Maxwellian are defined by a finite number (D+2, actually) of their velocity averages.

Then, the problem of proving strong compactness for the family g_ε reduces to proving strong compactness for its velocity averages. Therefore, it seems that the appropriate tool by which to obtain strong convergence is the following compactness result due to Golse, Lions, Perthame, and Sentis (see [13] and [14]), referred to as velocity averaging theorem and given here in an L^1 setting well adapted to the present problem from the L^2 version introduced in Section 2.

THEOREM 7.1. (VELOCITY AVERAGING) *Let $\mathcal{F}$ be a relatively compact subset of w-$L^1((1 + |v|^n)M dv\, dx)$ (where n is any positive number), and assume that the set $\{v\cdot\nabla_x f \ : \ f \in \mathcal{F}\}$ is relatively compact in w-$L^1(M dv\, dx)$. Then, for any measurable function $p \equiv p(v)$ satisfying the following growth condition for large v's:*

$$\frac{p(v)}{(1 + |v|^n)} \quad \textit{remains bounded as } |v| \to +\infty \, ,$$

the set

$$\{\langle f p\rangle(x) \ : \ f \in \mathcal{F}\}$$

is relatively compact in strong $L^1(dx)$.

The strategy sketched in (i)–(ii) is quite common to all the hydrodynamic limits of kinetic equations leading to a macroscopic equation with a diffusion term and has been used successfully in various contexts, such as radiative transfer (see [13]) and the kinetic theory of semiconductors (see [15]).

Pointwise Convergence of the Fluctuations

PROPOSITION 7.2. *The family g_ε is relatively compact in $L^1((1+|v|^2)dv\, dx)$. If g is the limit of a subsequence of g_ε, then for almost every $x \in \mathbb{T}^D$, $g(x, \cdot) \in \mathrm{N}(L)$, which means that g is of the form*

$$g(x, v) = \rho(x) + u(x)\cdot v + \theta(x) \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right) . \tag{7.5}$$

Proof: The asymptotic form (7.5) has already been proved in Proposition 3.7. According to decomposition (5.5), it suffices to prove that $\hat{g}_\varepsilon$ is relatively compact in $L^1((1 + |v|^2)dv\, dx)$. Observe that, by the Cauchy-Schwarz inequality

$$|L\hat{g}_\varepsilon|^2 \leqq |L|(1)\,|L|(\hat{g}_\varepsilon^2) \leqq \frac{3}{2}|L|(1)\,|L|(\tilde{g}_\varepsilon) \leqq C\,(1 + |\phi|)\ \ |L|(\tilde{g}_\varepsilon)$$

according to Proposition C (i) (see Appendix C). It follows from Proposition B (iii) and assumption (H2) that $|L\hat{g}_\varepsilon|^2$ is relatively compact in $w\text{-}L^1(Mdv\,dx)$. Using then Proposition 5.2 (i) proves that $|L\hat{g}_\varepsilon|^2 \to 0$ almost everywhere as ε tends to zero. Therefore $|L\hat{g}_\varepsilon|^2 \to 0$ in $L^1(Mdv\,dx)$.

Now write the orthogonal decomposition of $\hat{g}_\varepsilon$ as

$$\hat{g}_\varepsilon = w_\varepsilon + z_\varepsilon \quad \text{with } w_\varepsilon \in \mathrm{N}(L)^\perp\,, \quad z_\varepsilon \in \mathrm{N}(L)\,. \tag{7.6}$$

According to (H1), the interaction potential is strong. Therefore (see [7]) there exists a positive constant C such that

$$\langle \hat{g}_\varepsilon L\hat{g}_\varepsilon\rangle \geqq C\,\langle w_\varepsilon^2\rangle\ .$$

Then,

$$w_\varepsilon \to 0 \quad \text{in } L^2(Mdv\,dx)\,. \tag{7.7}$$

Now observe that

$$z_\varepsilon = \langle\hat{g}_\varepsilon\rangle + \langle v\hat{g}_\varepsilon\rangle\cdot v + \frac{2}{\mathrm{D}}\left\langle\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\hat{g}_\varepsilon\right\rangle\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\,. \tag{7.8}$$

First, one has

$$v\cdot\nabla_x\gamma_\varepsilon = \frac{1}{\varepsilon^2}\frac{Q(G_\varepsilon,G_\varepsilon)}{N_\varepsilon} + \varepsilon\frac{g_\varepsilon^{in} - g_\varepsilon}{N_\varepsilon}\,.$$

According to Corollary 3.2, Proposition 5.1, and the velocity averaging theorem 7.1,

$$\langle\gamma_\varepsilon\rangle\,,\quad \langle v\gamma_\varepsilon\rangle\,,\quad \left\langle\left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}\right)\gamma_\varepsilon\right\rangle$$

are relatively compact in $L^1(dx)$. It follows from Corollary 3.2 (1) and (7.5) that z_ε is relatively compact in $L^2(Mdv\,dx)$. Combining (7.7), (7.8), and decomposition (7.6) yields the announced result.

The Navier-Stokes Limit Theorems

The main results of the present section are

THEOREM 7.3. *Assume* (H1). *Let G_ε be a family of renormalized solutions of the time-discretized Boltzmann problem* (2.21) *with initial condition* (7.1) *and satisfying* (H0′) *and* (H2). *Assume that g_ε^{in} has a $w\text{-}L^1((1 + |v|^2)Mdv\,dx)$ limit g^{in}.*

Then the family g_ε is relatively compact in w-$L^1((1 + |v|^2)M dv\, dx)$ and any of its sequential limit points g is a local Maxwellian (2.7) *such that:*

(i) *The dissipation inequality* (4.24) *holds;*
(ii) $u = \langle vg \rangle \in \mathcal{V}_v$ *is a weak solution of the time discretized Navier-Stokes system*

$$u + \nabla_x \cdot (u \otimes u) + \nabla_x p = \nu \Delta_x u + u^{in} , \qquad \nabla_x \cdot u = 0 ,$$
$$u^{in} = P_v \langle v g^{in} \rangle ,$$

with the viscosity ν given by the formula (1.51).

As in Section 6, a careful choice of the initial data G_ε^{in} allows us to characterize the limiting fluctuations g more accurately. Observe that the restriction bearing on the dimension is related to the fact that H^1 solutions of the time discretized Navier-Stokes equation verify the energy equality in (2.26) for dimension D $\leqq$ 4. Also, the defect of convergence (that is, the nonuniqueness of the sequential limit points of g_ε) is exactly measured by the lack of uniqueness of a weak solution of the time discretized Navier-Stokes system.

Another point of interest is the scaled collision integrand

$$q_\varepsilon = \frac{1}{\varepsilon^2} \left(G'_{\varepsilon 1} G'_\varepsilon - G_{\varepsilon 1} G_\varepsilon \right) .$$

As in Section 6, the limiting form of the normalized scaled collision integrand follows from the energy equality.

THEOREM 7.4. (THE STRONG NAVIER-STOKES LIMIT) *Assume* (H1) *and* D $\leqq$ 4. *Let $u^{in} \in \mathcal{H}_v$ and define the initial infinitesimal Maxwellian by*

$$g^{in} = u^{in} \cdot v .$$

Let $G_\varepsilon^{in} = 1 + \varepsilon g_\varepsilon^{in} \geqq 0$ be any sequence of initial data such that g_ε^{in} converges to g^{in} entropically of order ε. Let $G_\varepsilon = 1 + \varepsilon g_\varepsilon \geqq 0$ be any family of renormalized solutions of the corresponding time-discretized Boltzmann equation (2.21) *satisfying* (H0′) *and* (H2). *Then the family g_ε is relatively compact in w-$L^1((1 + |v|^2)M dv\, dx)$ and for any convergent subsequence (again denoted g_ε)*

$$g_\varepsilon \to g = u \cdot v \quad \textit{entropically of order } \varepsilon ,$$

where $u \in \mathcal{V}_v$ is a weak solution of the time-discretized Navier-Stokes system

$$u + \nabla_x \cdot (u \otimes u) + \nabla_x p = \nu \Delta_x u + u^{in} , \qquad \nabla_x \cdot u = 0 ,$$

with the viscosity ν *given by the formula* (1.51). *Moreover, the normalized scaled collision integrands converge strongly to* q*:*

$$\frac{q_\varepsilon}{N_\varepsilon} \to q = \left(\nabla_x u + (\nabla_x u)^T\right) : \Phi \quad \text{in } L^1\left(\left(1 + |v|^2\right) d\mu\, dx\right) ,$$

where $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$.

Proof of Theorem 7.3: First, notice that the incompressibility relation (4.7) follows in the time-discretized case of Proposition 4.2 and the remark following Proposition 4.1. The dissipation inequality was proved in Section 4 and stated as Proposition 4.8.

The essential step in the proof, and what makes it actually different from that of Proposition 6.1 (in the Stokes case) is

LEMMA 7.5. *Let* g *be a* $L^1((1+|v|^2)M dv\, dx)$ *limit of a converging subsequence of fluctuations* g_ε. *As* ε *tends to zero,*

$$Q(\hat{g}_\varepsilon, \hat{g}_\varepsilon) \to Q(g, g) = \frac{1}{2} L(g^2) , \quad \text{in } L^1\left(\left(1 + |\phi|\right) M dv\, dx\right) . \tag{7.9}$$

Proof of Lemma 7.5: The fact that $Q(g, g) = \frac{1}{2}L(g^2)$ comes from the special form (7.5) of g and was proved in [3].

Observe that

$$|\hat{g}_{\varepsilon 1}\hat{g}_\varepsilon - g_1 g| \leqq |\hat{g}_\varepsilon|\, |\hat{g}_{\varepsilon 1} - g_1| + |g_1|\, |\hat{g}_\varepsilon - g| . \tag{7.10}$$

Using the fact that $\hat{g}_\varepsilon$ is a bounded family of $L^2(M dv\, dx)$, that $g \in L^2(M dv\, dx)$ and Corollary 3.2 (1) together with Proposition 7.2, one obtains that

$$|\hat{g}_{\varepsilon 1}\hat{g}_\varepsilon - g_1 g| \to 0 \quad \text{in } L^1(M_1 dv_1 M dv\, dx) .$$

In fact, this convergence holds in $L^1((1 + |v|^2)(1 + |v_1|^2)M_1 dv_1 M dv\, dx)$ because of assertion 1 of Lemma 5.3. Following exactly the same route as in Step 1 of Proposition 5.3, one obtains first that

$$Q^-(\hat{g}_\varepsilon, \hat{g}_\varepsilon) \to Q^-(g, g)$$

and, exchanging primed and unprimed velocities in the collision integral,

$$Q^+\left(\hat{g}_\varepsilon, \hat{g}_\varepsilon\right) \to Q^+(g, g)$$

in $L^1((1 + |\phi|)M dv\, dx)$. Lemma 7.5 is essential to obtain the asymptotic form of the local momentum conservation. Let $w = w(x)$ be any smooth divergence free vector field. Multiplying the local momentum conservation (7.4) by w and integrating with respect to x gives:

$$\begin{aligned}
&\int w\cdot\langle v g_\varepsilon\rangle\,dx - \int (\nabla_x w)^T : \left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\right\rangle dx \\
&= -\int (\nabla_x w)^T : \left\langle \phi \frac{1}{\varepsilon^2}\frac{Q(G_\varepsilon, G_\varepsilon)}{N_\varepsilon}\right\rangle dx + \int w\cdot\langle v g^{in}\rangle\,dx \\
&\quad + \int (\nabla_x w)^T : \left\langle \phi \frac{1}{\varepsilon}\left(1 - \frac{1}{N_\varepsilon}\right) L g_\varepsilon\right\rangle dx .
\end{aligned} \tag{7.11}$$

Using now Proposition 5.2 (ii) and Lemma 7.5 shows that

$$\int (\nabla_x w)^T : \left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\right\rangle dx \to \int (\nabla_x w)^T : \langle \phi\, Q(g,g)\rangle\, dx . \tag{7.12}$$

Proceeding as in [3] shows that

$$\begin{aligned}
\int (\nabla_x w)^T : \langle \phi\, Q(g,g)\rangle\, dx &= \int (\nabla_x w)^T : \left\langle \phi \frac{1}{2} L g^2 \right\rangle dx \\
&= \int (\nabla_x w)^T : \frac{1}{2}\, \langle g^2 L\phi\rangle\, dx .
\end{aligned} \tag{7.13}$$

Now, using the form (7.5) of g shows that

$$\begin{aligned}
\left\langle g^2 (L\phi)_{ij}\right\rangle &= \left\langle (L\phi)_{ij}(L\phi)_{kl}\right\rangle u_k u_l + \left\langle (L\phi)_{ij}\frac{1}{4}|v|^4\right\rangle \theta^2 \\
&= 2u_i u_j - \frac{2}{\mathrm{D}}|u|^2\delta_{ij} .
\end{aligned} \tag{7.14}$$

Combining then (7.12), (7.13), and (7.14) yields:

$$\int (\nabla_x w)^T : \left\langle \phi \frac{Q(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\right\rangle dx \to \int \nabla_x w : u\otimes u\, dx , \tag{7.15}$$

using the fact that the vector field w is divergence free.

Proceeding as in Section 6 and taking the limits in (7.11) in the various terms involved as $\varepsilon \to 0$ shows that the limiting vector field u satisfies the following relation

$$\int w\cdot u\, dx - \int (\nabla_x w)^T : (u\otimes u)\, dx = \nu \int u\cdot \Delta_x w\, dx + \int w\cdot u^{in}\, dx . \tag{7.16}$$

The fact that $u \in \mathcal{V}_\nu$ follows easily from the dissipation inequality (4.24). Proposition 7.3 is therefore proved.

Proof of Theorem 7.4: The restriction to initial data such that g_ε^{in} converges to g^{in} entropically ensures that the sharpened dissipation inequality (4.37) holds. Let g be a sequential limit point of g_ε; it follows from Theorem 7.3 that g is a local infinitesimal Maxwellian form (2.7) where $u \in \mathcal{V}_v$ is a weak solution of the time-discretized Navier-Stokes system. The assumption that $\mathrm{D} \leqq 4$ allows us to write the energy equality in (2.26) recalled in Section 2. Substracting the energy equality form of (2.26) from the sharpened dissipation inequality (4.37) yields

$$\int \frac{1}{2}\left(\rho^2 + \frac{\mathrm{D}}{2}\theta^2\right) dx + \int \frac{\mathrm{D}+2}{2}\kappa|\nabla_x\theta|^2\, dx \leqq \int \frac{1}{2}\left(\rho^{in}\rho + \frac{\mathrm{D}}{2}\theta^{in}\theta\right) dx = 0\,,$$

according to the choice of the initial fluctuation $g^{in} = u^{in}\cdot v$. Therefore, $\rho = \theta = 0$.

The convergence assertions of Theorem 7.4 follow from a squeezing argument in the style of Theorem 6.2. Again using the energy equality in (2.26) when $\mathrm{D} \leqq 4$, one infers the existence of the limits

$$\lim_{\varepsilon\to 0}\int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon)\right\rangle dx = \int \frac{1}{2}|u|^2\, dx\,,$$

$$\lim_{\varepsilon\to 0}\int \frac{1}{4}\left\langle\!\!\left\langle \frac{1}{\varepsilon^{2m+2}} r\left(\frac{\varepsilon^{m+1}q_\varepsilon}{G_{\varepsilon 1}G_\varepsilon}\right) G_{\varepsilon 1}G_\varepsilon \right\rangle\!\!\right\rangle dx = \int \frac{1}{2}\nu|\nabla_x u + (\nabla_x u)^T|^2\, dx\,.$$

The first limit above implies the entropic convergence of g_ε, while the second will yield the convergence of the scaled collision integrands as in the proof of Theorem 6.2, whence Theorem 7.4 holds.

8. Conclusions

As announced in Section 1, we have been unable to fulfill the program outlined there. The goal of the present section is to comment upon the various difficulties encountered in trying to do so. In particular, we wish to make it clear which ones we consider fundamental and which ones we consider more technical.

This article relies as much as possible on known physical estimates (like conservation laws or the entropy inequality). With the exception of assumptions (H0)-(H1)-(H2), no additional hypothesis on the regularity or the size of the initial data is needed. Therefore, this article deals with solutions of the Boltzmann or Navier-Stokes equations in the weakest possible sense that is compatible with those basic physical properties.

Some of our shortcomings are due to the fact that the DiPerna-Lions renormalized solutions of the Boltzmann equation lack local conservation laws of momentum and energy. This is due to the lack of control of the high velocity tails in the corresponding fluxes. Such controls were recently obtained on simplified Bathganar-Gross-Krook (BGK) models, first by B. Perthame (see [17]) in the whole space, and then by E. Ringeisen (see [18]) for more general domains and boundary conditions. These controls would be of little use, however, to get a temperature equation for the incompressible limit. Indeed, the incompressible limit calls for high velocity controls on fluctuations of the number density, something that no conservation law can provide.

The fact that was used in Section 3 to provide control of the $|v|^2$ moments of these fluctuations was the structure of the integrands of the entropy and dissipation functionals; particularly, it was the exponential asymptotic behavior of h^* and r^* as $y \to \infty$. It is clear on the basis of this remark that no control of moments of order higher than two on the fluctuations can be attained by these methods.

On the other hand, it can be argued that local conservation of momentum and energy for the renormalized solutions of the Boltzmann equation are only needed in the hydrodynamic limit. It is possible that both of these local conservation laws (or at least that of momentum) can be recovered at the level of hydrodynamics; that is, in the limit when ε tends to zero. If so, then hypothesis (H0) could be dropped and our program would be complete for the Stokes limit.

In addition to the local momentum conservation, two additional ingredients were required for the Navier-Stokes limit. The first was assumption (H2), the w-L^1 compactness of $(1 + |v|^2) g_\varepsilon^2 / N_\varepsilon$. Observe that the result in Proposition 3.3 does not miss by much, and might very well be sharpened. The second was our introduction of the discrete time problem for this limit. This was necessitated by the fact that the time regularity provided by the velocity averaging theorem and the equicontinuity arguments is not sufficient to get the evolution Navier-Stokes equation in the limit as ε tends to zero. More precisely, since the equicontinuity argument in Proposition 6.1 also goes through for the $m = 1$ (Navier-Stokes) scaling, what is presently lacking is a time regularity result for the acoustic component of the hydrodynamic modes (see [6]). This regularity could have easily been added as a hypothesis in order to obtain the continuous time Navier-Stokes equation in the limit as was asserted in [2]. Such a hypothesis seems rather strong, however, when compared to the ones we have already made, in that formal obstructions exist to an equicontinuity argument and the techniques used to prove weaker notions of regularity are subtler.

The set of technical assumptions (H1) on the Boltzmann kernel $b(v_1 - v, \omega)$ enjoy (as was emphasized previously) the following status:

(i) They are natural and consistent with power law hard potentials by formal scaling arguments; at this time, however, we have no proof that (H2) holds in this case.
(ii) They hold for the so-called Maxwell potentials.

There is clearly room for improvement in this direction, maybe at the expense of a detailed analysis of the linearized collision operator, a task that we leave for future work.

Finally, we remark that the horizon of the whole program can be extended considerably. For example, the temperature equation obtained formally with the above scaling (see [2]) does not correspond to a dominant balance asymptotics (it is easily seen that the viscous heating term is missing). Dominant balance is obtained when the density and temperature fluctuations are of order ε^2 while the bulk velocity fluctuations are of order ε (see [5]). Although a formal moment calculation in the style of [3] can be carried out, say, by splitting the fluctuations of the number density G_ε into odd and even components with respect to v, there is no analog of the entropy estimate that can be used to globally maintain the assumed scale separation of these odd and even components. Since the stability of this asymptotic regime is well verified by experience, it should be understood through basic physical estimates.

Appendix A. The Notation Regarding Spaces

Throughout this article, many basic topological linear spaces are employed. Some of our notation regarding these spaces is standard while some of it is less so. These spaces, as well as our notation for them, are described below. A comprehensive treatment of them can be found in many standard references; for example, see [11].

Let E be any normed linear space; $\|\cdot\|_E$ denotes its norm and E^* denotes its dual space. We shall use the notation w-E to indicate the space E equipped with its weak topology, that is the coarsest topology on E for which each of the linear forms

$$u \mapsto \langle w\,; u\rangle_{E^*,E} \quad \text{for } w \in E^*\ ,$$

is continuous. Here $\langle\cdot\,;\,\cdot\rangle_{E^*,E}$ is the natural bilinear form relating E^* and E.

Let X be a locally compact topological space and E a normed linear space. We shall use the usual notation $C(X; w\text{-}E)$ to indicate the space of continuous functions from X to w-E; that is the set of fuctions u for which

$$x \mapsto \langle w\,; u(x)\rangle_{E^*,E} \quad \text{is in } C(X) \quad \text{for each } w \in E^*\ .$$

We note that an Arzela-Ascoli theorem holds for such spaces.

Let $(Y, \mathcal{M}, dm)$ be a measure space and E a normed linear space. For every $1 \leqq p \leqq \infty$ we shall use the abbreviated notation $L^p(dm; E)$ for the Bochner space $L^p((Y, \mathcal{M}, dm); E)$ whenever there is no possible confusion; we shall also use $L^p(dm)$ to denote the same space whenever E is a power of $\mathbb{R}$, unmistakeable in its context. For $1 \leqq p < \infty$, the dual space of $L^p(dm; E)$ is $L^{p^*}(dm; E^*)$ where $p^* = p/(p-1)$. Only $p = 1, 2,$ or ∞ arise in this article.

When Y is locally compact and dm is a Borel measure, we shall denote by $L^p_{\text{loc}}(dm; E)$ (or $L^p_{\text{loc}}(dm)$) the space determined by the family of seminorms

$$u \mapsto \left(\int_K \|u(y)\|_E^p \, dm(y)\right)^{1/p} \quad \text{for compact } K \subset Y\ .$$

For every $1 \leqq p < \infty$ we shall use the notation $w\text{-}L^p(dm; w\text{-}E)$ (or $w\text{-}L^p(dm)$) to denote the space $L^p(dm; E)$ equipped with its weak topology, that is the coarsest topology on $L^p((Y, \mathcal{M}, dm); E)$ for which each of the linear forms

$$(\text{A.1}) \qquad u \mapsto \int \chi(y) \left\langle w\, ; u(y) \right\rangle_{E^*,E} dm(y) \quad \text{for } w \in E^* \quad \text{and} \quad \chi \in L^{p^*}(dm)\,,$$

is continuous.

Finally, in the case where Y is locally compact, we denote by $w\text{-}L^p_{\text{loc}}(dm; w\text{-}E)$ (or $w\text{-}L^p_{\text{loc}}(dm)$) the space $L^p_{\text{loc}}(dm; E)$ equipped with its weak topology; that is, the coarsest for which all the linear forms (A.1) are continuous, where the functions a are restricted to have compact support in Y.

Appendix B. The Product Limit Theorem

In the body of this article extensive use has been made of the so-called product limit theorem to establish certain limits of products of sequences of functions. Since this result is not given in most standard references on integration theory, we present it here.

Let $(X, \mathcal{M}, dm)$ be a measurable space equipped with a positive finite measure dm. The result is a consequence of the classical Egorov theorem (see [11]), which is restated here for the sake of completeness.

EGOROV THEOREM. *Let g_n be a bounded sequence in $L^\infty(dm)$ such that $g_n \to g$ almost everywhere. Then, for any $\delta > 0$ there exists a measurable set $E \subset X$ such that*

$$(\text{B.1}) \qquad \begin{aligned} &\text{(i) } m(E) < \delta\,, \\ &\text{(ii) } g_n(x) \to g(x) \text{ as } n \to +\infty \quad \text{uniformly over } x \in X - E\,. \end{aligned}$$

PRODUCT LIMIT THEOREM. *Consider two sequences of real-valued measurable functions defined on X denoted f_n and g_n.*

(i) *If g_n is bounded in $L^\infty(dm)$ such that $g_n \to 0$ almost everywhere and $f_n \to f$ in $w\text{-}L^1(dm)$ then $f_n g_n \to 0$ in $L^1(dm)$.*
(ii) *If g_n is bounded in $L^\infty(dm)$ such that $g_n \to g$ almost everywhere and $f_n \to f$ in $w\text{-}L^1(dm)$ then $f_n g_n \to fg$ in $w\text{-}L^1(dm)$.*

Proof: Writing $f_n g_n - fg = f_n(g_n - g) + g(f_n - f)$ shows that (ii) is a consequence of (i) with $g_n - g$ in place of g_n. Let $\varepsilon > 0$ be some arbitrary positive number. The sequence f_n, being relatively compact in $L^1(dm)$, is equi-integrable ; see [10]. Thus, by picking $\delta > 0$ small enough, one has that for every measurable set E

$$m(E) < \delta \quad \text{implies} \quad \int |f_n - f|\, \mathbf{1}_E\, dm < \varepsilon \quad \text{uniformly in } n\,.$$

Now fix $E \subset X$ to be as given by the Egorov theorem in (B.1). Then

$$\int |(f_n - f)g_n|\, dm = \int |(f_n - f)g_n|\, \mathbf{1}_E\, dm + \int |(f_n - f)g_n|\, \mathbf{1}_{X-E}\, dm \,. \tag{B.2}$$

The first term in the right side of (B.2) is treated as follows:

$$\int |(f_n - f)g_n|\, \mathbf{1}_E\, dm \leqq \sup_k \{ \|g_k\|_{L^\infty} \} \int |(f_n - f)|\, \mathbf{1}_E\, dm \,. \tag{B.3}$$

Then, using the equi-integrablity of $f_n - f$, (B.3) shows that (B.2) implies

$$\int |(f_n - f)g_n|\, dm \leqq \varepsilon \sup_k \{ \|g_k\|_{L^\infty} \} + \sup_k \{ \|f_k - f\|_{L^1} \} \, \|g_n \mathbf{1}_{X-E}\|_{L^\infty} \,. \tag{B.4}$$

Taking the limit as $n \to \infty$ in (B.4) shows that

$$\limsup_{n \to \infty} \int |(f_n - f)g_n|\, dm \leqq \varepsilon \sup_k \{ \|g_k\|_{L^\infty} \} \;;$$

but ε was arbitrary, whence assertion (i) holds.

Appendix C. On the Linearized Collision Operator

This appendix presents some technical estimates regarding the linearized collision operator that are used extensively in the main body of this article in order to control the high velocity tails of various distributions.

The following estimates bear on the operator $|L|$ which was defined in Section 6 by

$$|L|(f) = \iint (f_1' + f' + f_1 + f)\, b(v_1 - v, \omega) M_1 dv_1\, d\omega \,, \tag{C.1}$$

where f is any function of v for which the integrals make sense.

PROPOSITION C. *The operator $|L|$ satisfies the following two properties.*

(i) *Let f and g be two functions of v. Then*

$$|L(fg)|^2 \leqq \left(|L|(|fg|)\right)^2 \leqq |L|(f^2)\, |L|(g^2) \,. \tag{C.2}$$

(ii) *Assume* (H1). *Then, there exists a constant C such that*

$$|L|(1) \leqq C(1 + |\phi|) \,. \tag{C.3}$$

Moreover, the following continuity property holds:

(iii) *Assume* (H1). *Then, the operators* L *and* $|L|$ *are continuous from* $L^1((1+|v|^2)Mdv)$ *into* $L^1((1+|\phi|)Mdv)$.

Proof: The proof of assertion (i) is nothing but the Cauchy-Schwarz inequality, and makes no use of assumption (H1).

To prove assertion (ii), first weaken the first statement in (H1) into the inequality

$$|\phi(v)|\, b(v_1 - v, \omega) \leqq C\left(1 + |v|^2 + |v_1|^2\right) .$$

This inequality is then integrated with respect to the measure $M_1 dv_1\, d\omega$, which yields

$$\begin{aligned} |\phi(v)|\, |L|(1) &= 4|\phi(v)| \iint b(v_1 - v, \omega) M_1 dv_1\, d\omega \\ &\leqq C \iint (1 + |v|^2 + |v_1|^2) M_1 dv_1\, d\omega \leqq C(1 + |\phi|)^2 \end{aligned}$$

according to assumption (H1). The announced conclusion then follows.

To prove (iii), observe that, by (H1)

$$\begin{aligned} (1 + |\phi|)\, |L(f)| &\leqq (1 + |\phi|)\, |L|(|f|) \\ &\leqq C \iint (|f_1'| + |f'| + |f_1| + |f|)\, \left(1 + |v_1|^2 + |v|^2\right) M_1 dv_1 d\omega , \end{aligned}$$

whence, after integrating with respect to v and using the $d\mu$-symmetries (1.24), the following inequality holds:

$$\begin{aligned} \int (1 + |\phi|)\, |L(f)|\, Mdv &\leqq \int (1 + |\phi|)\, |L|(|f|) Mdv \\ &\leqq C \iiint (|f_1| + |f|)(1 + |v_1|^2 + |v|^2) Mdv M_1 dv_1\, d\omega \\ &\leqq C \iint |f|\, (1 + |v|^2) Mdv . \end{aligned}$$

Assertion (iii) follows upon this last inequality.

The fact that Maxwell potentials satisfy assumption (H1) relies on a well-known observation, probably going back to Maxwell, that the entries of the tensor ϕ are eigenfunctions of the operator L. Therefore, $|\phi(v)| = O(|v|^2)$ and the second inequality in (H1) obviously holds. As for the first inequality in (H1), it stems from the fact that, in the case of a Maxwell potential, $b(V, \omega) = O(1)$ uniformly in ω as $|V|$ tends to infinity. Therefore, the class of potentials satisfying assumption (H1) is nonempty. It is very likely that all cut-off hard potentials such that $b(V,\omega) = O(|V|^\gamma)$ as $|V|$ tends to infinity also satisfy (H1). Although we have not been able to prove this fact rigorously, a simple homogeneity argument at large v seems to indicate that (H1) also holds in this case.

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