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Steady flows of a rarefied gas around arbitrary obstacle distributions

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ABSTRACT. — This paper is devoted to the mathematical study of a highly rarefied gas around a group of bodies which are at rest. Emphasis is put on the relations existing between the mathematical structure of the problem and its physical properties.

1. Introduction

In two papers which appeared in the "Journal de Mécanique Théorique et Appliquée", Y. Sone [1985] gave an explicit construction for the description of a highly rarefied gas around bodies at rest with various temperature distributions. The purpose of the present note is to give a functional analysis proof of the existence and uniqueness of such a solution and to study the basic properties of the problem. This will produce a different proof of existence and uniqueness from that which was given in [S, 1985]. For closed systems the hypothesis and conclusions turn out to be similar. In particular we assume that the temperature of the wall and the accommodation coefficient are uniformly bounded away from zero. These assumptions are instrumental both for energy estimates (in the present paper) and to prove that the series expansion introduced in [S, 1985] is uniformly convergent. Section 3 is devoted to the open system (two reservoirs connected by a pipe with complicated structure). Unlike [S, 1985] it is assumed that the pipe which connects the two reservoirs is contained in a bounded domain. On the other hand more general flows are considered (in particular the case in which the macroscopic velocity of the flow is not identically equal to zero can be treated).

The methods presented in this article can be extended to very general situations (use a compactness argument as in [Babovsky *et al.*, 1991] and the Krein and Rutman Theorem as it is described in [Smoller, 1982] page 122). However the proofs given in [B, 1991] are incomplete; they will be completed in a forthcoming paper [Bardos *et al.*, to appear], and it turns out that in the present case direct simple proofs are available and will be described in the paper.

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2. The closed bounded vessel

We denote by X a vessel with boundary ∂X and outward unit normal vector $n(x)$. The shape of this vessel may be arbitrary but for simplicity we shall assume that its boundary is locally smooth: it may be composed of several disjoint connected components

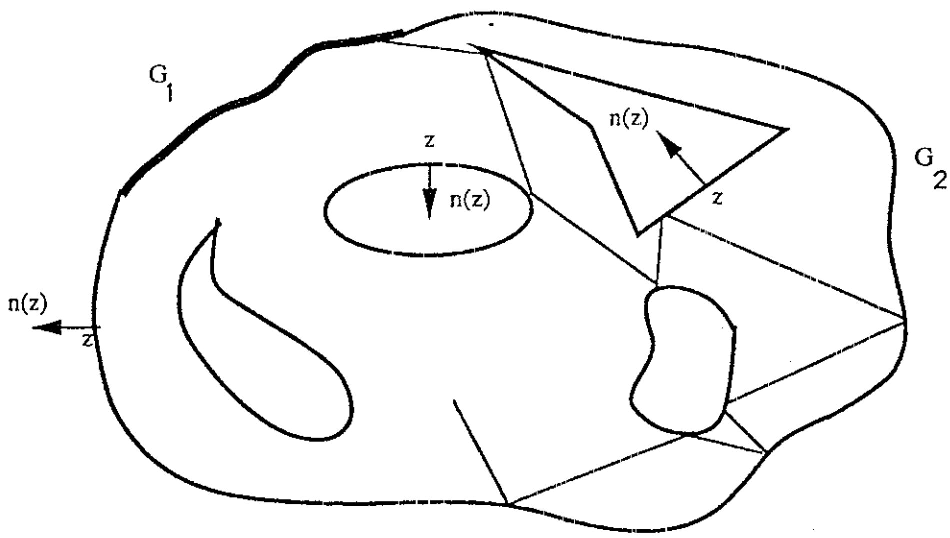

Fig. 1. – Bounded vessel of arbitrary shape.

(*see Fig.* 1) with boundary a union of C^1 submanifolds of dimension two. The outward normal $n(x)$ is well defined everywhere on ∂X with the exception of a submanifold of dimension one corresponding to the edges and corners of the boundary. The solution is a function defined on one of the following two phase spaces $X \times \mathbf{R}_v^3$ or $X \times \mathbf{S}^2$ ($\mathbf{S}^2$ denoting the unit sphere in $\mathbf{R}_3$).

As usual we shall denote by $\Gamma_{\pm}^u$ the set of points of $\partial X \times \mathbf{S}^2$ such that $\pm n(x).\Omega > 0$ and by $\Gamma_{\pm}$ the set of points of $\partial X \times \mathbf{R}_v^3$ such that $\pm n(x).v > 0$

The description of the specular reflexion involves the reflected value of the velocity at the boundary. For any $(x, \Omega) \in \Gamma_{-}^u$ we denote by Ω^R the vector

$$\Omega^R = \Omega - 2(\Omega . n(x)) n(x). \tag{1}$$

Similarly $v^R = |v| \Omega^R$ is defined by

$$v^R = v - 2(v . n(x)) n(x). \tag{2}$$

Whenever this makes sense (classical trace theorems being invoked for this purpose (*see* [Bardos, 1969]; [Cessenat, [1985], or [Dautray & Lions, [1968]), one introduces the restriction to $\Gamma_{\pm}^u$ (resp. $\Gamma_{\pm}$) of a function f defined in $X \times \mathbf{S}^2$ (resp. in $X \times \mathbf{R}_v^3$); this restriction is denoted by $f_{\pm}$. For $(x, \Omega) \in X \times \mathbf{S}^2$ denote by $t(x, \Omega) > 0$ the first positive time at which the ray $\{x - t\Omega \text{ s.t. } t > 0\}$ intersects the boundary ∂X. Then, for $(x, \Omega) \in \Gamma_{-}^u$ we denote by $j_1(x, \Omega)$ the point

$$j_1(x, \Omega) = (x_1, \Omega_1) = (x - t(x, \Omega^R) \Omega^R, \Omega^R) \tag{3}$$

Observe that j_1 defines a map from Γ^u_- into Γ^u_- and also from Γ_- into Γ_-. Given n unit vectors Ω_k $1 \leqq k \leqq n$, one defines a broken ray of order n with end points x and y by the formulaes

(4) $$x = x_0, \qquad x_k = x_{k-1} - t(x_{k-1}, \Omega_k)\,\Omega_k, \qquad 1 \leqq k \leqq n, \qquad y = x_n.$$

An ergodicity assumption **Ergodicity Hypothesis** is introduced: For any pair $(x, y) \in \partial X \times \partial X$ there existes a broken ray of the above type with end points x and y (*see Fig.* 1). This assumption is used to obtain the uniqueness of the solution and, since the Ω_k can take any value, it coincides with the "bridge" hypothesis of [S, 1985].

The gas is so highly rarefied that collisions between gas molecules may be neglected (this is the regime of free molecular – or Knudsen gas – flow). Therefore the evolution of the gas density is governed by the formula

(5) $$\partial_t f + v \,.\, \nabla_x f = 0$$

$f(x, v, t)$ is a non-negative function of the variables $(x, v, t) \in X \times \mathbf{R}^3 \times \mathbf{R}^+$. At each point of the boundary an accommodation coefficient $0 < \alpha < \alpha(x) \leqq 1$ is given as well as a temperature $T(x)$. The notation $\beta(x) = 1/2T(x)$ will be adopted in the sequel. The functions $\alpha(x)$ and $\beta(x)$ are assumed continuous on ∂X. Consider the following boundary condition:

(6) $$f(x, v)_{|\, v.n(x) < 0} = (1 - \alpha(x))\, f(x, v^R) + \alpha(x)\,\beta(x)^2 \exp(-\beta(x)\,|v|^2)\,\frac{2}{\pi}\int_{v'.n(x) > 0} f(x, v')\, v'.n(x)\, dv'.$$

This is a linear combination of the specular reflection and diffuse reflection (Maxwell type boundary condition). The evolution Eq. (5) with the boundary condition (6) can be written in the abstract form $\partial_t f + A f = 0$ where $-A$ is a closed unbounded operator, which is the generator of a strongly continuous semigroup in any

$$L^p(X \times \mathbf{R}^3_v)\ (1 \leqq p < \infty).$$

Its adjoint $-A^*$ is an unbounded operator defined in $L^{p'}(X \times \mathbf{R}^3_v)$, $1/p + 1/p' = 1$. For $1 < p < \infty$, its adjoint is also the generator of a strongly continuous semigroup. However for $p' = \infty$, $D(A^*)$ is not dense in $L^\infty(X \times \mathbf{R}^3_v)$; $\exp(-tA^*)$ is only a weakly continuous contraction semigroup and only its restriction to the closure of $D(A^*)$ is strongly continuous. In any case A^* is defined for $1 \leqq p \leqq \infty$ by the formulae

(7) $$\begin{cases} D(A^*) = \{ f^*(x, v) \in L^{p'}(X \times \mathbf{R}^3_v) \text{ s.t. } v \,.\, \nabla_x f \in L^{p'}(X \times \mathbf{R}^3_v) \}, \\ A^* f^* = -v \,.\, \nabla_x f^* \end{cases}$$

with the adjoint boundary condition

(8) $$f^*(x, v)_{|\, v.n(x) > 0} = (1 - \alpha(x))\, f^*(x, v^R) + \alpha(x)\,\frac{2}{\pi}\int_{v'.n(x) < 0} f^*(x, v')\,\beta^2(x) \exp(-\beta(x)\,|v'|^2)\, v'.n(x)\, dv'.$$

In particular, a simple explicit computation shows that

$$(9)\qquad \frac{2}{\pi}\int_{v'.n(x)>0} \beta^2(x)\exp(-\beta(x)|v'|^2)\, v'.n(x)\,dv' = 1.$$

In the present section we consider the homogeneous solution of the stationary problem

$$(10)\qquad v.\nabla_x f = 0,$$

and

$$(11)\qquad f(x, v)_{|v.n(x)<0} - (1-\alpha(x)) f(x, v^R) - \alpha(x)\frac{2}{\pi}\int_{v'.n(x)>0} f(x, v)\,\beta^2(x)\exp(-\beta(x)|v|^2)\, v'.n(x)\,dv' = 0.$$

The problem (10) and (11) corresponds to the study of the nullspace of the operator A. The relations (7), (8) and (9) show that 0 belongs to the spectrum of A* and that $f^*(x, v) \equiv 1$ is an associated eigenvector. Therefore 0 also belongs to the spectrum of A and it is natural to discuss the existence of non-trivial solutions of (10) and (11). These would correspond to steady molecular flows. Finally it will be convenient to introduce for any function f defined on $X \times \mathbf{R}_v^3$ the function U_f defined in $X \times \mathbf{S}^2$ by the formula

$$(12)\qquad U_f(x, \Omega) = \int_0^\infty r^3 f(x, r\Omega)\,dr$$

and the Banach space **E** of functions satisfying the estimate

$$|||f|||_{\mathbf{E}} = \sup_{(x, \Omega)\in X\times S^2} \int_0^\infty r^3 |f(x, r\Omega)|\,dr < \infty$$

equipped with the corresponding norm.

THEOREM 1. – *For any bounded domain* X *with piecewise* C^1 *boundary, there exists a unique (up to a multiplicative constant) non-trivial, non-negative solution* $f \in E$ *to the problem*

$$(13)\qquad v.\nabla_x f = 0,$$

and

$$(14)\qquad f(x, v)_{|v.n(x)<0} - (1-\alpha(x))\, f(x, v^R) - \alpha(x)\,\beta^2(x)\exp(-\beta(x)|v|^2)\frac{2}{\pi}\int_{v'.n(x)>0} f(x, v')\, v'.n(x)\,dv' = 0.$$

In particular, for any positive constant ρ *there exists a unique solution such that*

$$\iint f(x, v)\,dx\,dv = \rho.$$

Proof. – Observe the relation

$$2\int_0^\infty \rho^3 \beta(x)^2 e^{-\beta(x)\rho^2} d\rho = 1 \tag{15}$$

Fact n° 1. – If f is a solution of (13)-(14), then the function $U_f \in L^\infty(X \times S^2)$ defined by (12) is a solution of the "reduced problem"

$$\Omega \nabla_x U_f(x, \Omega) = 0 \tag{16}$$

with the boundary condition

$$U_f(x, \Omega)_{|n(x).\Omega<0} = (1-\alpha(x))\, U_f(x, \Omega^R) + \alpha(x) \frac{1}{\pi} \int_{n(x).\Omega>0} n(x).\Omega\, U_f(x, \Omega)\, d\Omega. \tag{17}$$

Define the operator $Z: L^\infty(\Gamma_-) \to L^\infty(X \times \mathbf{R}_v^3)$ by the formula

$$(Zh)(x, v) = h\left(x - t\left(x, \frac{v}{|v|}\right)\frac{v}{|v|}, v\right). \tag{18}$$

For any $h \in L^\infty(\Gamma_-)$, $f = Zh$ is the unique solution of

$$v.\nabla_x f = 0; \qquad f(x, v) = h(x, v) \quad \text{for} \quad (x, v) \in \Gamma_-. \tag{19}$$

Introduce the operator J defined by the formula

$$J\,\phi(x, \Omega) = \phi(x - t(x, \Omega^R), \Omega^R) \tag{20}$$

which is an isometry on the space $L^\infty(\Gamma_-)$. Therefore the operator $1-(1-\alpha(x))\,J$ is invertible.

Fact n° 2. – Then, $f \in E$ is a solution of (13)-(14) if and only if

$$f = Z[1-\alpha(x))\,J]^{-1} \times \left(\alpha(x)\beta(x)^2 \exp(-\beta(x)|v|^2) \frac{2}{\pi} \int_{\Omega'.n(x)>0} U_f(x, \Omega')\, \Omega'.n(x)\, d\Omega'\right). \tag{21}$$

Existence. Let

$$f = Z[1-(1-\alpha(x))\,J]^{-1}(2\,\alpha(x)\,\beta(x)^2 \exp(-\beta(x)|v|^2)). \tag{22}$$

Then f satisfies

$$\begin{cases} v.\nabla_x f = 0, \\ f(x, v) = (1-\alpha(x))\, f(x, v^R) + 2\,\alpha(x)\,\beta(x)^2 \exp(-\beta(x)|v|^2) \quad (x, v) \in \Gamma_- \end{cases} \tag{23}$$

Observing that

$$\int_{\Omega'.n(x)>0} \Omega'.n(x)\, d\Omega' = \pi, \tag{24}$$

it follows easily from (23)-(24) that

$$(25)\qquad \Omega . \nabla_x \mathrm{U}_f = 0, \qquad \mathrm{U}_f(x, \Omega) = (1-\alpha(x))\,\mathrm{U}_f(x, \Omega^{\mathrm{R}}) + \alpha(x), \qquad (x, \Omega) \in \Gamma^u_-.$$

Use of the fact that J is an isometry on $L^\infty(\Gamma^u_-)$, shows that (25) has a unique solution, which turns out to be $\mathrm{U}_f \equiv 1$. It follows from the definition (22) of f that f satisfies Eq. (21) or, in other words, that f is a solution of (13)-(14) in E (Fact n° 2).

Uniqueness (up to a multiplicative constant). – Let f_1 and f_2 be two solutions of (13)-(14). It follows from Fact n° 1 that U_{f_1} and U_{f_2} are solutions of (16)-(17). Following (21), it suffices to show uniqueness (up to a multiplicative constant) of the solution of (16)-(17), or more precisely that any solution U of (16)-(17) is constant.

Let U be a solution of (16)-(17). Multiplying (16) by U, integrating over $X \times S^2$ and applying Green's formula leads to

$$(26)\qquad \iint_{\Gamma^u} \mathrm{U}(x, \Omega)^2\, \Omega . n(x)\, d\Omega\, dx = 0.$$

This last equality is split as follows:

$$(27)\qquad \iint_{\Gamma^u} \mathrm{U}(x, \Omega)^2\, \Omega . n(x)\, d\Omega\, dx = \mathrm{I}_1 - \mathrm{I}_2$$

where

$$\mathrm{I}_1 = \iint_{\Gamma^u_+} \mathrm{U}(x, \Omega)^2\, \Omega . n(x)\, d\Omega\, dx$$

and

$$\mathrm{I}_2 = \iint_{\Gamma^u_-} \mathrm{U}(x, \Omega)^2\, |\Omega . n(x)|\, d\Omega\, dx.$$

Using the boundary condition (17) transforms the above integral into

$$(28)\qquad \mathrm{I}_2 = \iint_{\Gamma^u_-} |\Omega . n(x)| \Big[(1-\alpha(x))\,\mathrm{U}(x, \Omega^{\mathrm{R}}) + \alpha(x)\frac{1}{\pi}\int_{\Omega' . n(x) > 0} \Omega' . n(x)\,\mathrm{U}(x, \Omega')\, d\Omega'\Big]^2 d\Omega\, dx.$$

Expanding the square in (28) leads to a new form of equality (26):

$$(29)\qquad \int_{\partial X}\int_{\Omega . n(x) > 0} (2\alpha(x) - \alpha(x)^2)\left(\int_{\Omega' . n(x) > 0} \Omega' . n(x)\,\mathrm{U}(x, \Omega')\frac{d\Omega'}{\pi}\right)^2 dx$$
$$= \int_{\partial X}\int_{\Omega . n(x) > 0} (2\alpha(x) - \alpha(x)^2)\left(\int_{\Omega' . n(x) > 0} \Omega' . n(x)\,\mathrm{U}(x, \Omega')^2\frac{d\Omega'}{\pi}\right) dx.$$

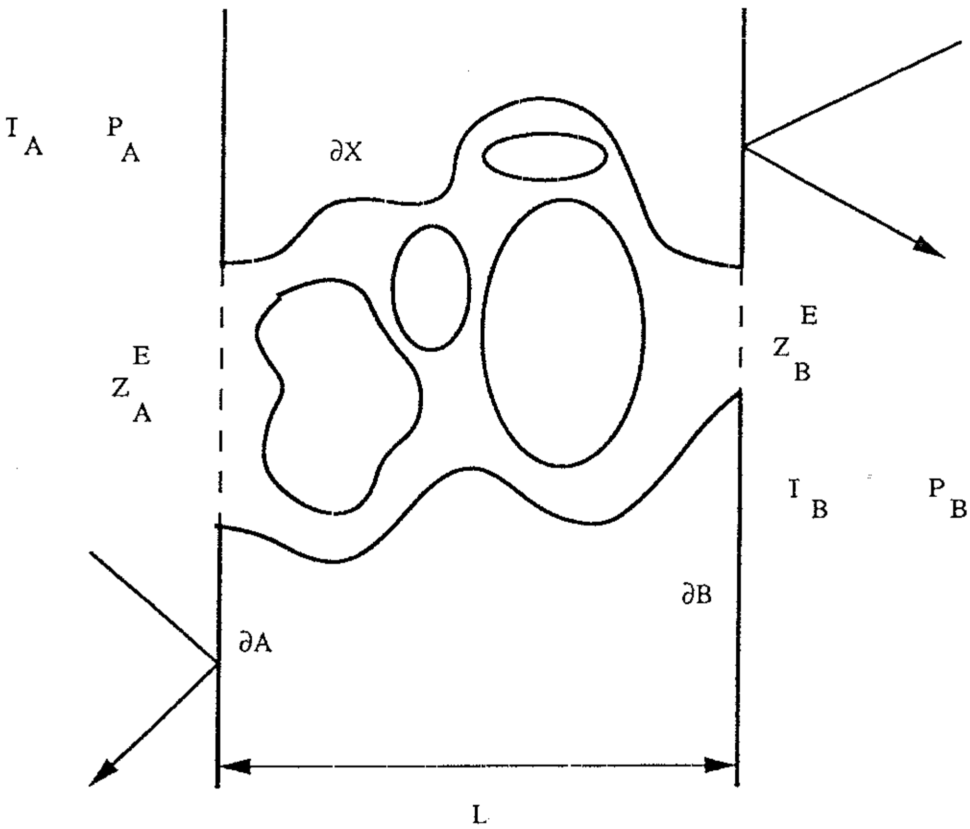

Fig. 2 — Pipe between two reservoirs.

It follows from (29) that

$$(30) \qquad \left(\int_{\Omega' . n(x)>0} \Omega' . n(x)\, U(x, \Omega') \frac{d\Omega'}{\pi}\right)^2 = \int_{\Omega' . n(x)>0} \Omega' . n(x)\, U(x, \Omega')^2 \frac{d\Omega'}{\pi}.$$

Observe that this is precisely the equality case in the Cauchy-Schwartz inequality for the probability measure $\Omega' . n(x)\, d\Omega'/\pi$. Therefore the trace of U on ∂X does not depend on the angle Ω:

$$(31) \qquad U(x, \Omega) = \psi(x), \qquad x \in \partial X, \qquad \Omega \in S^2.$$

Since $U(x+t\Omega, \Omega) = U(x, \Omega)$ whenever x and $x+t\Omega$ belong to X, it follows from the *Ergodicity Hypothesis* that the trace of U on ∂X is a constant and thus that U itself is a constant. //

COROLLARY 2. — *The solutions f of* (13)-(14) *defined in Theorem* 1 *have zero bulk velocity*:

$$\rho u_f = \int_{\mathbf{R}^3} v f(x, v)\, dv = 0.$$

Proof. — Observe that

$$\int_{\mathbf{R}^3} vf(x, v)\, dv = \int_{S^2} \Omega\left(\int_0^\infty r^3 f(x, r\Omega)\, dr\right) d\Omega = \int_{S^2} \Omega\, U_f(x, \Omega)\, d\Omega = 0$$

since U_f is a constant according to Theorem 1. //

3. Existence and uniqueness of the solution in the case of an open system.

We now consider two reservoirs

$$A = \{(x, y, z)/-\infty < x < 0\} \quad \text{and} \quad B = \{(x, y, z)/L < x < \infty\}$$

joined by a pipe of length L (*see Fig.* 2), treated as an example of the general solution in S, 1985]. The pipe denoted by X has boundary composed of two parts, the physical boundary and an artificial boundary corresponding to the inlet ($x=0$ denoted by Z_E^A) and outlet ($x=L$ denoted by Z_E^B). The distributions of molecules starting at the infinity of A and B are given by the formulae

$$f_A = C_A \beta_A^2 \exp(-\beta_A |v|^2) \quad \text{and} \quad f_B = C_B \beta_B^2 \exp(-\beta_B |v|^2) \tag{32}$$

The pipe can be viewed as a closed system (but of course on $Z_E^A \cup Z_E^B$ the boundary conditons will have to be modified as described below). Therefore we begin with a theorem concerning the inhomogeneous boundary value for the stationary problem. Unlike the case dealt with in the previous section, this is not a spectral problem, provided some **Transport Hypothesis** (to be defined below), is satisfied.

THEOREM 3. — *Let* X *denote a bounded open set of* $\mathbf{R}^3$ *with piecewise* C^1 *boundary* ∂X *and exterior normal denoted by* $n(x)$. *The boundary* ∂X *admits a non-trivial (in the measure sense) partition*

$$\partial X = G_1 \cup G_2$$

which induces partitions of Γ *and* Γ^u *denoted*

$$\Gamma = \Gamma_1 \cup \Gamma_2 \quad \text{and} \quad \Gamma^u = \Gamma_1^u \cup \Gamma_2^u.$$

Assume the following **Transport Hypothesis**: *For any* $x \in G_2$ *there exists a broken ray of finite order,* N *say, with end points* x *and* $y \in G_1$. *Let* $g(x, v)$ *be a function defined on* Γ_1 *such that*

$$\int_{v \cdot n(x) < 0} |g(x, v)|\, |v \cdot n(x)|\, dv \in L^\infty(G_1).$$

Then there exists a unique solution f *of the following boundary value problem*:

$$\forall (x, v) \in X \times \mathbf{R}^3 \qquad v \cdot \nabla_x f = 0, \tag{33}$$

$$\forall (x, v) \in \Gamma_{1-} \qquad f(x, v) = g(x, v). \tag{34}$$

$$\forall (x, v) \in \Gamma_{2-} \qquad f(x, v)_{|v \cdot n(x) < 0} = (1-\alpha(x)) f(x, v^R) + \alpha(x)\, \beta^2(x) \exp(-\beta(x)|v|^2) \frac{2}{\pi} \int_{v' \cdot n(x) > 0} f(x, v')\, v' \cdot n(x)\, dv'. \tag{35}$$

Proof. – As in the previous section, the existence and uniqueness of the solution f of (33)-(34)-(35) can be reduced to the existence and uniqueness of the solution U_f of the following:

$$\forall (x, \Omega) \in X \times S^2 \qquad \Omega . \nabla_x U = 0, \tag{36}$$

$$\left\{\begin{array}{l} U(x, \Omega) = U_g(x, \Omega), \qquad (x, v) \in \Gamma^u_{1-}, \\ U(x, \Omega) = (1-\alpha(x)) U(x, \Omega^R) + \alpha(x) \dfrac{1}{\pi} \displaystyle\int_{\Omega . n(x) > 0} U(x, \Omega) \Omega . n(x) d\Omega \end{array}\right. \tag{37}$$

$$(x, \Omega) \in \Gamma^u_{2-}. \tag{38}$$

The existence and uniqueness of the solution of (36)-(37)-(38) is achieved with the help of the following lemma.

LEMMA 4. – *We keep the notations and assumptions of Theorem* 3. *Let* $U_\lambda(x, \Omega) \in L^\infty(X \times S^2)$ *be the solution of* (37)-(38)-(39) *with Eq.* (39) *given by*

$$\lambda U_\lambda + \Omega . \nabla_x U_\lambda = 0, \qquad (x, \Omega) \in X \times S^2, \tag{39}$$

(the existence and uniqueness of such a U_λ is classical: *see* [D & L, 1968]) Then the following estimate holds *uniformly* for $\lambda \geqq 0$:

$$\| U_\lambda \|_{L^\infty (X \times S^2)} \leqq \| g \|_{L^\infty (\Gamma^u_{1-})}. \tag{40}$$

The above lemma proves Theorem 3 as follows. Indeed, uniqueness is obtained by setting $\lambda = 0$ and $g \equiv 0$ in the lemma. Existence is obtained by letting $\lambda \to 0$ after extraction of a subsequence of U_λ converging in $L^\infty(X \times S^2)$ weak-*. //

Proof of Lemma 4. – Let $M = \| U_g \|_{L^\infty}$. Multiply (39) by $2(U_\lambda - M)_+$ (we denote $\sup(z, 0)$ by z_+) and integrate over the domain $X \times S^2$. Green's formula gives

$$2\lambda \iint_{X \times S^2} (U_\lambda - M)^2_+ \, d\Omega \, dx + \iint_{\Gamma^u} (U_\lambda - M)^2_+ \, \Omega . n(x) \, d\Omega \, dx \leqq 0 \tag{41}$$

The second integral in (41) is split as

$$\begin{aligned} &\iint_{\Gamma^u} (U_\lambda - M)^2_+ \, \Omega . n(x) \, d\Omega \, dx \\ &= \iint_{\Gamma^u_{1+}} (U_\lambda - M)^2_+ \, \Omega . n(x) \, d\Omega \, dx + \iint_{\Gamma^u_{2+}} (U_\lambda - M)^2_+ \, \Omega . n(x) \, d\Omega \, dx \\ &- \iint_{\Gamma^u_{2-}} \left((1-\alpha(x)) (U(x, \Omega^R) - M) + \frac{\alpha(x)}{\pi} \int_{\Omega' . n(x) > 0} (U(x, \Omega') - M) \Omega' . n(x) \, d\Omega' \right)^2_+ \\ &\qquad \times | \Omega . n(x) | \, d\Omega \, dx. \end{aligned} \tag{42}$$

In treating the second integral on the right hand side of (42), one should observe the following three simple inequalities:

a) For all $f \in \mathbf{R}$ and $0 \leqq \chi \leqq 1$, $f\chi \leqq f_+$;

b) $(A+B)_+ \leqq A_+ + B_+$;

c) if μ denotes a probability measure, $0 \leqq \chi \leqq 1$ and $f \in L^1(\mu)$, Jensen's inequality shows that $\chi \int f d\mu \leqq \int f_+ \, d\mu$.

Then it follows from (42) that

$$(43)\quad \iint_{\Gamma^u} (U_\lambda - M)_+^2 \, \Omega . n(x) \, d\Omega \, dx$$
$$\geqq \iint_{\Gamma_{1+}^u} (U_\lambda - M)_+^2 \, \Omega . n(x) \, d\Omega \, dx + \iint_{\Gamma_{2+}^u} (U_\lambda - M)_+^2 \, \Omega . n(x) \, d\Omega \, dx$$
$$- \iint_{\Gamma_{2-}^u} \left((1-\alpha(x)) (U(x, \Omega^R) - M)_+ + \frac{\alpha(x)}{\pi} \int_{\Omega' . n(x) > 0} (U(x, \Omega') - M)_+ \, \Omega' . n(x) \, d\Omega' \right)$$
$$\times |\Omega . n(x)| \, d\Omega \, dx.$$

Expanding the sum of squares above leads to

$$(44)\quad \iint_{\Gamma^u} (U_\lambda - M)_+^2 \, \Omega . n(x) \, d\Omega \, dx$$
$$\geqq \iint_{\Gamma_{1+}^u} (U_\lambda - M)_+^2 \, \Omega . n(x) \, d\Omega \, dx + \iint_{\Gamma_{2+}^u} (2\alpha(x) - \alpha(x)^2)(U_\lambda - M)_+^2 (x, \Omega) \, \Omega . n(x) \, d\Omega \, dx$$
$$- \int_{G_2} \pi (2\alpha(x) - \alpha(x)^2) \left(\int_{\Omega' . n(x) > 0} (U_\lambda - M)_+ (x, \Omega') \, \Omega' . n(x) \frac{d\Omega'}{\pi} \right)^2 |\Omega . n(x)| \, d\Omega \, dx$$

Inequality (41) employed with (44) shows that

$$(45)\quad U_\lambda(x, \Omega) \leqq M \qquad (x, \Omega) \in \Gamma_1^u,$$

and (since $\alpha > 0$)

$$(46)\quad \left(\int_{\Omega' . n(x) > 0} (U_\lambda - M)_+ (x, \Omega') \, \Omega' . n(x) \frac{d\Omega'}{\pi} \right)^2$$
$$= \int_{\Omega' . n(x) > 0} (U_\lambda - M)_+^2 (x, \Omega') \, \Omega' . n(x) \frac{d\Omega'}{\pi}.$$

The equality case in the Cauchy-Schwartz inequality shows that

$$(U_\lambda(x, \Omega) - M)_+ = \phi(x) \text{ for almost every } (x, \Omega) \in \Gamma_{2+}^u,$$

which, by the boundary condition (38), shows that

(47) $$(U_\lambda(x, \Omega)-M)_+=\varphi(x) \text{ for almost every } (x, \Omega)\in\Gamma^u_{2-}.$$

The equation governing $(U_\lambda-M)_+$ is obtained by multiplying (39) by $\text{sign}_+(U_\lambda-M)$:

(48) $$\lambda(U_\lambda-M)_+ +\Omega.\nabla_x(U_\lambda-M)_+=-\lambda M\,\text{sign}_+(U_\lambda-M)\leqq 0.$$

Eq. (48) shows that the map

$$t\mapsto e^{\lambda t}(U_\lambda-M)_+(x+\Omega t, \Omega)$$

is non-increasing for almost every $(x, \Omega)\in X\times S^2$. Let $x\in G_2$, then in accordance with the transport hypothesis a broken ray of finite order with end points $(x, \Omega)\in\Gamma^u_{2+}$ and $(y, \Omega_n)\in\Gamma^u_{1-}$ can be introduced, and the relation (40) implies that along this broken ray the quantity $(U_\lambda-M)_+$ is equal to zero. Therefore $\phi\equiv 0$ on ∂X and thus $(U_\lambda-M)_+\equiv 0$ everywhere in X. This shows that $U_\lambda\leqq M$. To prove that $|U_\lambda|\leqq M$ one need only apply the result to $-U_\lambda$. //

COROLLARY 5. — *Assume the following* **Transport Hypothesis:** *For any $x\in\partial X$ there exist an integer* N *and a unit vector* Ω *such that the point* x_N *defined by* (3) *and* (4) *belongs to* $Z^A_E\cup Z^B_E$. *Then there exists a unique function f such that*

(49) $$v.\nabla_x f=0 \quad \textit{in } X\cup A\cup B\times R^3$$

and the following boundary conditions are satisfied.

(50) $$f(x, v)=C_A\,\beta_A^2\exp(-\beta_A|v|^2)\ \forall x\in A \quad \textit{and} \quad v_1>0$$

(51) $$f(x, v)=C_B\,\beta_B^2\exp(-\beta_B|v|^2)\ \forall x\in B \quad \textit{and} \quad v_1<0$$

(52) $$\forall x\in\partial X \quad f(x, v)_{|v.n(x)<0}=(1-\alpha(x))f(x, v^R) +\alpha(x)\frac{2}{\pi}\int_{v.n(x)>0} v.n(x)f(x, v)\,dv\,\beta(x)^2\exp(-\beta(x)|v|^2)$$

(53) $$f(x, v)=f(x, v^R)\ \forall x\in\partial A \quad \textit{and} \quad \forall x\in\partial B.$$

Proof. — The "physical boundary" of the tube ∂X is identified with the subset G_2 of Theorem 3 and the "artificial boundary" $Z^A_E\cup Z^B_E$ is identified with the subset G_1. On this subset the incoming density of particles $g(x, v)$ is defined by

(54) $$\begin{cases} g(x, v)=C_A\,\beta_A^2\exp(-\beta_A|v|^2) & \text{if } x\in Z^A_E \text{ and } v_1>0\\ g(x, v)=C_B\,\beta_B^2\exp(-\beta_B|v|^2) & \text{if } x\in Z^B_E \text{ and } v_1<0\end{cases}$$

Theorem 3 shows that there exists a unique solution $f_X(x, v)$ which is defined in X. This solution is extended to A by the following explicit construction:

(55) $$f(x, v)=C_A\,\beta_A^2\exp(-\beta_A|v|^2) \quad \text{if } v_1>0, \quad x\in A.$$

For $x \in A$ and $v_1 < 0$ there exists a unique $t > 0$ such that $x - tv \in Z_E^A \cup \partial A$. Then

$$(56) \qquad \begin{cases} f(x, v) = C_A \beta_A^2 \exp(-\beta_A |v|^2) & \text{if } x - tv \in \partial A \\ f(x, v) = f_X(x - tv, v) & \text{if } x - tv \in Z_E^A. \end{cases}$$

A similar construction if done in B. The function f obtained in this way is obviously the unique solution of the problem (49)-(53).

With the thermodynamical equilibrium states given by (32), the temperature and the pressure in regions A and B are defined by the formulae

$$(57) \quad T_A = \frac{1}{2}\beta_A, \qquad p_A = C_A \beta_A^2 \left(\frac{2}{\pi} T_A\right)^{3/2} T_A, \qquad T_B = \frac{1}{2}\beta_B, \qquad p_B = C_B \beta_B^2 \left(\frac{2}{\pi} T_A\right)^{3/2} T_B.$$

PROPOSITION 6. — *Under the relation $p_A/p_B = (T_B/T_A)^{1/2}$ there is no flow induced in the system. More precisely at any point x of the domain the following relation holds:*

$$\rho u = \int v f(x, v)\, dv = 0$$

Proof. — The proof follows the same lines as in Section I. One introduces the function

$$U_f(x, \Omega) = \int_0^\infty r^3 f(x, r\Omega)\, dr$$

and $\rho u = \int \Omega\, U_f(x, \Omega)\, d\Omega$. Now $U_f(x, \Omega)$ is a solution of the problem $\Omega . \nabla_x U_f(x, \Omega) = 0$ with the boundary condition

$$(58) \quad U_f(x, \Omega)|_{\Omega . n(x) < 0} = (1 - \alpha(x))\, U_f(x, \Omega^R) + \alpha(x) \left[\frac{2}{\pi} \int_{v . n(x) > 0} v . n(x) f(x, v)\, dv\right] \int_0^\infty r^3 \beta^2(x) \exp(-\beta(x) r^2)\, dr \quad \text{on } \partial X$$

which, as above, reduces to the relation

$$(59) \quad U_f(x, \Omega)|_{\Omega . n(x) < 0} = (1 - \alpha(x))\, U_f(x, \Omega^R) + \alpha(x) \int_{\Omega . n(x) > 0} \Omega . n(x)\, U_f(x, \Omega)\, d\Omega.$$

On Z_E^A and Z_E^B it satisfies the relations

$$U_f(x, \Omega) = C_A \quad \text{if } x \in Z_E^A \quad \text{and} \quad \Omega_1 > 0,$$
$$U_f(x, \Omega) = C_B \quad \text{if } x \in Z_E^B \quad \text{and} \quad \Omega_1 < 0$$

With Theorem 3 it is known that this problem has a unique solution which will be equal to $C = C_A = C_B$ whenever these two numbers coincide. According to (57), this is equivalent to the relation

$$p_A/p_B=(T_B/T_A)^{1/2}.$$

We conclude as in the Corollary 2.

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