Algebraic dynamics of polynomial maps: the dynamical Manin-Mumford conjecture

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Distribution of preperiodic points

The set up.

- $f : X \to X$ a regular dominant map on an algebraic variety $/\mathbb{C}$;
- $\mathrm{Per}(f) = \{x \in X,\ f^n(x) = x \text{ for some } n \geq 1\}$;
- $\mathrm{PrePer}(f) = \{x \in X,\ f^n(x) = f^m(x) \text{ for some } n > m \geq 0\}$.

The problem.

- Describe the distribution of $\mathrm{Per}(f)$ (and/or $\mathrm{PrePer}(f)$) in X.
- In the euclidean topology: look at the limits of atomic measures equidistributed over $\{f^n = \mathrm{id}\}$;

$$\begin{cases} \text{Bedford-Smillie: automorphisms of } \mathbb{A}^2_{\mathbb{C}}; \\ \text{Lyubich, Briend-Duval: endomorphisms of } \mathbb{P}^d_{\mathbb{C}}; \\ \text{Many other cases: Dinh, Sibony, etc...} \end{cases}$$

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The abelian case (the Manin-Mumford conjecture)

- X abelian variety (compact complex torus that is projective);
- $f(x) = k \cdot x = \underbrace{x + \cdots + x}_{k \text{ times}}$ with $k \geq 2$;
- $\mathrm{PrePer}(f) = \mathrm{Tor}(X)$.

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Theorem (Raynaud)

Pick $V \subset X$ irreducible s.t. $\mathrm{Tor}(f) \cap V$ is Zariski dense. Then V is a translate by a torsion point of an abelian subvariety.

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Towards a DMM conjecture?

Question (The DMM conjecture)

Pick $V \subset X$ irreducible s.t. $\mathrm{PrePer}(f) \cap V$ is Zariski dense. Does this imply V to be preperiodic?

- Wrong!!! Counterexamples for endomorphisms of $\mathbb{P}^2$ (Ghioca-Tucker-Zhang, Pazuki)
- True in some cases: a very general endomorphism of $\mathbb{P}^d_{\mathbb{C}}$ (Fakhruddin)

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Variations on the DMM problem

Question

Given f, describe all irreducible subvarieties $V \subset X$ s.t. $\mathrm{PrePer}(f) \cap V$ is Zariski dense.

Question

Describe the maps f for which the DMM conjecture has a positive/negative answer.

The DMM problem for polynomial automorphisms

$f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ an automorphism.

- When f is affine or elementary $(x, y) \mapsto (ax + b, cy + P(y))$ the DMM problem has a positive answer (exercice).

In the sequel suppose

$$f(x, y) = (ay, x + P(y)), \deg(P) \geq 2$$

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First reductions

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- Assumption:
  $V \cap \mathrm{PrePer}(f)$ is Zariski dense
- Conclusion:

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Question

Is the set $\mathrm{Per}(f) \cap V$ *is finite for any irreducible curve* V*?*

Main theorem

Theorem (Dujardin-Favre)

Suppose $f(x, y) = (ay, x + P(y))$ *with* $|\mathrm{Jac}(f)| = |a| \neq 1$*. Then the set* $\mathrm{Per}(f) \cap V$ *is finite for any irreducible curve* V*.*

Counter-examples: reversible maps

- $f(x,y) = (y, -x + y^2)$, $f^{-1} = (-y + x^2, x)$;
- $f^{-1} = \sigma \circ f \circ \sigma$ with $\sigma(x,y) = (y,x)$;
- $\Delta = \{(x,x)\}$, $\Delta \cap f^n(\Delta) \subset \mathrm{Fix}(f^{2n})$;

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Proposition

$$|\Delta \cap f^n(\Delta)| \to \infty.$$

Proof.

Use Arnold's result: $\mathrm{mult}_{(x,x)}\,(f^n(\Delta), \Delta) = O(1)$. □

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Proposition

$$|\Delta \cap f^n(\Delta)| \simeq 2^n.$$

Proof.

The image $f^n(\Delta)$ converges to a laminar current. □

Some conjectures

Conjecture

Suppose $\mathrm{Per}(f) \cap V$ *is infinite. Then* $f^{-n} = \sigma \circ f^n \circ \sigma$ *for some* $n \geq 1$ *and some involution* σ.

Conjecture (Weak form)

Suppose $\mathrm{Per}(f) \cap V$ *is infinite. Then* $\mathrm{Jac}(f)$ *is a root of unity.*

Conjecture (Effective bounds)

Fix f *for which the DMM conjecture has a positive answer. Give a bound on* $\mathrm{Per}(f) \cap V$ *in terms of* $\deg(V)$.

Szpiro-Ullmo-Zhang' strategy

Reduce to the case $f(x, y) = (ay, x + y^2 + c)$, $V = \{Q = 0\}$ with $a, c \in \mathbb{Q}$, $Q \in \mathbb{Q}[x, y]$.

Assumption: $V \cap \mathrm{Per}(f)$ is infinite.

Conclusion: $|a| = 1$?

- Step 1: describe the distribution of periodic point on V to get $\mu_V^+ = \mu_V^-$.
- Step 2: exploit the equality of measures and use a renormalization argument to conclude.

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The Green function

$f(x, y) = (x_1, y_1) = (ay, x + y^2 + c)$,
$\|(x, y)\| = \max\{|x|, |y|\}$

- if $|y| \geq |x| \geq R \gg 1$, then $|y_1| = |y|^2 \geq |y| = |x_1| \geq R$.
- $\frac{1}{2^n} \log \max\{\, 1, \|f^n(x, y)\|\}$ converges when $n \to +\infty$ uniformly to a Green function G^+

Properties:

- $G^+ \geq 0$, G^+ is continuous;
- $G^+ \circ f = 2G^+$
- $\{G^+ = 0\} = \{\, (x, y), \sup_{n \geq 0} \|f^n(x, y)\| < +\infty\}$

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Distribution of periodic points

Theorem

Suppose $z_n \in V$ is a sequence of distinct periodic points. Then

$$\frac{1}{\deg(z_n)} \sum_{w \text{Galois conj. to } z_n} \delta_w \longrightarrow \mu_V^+ := c_+ \, \Delta(G^+|_V) \, .$$

Corollary

$$G^+|_V = c \, G^-|_V$$

Proof

- Build G_p^+ over any $\mathbb{C}_p$ for any prime;
- Sum them up to get a height:

$$h(z) := \frac{1}{\deg(z)} \sum_{w \text{ Galois conj. to } z} \sum G_p^+(z) \; .$$

- $h(z) = 0$ when z is periodic
- The height function $h|_V$ is a good height: one can apply Autissier' result to conclude.

The renormalization argument

Assumptions:

- $z \in V_{\text{reg}}$ hyperbolic fixed point;
- $W^u_{\text{loc}}(z)$ and $W^s_{\text{loc}}(z)$ cut V transversally

$$df(z) = \begin{bmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{bmatrix}, \ |\lambda^+| > 1 > |\lambda^-|$$

Main idea: compute the Hölder exponent $\kappa^\pm$ of $G^\pm|_V$ near z.

- Transversality implies $G^+|_V$ and $G^+|_{W^u_{\text{loc}}}$ have the same exponent
- Linearization: $f|_{W^u_{\text{loc}}(z)}(t) = \lambda^+ t$
- $G^+(t) \asymp |t|^{\kappa^+}$
- $G^+ \circ f(t) = 2G^+ \Longrightarrow 2 = |\lambda^+|^{\kappa^+}$