# Algebraic dynamics of polynomial maps: degree growth

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# The general setup

$X$ is an algebraic variety defined over a field $k$

- $f : X \to X$ is a regular (dominant) map
- $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$

Ask questions of algebraic nature on this dynamical system. Recent sport motivated by:

- the study of holomorphic dynamical systems in arbitrary dimensions;
- the arithmetic of torsion points on abelian varieties (these are preperiodic points for the doubling map).

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# The restricted setup



Focus on (dominant) polynomial maps

$$f(x,y) = (P(x,y), Q(x,y)) : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}} .$$

- This is a non-trivial class of examples: Hénon maps

  $$(x,y) \mapsto (ay, x + P(y))$$

  have been studied in depth (over $\mathbb{C}$ and $\mathbb{R}$), and their dynamics is complicated (positive entropy).
- It is easier to deal with than arbitrary maps: small dimension, simple geometry.

# The program

1. Construction of projective compactifications adapted to the dynamics (Favre-Jonsson).
2. The dynamical Mordell-Lang conjecture (Xie).
3. The dynamical Manin-Mumford problem (Dujardin-Favre).

# Degree growth



- $\deg(f) = \max\{\deg(P), \deg(Q)\} \in \mathbb{N}^*$;

## Problem

*Describe the sequence* $\deg(f^n)$*:*

- *give an asymptotic;*
- *compute all degrees.*

Motivation: in $(\mathbb{P}^2, \omega_{FS})$ the entropy is bounded by

$$h_{top}(f) \overset{\text{Gromov}}{\leq} \sup_C \limsup_n \frac{1}{n} \log \mathrm{vol}(f^{-n}(C)) = \max\left\{ e(f), \limsup_n \frac{1}{n} \log \deg(f^n) \right\}.$$

$e(f) = \# f^{-1}\{p\} =$ topological degree of $f$.

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**Proof.**

If $f = (P, Q)$, $g = (R, S)$, then we have
$f \circ g = (P(R, S), Q(R, S))$. □

# Basics on degrees



- $\deg(f \circ g) \leq \deg(f) \times \deg(g)$;

Invariance under conjugacy

- if $g = h^{-1} \circ f \circ h$, for some $h \in \mathrm{Aut}[\mathbb{A}^2_k]$ then

$$0 < \frac{1}{C} \leq \frac{\deg(g^n)}{\deg(f^n)} \leq C < \infty \ .$$

Dynamical degree

- The limit $\lambda(f) := \lim_n \deg(f^n)^{1/n}$ exists.

Upper bound

- By Bezout $e(f) \leq \lambda(f)^2$.

# Some examples: automorphisms



By Jung and Friedland-Milnor any $f \in \mathrm{Aut}[\mathbb{A}^2_{\mathbb{C}}]$ is conjugated to

- affine map or elementary map

$$f(x, y) = (ax + b, cy + P(x))$$

  in which case $\deg(f^n) \leq \deg(f)$ for all $n$.

- Hénon-like map $f = h_1 \circ \cdots \circ h_k$ with

$$h_i = (a_i y, x + P_i(y))$$

  $d_i := \deg(P_i) \geq 2$, in which case $\deg(f^n) = \deg(f)^n = (\prod_i d_i)^n$ for all $n$.

hence $\lambda(f)$ is an integer.

# Some examples: monomial maps



- if $f(z) = f(x, y) = (x^a y^b, x^c y^d) = z^M$ with

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \; ad \neq bc, \; a, b, c, d \in \mathbb{N}$$

  then $f^n(z) = z^{M^n}$, and $\lambda(f)$ is the spectral radius of $M$.

hence $\lambda(f)$ is a quadratic integer.

# Algebraic stability



- There is a simple geometric condition under which $\deg(f^n)$ can be controlled (Fornaess-Sibony).

### Definition

*A rational map $f : X \dashrightarrow X$ is algebraically stable iff for any irreducible curve $E \subset X$, the image variety $\check{f}^n(E)$ is not a point of indeterminacy for any $n \geq 1$.*

### Definition

*A projective surface $X \supset \mathbb{A}^2_{\mathbb{C}}$ is a good dynamical compactification for $f$ if the (rational) extension of $f$ to $X$ is algebraically stable.*

# Algebraic stability: examples and consequences



- Affine map and Hénon-like maps are alg. stable in $\mathbb{P}^2$;
- an elementary map $(x, y + P(x))$ is alg. stable in a suitable Hirzebruch surface;
- a monomial map is alg. stable in a suitable product of weighted projective lines.

**Fact**

*When $f$ is alg. stable in $X$, then $(f^{n+m})^* = (f^n)^* \circ (f^m)^*$ for the natural actions of $f^n$ on the (real) Neron-Severi space of $X$.*

- *$\lambda(f)$ is an algebraic integer;*
- *$\sum_{n \geq 0} \deg(f^n) T^n \in \mathbb{Z}(T)$ (if $X$ dominates $\mathbb{P}^2$)*

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# Main results



**Theorem**

*Any polynomial map of $\mathbb{A}^2_k$ admits an iterate for which there exists a good dynamical compactification $X \supset \mathbb{A}^2_k$.*

# Main results: precise form



## Theorem

*When $e(f) < \lambda(f)^2$, one can choose $X$ s.t.*

1. $H_\infty := X \setminus \mathbb{A}^2_k$ *is irreducible and not contracted by* $f$;
2. $H_\infty$ *is irreducible and contracted to a smooth point of* $X$ *that is fixed by* $f^N$, $N \gg 1$;
3. $H_\infty$ *has two components intersecting transversally at a fixed point that are contracted to that point by* $f^N$.

## Corollary

*For any polynomial map of $\mathbb{A}^2_k$, the real number $\lambda(f)$ is a quadratic integer.*

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*For any polynomial map of $\mathbb{A}^2_k$, the real number $\lambda(f)$ is a quadratic integer.*

# First tentative



Optimistic hope:

- find $X = \mathbb{A}^2_{\mathbb{C}} \sqcup E$ with $E$ irreducible and $\check{f}(E) = E$;
- if $E$ exists, the divisorial valuation $\mathrm{ord}_E : \mathbb{C}[x, y] \to \mathbb{Z}$ is $f_*$-invariant in the sense

$$f_*(\mathrm{ord}_E)(P) := \mathrm{ord}_E(P \circ f) = \lambda(f)\, \mathrm{ord}_E(P) \ .$$

Difficulties.

- How to find a fixed point for the projective action of $f_*$ on divisorial valuations?
- If a divisorial valuation $\nu$ is fixed, is it possible to compactify $\mathbb{A}^2_{\mathbb{C}}$ by adding one irreducible component $E$ at infinity such that $\nu = \mathrm{ord}_E$?

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# Identify good valuations



## Definition

*A good divisorial valuation is a one proportional to* $\mathrm{ord}_E$ *where* $\mathbb{A}^2_k \sqcup E$ *is a compactification.*

- $X = \mathbb{A}^2_k \sqcup D$, with $D = E_1 \cup \cdots \cup E_r$, and $\nu_i = \mathrm{ord}_{E_i}$.
- Dual divisor: $\check{E}_i \cdot E_j := \delta_{ij}$

## Fact

$\nu_i$ *is good iff* $\check{E}_i \cdot \check{E}_i > 0$.

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## Theorem

$\nu_i$ *is good* $\Leftrightarrow \check{E}_i \cdot \check{E}_i > 0 \Leftrightarrow \check{E}_i$ *is nef and big*

## Remark

$\check{E}_i \cdot \check{E}_i$ *only depends on* $\nu_i$ *not on the choice of a model*

# The space of good valuations I



## Definition

*Let $\mathcal{V}_1$ be the space of good divisorial valuations on $\mathbb{C}[x, y]$, i.e. of the form $t\,\mathrm{ord}_E$ with $t > 0$ and $E$ is a component at infinity in some compactification such that $\check{E} \cdot \check{E} > 0$.*

## Remark

*A valuation $\nu \in \mathcal{V}_1$ is close to $-\deg$ since $\nu(P) < 0$ for all non constant polynomials.*

# The space of good valuations II

To get a space amenable to a fixed point theorem:

## Definition

*Let $\mathcal{V}_2$ be the closure of $\mathcal{V}_1$ in the space of all (non-trivial) valuations $\nu : \mathbb{C}[x, y] \to \mathbb{R}_-$.*

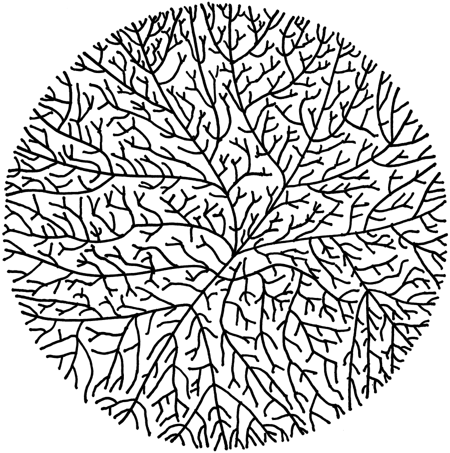
## Theorem

*The space $\mathcal{V}_2$ is a cone over*

$$\mathcal{V}_2' := \{\nu \in \mathcal{V}_2, \min\{\nu(x), \nu(y)\} = -1\} ,$$

*and $\mathcal{V}_2'$ is a compact $\mathbb{R}$-tree.*

# A tree dream



# The space of good valuations III



For technical reason, and get a better description of the end points of the tree:

## Definition

*Let $\mathcal{V}_3$ be the closure of the set of good divisorial valuations $t\,\mathrm{ord}_E$ such that*

$$A(t\,\mathrm{ord}_E) := t\,(1 + \mathrm{ord}_E(dx \wedge dy)) < 0\ .$$

## Theorem

*The space $\mathcal{V}_3$ is a cone over*

$$\mathcal{V}_3' := \{\ \nu \in \mathcal{V}_3,\ \min\{\nu(x), \nu(y)\} = -1\}\ ,$$

*and $\mathcal{V}_3'$ is an $\mathbb{R}$-tree whose divisorial end points are either good or associated to a rational pencil.*

# Existence of the fixed point



## Theorem

*A polynomial map induces a natural continuous map $f_\bullet$ on the $\mathbb{R}$-tree $\mathcal{V}_3'$.*
*This map admits a fixed point which attracts all good divisorial valuations when $e(f) < \lambda(f)^2$.*

- Invariance of $\mathcal{V}_3'$ is by invariance of nef divisors and the jacobian formula.
- Existence of the fixed point follows from a tracking argument.
- The attraction property is deeper: $\frac{1}{\sqrt{e(f)}} f^*$ is an isometry on the hyperbolic space $\varinjlim_X \mathrm{NS}_{\mathbb{R}}(X)$.

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# Construction of the compactification



If the invariant valuation $\nu$ is

- divisorial $\mathrm{ord}_E$: either it is good (pick $\mathbb{A}^2_k \sqcup E$) or associated to an rational invariant fibration (pick a suitable Hirzebruch surface);
- not divisorial: allows to construct by induction a sequence of blow ups $X_{n+1} \to X_n \to \mathbb{P}^2$, and $f^N$ is alg. stable in $X_n$ for some $n, N \gg 1$.