Dynamical system on valuation space

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Sevilla, December 2005.

Remember yesterday

- $P : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial, dominant.
- $d_n = \deg(P^n)$, $d_\infty = \lim_n d_n^{1/n}$

Theorem

- *Either* $P = (Q(x), R(x, y))$ *is a skew product;*
- *Or* $d_\infty^n \leq d_n \leq C \cdot d_\infty^n$

d_∞ *is a quadratic integer.*

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Method

- $\mathcal{V}_1 = \overline{\{\nu : \mathbb{C}[x, y] \to \mathbb{R} \text{ centered at } \infty,\ \nu(\phi) < 0,\ A(\nu) < 0\}}$
- $P_*\nu(\phi) = \nu(\phi \circ P)$.

Study the dynamics of $P_* : \mathcal{V}_1 \to \mathcal{V}_1$

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Key results

Theorem

$\mathcal{V}_1$ is a tree

Theorem (Eigenvaluation)

$P_*\nu = \lambda\nu$ for some $\nu \in \mathcal{V}_1$

Theorem (Structure of valuations in $\mathcal{V}_1$)

Suppose $\nu \in \mathcal{V}_1$

- *Either $C_1(-\deg) \leq \nu \leq C_2(-\deg)$*
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Valuations

Definition

Valuation $\nu : \mathbb{C}[x, y] \setminus \{0\} \to \mathbb{R}$

- $\nu|_{\mathbb{C}^*} \equiv 0$;
- $\nu(\phi_1\phi_2) = \nu(\phi_1) + \nu(\phi_2)$;
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- ν centered at infinity.

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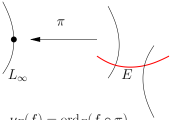
- $\min\{\nu(X), \nu(Y)\} < 0$

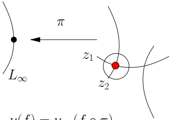
Examples

Divisorial valuation



$\nu_E(f) = \mathrm{ord}_E(f \circ \pi)$

Quasimonomial valuation



$\nu(f) = \nu_{z,s}(f \circ \pi)$

- $-\deg$
- Monomial valuation. $\nu_s(\sum a_{ij} x^i y^j) = \min\{is_1 + js_2,\ a_{ij} \neq 0\}$
- Divisorial valuation
- Quasimonomial or Abhyankhar valuations.
- Zariski or infinitely singular valuations

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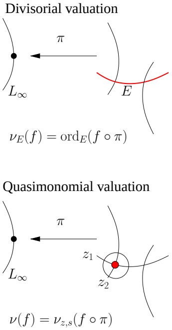

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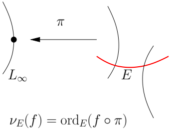

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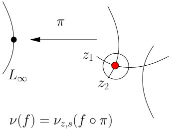

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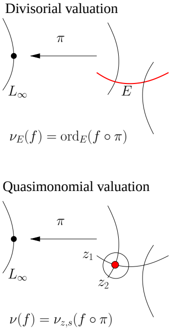

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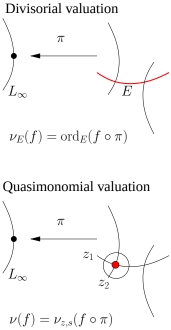

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Pencil valuations

- C with one place at infinity $C = P^{-1}(0)$
- Moh $\Rightarrow C_\lambda = P^{-1}(\lambda)$ has one place at infinity.
- $\nu_C(Q) = -\frac{(C \cdot Q^{-1}(0))_{\mathbb{C}^2}}{\deg(C)}$ curve valuation
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- $\mathcal{V} = \overline{\{\text{ normalized valuations } \min\{\nu(x), \nu(y)\} = -1\}}$
- Order relation $\nu \leq \mu \Leftrightarrow \forall \phi, \nu(\phi) \leq \mu(\phi)$
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$(\mathcal{V}, \leq)$ *is a tree:*

- $-\deg$ *is the unique minimal element;*
- $(\{-\deg \leq \cdot \leq \nu\}, \leq) \simeq ([0,1], \leq)$

Quasimonomial segments: $\{\pi_* \nu_{(s_1,s_2)}$ s.t. $a_1 s_1 + a_2 s_2 = -1\}$.

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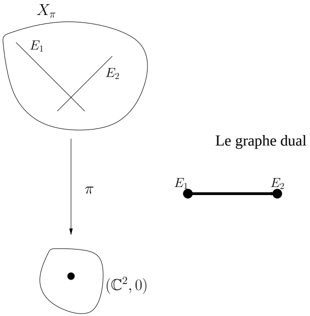

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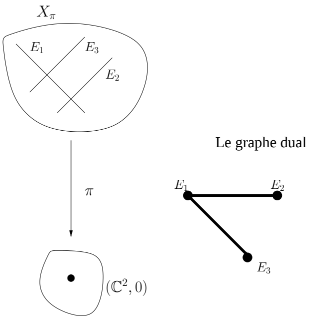

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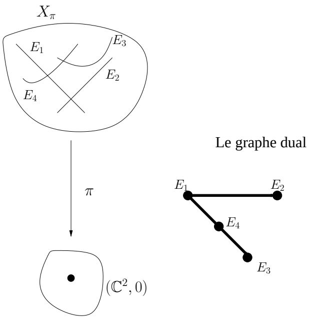

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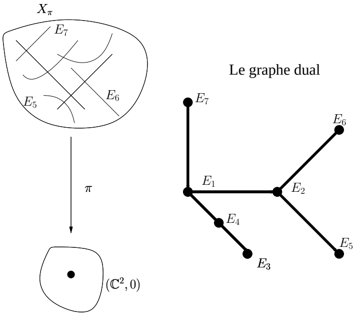

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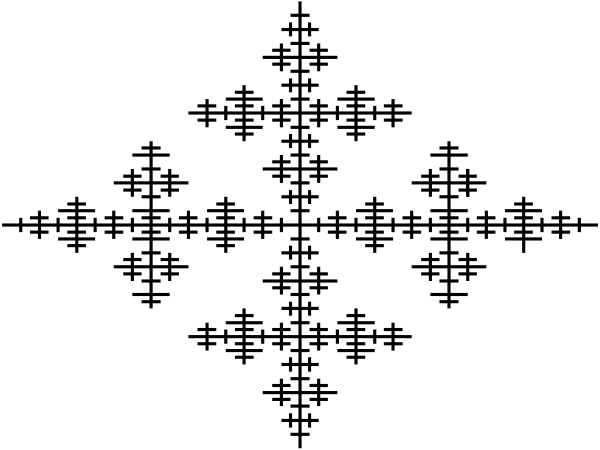

Idea

Ex. $P(x, y) = (x, xy)$, $P_*\nu_{s,t} = \nu_{s,s+t}$ hence $P_*\mathcal{V} \not\subset \mathcal{V}$

- Valuations are local object
- Some carry global informations:
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  - pencil valuation

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A new valuation space

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$$\mathcal{V}_1 = \overline{\{\nu \in \mathcal{V},\ \nu(\phi) < 0\ \forall\phi,\ \text{and } A(\nu) < 0\}}$$

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$\mathcal{V}_1$ is a closed subtree of $\mathcal{V}$.

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Theorem

For all $\nu \in \mathcal{V}$, $A : [-\deg, \nu] \to [-2, A(\nu)]$ is a bijection

Proof

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Proof.

- $\nu \mapsto \nu(\phi)$ is increasing
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Idea of proof. $p =$ center of ν.

- $\nu \rightsquigarrow \{P_k\}$ key polynomials
  - Monomial $\rightsquigarrow \{X, Y\}$
  - Quasim. $\rightsquigarrow \{X, Y, X^p + \theta Y^q, \cdots, P_N\}$
- $P_k \in \mathbb{C}[x, y]$ irreducible, analytically irreducible at p

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Basics

- $P_*\nu(\phi) = \nu(\phi \circ P)$
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- $P_*\nu = d(P,\nu) \times P_\bullet \nu$
  - $P_\bullet : \mathcal{V}_1 \to \mathcal{V}_1$ continuous
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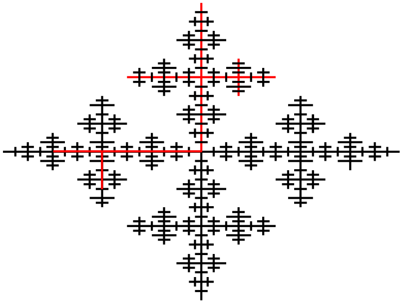

Fixed pt thm

Theorem (Eigenvaluation)

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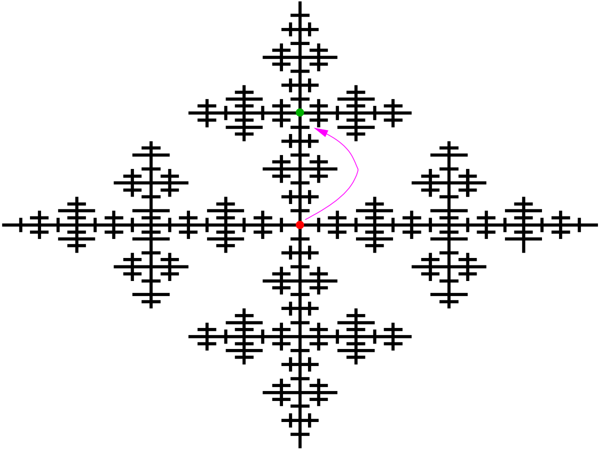

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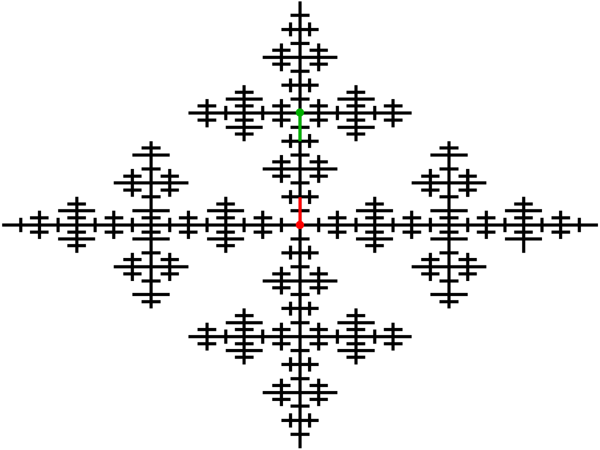

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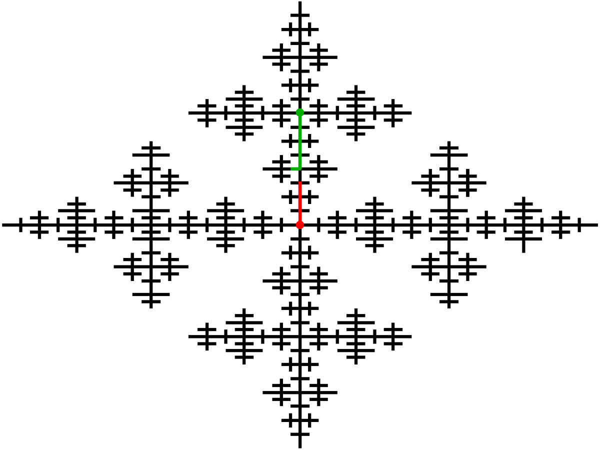

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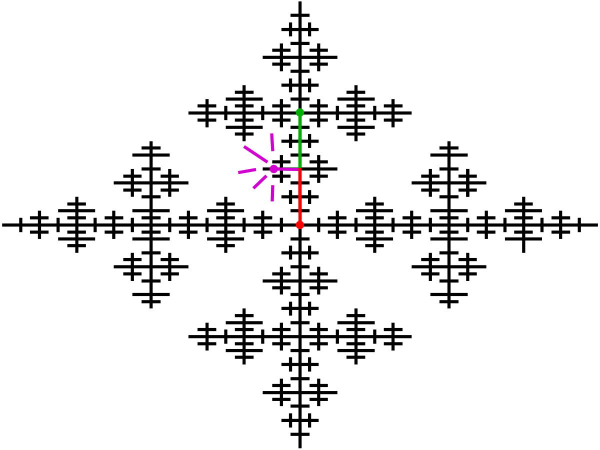

Summary

- $P_*\nu = \lambda\nu$
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  - $d_\infty = \lambda$

Why d_∞ is a quadratic integer?

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There exists $U \ni \nu_\star$ such that $P_\bullet^n \nu \to \nu_\star$ for all $\nu \in U$.

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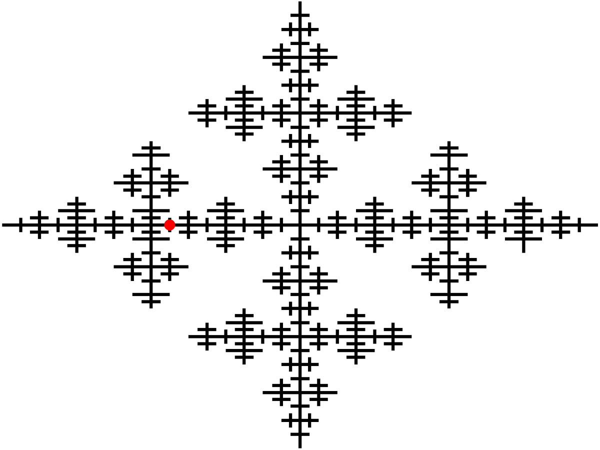

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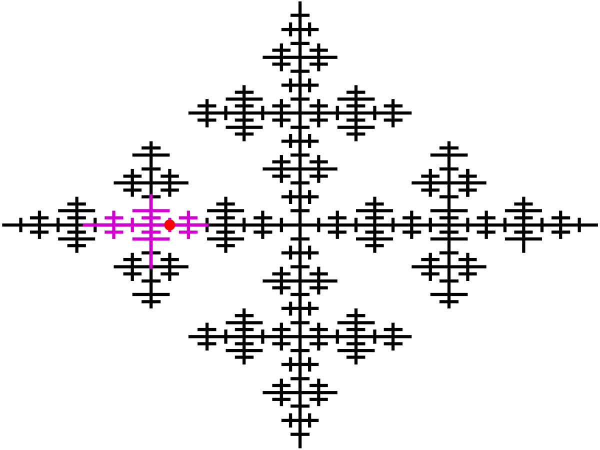

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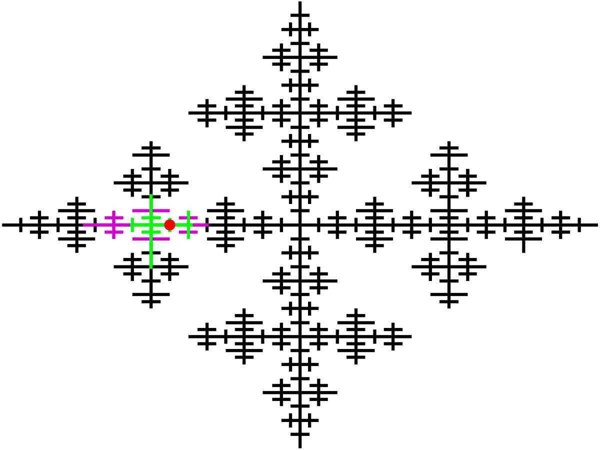

Idea of proof

For a suitable parameterization on $\mathcal{V}_1$

$$P_\bullet : \alpha \mapsto \frac{a\alpha + b}{c\alpha + d}$$

with $a, b, c, d \in \mathbb{N}$.

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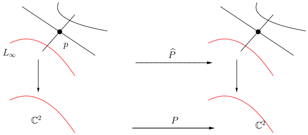
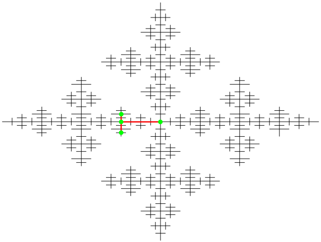
Consequences in picture





Towards monomialization

Theorem

There exists $\pi : X \to \mathbb{P}^2$ and $p \in \pi^{-1}(L_\infty)$ such that

- *P is holomorphic at p*
- *Critical set of P is included in $\pi^{-1}(L_\infty)$ and contracted to p.*

Normal form for (P, p)

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Rigid germs

Definition

$f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is rigid if $\bigcup_n \mathrm{Crit}\,(f^n)$ has normal crossing singularities.

Theorem (Favre 2000)

Suppose f rigid and not invertible. Then $f =$

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- $(\alpha z, w^p)$
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Conclusion

- When $\nu_\star$ is divisorial or an end point then $d(P, \nu_\star)$ is an integer
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  (P, p) is locally monomial
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