(10) The affine Berkovich line

$(K, |\cdot|)$ alg. closed complete metrized field $\left\{ \begin{array}{l} \text{non trivially valued} \\ \text{non archimedean} \end{array} \right.$

$\mathbb{A}^{1,an}_K$ = Berkovich affine line $\overset{def}{=}$ $\{$ multiplicative semi-norms on $K[T]$ whose restriction to K is equal to $|\cdot|$ $\}$ $= \{ \varphi : K[T] \to \mathbb{R}_+ \quad \varphi(a) = |a| \ \forall a \in K$
$\varphi(PQ) = \varphi(P)\varphi(Q) \qquad \varphi(P+Q) \leq \max\{\varphi(P), \varphi(Q)\} \}$.

<u>Example</u> $\quad z \in K \qquad P \longmapsto |P(z)|$ is an element of $\mathbb{A}^{1,an}_K$

such points are called rigid or type 1 points.

<u>Notation</u> $\quad x \in \mathbb{A}^{1,an}_K \qquad P \in K[T]$

write $|P|_x$ or $|P(x)|$ or $|P|(x)$

confusing when x is rigid!

Topology on $\mathbb{A}^{1,an}_K$ = pointwise convergence.

basis of open sets given by $\displaystyle\bigcap_{j=1}^{k} \{x, \ |P_j|(x) \in I_j\}$

I_j = ~~open~~ intervals in $[0, +\infty[$
$P_j \in K[T]$

<u>Prop.</u> $\mathbb{A}^{1,an}_K$ is locally compact.

<u>proof</u>. by definition $\mathbb{A}^{1,an}_K \subseteq \displaystyle\prod_{P \in K[T]} [0, +\infty[\ = \mathbb{R}^{K[T]}$

- closed subset when $\mathbb{R}^{K[T]}$ endowed with the product topology.

- $r > 0 \quad \overline{E}(r) = \{ x \in \mathbb{A}^{1,an}_K, \ |T|(x) \leq r \}$ "looks like a closed ball".

$E(r) = \{ \qquad < r \}$

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$P = \sum a_i T^i \qquad x \in \overline{E}(r) \Rightarrow |P(x)| \leq \max |a_i| r^i$

$$\overline{E}(r) \subseteq \prod_{P = \sum a_i T^i} [0, \max |a_i| r^i]$$

$\rightsquigarrow$ Tychonoff $\overline{E}(r)$ is compact.

$\rightsquigarrow$ $E(r)$ is open $\Rightarrow$ any point admits a basis of nghd that are relatively compact ///

Remark + $\overline{E}(r)$ is also sequentially compact.

exercice when K has a countable dense subset. (e.g. $K = \mathbb{C}_p$)

harder in general

+ $\mathbb{A}^{1,an}_K$ is not metrizable at least when $\tilde{K}$ is uncountable

(see Baker-Rumely's book §1.5)

The space $\mathbb{A}^{1,an}_K$ is close to the space of balls in K !

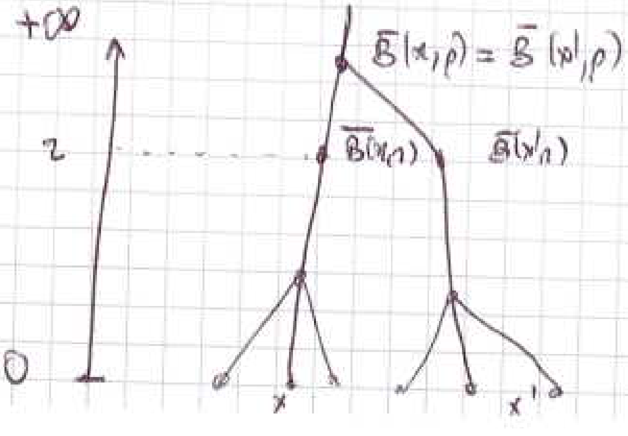
recall = closed ball is $\overline{B}(x,r) = \{ z \in K,\ |x-z| \leq r \}$.

$\rightarrow y \in \overline{B}(x,r) \Rightarrow \overline{B}(x,r) = \overline{B}(y,r)$

$\rightarrow \overline{B}(x,r)\quad \overline{B}(x',r')$ 2 balls with $r \leq r'$

$$\begin{cases} \text{either} & \overline{B}(x,r) \cap \overline{B}(x',r') = \emptyset \\ \text{or} & \overline{B}(x,r) \subseteq \overline{B}(x',r') \end{cases}$$

Picture for the set of closed balls



$\rho = |x - x'|$.

Obs : $r \mapsto x_{\overline{B}(x,r)}$ is continuous !

(follows from next proposition)

K.

34/ $\bar{B} = \bar{B}(x,r)$ a closed ball

ordr $|P|_{\bar{B}} = \sup_{z\in\bar{B}} |P(z)|$

lemma $|\cdot|_{\bar{B}} \in \mathbb{A}^{1,an}_K$

$|\cdot|_{\bar{B}} = |\cdot|_{\bar{B}'} \iff \bar{B} = \bar{B}'$

obs. the set of closed balls is naturally included in $\mathbb{A}^{1,an}_K$

proof. ① only need to check the multiplicativity WLOG $x=0$.
$|P+Q|_{\bar{B}} \le \max |P|_{\bar{B}}, |Q|_{\bar{B}}$

$r>0$ real

if $r \notin |K^\times|$ $P = \sum a_i T^i$ $|a_i T^i|_{\bar{B}} = |a_i| r^i$ (since K alg closed & nontrivial $|K^\times|$ = dense in $\mathbb{R}$)

$\exists$ unique i_0 $|a_{i_0}| r^{i_0} > \max_{i\neq i_0} |a_i| r^i$

$Q = \sum b_j T^j$ idem $\leadsto j_0$

$$PQ = (a_{i_0} T^{i_0})(b_{j_0} T^{j_0}) + P \underbrace{\sum_{j\neq j_0} b_j T^j}_{} + Q \sum_{i\neq i_0} a_i T^i$$

$\wedge$ $|a_{i_0}| r^{i_0} \max_{j\neq j_0} |b_j| r^j$ $\wedge$ $|b_{j_0}| r^{j_0} \max_{i\neq i_0} |a_i| r^i$

$\sup = |a_{i_0} r^{i_0}| \, |b_{j_0} r^{j_0}|$

$\Rightarrow |PQ|_{\bar{B}} = |P|_{\bar{B}} |Q|_{\bar{B}}$

if $r \in |K^\times|$ the argument does not apply but there exists $y \in K^\times$
$r = |y|$.

claim $P \in K[T]$ $\exists\, \mathcal{E}_P \subseteq \tilde{K}$ finite such that
$$|P(yz)| = |P|_{\bar{B}} \quad \text{for all } z \text{ st } \tilde{z} \notin \mathcal{E}_P.$$

$\leadsto$ implies multiplicativity $z \in K$ $\tilde{z} \notin \mathcal{E}_P \cup \mathcal{E}_Q$ $|PQ(yz)| = |P|_{\bar{B}} |Q|_{\bar{B}}$

proof of claim $P = \sum a_i T^i$ $R = \max |a_i| r^i$ $I = \{i, |a_i| r^i = R\}$
$P = \sum_I + \sum_{\bar{I}}$

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$$\underline{P}(y z) = \sum_I (a_i y^i) \underset{\underline{P}_I}{\underset{"}{z^i}} + \sum_J (a_j y^j) \underset{\underline{P}_J}{\underset{"}{z^i}}.$$

Pick $y_1 \in K^\times$ $|y_1| = R$.

$$\underset{I}{\tilde{\underline{P}}}(T) \overset{\text{def}}{=} \sum_I^{'} \left(\frac{\overline{a_i y^i}}{y_1}\right) T^i \in \tilde{K}[T]$$

$\tilde{\mathcal{E}}_P =$ zeroes of $\tilde{P}_I$.

if $\tilde{z}$ is not a root of $\tilde{P}$

$$\tilde{P}_I(\tilde{z}) = \overbrace{\left[\frac{1}{y_1} P_I(z)\right]} \neq 0 \Rightarrow \left|\frac{P_I(z)}{y_1}\right| = 1$$

ie $|P_I(z)| = R > |P_J(z)|$

② $\sup |T-\alpha| = \max\{|x-\alpha|, r\}$

$\overline{B}(x,r)$ $\quad \Sigma$ arguments of ①.

$$|\cdot|_{\overline{B}(x,r)} = |\cdot|_{\overline{B}(x',r')} \Rightarrow \forall \alpha \quad \max\{|x-\alpha|, r\} = \max\{|x'-\alpha|, r'\}$$

$\underline{\alpha = x}$ $\quad r = \max\{|x'-x|, r'\}$. $\quad |x'-x| \leq r$ (hence $\leq r'$)

hence may choose $x = x'$.

$\Rightarrow r = r'$ since $|K^\times|$ is dense $\quad$ ///.

<u>**Thm**</u> For any $x \in A^{1,an}_K$ there exists a family of closed Balls $\overline{B}_n$ such that $\overline{B}_{n+1} \subseteq \overline{B}_n$ and

$$|f|_x = \inf_n |f|_{\overline{B}_n}.$$

Observation = if $\operatorname{diam}(\overline{B}_n) \to 0$ by non-arch and completeness

$\bigcap \overline{B}_n = \{x\}$ for some $x \in K$ and $|f|_x$ is rigid

otherwise either $\bigcap \overline{B}_n$ is a closed ball or is empty

The latter case may appear e.g. in $\mathbb{C}_p$ (see $\quad$)

<u>Notation</u> type(1) $\Leftrightarrow$ $\overline{B}(x,r)$ $r=0$ $\quad$ type(2) $\Leftrightarrow$ $\overline{B}(x,r)$ $r \in |K^\times|$ $\quad$ type(3) $\Leftrightarrow$ $\overline{B}(x,r)$ $r \notin |K^\times|$ $\quad$ type(4) $\Leftrightarrow$ other

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<u>proof of the theorem</u> (Berkovich)

$x \in \mathbb{A}^{1,an}_K \qquad \overline{B}_x(a) \overset{def}{=} \overline{B}(a, |T-a|(x)) \qquad a \in K.$

<u>obs</u> if $|T-a|(x) \le |T-b|(x)$ then $|a-b| = |(T-a)-(T-b)|(x) \le |T-b|(x)$

in particular $\overline{B}_b = \overline{B}(a, |T-b|(x))$

and $\overline{B}_a \subseteq \overline{B}_b$

$t \in I = \{ |T-a|(x),\ a \in K \}$ set $\overline{B}(t) \overset{def}{=} \overline{B}_a$ for any $a \in K$, $|T-a|(x) = t$

$t \mapsto \overline{B}(t)$ increasing.

<u>claim</u> $\forall a \in K \qquad |T-a|(x) = \inf_{t \in I} |T-a|_{\overline{B}(t)}$

- $t = |T-a|(x) \qquad |T-a|_{\overline{B}(t)} = \sup_{\overline{B}(a, |T-a|(x))} |T-a| = |T-a|(x).$

- $t' < t \qquad \overline{B}(t') = \overline{B}(a', |T-a'|(x)) \quad |T-a'|(x) < |T-a|(x)$

$\Rightarrow \quad |a-a'| = |T-a|(x)$

$\sup_{\overline{B}(t')} |T-a| \ge |T-a|(x)$ end of claim!

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<u>Notation</u> $\overline{B}$ closed ball $\quad x_{\overline{B}}$ = point associated to $\overline{B}$

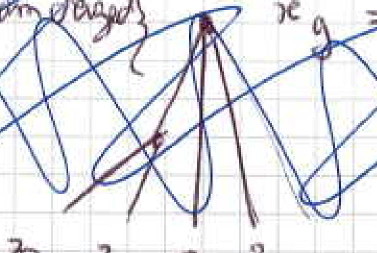
<u>Consequences</u> (heuristic picture)

$\mathbb{A}^{1,an}_K$ is obtained from the tree of closed balls by adding some end points (those of type 4)

$\rightsquigarrow$ it remains an $\mathbb{R}$-tree

~~Branches parametrized by $\mathbb{P}^1$~~ ~~x_g = Gauss point = $x_{\overline{B}(0,1)}$~~

~~$|f(x_g)| = \max |a_i|$ for $\sum a_i T^i$~~



~~$\{|z_0|, |z_1|, |z_2|, |z_3|\} \le 1$~~

~~$\overline{z_0} = \overline{z_1} \neq \overline{z_2} \neq \overline{z_3}$~~

$z_0 \quad z_1 \quad z_2 \quad z_3 \quad \dots$

37/ ~~[illegible]~~

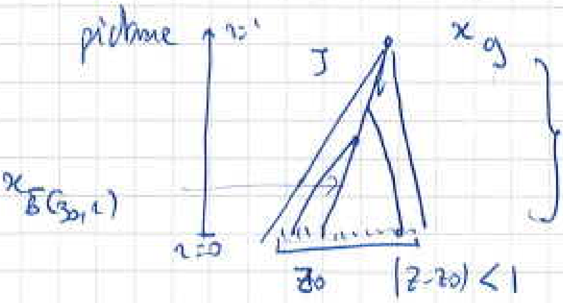
The geometry in a neighborhood of the Gauß point $x_g = x_{\overline{B}(0,1)}$

For any $\zeta \in \tilde{k}$ let $E(\zeta)$ be the closure in $\mathbb{A}^{1,an}_K$ of the set $z \in K^0$ whose reduction is equal to ζ. Pick $z_0 \in K^0$ $\bar{z}_0 = \zeta$.

$\to E(\zeta) \cap K = \{z, |z-z_0| < 1\}$

Terminology = $E(\zeta)$ is an open ball in the Berkovich space

picture



$E(\zeta)$ Prop connected $\mathbb{R}$-tree but not compact.

proof need to prove that for all $0 \le r < 1$ $x_{\overline{B}(z_0,r)} \in E(\zeta)$.

Fix $r_n \to r$ $r_n \in |K^\times|$ (possible to find such sequence since K is alg. closed).
$r_n < 1 \ \forall n$

for each n choose z_n s.t. $|z_n| = r_n$.

then $z_n \to x_{\overline{B}(z_0,r)}$.

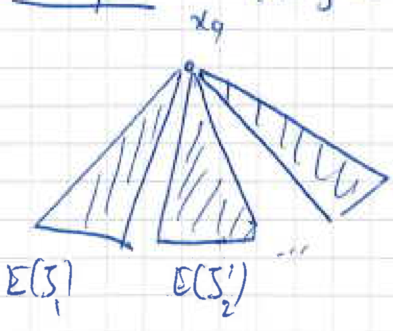
Only need to check that for any $\alpha \in K$

$$|z_n - \alpha| \longrightarrow |(T-\alpha)(x_{\overline{B}(z_0,r)})| = \max\{|z_0-\alpha|, r\}$$ //

Consequence $\{\zeta_i\} = \tilde{k}$

$$\overline{B(0,1)} = \{x_g\} \bigsqcup_{\zeta \in \tilde{k}} E(\zeta).$$

closure of k^0 in $\mathbb{A}^{1,an}_K$



Order relation = $x \le y$ in $\mathbb{A}^{1,an}_K \iff |f(x)| \le |f(y)|$ for all x,y.

~~[illegible]~~ obs: $x_B \le x_{B'} \iff B \subseteq B'$.

$\Leftarrow$ when $B \subseteq B'$ then $|f(x_B)| = \sup_B |f| \le \sup_{B'} |f| = |f(x_{B'})|$.

$\Rightarrow$ $B \not\subseteq B'$ nothing as if $B \subseteq B'$ or $B' \subseteq B$ we have $x_B \le x_{B'}$ or $x_{B'} \le x_B$

otherwise $B \cap B' = \emptyset$ and x_B & $x_{B'}$ are not comparable

$\longrightarrow$ TSVP

$B = \overline{B}(z, r)$ $\quad B' = \overline{B}(z', r')$ $\quad |z - z'| > \max\{r, r'\}$

$\ell_1 = T_z$ $\quad |\ell_1(x_B)| = r < |\ell_1(x_{B'})| = |z - z'|$

$\ell_2 = T_{z'}$ $\quad |\ell_2(x_{B'})| < |\ell_2(x_B)|$

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