COMPACTIFICATION OF SPACES OF REPRESENTATIONS AFTER CULLER, MORGAN AND SHALEN

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Introduction

This paper stems from notes of a course given at the "Ultrametric Dynamical Days" in Santiago de Chile. The purpose of this course was to explain the compactifications of spaces of representations, as this tool applies for questions in low-dimensional topology and in hyperbolic geometry.

The use of methods from algebraic geometry for studying spaces of representations originates in [10]. Marc Culler and Peter Shalen were motivated by questions from 3-dimensional topology, in particular by the *Property P* (if $K \subset S^3$ is a nontrivial knot and $K_{p/q}$ denotes the 3-manifold obtained by p/q-Dehn surgery along K, then the fundamental group $\pi_1(K_{p/q})$ is nontrivial when $q \neq 0$.) The theory of Culler and Shalen can be very briefly summarized as follows. Let G be the fundamental group of the complement of an hyperbolic knot. Let $\mathcal{R}(G)$ the set of representations of G into $\mathrm{SL}(2,\mathbb{C})$ and let $X(G)$ the quotient of $\mathcal{R}(G)$ by the action of $\mathrm{SL}(2,\mathbb{C})$ by conjugacy: both spaces are affine algebraic sets defined over $\mathbb{Q}$. A theorem of Thurston says that $X(G)$ contains an irreducible component which is an affine curve C. Let R be an irreducible component of $\mathcal{R}(G)$ above C. Let $\mathbb{Q}(C)$ and $\mathbb{Q}(R)$ denote the fields of rational functions on C and on R respectively. Any point at infinity of C can be interpreted as a valuation on $\mathbb{Q}(C)$. There is also a "tautological" representation of G into $\mathrm{SL}(2,\mathbb{Q}(R))$. A classical construction due to Serre associates to such a valuation v a simplicial tree with an action of $\mathrm{SL}(2,\mathbb{Q}(R))$, and therefore with an action of G. Transversality constructions permit then to deduce an incompressible surface in the knot complement. The surfaces obtained on that way give important topological informations on the knot. For instance one important achievement of this theory is the Cyclic Surgery Theorem of Culler et al. [8]: *if surgery on a nontrival knot produces a manifold with cyclic fundamental group, then the surgery slope is an integer.*

This interpretation of the points at infinity of C was extended by John Morgan and Peter Shalen when G is a general finitely generated discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ to define a compactification of $X(G)$ [17]. It led first to a new proof of the following Compactness Theorem of Thurston. Let G denote the fundamental group of an acylindrical hyperbolic 3-manifold with incompressible boundary (in terms of the limit set of a convex cocompact model of the manifold, these hypothesis simply mean that this limit set is connected and cannot be disconnected by removing two points). Denote by $\mathcal{DF}(G)$ the space of discrete and faithfull representations up to conjugacy of G into $\mathrm{SL}(2,\mathbb{C})$. *Then $\mathcal{DF}(G)$ is compact.* The approach of Morgan-Shalen of this fundamental result can be sketched as follows. They argue by contradiction and consider an irreducible component C of $X(G)$ such that $C \cap \mathcal{DF}(G)$ is not compact. To an unbounded sequence (x_i) in this intersection, they associate a valuation v on $\mathbb{Q}(C)$. Since C is not a curve in general, v is not necesarily discrete: its value group is a totally ordered abelian group Λ. The Serre construction can be adapted to this situation and produces *a Λ-tree $\mathcal{T}_v$*. When Λ is isomorphic to $\mathbb{Z}$, this is an equivalent notion to that of a simplical tree; when more generally Λ is archimedean, this determines a $\mathbb{R}$-tree. When Λ is not archimedean, it has always a non-trivial archimedean quotient and a non-trivial $\mathbb{R}$-tree $\mathcal{T}$ can always be determined from $\mathcal{T}_v$. The group G acts by isometries on $\mathcal{T}$ minimally and furthermore with the property that any subgroup which stabilizes a non degenerate segment is virtually abelian. The proof by Morgan-Shalen of Thurston's Compactness Theorem reduces then to the theorem from the domain of $\mathbb{R}$-trees that no action with property exists when G is the fundamental group of an hyperbolic 3-manifold which is boundary-incompressible and acylindrical. They proved this using tools from 3-dimensional topology and foliations. This theorem has been now widely generalized by Eliyahu Rips beyond the context of 3-manifolds [2] [25] under the hypothesis that G is not an amalgamated product over a virtually abelian subgroup. Morgan and Shalen applied also their theory to the case when G is the fundamental group of a closed surface S. Then some component of the space of real points of $X(S)$ can be identified with the Teichmüller space of S. Their construction gives a compactification where the points at infinity correspond to actions of $\pi_1(S)$ on $\mathbb{R}$-tree with the property that the subgroups which stabilize non-degenerated segments are cyclic. Such a tree is isometric to the dual tree to a measured lamination on S ([17], [16], [31]). Thus their theory offered a different perspective on Thurston's compactification of Teichmüller space by measured laminations.

The paper is organized as follows. In chapter 1 we describe, following Culler-Shalen, the explicit structure of an affine algebraic set on *the space of characters* of a group finitely generated group G. Then we describe the general construction of Morgan-Shalen of particular compactifications of affine algebraic sets.

In chapter II, we recall classical properties of valuations and we explain the general procedure of Morgan-Shalen for associating to any unbounded sequence of points on an affine algebraic variety X defined over $\mathbb{Q}$ a valuation on $\mathbb{Q}(X)$.

Chapter III presents certain basic properties of Λ-trees. These are metric spaces which share many common properties with $\mathbb{R}$-trees but the distance takes value in an abelian totally ordered group Λ. The basic examples are provided by the Bass-Serre tree of $\mathrm{SL}(2, F)$ when the field F is endowed with a valuation with value group Λ.

In Chapter IV, we explain how Morgan and Shalen interpret the valuation v on the $\mathbb{Q}(X)$ associated in chapter II to an unbounded sequence (x_i) on X, one component of the space of characters of a group G: the actions of G on $\mathbb{H}^3$ converge (in an appropriate sense) to an action by isometries of G on a Λ-tree (this tree is a subtree of the Bass-Serre tree of $\mathrm{SL}(2, F)$ with the valuation v). When the sequence (x_i) consists of characters of discrete representations of G, the action has the important property that any subgroup which stabilizes a non degenerate segment is "small", i.e. contains an abelian subgroup of finite index.

Chapter V exhibits geometric examples of Λ-trees: they arise from codimension 1 laminations of a closed manifold with a "Λ-valued transverse measure". We sketch the proof of the theorem of Skora that any minimal action of a surface group on an $\mathbb{R}$-tree such that the segment stabilizers are small is geometric.

The presentation given here follows closely the papers [10] and [17]. There are other approaches of the convergence of sequences of representations of a group to an action of the group on a tree: Betsvina [1] and Paulin [24] for a geometric proof of the convergence to an $\mathbb{R}$-tree in much borader context, [6] for a proof using non standard analysis of the convergence to a Λ-tree.

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Notations. Throughout this paper we follow the conventions of [12, Chapter 1]. We let $\mathbb{A}^n_{\mathbb{C}}$ be the standard complex affine space of dimension n. Its ring of regular functions is $\mathbb{C}[X_1, ..., X_n]$, and the set of its complex points is $\mathbb{C}^n$. An *affine variety* X is an *irreducible* algebraic subspace of some $\mathbb{A}^n_{\mathbb{C}}$. Its set of complex points $X_{\mathbb{C}} = X(\mathbb{C})$ is in bijection with the zero locus of a finite family of polynomials $P_1, ..., P_k \in \mathbb{C}[X_1, ..., X_n]$. When all polynomials have coefficients in a field $k \subset \mathbb{C}$, one says that X *is defined over* k.

1. The space of characters of a group

We first review the construction of the space of representations of a finitely generated group G into SL(2,$\mathbb{C}$) as an affine algebraic set. We follow Culler-Shalen for the presentation of *the space of characters of* G, which is an explicit model for the algebraic quotient of the space of representations and make then a link between the two presentations. We then describe the Morgan-Shalen compactification of a general affine variety.

1.1. The space of characters as an affine algebraic set.

Definitions. Let G be a finitely generated group generated by n elements.

Denote by $\mathcal{R}_{\mathbb{C}}(G)$ the set of representations (i.e. of group morphisms) of G into SL(2,$\mathbb{C}$). Each point in $\mathcal{R}_{\mathbb{C}}(G)$ is determined by its value on the elements of a generating family of G, that is by a point in $(\mathrm{SL}(2,\mathbb{C}))^n$ which is an algebraic subset of the affine space $\mathbb{A}^{4n}_{\mathbb{C}}$. The points in $\mathcal{R}_{\mathbb{C}}(G)$ hence naturally form an affine algebraic set $\mathcal{R}(G)$ defined by the vanishing of a family of polynomials with integer coefficients. In this way, $\mathcal{R}(G)$ can be identified with an affine algebraic set defined over $\mathbb{Q}$. This algebraic set does not depend on the particular choice of a generating family: different choices of generating systems lead to isomorphic algebraic sets.

In the sequel we shall be interested in the quotient of $\mathcal{R}(G)$ under the action of SL(2) by conjugacy, that is its quotient where one identifies any two representations ρ_1 and ρ_2 when $\rho_2 = M \circ \rho_1 \circ M^{-1}$ for $M \in \mathrm{SL}(2)$. This quotient space has a natural algebraic structure. We will first describe the explicit construction due to Culler and Shalen of the *space of characters*. Then we will indicate how this construction enters the general theory of algebraic quotients.

For each $g \in G$ the function $\mathcal{R}(G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}(\rho(g))$ is a regular function on $\mathcal{R}(G)$, that is an element of the ring $\mathbb{Q}[\mathcal{R}(G)]$.

Proposition 1. *The ring* $\mathbb{Q}[\mathrm{tr}(\rho(\mathrm{g})), \mathrm{g} \in \mathrm{G}]$ *is finitely generated.*

Proof. The proposition follows from the classical identity satisfied by all $A, B \in$ SL(2,$\mathbb{C}$) : $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B)$. □

Definition 2. *Choose a set of generators* (X_i) *of* $\mathbb{Q}[\mathrm{tr}(\rho(\mathrm{g})), \mathrm{g} \in \mathrm{G}]$, $X_i = \mathrm{tr}(\rho(\mathrm{g_i}))$, $i = 1, ...N$. *For any element* $g \in G$, *we denote* $\mathcal{T}_g$ *the polynomial in the variables* X_i *such that* $tr(\rho(g)) = \mathcal{T}_g(X_1, \cdots, X_N)$. *Consider the regular map* $t : \mathcal{R}(G) \to \mathbb{C}^N$, $\rho \mapsto t(\rho) = (X_i(\rho))$*: its image, denoted by* $X(G)$*, is the space of characters of* G.

The space of representations $\mathcal{R}(G)$ is clearly an affine algebraic set defined over $\mathbb{Q}$; the same property holds for the space of characters.

Proposition 3. [17] *The space* $X(G)$ *is an algebraic set defined over* $\mathbb{Q}$.

Before starting the proof we recall a few definitions.

Reducible representations. A representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ is *reducible* when there exists a 1-dimensional subspace of $\mathbb{C}^2$ that is invariant by any element of $\rho(G)$. This is equivalent to saying that the representation can be conjugated to take value in the group of upper-triangular matrices. Recall that the group of isometries of the hyperbolic space $\mathbb{H}^3$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$. In the model of the upper half-space, the *ideal boundary* of $\mathbb{H}^3$ is identified to $\mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$ with its natural conformal structure. On this boundary, $\mathrm{PSL}(2,\mathbb{C})$ acts by Möbius transformations. The group $\mathrm{SL}(2,\mathbb{C})$ also acts on $\mathbb{H}^3$ via the quotient map $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$. In terms of the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3$ a representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ is reducible if and only if $\rho(G)$ has a fixed point in $\mathbb{C}P^1$.

Lemma 4. *A representation $\rho \in \mathcal{R}(G)$ is reducible if and only if for any element g in the commutator subgroup $[G,G]$, one has $tr(\rho(g)) = 2$.*

Proof. Since a reducible representation is conjugated to a group of upper-triangular matrices, the trace of any element in $\rho([G,G])$ is equal to 2.

Conversely suppose that each element of $\rho([G,G])$ has trace equal to 2. Suppose also that this group is not reduced to the identity element Id (if not $\rho(G)$ would be abelian and would then leave a one-dimensional subspace). Let $h \in \rho([G,G])$, $h \neq \mathrm{Id}$; since $\mathrm{tr(h)} = 2$, h leaves invariant a unique point $p \in \mathbb{C}P^1$. If any element of $\rho([G,G])$ leaves p invariant, then p is also invariant by the entire group $\rho(G)$, since $\rho([G,G])$ is a normal subgroup. Thus we can suppose that some element k of $\rho(G)$ does not fix p. Then k and h are two parabolic elements which fix distinct points of $\mathbb{C}P^1$. The *Ping-pong Lemma* (see [26]) produces then an element of $< h, k >$ which is hyperbolic, and in particular has a trace $\neq 2$. □

Lemma 4 says that the set of the characters of reducible representations is the algebraic subset defined by the vanishing of the polynomials $\mathcal{T}_g - 2$ for $g \in [G,G]$.

The proof of Proposition 3 is based essentially on the following result of independent interest.

Proposition 5. *Let (ρ_i) be a sequence of representations in $\mathcal{R}_{\mathbb{C}}(G)$. Suppose that for any $g \in G$, the sequence $(\mathrm{tr}(\rho_i(g)))$ is bounded. Then one can conjugate ρ_i so that the sequence (ρ_i) is bounded, i.e. for all $g \in G$, $(\rho_i(g))$ stays in a compact set of* $\mathrm{SL}(2,\mathbb{C})$.

Proof. It is sufficient to deal with the following two different cases: either all representations are reducible or they are all irreducible.

— Suppose we are in the first case. By conjugating each ρ_i by a suitable matrix, we may also assume that all representations take their values in the subgroup of upper-triangular matrices. By assumption, for each element g of G, the diagonal terms of $\rho_i(g)$ are uniformly bounded. Conjugating ρ_i by a suitable diagonal matrix, one can also get a uniform bound on the upper-right term of $\rho_i(g)$ also. Applying this to a finite set of generators g_j of G, one concludes that up to conjugacy the family $(\rho_i(g_j))_{i,j}$ is bounded, thus proving Proposition 5 in this case.

—Suppose now that each representation ρ_i is irreducible. We proceed by induction on the number of generators of G and use the geometric action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3$.

Pick $g \in \mathrm{PSL}(2,\mathbb{C})$, $R \geq 0$, and denote by $C_R(g)$ the following subset of $\mathbb{H}^3$:

$$C_R(g) = \{x \in \mathbb{H}^3 |\, d(x, gx) \leq R\} \ .$$

Each set $C_R(g)$ is closed, convex and invariant under the normalizer of g in $\mathrm{PSL}(2,\mathbb{C})$. Therefore $C_R(g)$ can be described as follows when $g \neq \mathrm{Id}$. If g is hyperbolic $C_R(g)$ is a neighborhood of constant radius of the axis of g. In the model of the upper half space, and if the axis of g points towards infinity, then $C_R(g)$ is a circular cone based at the finite endpoint of the axis. When g is parabolic $C_R(g)$ is an horoball centered at the fixed point of g on $\partial\mathbb{H}^3$; when g is elliptic $C_R(g)$ is a neighborhood of constant radius of the (axis of) fixed points of g. We shall make use of the following observation.

Claim 6. *For $x \in \partial C_R(g)$, the geodesic segment $x.gx$ is contained in $C_R(g)$ and makes an angle with $\partial C_R(g)$ which tends to $\pi/2$ as $R \to \infty$; furthermore, this convergence is uniform in g, as long as the modulus of* $\mathrm{tr(g)}$ *is bounded from above.*

Suppose that G is generated by n elements $g_1, ..., g_n$. Saying that a sequence of representations $\rho_i : G \to \mathrm{SL}(2,\mathbb{C})$ can be conjugated inside $\mathrm{SL}(2,\mathbb{C})$ to become bounded is equivalent to the existence of a constant $R > 0$ such that the intersection of the neighborhoods $C_R(\rho_i(g_j))$ is non-empty: $C_R(\rho_i(g_1), ..., \rho_i(g_n)) = \cap_j C_R(\rho_i(g_j)) \neq \emptyset$. To see that, pick for each i a point p_i in this intersection and conjugate ρ_i by an element of $\mathrm{PSL}(2, \mathbb{C})$ mapping p_i to the origin in $\mathbb{H}^3$. Then all isometries $\rho_i(g_j)$ belong to a compact set of $\mathrm{PSL}(2, \mathbb{C})$.

We now observe that each set $C_R(g_1, \cdots, g_k)$ is convex since each $C_R(g_j)$ is. The boundary of $C_R(g_1, \cdots, g_k)$ is also contained in the union of the boundaries of the tubes $C_R(g_j)$. Suppose that Proposition 5 has been proven for all groups generated by $\leq n-1$ elements, and let G be a group generated by n elements $g_1, ..., g_n$. By the induction hypothesis there is an $R > 0$ such that $C_R(\rho_i(g_2), \cdots, \rho_i(g_n))$ and $C_R(\rho_i(g_1))$ are non-empty. Using Claim 6 and the fact that the traces are bounded, we may also choose R sufficiently large so that the angles between $x.\rho_i(g_j)x$ and $\partial C_R(\rho_i(g_j))$ are uniformly close to $\pi/2$ for all $x \in \partial C_R(\rho_i(g_j))$.

For any i, set $d_i = \inf\{d(x, y) | x \in C_R(\rho_i(g_1)), y \in C_R(\rho_i(g_2), \cdots, \rho_i(g_n))\}$. Let us first show that for all i this infimum is attained. Pick $x_k \in C_R(\rho_i(g_1))$ and $y_k \in C_R(\rho_i(g_2), \cdots, \rho_i(g_n))$ such that $d(x_k, y_k)$ tends to d_i. Suppose, by contradiction, that d_i is not a minimum. Then x_k and y_k tend to ∞ in $\mathbb{H}^3$. Up to extracting a subsequence we may assume that they converge in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ to the same limit x. Since $x_k \in C_R(\rho_i(g_1))$, $x = \lim_k x_k$ is fixed by $\rho_i(g_1)$. In the same way $y_k \in C_R(\rho_i(g_j))$ for $j \geq 2$, hence x is fixed by all $\rho_i(g_j)$'s for $j \geq 2$. We conclude that x is fixed by $\rho_i(G)$, hence the representation ρ_i is reducible, contradicting the hypothesis.

We may thus find points $e_i \in C_R(\rho_i(g_1))$ and $f_i \in C_R(\rho_i(g_2), \cdots, \rho_i(g_n))$ such that $d_i = d(e_i, f_i)$.

If the sequence (d_i) is bounded, say less than A, then $C_{R+A}(g_1, \cdots, g_n)$ is non-empty which proves Proposition 5 in that case.

Suppose that d_i tends to ∞. Denote by k_i the geodesic in $\mathbb{H}^3$ with endpoints e_i and f_i. Since the boundary $\partial C_R(\rho_i(g_1))$ is smooth, k_i is orthogonal to $\partial C_R(\rho_i(g_1))$ at e_i, by the first variation formula. The tangent vector to k_i at e_i is thus equal to the outward-pointing normal to $\partial C_R(\rho_i(g_1))$. Observe that the hyperplane orthogonal to k_i at f_i separates $\mathbb{H}^3$ into two half-spaces, one of which contains k_i, and the other contains $C_R(\rho_i(g_2), \cdots, \rho_i(g_n))$. This implies that the tangent vector to k_i at f_i makes an angle smaller than $\pi/2$ with the inward-pointing normal of $C_R(\rho_i(g_j))$'s for some $j \geq 2$, say of $C_R(\rho_i(g_2))$.

Consider the piecewise geodesic γ which is the concatenation of the four geodesics $\rho_i(g_1^{-1})e_i.e_i$, k_i, $f_i.\rho_i(g_2)f_i$ and $\rho_i(g_2)k_i$. The choice of R implies that for any $x \in \partial C_R(\rho_i(g_j))$, the geodesic $x.\rho_i(g_j)x$ is almost orthogonal to $C_R(\rho_i(g_j))$ at its endpoints. In particular, the angles of two consecutive geodesic segments of γ are almost flat. The same remark implies that the union of the segments $\rho_i(g_1g_2)^l(\gamma)$, $l \in \mathbb{Z}$ is a quasi-geodesic. This quasi-geodesic is invariant under $\rho_i(g_1g_2)$ with fundamental domain γ. It is a classical result [26] that in these circumstances $\rho_i(g_1g_2)$ is an hyperbolic element of $\mathrm{PSL}(2, \mathbb{C})$ with translation distance comparable to the length of γ. Since d_i tends to ∞ this contradicts that the trace of $\rho_i(g_1g_2)$ is bounded. □

Proof of Proposition 3. The algebraic set $X(G)$ is the union of the images of the irreducible components of $\mathcal{R}(G)$ by the polynomial map t. We show that the image of any irreducible component is an algebraic variety defined over $\mathbb{Q}$. We use the following classical result from Elimination Theory (cf. [22]).

Lemma 7. *Let $S \subset \mathbb{C}^n$ be an algebraic variety defined over a field $k \subset \mathbb{C}$ and $P : S \to \mathbb{C}^q$ be a polynomial map with coefficients in k. Then the Zariski closure $\overline{P(S)}$ of the image $P(S)$ is an algebraic variety defined over k and there is an algebraic set W strictly contained in $\overline{P(S)}$ such that $P(S) \cup W = \overline{P(S)}$.*

To show that $X(G)$ is an algebraic set, it suffices to prove that for each irreducible component $\mathcal{R}_0$ of $\mathcal{R}(G)$, the algebraic set $W = \overline{t(\mathcal{R}_0)} - t(\mathcal{R}_0)$ provided by Lemma 7 is empty. Let $x \in W$. Let (x_i) be a sequence in $t(\mathcal{R}_0)$ which converges to x. There is a sequence (ρ_i) in $\mathcal{R}_0$ with $t(\rho_i) = x_i$. By Proposition 5, we can conjugate ρ_i so that some subsequence of (ρ_i) converges to a representation ρ_∞. By continuity $t(\rho_\infty) = \lim t(\rho_i) = x$. Therefore $x \in t(\mathcal{R}_0)$. □

Relations with the Geometric Invariant Theory.

Another way to describe the algebraic structure of the quotient of $\mathcal{R}(G)$ under the action of $SL(2)$ relies on Geometric Invariant Theory (cf. [21]). The group SL(2) is reductive and acts rationally in the sense of [21] on $\mathcal{R}(G)$. By a theorem of Hilbert the ring $\mathbb{Q}[\mathcal{R}(G)]^{\mathrm{SL}(2)}$ of regular functions on $\mathcal{R}(G)$ which are invariant by SL(2) is thus finitely generated. It is the ring of regular functions of an algebraic variety over $\mathbb{Q}$ which we denote by $\mathcal{R}(G)//\mathrm{SL}(2)$. The canonical map $\mathcal{R}(G) \to \mathcal{R}(G)//\mathrm{SL}(2)$ which is dual to the inclusion of coordinates rings is onto: this is a particular case of a theorem of D. Mumford [21]. The complex points of $\mathcal{R}(G)//\mathrm{SL}(2)$ are in bijection with the closed orbits of $\mathrm{SL}(2,\mathbb{C})$ in $\mathcal{R}_{\mathbb{C}}(G)$. It is important to note that it is possible that two representations in $\mathcal{R}(G)$ are not conjugated one to another, but are mapped however to the same point in $\mathcal{R}(G)//\mathrm{SL}(2)$. This is the case for instance (when $G =< g >$) for representations of the form:

$$\rho_1(g) = \begin{pmatrix} \lambda(g) & b(g) \\ 0 & \lambda^{-1}(g) \end{pmatrix} \text{ and } \rho_2(g) = \begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^{-1}(g) \end{pmatrix},$$

the representation ρ_2 is in the closure of the SL(2)-orbit of ρ_1.

In our case the ring of invariant regular functions is generated by the traces, i.e. $\mathbb{Q}[\mathcal{R}(G)]^{\mathrm{SL}(2)} = \mathbb{Q}[\mathrm{tr}(\rho(g)), g \in G]$ [17]. Let $X_i = \mathrm{tr}(\rho(g_i))$ for $i \leq N$, be a basis of this ring (cf. Proposition 1). Let I be the ideal generated by all polynomials P in N variables such that $\rho \mapsto P(\mathrm{tr}(\rho(g_1)), ..., \mathrm{tr}(\rho(g_N)))$ is identically zero. Then $\mathcal{R}(G)//\mathrm{SL}(2)$ is isomorphic to the algebraic subset of $\mathbb{A}^N_{\mathbb{C}}$, $Y(G) = \{(X_i) | P(X_1, ..., X_N) = 0, \forall P \in I\}$. Clearly $X(G) \subset Y(G)$ and the trace map t is equal to the canonical map. It follows from the Mumford Theorem quoted above that $X(G) = \mathcal{R}(G)//\mathrm{SL}(2)$.

Remark. In general $\mathcal{R}(G)$ is reducible and the dimension of its irreducible components may vary. The Hyperbolic Dehn Surgery Theorem of Thurston states (among other things) that if G is the fundamental group of a finite volume hyperbolic manifold with k boundary components, then one irreducible component of $X(G)$ is k-dimensional. This component actually coincides with the set of all discrete and faithfull representations $\mathcal{DF}(G)$. But one can construct examples where the group G maps onto the free group $\mathbb{F}^2$ with two generators. Since any representation of $\mathbb{F}^2$ induces a representation of G we have an inclusion of $X(\mathbb{F}^2)$ into $X(G)$ and its image is a component disjoint from $\mathcal{DF}(G)$.

1.2. **The tautological representation of G.**

This representation will be an important tool in Chapter 4 for the construction of trees.

We identified representations of $G \to \mathrm{SL}(2,\mathbb{C})$ with the complex points in the affine algebraic set $\mathcal{R}(G)$. Fix an element of $g \in G$, and consider the matrix

$$\rho(g) = \begin{pmatrix} a_g(\rho) & b_g(\rho) \\ c_g(\rho) & d_g(\rho) \end{pmatrix}.$$

The coefficients of this matrix are polynomials with integers coefficients in the coordinates of the affine space $(\mathbb{A}^4)^n$ which contains the algebraic set $\mathcal{R}(G)$. Therefore $\rho(g)$ is an element of SL(2,$\mathbb{Q}[\mathcal{R}(G)]$). The map $g \to \rho(g)$ defines a representation of G into SL(2,$\mathbb{Q}[\mathcal{R}(G)]$), called *the tautological representation*. If $\mathcal{R}_0$ is an irreducible component of

$\mathcal{R}(G)$, we have an obvious restriction map $\mathbb{Q}[\mathcal{R}(G)] \to \mathbb{Q}[\mathcal{R}_0]$. We can therefore consider, for each irreducible component, the representation of G with values in $\mathrm{SL}(2, \mathbb{Q}[\mathcal{R}_0])$. It will be more convenient in the applications to think of it as a representation into $\mathrm{SL}(2, \mathbb{Q}(\mathcal{R}_0))$.

1.3. **A compactification of affine algebraic sets.**

Let $X \subset \mathbb{A}^N_{\mathbb{C}}$ be an affine algebraic set defined over a countable field $k \subset \mathbb{C}$. Let $k[X]$ be the ring of the regular functions on X: it is a countable set. Let $\mathcal{F} \subset k[X]$ denote any finite or countable set which generates $k[X]$ as a ring.

Notations. Consider the direct product $[0, \infty[^{\mathcal{F}}$ with its product topology. The *projective space* $\mathbb{P}^{\mathcal{F}}$ is defined as the quotient of $[0, \infty[^{\mathcal{F}} - \{0\}$ by the equivalence relation which identifies any sequence (t_f) with the sequence (αt_f) for $\alpha \in \mathbb{R}^*_+$. We denote by π the natural projection $\pi : [0, \infty[^{\mathcal{F}} - \{0\} \longrightarrow \mathbb{P}^{\mathcal{F}}$.

Define a map $\theta_0 : X \to [0, \infty[^{\mathcal{F}}$ by sending x to $\theta_0(x) = (\log(|f(x)| + 2))_{f \in \mathcal{F}}$, and write $\theta = \pi \circ \theta_0$. The projective space $\mathbb{P}^{\mathcal{F}}$ is not compact, however the following holds:

Proposition 8. *The closure of the image $\theta(X)$ in $\mathbb{P}^{\mathcal{F}}$ is compact and metrizable.*

Proof: This is a consequence of the next easy fact.

Claim 9. *Let $h_1, \cdots, h_m \in \mathcal{F}$ be a finite set of functions which generate $k[X]$. Then for any $f \in \mathcal{F}$, there is a constant c_f such that*

$$\log(|f(x)| + 2) \leq c_f \max \log(|h_j(x)| + 2).$$

Therefore $\tilde{\theta}(x) = \theta_0(x) / \max \log(|h_j(x)| + 2)$ is contained in the product $[0, c_f]^{\mathcal{F}}$, which is compact and metrizable. In particular, $\tilde{\theta}(\mathcal{F})$ has compact closure in $[0, \infty[^{\mathcal{F}}$. This closure does not contain the point $\{0\}$: indeed by definition of $\tilde{\theta}$, one of the coordinates of $\tilde{\theta}(x)$ with index $1, \cdots, m$ is equal to 1. Since $\theta(X) = (\pi \circ \tilde{\theta})(X)$, the closure of $\theta(X)$ in $\mathbb{P}^{\mathcal{F}}$ is thus compact and metrizable. □

Notice that the closure $\overline{\theta(X)}$ might not be a compactification of X since $\theta : X \to \mathbb{P}^{\mathcal{F}}$ is in general not injective. In order to avoid this difficulty we introduce the one-point-compactification $\hat{X}$ of X and take the fibered product. Concretely let us define $\hat{\theta} : X \to \hat{X} \times \mathbb{P}^{\mathcal{F}}$ by $\hat{\theta}(x) = (x, (\pi \circ \theta)(x))$. The map $\hat{\theta}$ is clearly injective. The closure of $\hat{\theta}(X)$ in $X \times \mathbb{P}^{\mathcal{F}}$ is then a metric space which contains a dense open subset homeomorphic to X.

Definition. The *compactification of X associated to the family $\mathcal{F}$* is $\overline{X}^{\mathcal{F}} = \overline{\hat{\theta}(X)}$. The *boundary of* X in $\overline{X}^{\mathcal{F}}$ is by definition $B_{\mathcal{F}}(X) = \overline{\hat{\theta}(X)} - X$ and can be identified with the set of cluster values of $\theta(X)$ in $\mathbb{P}^{\mathcal{F}}$.

Example. Although we shall be more interested in the case where $\mathcal{F}$ is countable, the following simple example sheds some light on the general structure of the compactification described above. Take $X = \mathbb{C}^n$, and let $\mathcal{F} = \{X_1, ..., X_n\}$ be a set of coordinates. Then $\mathbb{P}^{\mathcal{F}}$ is naturally homeomorphic to the set of $(s_i) \in [0, 1]^n$ such that $\max s_1 = +1$: it is a piecewise real affine space of dimension $n - 1$ homeomorphic to the closed unit ball in $\mathbb{R}^{n-1}$. The map θ is surjective but not injective. The space $\overline{X}^{\mathcal{F}}$ is homeomorphic to the sphere of real dimension $2n$.

1.4. **Thurston's compactification of Teichmüller space.**

The previous compactification can be compared with *Thurston's compactification of Teichmüller space.* Fix a hyperbolic surface S of finite volume and consider its Teichmüller space $\mathcal{T}_S$. It is the set of orientation-preserving homeomorphisms $\phi : S \to S'$ where S' is a hyperbolic surface of finite volume modulo the equivalence relation $(S_1, \phi_1) \simeq (S_2, \phi_2)$ when $\phi_2 \circ \phi_1^{-1} : S_1 \to S_2$ is isotopic to an isometry.

Now choose a base point in S, and consider the fundamental group $\pi_1(S)$. One has a natural map from $\mathcal{T}(S)$ to the set of real points in $X_{\mathbb{R}}(\pi_1(S))$ which is constructed as follows. Pick a point $(S', \phi) \in \mathcal{T}(S)$. Take a universal cover $\mathbb{H} \to S'$ by the upper half-plane. Then $\pi_1(S')$ acts by isometries on $\mathbb{H}$, giving us a representation of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$. Note that changing the universal cover or the base point amounts to conjugate the representation. Also this representation lifts to a representation with values in $\mathrm{SL}(2, \mathbb{R})$, so that we get a well-defined map from $T(S)$ to $X_{\mathbb{R}}(\pi_1(S))$, the real points of $R(\pi_1(S))//SL(2)$. This map is injective and it is a theorem that when S is compact, $\mathcal{T}_S$ is isomorphic to a connected component of $X_{\mathbb{R}}(\pi_1(S))$; when S has finite volume, $\mathcal{T}(S)$ gets identified with a connected component of an analytic subset of $X_{\mathbb{R}}(\pi_1(S))$, defined by the equality of the traces of $\rho(g)$ to ± 2 for all punctures of S.

Consider now the collection $\mathcal{S}$ of all homotopy classes of simple closed curves on S. There is a natural map $\theta : X_{\mathbb{R}}(\pi_1(S)) \to \mathbb{P}^{\mathcal{S}}$, defined like in section 1.3 whence a map $\theta_{\mathcal{T}} : \mathcal{T}_S \to \mathbb{P}^{\mathcal{S}}$.

One can construct another map $\mathcal{L} : \mathcal{T}(S) \to \mathbb{P}^{\mathcal{S}}$ as follows. Fix $\sigma = (S', \phi) \in T(S)$, and let σ be the pull-back by ϕ of the Poincaré metric on S'. For any $\gamma \in \mathcal{S}$, we denote by $\mathrm{length}_\sigma(\gamma)$ the shortest length on S' of a representative of the homotopy class $\phi(\gamma)$. Set $\mathcal{L}(\sigma) = (\mathrm{length}_\sigma(\gamma))$. One can prove that this map is injective and that its image is relatively compact in $\mathbb{P}^{\mathcal{S}}$. The set $\mathcal{L}(\mathcal{T}(S))$ is Thurston's compactification of the Teichmüller space of S.

Pick a point $\sigma \in \mathcal{T}(S)$ and consider the representation $\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbb{R})$ associated to it. Then for any $\gamma \in \mathcal{S}$, one has:

$$c_1 \,\mathrm{length}_\sigma(\gamma) \leq \log(|\mathrm{tr}(\rho(\gamma))| + 2) \leq c_2 \,\mathrm{length}_\sigma(\gamma),$$

the constants c_1 and c_2 being independent of σ and γ. Therefore the boundary of $\mathcal{L}(\mathcal{T}_S)$ in $\mathbb{P}^{\mathcal{S}}$ coincides with the boundary of $\theta_T(\mathcal{T}_S)$.

2. The compactification of an affine algebraic variety by valuations.

Morgan and Shalen have shown that points in the boundary $B_{\mathcal{F}}(X)$ of the compactification of the previous chapter can be described in terms of valuations on the field of rational functions $k(X)$. We first recall some elements of valuation theory and then prove their result, Proposition 17 below.

2.1. Valuations.

For any field k, write $k^* = k \setminus \{0\}$.

Definition. Let F/k be a field extension. A *valuation* on F/k is a group homomorphism $v : F^* \to \Lambda$ where Λ is an ordered abelian group such that

(i) for all f, $g \in F^*$, such that $f + g \in F^*$, then $v(f + g) \geq \min(v(f), v(g))$; and
(ii) the restriction $v|k^* = 0$.

The group Λ is called *the value group* of v.

Two valuations $v : F^* \to \Lambda$ and $v' : F^* \to \Lambda'$ are *equivalent* if there is an isomorphism of ordered groups $i : v(F^*) \to v'(F^*)$ such that $v' = i \circ v$.

Let v be a valuation on K/k. The *valuation ring of* v is

$$\mathfrak{o}_v = \{f \in F |\, f = 0 \quad \text{or} \quad v(f) \geq 0\}.$$

It contains the field k and possesses a unique maximal ideal

$$\mathfrak{m}_v = \{f |\, f = 0 \quad \text{or} \quad v(f) > 0\}.$$

The subadditivity property (i) has the following consequence. If $f_1, \cdots, f_n$ are elements of F^* such that the minimum of the valuations $v(f_i)$ is reached by a single function f_j, one has : $v(f_1 + \cdots + f_n) = v(f_j)$. One deduces from this that if a non-trivial sum $\sum f_j$ is 0, then there exist two distinct indexes k and l such that $v(f_k) = v(f_l)$.

We give examples of valuations in section 2.4 below. There are three basic invariants attached to a valuation of F/k.

The rational rank of a valuation. This is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes \Lambda$, and is denoted by rat.rk(v). The last remark in the paragraph above implies that rat.rk(v) is bounded from above by the transcendence degree of the extension F/k.

The rank of a valuation. Let Λ be an ordered abelian group. A subgroup Λ' of Λ is said to be *convex* if for all $x > 0$ in Λ' the interval $[0, x] = \{0 \leq y \leq x\}$ is contained in Λ'. When Λ is finitely generated, its convex subgroups form a finite sequence $\Lambda_0 = \{0\} \subset \Lambda_1 \subset \cdots \subset \Lambda_r = \Lambda$. The integer r is called the *rank* of v and is denoted by $\mathrm{rk}(v)$. By induction, one proves that r is less than the rational rank of v. Each successive quotient group Λ_{j+1}/Λ_j carries a natural order relation, the *quotient ordering* and is *archimedean* for that ordering, i.e. when x and y are > 0 then there is $n > 0$ such that $ny > x$. It is well-known that any archimedean totally ordered group is isomorphic to a subgroup of $\mathbb{R}$. The isomorphism is well-defined up to multiplication by a positive real number. In particular any valuation of rank 1 is equivalent to a valuation taking values in $\mathbb{R}$.

The transcendence degree of a valuation. The quotient of the valuation ring $\mathfrak{o}_v$ by its unique maximal ideal $\mathfrak{m}_v$ is a field k_v called *the residue field of* v, which naturally contains k. The *transcendence degree of* v is by definition the transcendence degree of the field extension k_v/k, and is denoted by deg.tr(v). It is also bounded from above by the transcendence degree of F/k.

Abhyankar inequality. Although we shall not use it in the sequel, we mention that the three invariants above satisfy the following fundamental inequality which is usually referred to as Abhyankar inequality:

$$\mathrm{rk}(v) + \mathrm{deg.tr}(v) \leq \mathrm{rat.rk}(v) + \mathrm{deg.tr}(v) \leq \mathrm{deg.tr}(F/k) \ .$$

This inequality is actually valid in a much broader context of ring extensions. For fields it is due to Zariski. The case of equality in the inequalities above is realized by valuations of simple nature. We refer to [,] for a geometric description of these valuations.

Places. Let v be a valuation of F/k. If $f \notin \mathfrak{o}_v$ then $v(f) < 0$; therefore $1/f \in \mathfrak{m}_v$. This motivates the following definition.

Definition. A *place of* F/k is the data of a ring $\mathfrak{o} \subset F$ containing k and a maximal ideal $\mathfrak{m} \subset \mathfrak{o}$ such that if $f \in F$ and $f \notin \mathfrak{o}$, then $1/f \in \mathfrak{m}$.

Remark. The notions of place and valuation are equivalent. The valuation ring of a valuation is a place. Conversely, for any place there is a valuation v such that $\mathfrak{o}_v = \mathfrak{o}$, and this valuation is unique up to equivalence. One can indeed prove that $\Lambda = F^*/(\mathfrak{o} - \mathfrak{m})$ is an ordered (multiplicative) abelian group [37, p. 35]: positive elements are in bijection with cosets $m(\mathfrak{o} - \mathfrak{m})$, with $m \in \mathfrak{m}^*$. The valuation v is the quotient map $F^* \to F^*/(\mathfrak{o} - \mathfrak{m})$.

However even if both concepts equivalent, they offer two different perspectives on the same object. A place $\mathfrak{m} \subset \mathfrak{o}$ of F/k determines an homomorphism of $\mathfrak{o}$ to a field, namely to the field $\mathfrak{o}/\mathfrak{m}$. Viewed on this way a place can be compared with the "evaluation of a function at a point"; it assigns to $f \in F$ an element in $\mathfrak{o}/\mathfrak{m}$ if $f \in \mathfrak{o}$ or ∞ if $f \notin \mathfrak{o}$. While the valuation associated to this place rather measures "the order of vanishing of f at the same point".

2.2. The Riemann-Zariski space of F/k.

Definition 10. *Let F/k be a field extension. The Riemann-Zariski space of F/k is the set $\mathcal{V}(F/k)$ of valuations on F/k. We put on $\mathcal{V}(F/k)$ the topology generated by the open sets $\{v|\, v(f) \geq 0\}$ as f ranges over F^*.*

Theorem 11. *The Riemann-Zariski space of F/k is quasi-compact: from any open cover one can extract a finite cover.*

Note however that the Riemann-Zariski space of F/k is never Hausdorff except when F has transcendence degree 1 over k.

Proof [33]. Consider the product of discrete spaces $Z = \{-, 0, +\}^{F^*}$ with the product topology of the topologies whose open sets are $\emptyset$, $\{-, 0, +\}$ and $\{0, +\}$. The Riemann-Zariski surface $\mathcal{V}(F/k)$ can be mapped into Z using the map $v \to (s(f))_{F^*}$ where $s(f)$ is the sign of $v(f)$. This map is injective and its image is a closed set. One concludes using Tychonoff's Theorem. □

Remark. The definition of the Riemann-Zariski space and Theorem 11 are due to Zariski [36]. It is an interesting fact that the Zariski topology (for algebraic set) was in fact designed to prove the aforementioned theorem. For the same reason, Zariski was naturally led to enrich the set of points of a fixed algebraic set in order to include all irreducible subvarieties as well. This feature is a characteristic of scheme theory, and was later formalized by Grothendieck in a much broader context.

The center of a valuation on a projective model.

Notations. Let us first review some basic aspects of complex algebraic geometry as developed in [12, Chapter 1].

A *projective variety* $V \subset \mathbb{C}P^N$ is by definition the set of common zeroes of all polynomials lying in a fixed prime homogeneous ideal I_V of the ring of homogeneous polynomials in $N+1$ variables. In other words, it is an irreducible closed subset of $\mathbb{C}P^N$ endowed with its Zariski topology. The *function field of* V is the quotient ring of the ring of rational functions P/Q, where P and Q are homogeneous of the same degree and $Q \notin I_V$ by the ideal of the functions P/Q with $P \in I_V$. We denote it $k(V)$.

Let V, M be any two projective varieties. Then $V \subset M$ if and only if $I_M \subset I_V$. We denote by $\mathfrak{o}_V$ the subring of $k(M)$ which consists of all rational functions P/Q with P and $Q \notin I_V$. This ring is called *the local ring of M at V*. Its maximal ideal $\mathfrak{m}_V$ consists of rational functions P/Q, with $P \in I_V$. The quotient $\mathfrak{o}_V/\mathfrak{m}_V$ is isomorphic to $k(V)$, the function field of V.

Let F/k be any extension of finite transcendence degree. Suppose that k is a subfield of $\mathbb{C}$. A *projective model* of F/k is a pair (M, i) where M is a projective variety $M \subset \mathbb{C}P^N$ defined over k and $i : k(M) \to F$ is an isomorphism between the field of rational functions $k(M)$ and F.

We now explain the definition of Zariski [35, p. 497] of *the center of a valuation* on the function field of a variety.

Lemma 12. *Let M be a projective variety and let v be any non-trivial valuation on $k(M)/k$. Then, there exists a unique proper variety $W \subset M$ whose valuation ring and maximal ideal satisfy : $\mathfrak{o}_W \subset \mathfrak{o}_v$ and $\mathfrak{m}_W \subset \mathfrak{m}_v$.*

The variety $W = W_{M,v}$ is called *the center of the valuation v in M*.

Proof. Let $(X_0, \cdots, X_N)$ be a system of homogeneous coordinates of the projective space which contains M. Consider a coordinate function $X = X_i$ which is minimal in the sense that $v(X_j/X) \geq 0$ for all $j = 0, \cdots, N$.

Since v is a valuation on $k(M)/k$, for each homogeneous polynomial P of degree m, one has $v(P/X^m) \geq 0$. Now define the subset $I \subset k[X_0, \cdots, X_N]$ containing 0 and all polynomials P such that $v(P/X^m) > 0$ with m equal to the degree of P. Then I is an ideal that contains I_M. It is even a prime ideal since if P and Q are homogeneous polynomials with respective degrees r and s such that $v(PQ/X^{r+s}) > 0$, then either $v(P/X^r) > 0$ or $v(Q/X^s) > 0$. It contains I_M strictly since v is a non-trivial valuation. So there is a proper variety $W \subset M$ such that I is the defining ideal I_W of W.

Let $P/Q \in \mathfrak{o}_W$. Then $v(P/X^d) + v(X^d/Q) \geq 0$ since $Q \notin I_W$. Therefore $\mathfrak{o}_W \subset \mathfrak{o}_v$. The same argument shows that $\mathfrak{m}_W \subset \mathfrak{m}_v$.

For the uniqueness, let $W' \subset M$ be any variety such that $\mathfrak{o}_{W'} \subset \mathfrak{o}_v$ and $\mathfrak{m}_{W'} \subset \mathfrak{m}_v$. Then for any P and any $Q \notin I_{W'}$ both of degree d, one has $v(P/X^d) + v(X^d/Q) \geq 0$, hence $v(X^d/Q) \geq 0$. By the definition of I_W, we infer $Q \notin I_W$, thus $W' \subset W$. Suppose $W \neq W'$. Then one can find a homogeneous polynomial P in $I_{W'}$ but not in I_W. The rational function $P/X^d \in \mathfrak{m}_{W'}$ has positive valuation, but the valuation of its inverse X^d/P is also non negative, since this element lies in $\mathfrak{o}_W$. This is impossible. Therefore $W = W'$. □

Remark. Any rational function $f = P/Q \in F = k(M)$ defines a regular map with values in $\mathbb{C}P^1$ outside the intersection of the zero loci of P and Q. This intersection is called the indeterminacy locus of f. When this locus does not contain the center W of v in M, then $f \in \mathfrak{o}_v$ if and only if f takes finite values on an open set of W; and $f \in \mathfrak{m}_v$ if and only if f vanishes on a Zariski open subset of W. Note that when the indeterminacy locus of f contains W one cannot see directly whether or not f belongs to $\mathfrak{o}_v$.

The inverse system of the projective models of F/k.

We now suppose that X is an affine variety defined over k and set $F = k(X)$. Two models (M, i) and (M', i') of F/k are said to be *equivalent* if and only if there is an isomorphism $M \to M'$ which induces the isomorphism $i \circ (i')^{-1}$ between the function fields $k(M)$ and $k(M')$. We denote by $\mathcal{M}$ the set of equivalence classes of models of F/k. Given any two models M and M', there is a birational map which induces $i' \circ (i)^{-1}$ on the function fileds: we denote this map by $j_{M',M} : M' \to M$. The set $\mathcal{M}$ thus carries a natural partial order, defined by $(M, i) \leq (M', i')$ when $j_{M',M}$ is a regular map. When $(M, i) \leq (M', i')$, we say then that M' *dominates* M.

In order to state and prove properly Theorem 13 below it is necessary to work with the scheme-theoretic description of the variety associated to M. Concretely this amounts to replacing M by the set of all its proper subvarieties which are defined over k, and put the Zariski topology on this set (a closed set consists in all proper subvarieties included in a fixed algebraic subset of M). To lighten notations we keep the same letter M for this object. Note that a subvariety is now a point.

If $(M, i) \leq (M', i')$ the regular map $j_{M',M} : M' \to M$ is continuous for this topology. So we can consider the projective limit $\mathbf{M} = \varprojlim_{\mathcal{M}} M$ with the projective limit topology. A point $w \in \mathbf{M}$ is the data for each $M \in \mathcal{M}$ of a subvariety $W_M \subset M$ such that whenever $M \leq M'$, then the regular map $j_{M',M} : M' \to M$ maps $W_{M'}$ to W_M. The set of subvarieties of M with the Zariski topology is quasi-compact. Therefore the projective subset $\mathbf{M}$, as a closed subset of the direct product of all models of F is quasi-compact also. The following theorem is due to Zariski (cf. [37]).

Theorem 13. *The projective limit* $\mathbf{M}$ *is homeomorphic to the Riemann-Zariski space* $\mathcal{V}$ *of* $k(X)/k$.

Proof. one can define a natural map $\mathcal{W} : \mathcal{V} \to \mathbf{M}$ as follows. Pick a valuation $v \in \mathcal{V}$, and for each projective model M, consider the center $W_{M,v}$ of v in M. Suppose that M' dominates M. Then $j_{M',M}$ is regular and maps $W_{M',v}$ to a subvariety $W \subset M$. The local ring $\mathfrak{o}_W$ is mapped to the local ring of $W_{M',v}$ and its maximal ideal $\mathfrak{m}_W$ is mapped to the maximal ideal of $W_{M',v}$. By the characterization of the center given in Lemma 12, one has : $W = W_{M,v}$. This shows the collection of subvarieties $(W_{M,v})$ is a point in $\mathbf{M}$. In this way, we get a map $\mathcal{W}$ which is easily seen to be continuous. Since $\mathcal{V}$ is quasi-compact it is enough to prove that this map is bijective to conclude.

— $\mathcal{W}$ is surjective. Let $w = (W_M) \in \mathbf{M}$, and write $\mathcal{R}_w = \cup \mathfrak{o}_{W_M}$. It is a ring contained in $k(X)$. Let us show that it is a valuation ring. The ideal $\mathfrak{m}_w = \cup \mathfrak{m}_{W_M}$ is a maximal ideal of $\mathcal{R}_w$. We need to show that for any $f \in F^*$, either $f \in \mathcal{R}_w$ or $1/f \in \mathfrak{m}_w$. For this, we use the simple fact that for any model M, there exists a (not necessarily smooth) model M' which dominates M and on which f is a regular map $M \to \mathbb{C}P^1$ (this can be seen by taking M' to be the closure of the graph of f in $M \times \mathbb{C}P^1$). Now either f is infinite on $W_{M'}$, in which case $1/f \in \mathfrak{o}_{W_{M'}} \subset \mathcal{R}_w$; or f is finite on a Zariski dense open set of $W_{M'}$, in which case $f \in \mathfrak{m}_{W_{M'}} \subset \mathfrak{m}_w$. This shows that $\mathcal{R}_w$ is the ring of a valuation v. By the Lemma 12 one has: $\mathcal{W}(v) = w$.

— $\mathcal{W}$ is injective. If $\mathcal{W}(v) = w$ then the local ring $\mathcal{R}_w$ constructed above is necessarily included in R_v. But we saw that $\mathcal{R}_w$ was a valuation ring. Whence $\mathcal{R}_w = R_v$ and v has a unique preimage, so that $\mathcal{W}$ is injective. □

Valuations centered at ∞. Suppose that F is the function field of an affine variety $V \subset \mathbb{A}^N_{\mathbb{C}}$ defined over k. Then one says that a valuation $v \in \mathcal{V}(K/k)$ is *centered at* ∞ when the coordinate ring $k[X]$ is *not* contained in the valuation ring $\mathfrak{o}_v$. This is equivalent to saying that the center of v on the model $\overline{X} \subset \mathbb{C}P^N$ is contained in the hyperplane at infinity $\mathbb{C}P^N - \mathbb{A}^N_{\mathbb{C}}$. In particular, the valuations centered at ∞ form a closed subset of $\mathcal{V}(F/k)$.

2.3. Construction of valuations from sequences of points.

Let k be a countable field contained in $\mathbb{C}$, and let $X \subset \mathbb{A}^N_{\mathbb{C}}$ be a variety defined over k. Then $k(X)$ is countable.

We now describe a construction due to Morgan-Shalen of valuations of $k(X)/k$ from sequences of points in X. We shall see that all valuations on $k(X)/k$ arise in this way.

We say that a complex point x on a variety X is *k-generic on X*, if it is not contained in any proper subvariety of X defined over k, i.e. if it is dense in X for the Zariski topology. For instance a point in $\mathbb{A}^N_{\mathbb{C}}$ is k-generic if and only if its coordinates generate an extension of $\mathbb{Q}$ of transcendence degree equal to N.

Since X is irreducible a Baire category argument implies that the set of k-generic points is dense in X for the classical topology. Any element of $k(X)$ can be written as the ratio P/Q of two polynomials, where Q is not in the ideal defining X. If x is a k-generic point any meromorphic function in $k(X)$ can thus be evaluated at x.

Definition 14. *A sequence $(x_i)_{i\in\mathbb{N}}$ in X is a* valuating sequence *if*

(i) *each x_i is k-generic, and*
(ii) *for any $f \in k(X)$, $\lim_{i\to\infty} f(x_i)$ exists in $\mathbb{C} \cup \{\infty\}$.*

From any sequence of k-generic points which is contained in X, one can extract a valuating sequence using a Cantor diagonal argument.

The valuation associated to a valuating sequence.

Any valuating sequence (x_i) defines a place: the ring is the set $\mathfrak{o}$ of rational functions $f \in k(X)$ for which $\lim_{i\to\infty} f(x_i)$ belongs to $\mathbb{C}$, and its maximal ideal is the kernel of the natural homomorphism $\mathfrak{o} \to \mathbb{C}$, $f \mapsto \lim_{i\to\infty} f(x_i)$. This place corresponds to a valuation called *the valuation associated to the valuating sequence* (x_i). In the applications the sequence (x_i) will be unbounded. This implies that the associated valuation is non-trivial since for some coordinate X_j the rational function $1/X_j$ tends to 0 and therefore $X_i \notin \mathfrak{o}_v$.

However, for an arbitrary sequence this valuation could be trivial. This happens for instance when (x_i) tends to a k-generic point.

Let us give an application of Theorem 13.

Proposition 15. *Any valuation of $k(X)/k$ is associated to a valuating sequence (x_i) on X. The valuation is centered at infinity if and only if (x_i) tends to infinity in X.*

Proof. Let v be a valuation on $k(X)/k$. Let $w = (W_M)$ be the corresponding point in the projective limit $\mathbf{M}$. Since k is countable there is a sequence $M_0 = \overline{X}$, $M_n \leq M_{n+1}$ of totally ordered models of $k(X)/k$ such that any model M is dominated by some model M_j. Since the maps $j_{n+1} = j_{M_{n+1},M_n}$ are regular, in particular continuous, we can choose distances d_n on M_n such that the maps j_n are distance-decreasing. We define by induction a sequence of points $\xi_n \in M_n$ such that ξ_n is k-generic on W_{M_n} and $j_{n+1}(\xi_{n+1}) = \xi_n$. For each n we choose a point $x_n \in M_n$ which is k-generic on M_n and such that $d_n(x_n, \xi_n) \leq 1/n$. Then for any $n \geq m$, one has $j_{M_n,M_m}(x_n) \to \xi_m$ as $n \to \infty$. Define $x'_n = j_{M_n,M_0}(x_n)$. The point x'_n is k-generic on $\overline{X}$ hence belongs to X. Now pick $f \in k(X)$ and let $m \in \mathbb{N}$ be an integer such that f defines a regular map $M_m \to \mathbb{C}P^1$. By definition of W_m, $f \in \mathfrak{o}_v$ if and only if $f(\xi_m)$ is finite. This is equivalent to say that the sequence $(f(x_n))$ has a finite limit as $n \to \infty$. In the same way, by definition of j_{M_n,M_0}, we have $f(x_n) = f(x'_n)$. Therefore $f \in \mathfrak{o}_v$ if and only if the sequence $(f(x'_n))$ tends to a finite limit as $n \to \infty$. Hence (x_n) is a valuating sequence and the valuation it defines is v. □

In the next proposition we show that the valuation associated to a valuating sequence (x_i) which tends to infinity in X measures the growth rate of the sequence $f(x_i)$ when $f \in k(X)$. Before doing that, we need to define *the ratio* λ_1/λ_2 of two negative elements λ_1 and λ_2 in the value group Λ of v, when the group is not necessarily archimedean.

Let $\Lambda_0 = \{0\} \subset \Lambda_1 \cdots \subset \Lambda_r = \Lambda$ be the sequence of convex subgroups of Λ. Let $j \geq 1$ be the smallest index for which $\lambda_1 \in \Lambda_j$. Suppose first that j is also the smallest index for λ_2. Then λ_1 and λ_2 map to non-zero elements $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ in the quotient group Λ_j/Λ_{j-1}. Since this group is archimedean it can be embedded in $\mathbb{R}$ in a unique way up to multiplication by a positive real number, so that the ratio $\tilde{\lambda}_1/\tilde{\lambda}_2$ is a well-defined real number. We set $\lambda_1/\lambda_2 = \tilde{\lambda}_1/\tilde{\lambda}_2 \in \mathbb{R}$. When the smallest index for λ_2 is $< j$, we set $\lambda_1/\lambda_2 = 0$ and when it is $> j$, we set $\lambda_1/\lambda_2 = \infty$. Observe from the definition that if λ_1 and λ_2 are both < 0 and if r and s are positive integers, we have : $\lambda_1/\lambda_2 \leq r/s$ if and only if $r\lambda_2 \leq s\lambda_1$.

Proposition 16. *Let (x_i) be a valuating sequence on X and v be the valuation of $k(X)/k$ associated to it.*

1. *For any $f \in k(X)$, $v(f) \geq 0$ if and only if $\log|f(x_i)|$ is bounded from above.*
2. *Let f and $g \in k(X)$ such that $v(f) \leq 0$ and $v(g) < 0$. Then*

$$\lim_{i\to\infty} \frac{\log|f(x_i)|}{\log|g(x_i)|} = \frac{v(f)}{v(g)}.$$

Proof. The first statement is a rephrasing of the definition of v in terms of the valuating sequence. For the second statement, suppose first that $v(f) = 0$; then $\lim_{i\to\infty} |f(x_i)| \in \mathbb{C}^*$. Since $v(g) < 0$, $|g(x_i)| \to \infty$. So both terms in the formula are equal to 0 in this case. Suppose now that $v(f) < 0$. After exchanging the role of f and g we see that the proof will be complete if we show

$$\overline{\lim_{i\to\infty}} \frac{\log|f(x_i)|}{\log|g(x_i)|} \leq \frac{v(f)}{v(g)} .$$

Let r and s be positive integers with $v(f)/v(g) < r/s$. Then $rv(g) - sv(f) < 0$ so that $\lim_{i\to\infty} g^r(x_i)/f^s(x_i) = \infty$. In particular $\lim_{i\to\infty}(r|\log g(x_i)| - s|\log f(x_i)|) = \infty$. This implies the required result since this holds for all integers r and s with $r/s > v(f)/v(g)$. □

For any affine variety X defined over a countable field k, we defined in section 1.3 a compactification $\overline{X}^{\mathcal{F}}$ of X which depends on the choice of a family of polynomials $\mathcal{F} \subset k[X]$. We saw that the boundary of this compactification $B_{\mathcal{F}}(X) = \overline{X}^{\mathcal{F}} \setminus X$ can be identified with the set of cluster points of $\theta(X)$ inside $\mathbb{P}^{\mathcal{F}}$. The next proposition states that this boundary can be described in a precise way using valuations on $k(X)/k$.

Recall that the rank of a valuation is equal to 1 when the ordered group Λ is archimedean, in which case one can suppose $\Lambda \subset \mathbb{R}$. For any rank one valuation v on $k(X)/k$, denote by $U(v) \in \mathbb{P}^{\mathcal{F}}$ the point with homogeneous coordinates $(-\min(0, v(f)))$. We now extend this map to higher rank valuations as follows.

Let v be an arbitrary valuation on $k(X)/k$ centered at infinity. Let Λ denote its value group, and $\Lambda_0 = \{0\} \subset \Lambda_1 \cdots \subset \Lambda_r = \Lambda$ be the sequence of its convex subgroups. Let s be the smallest index such that for all polynomials $f \in k[X]$ either $v(f) \geq 0$ or $v(f) \in \Lambda_s$. We define a new valuation $\overline{v}$ by composing v with the quotient map $\Lambda \to \overline{\Lambda} = \Lambda/\Lambda_{s-1}$. Then $\overline{v}$ is a valuation on $k(X)/k$ which is still centered at infinity. By construction, $\overline{v}$ also enjoys the following property (P): for any polynomial $f \in k[X]$, either $\overline{v}(f) \geq 0$ or $\overline{v}(f)$ belongs to the largest convex archimedean subgroup $\overline{\Lambda}_1$ of $\overline{\Lambda}$.

Denote by $\mathcal{V}_0 \subset \mathcal{V}(k(X)/k)$ the set of the valuations on $k(X)/k$ which are centered at ∞ and satisfy the property (P). One can then copy the definition that we gave for valuations of rank 1 and define a natural map $U : \mathcal{V}_0 \to \mathbb{P}^{\mathcal{F}}$ by setting: $U(v) = (-\min(0, v(f)))$. Note that $U(v)$ does not depend on the choice of an embedding of $\overline{\Lambda}_1$ in $\mathbb{R}$.

Proposition 17. *The map U maps continuously and surjectively $\mathcal{V}_0$ onto $B_{\mathcal{F}}(X)$.*

Proof. It is clear that U is continuous. We claim that if (x_i) is a valuating sequence for a valuation v and $x_i \to \infty$, then $\lim_{i\to\infty} \theta(x_i) = U(\bar{v})$, where $\bar{v}$ is the quotient valuation defined by composing v with an homomorphism $\Lambda \to \Lambda/\Lambda_{s-1}$ as defined above.

Suppose that this claim is proved. Then pick $v \in \mathcal{V}_0$ and take a valuating sequence (x_i) for v. As v is centered at infinity the sequence (x_i) tends to infinity, and thus $U(v) = \lim_{i\to\infty} \theta(x_i)$ is a cluster point of $\theta(X)$. We conclude that $U(v) \in B_{\mathcal{F}}(X)$ which shows $U(\mathcal{V}_0) \subset B_{\mathcal{F}}(X)$. Conversely pick a point $\xi \in B_{\mathcal{F}}(X)$ and a sequence (x_i) tending to ∞ such that $\theta(x_i) \to \xi$. Using a Cantor diagonal argument we can approximate the sequence (x_i) by a valuating sequence (x'_i) contained in X such that $\theta(x'_i) \to \xi$. By what precedes we conclude $U(\overline{v}) = \xi$.

To prove the claim we proceed as follows. Choose a function $g \in \mathcal{F}$ such that $\overline{v}(g) < 0$. Then $|g(x_i)| \to \infty$ as $i \to \infty$. The point ξ is the limit of the sequence with homogeneous coordinates $(\log(|f(x_i)| + 2)/\log(|g(x_i)| + 2))$.

For a function $f \in \mathcal{F}$ such that $\lim\limits_{i\to\infty} f(x_i) \in \mathbb{C}$, then

$$\lim_{i\to\infty} \frac{\log(|f(x_i)| + 2)}{\log(|g(x_i)| + 2)} = 0$$

which is also $-\min(0, \overline{v}(f))/\overline{v}(g)$. When $\lim\limits_{i\to\infty} f(x_i) = \infty$, then $v(f) < 0$. By Proposition 16,

$$\lim_{i\to\infty} \frac{\log(|f(x_i)| + 2)}{\log(|g(x_i)| + 2)} = \frac{v(f)}{v(g)}.$$

From the definition of the ratio of two elements in an ordered group, this limit is also $\overline{v}(f)/\overline{v}(g)$. □

2.4. Examples of valuations.

Discrete valuations of rank 1. A valuation v on a field K is said *discrete of rank* 1 if its value group is equal to $\mathbb{Z}$. In a geometric context the main examples arise as follows.

Suppose X is an algebraic variety over $\mathbb{C}$ and D is an irreducible divisor on X. Then the function attaching to any rational function $f \in \mathbb{C}(X)$ its order of vanishing $\mathrm{ord}_D(f)$ along D defines a discrete valuation of rank 1. More generally if $\pi : Y \to X$ is a birational morphism and D an irreducible divisor in Y, then the function $\mathrm{ord}_D(f \circ \pi)$ is a discrete valuation of rank one on $\mathbb{C}(X)$. Such valuations are called *divisorial* valuations.

In fact any discrete valuation v of rank 1 on the field of rational functions of an algebraic variety X such that $\mathrm{tr.deg}(v) = \dim X - 1$ is divisorial (see []).

(Quasi)-monomial valuations. Let $X \subset \mathbb{A}^N_{\mathbb{C}}$ be an affine variety of dimension n defined over $\mathbb{C}$. Take local coordinates $x_1, ..., x_n$ at a point $p \in X$ and fix non-negative real numbers $s_1, ..., s_n \geq 0$. Pick any function f that is regular at p, and expand it locally in power series $f(x) = \sum_I a_I x^I$, with $a_I \in \mathbb{C}$, and $x^I = \prod x_j^{i_j}$ if $I = (i_1, ..., i_n)$. Denote $s.I = \sum_1^n s_k i_k$. Then the function $v_{s,x}(f) = \min\{s \cdot I, a_I \neq 0\}$ defines a valuation of rank 1 whose value group is equal to $\sum_{i=1}^n \mathbb{Z}s_i$. It is not difficult to check that any such valuation satisfies $\mathrm{rk}(v) + \mathrm{deg.tr}(v) = \dim(X)$. Conversely suppose v is a valuation of rank 1 on $\mathbb{C}(X)$ for which $\mathrm{rk}(v) + \mathrm{deg.tr}(v) = \dim(X)$. Then, using Hironaka's Desingularization Theorem one can show the existence of a birational morphism $\pi : Y \to X$, a point $p \in Y$, and local coordinates near p, such that the valuation $f \mapsto v_{s,x}(f \circ \pi)$ is equivalent to v (see []).

More complicated examples. In general, the structure of a valuation can be quite complicated, even on the field of rational functions of an algebraic variety X. If $X$is a curve then any valuation is discrete of rank 1. When X is a surface a complete classification can be obtained (see []). In higher dimension though, the picture is less clear (see however [] for recent progress).

Let us mention that given any sequence of integers $m_i \geq 1$ one can construct a valuation of rank 1 on the ring $\mathbb{C}[x_1, x_2]$ with value group $\sum_i \frac{1}{m_1 ... m_i}\mathbb{Z}$ (see [37, §15]). In particular choosing $m_i = p$ for all i, one may obtain a valuation with value group $\mathbb{Z}[\frac{1}{p}]$.

3. Λ-trees.

In the preceding two chapters we have constructed a compactification of the space of representations of a group G into SL(2,$\mathbb{C}$) and we interpreted the boundary points in terms of valuations. In the next chapter, we will describe a construction due to Morgan-Shalen of a geometric object associated to a valuation v on the character variety. This construction extended one by Tits, and also by Bass-Serre of a simplicial tree associated to a discrete valuation of rank 1. Here the geometric object is a Λ-*tree*, an object similar to a tree but the distance takes its values in a general abelian ordered group Λ. We review now the basic elements of this theory (cf [15], [5]).

3.1. Λ-trees.

Λ-metric spaces. Let Λ be an abelian ordered group; denote by Λ^+ the set of its positive elements. A Λ-*metric space* is a set Z with a map $d : Z \times Z \to \Lambda^+$ which satisfies the axioms of a distance: $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$; and the triangular inequality $d(x, z) \leq d(x, y) + d(y, z)$. The simplest example is given by the group Λ itself, which is a Λ-metric space with the distance $d(\lambda_1, \lambda_2) = |\lambda_2 - \lambda_1| = \max\{\lambda_2 - \lambda_1, \lambda_1 - \lambda_2\}$. The isometry group of Λ with this distance is generated by the translations $\lambda \to \lambda + \delta$ together with the involution $\lambda \to -\lambda$.

A subset $I \subset \Lambda$ is called *an interval* when it is *convex*, that is if for all $\lambda_1 < \lambda_2 \in I$ the subset $[\lambda_1, \lambda_2] = \{\lambda \in \Lambda | \lambda_1 \leq \lambda \leq \lambda_2\}$ is contained in I. A *closed interval* is an interval of the form $[\lambda_1, \lambda_2] = \{\lambda | \lambda_1 \leq \lambda \leq \lambda_2\}$ that is when it contains both its upper and lower bounds. One defines the analogous notions of open, left-, right-open interval and denote them in a natural way by $]\lambda_1, \lambda_2]$, $]\lambda_1, \lambda_2]$, etc. Note that in non-archimedean group Λ, a bounded interval is not always of this type. Take for instance $\{p\} \times \mathbb{Z}$ in $\mathbb{Z} \times \mathbb{Z}$ endowed with the lexicographic order.

Let Z be a Λ-metric space. A *segment* in Z is a subset isometric to an interval I contained in Λ; a segment is closed (resp. open) when the interval I is closed (resp. open). When the upper or lower bounds of I are contained in I, the corresponding points in Z are called *endpoints.* A segment is *non-degenerate* when it is not reduced at a single point.

Definition 18. *A Λ-tree is a Λ-metric space $\mathcal{T}$ which satisfies the following three axioms:*

(i) *$\mathcal{T}$ is "uniquely connected by segments", i.e. any two points are the endpoints of a closed segment; such a closed segment is unique and will be denoted $x.y$;*
(ii) *when two segments have a common endpoint, then their intersection is a segment;*
(iii) *when two segments $x.y$ and $y.z$ have only the point y in commun, then the union $x.y \cup y.z$ is a segment.*

Remark. When $\Lambda = \mathbb{Z}$, we get back the notion of simplicial tree: any $\mathbb{Z}$-tree is isometric to the set of vertices of a simplicial tree. The notion of Λ-tree in the case $\Lambda = \mathbb{R}$ coincides with the standard notion of $\mathbb{R}$-tree as defined in [13]. In the sequel, we shall be interested in Λ-trees when Λ is the value group of a valuation which is not necessarily of rank 1.

A broken segment is a map $\mu : [\lambda, \lambda'] \to \mathcal{T}$ such that there exists a subdivision of $[\lambda, \lambda']$, $\lambda_0 = \lambda < \lambda_1 < \cdots < \lambda_n = \lambda'$ for which the restriction μ to the interval $[\lambda_i, \lambda_{i+1}]$ is an isometry to a closed segment of $\mathcal{T}$.

Proposition 19. *Let $\mathcal{T}$ be a Λ-tree and let $\mu : [\lambda, \lambda'] \to X$ be a broken segment. Then*

(i) *the segment between $\mu(\lambda)$ and $\mu(\lambda')$ is contained in the image of μ;*
(ii) *if $d(\mu(\lambda), \mu(\lambda')) = \lambda' - \lambda$, then μ is an isometry to its image.*

Proof. We argue by induction on the integer n appearing in the definition of a broken segment. The case $n = 1$ is trivial. We consider the case $n = 2$. By assumption, $\mu([\lambda, \lambda_1])$ and $\mu([\lambda_1, \lambda'])$ are segments. By the property (ii) of Λ-tree, their intersection is a segment which can be equally parameterized by the restrictions $\mu|[\alpha, \lambda_1]$, for some $\alpha \in [\lambda, \lambda_1]$ or by $\mu|[\lambda_1, \beta]$ with $\beta \in [\lambda_1, \lambda']$. Then the images $\mu([\lambda, \alpha])$ and $\mu([\beta, \lambda'])$ are segments which intersect only at $\mu(\alpha) = \mu(\beta)$. By the property (iii) of a Λ-tree, the union of these segments is a segment, so it is equal to the segment $\mu(\lambda).\mu(\lambda')$. This proves Proposition 19 (i). Since the length of the segment $\mu(\lambda).\mu(\lambda')$ is equal to $\lambda' - \lambda - 2(\lambda_1 - \alpha)$, Proposition 19 (ii) follows also. The case $n > 2$ reduces to the case $n = 2$, using the induction hypothesis. □

3.2. **Classification of isometries of a Λ-tree.**

Let $\mathcal{T}$ be a Λ-tree. An isometry of $\mathcal{T}$ is a bijection $g : \mathcal{T} \to \mathcal{T}$ such that $d(gx, gy) = d(x, y)$ for all x, y in $\mathcal{T}$. Isometries of $\mathcal{T}$ fall into three categories: *elliptic isometries*, *phantom inversions*, and *hyperbolic isometries.* In this section we explain this trichotomy and describe with details the structure of hyperbolic isometries.

Let g be an isometry of $\mathcal{T}$. Pick a point $x \in \mathcal{T}$ and consider the segment $x.gx$. The image of this segment by g is the segment $gx.g^2x$. From the axioms of a Λ-tree we deduce that the intersection $x.gx \cap x.g^2x$ is a closed segment: it is equal to $gy.gx$ for some $y \in x.gx$. The classification of g will be done according to the value of $d(y, x)$ compared to that of $d(x, gx)$.

—$2d(x,y) = d(x,gx)$. Then y is the midpoint of $x.gx$ and it is fixed by g. One says that g is *an elliptic isometry.* Observe that since g is an isometry, the set $Fix(g)$ of its fixed points is a convex subset of $\mathcal{T}$: for any two points x and y in $Fix(g)$, the segment $x.y$ is contained in $Fix(g)$.

— $2d(y,x) > d(x,gx)$. Then y and gy are exchanged by g. Therefore the segment $y.gy$ is mapped to itself by g but g reverses the order on that segment. If the distance $d(x,gx)$ is divisible by 2 in Λ then g fixes this midpoint. In that case g is an elliptic isometry. If this distance is not divisible by 2 g has no fixed points. It is called a *phantom inversion.*

We now suppose that g has no fixed points and is not a phantom inversion.

— $2d(y,x) < d(x,gx)$. Then y and gy are distinct and $y \in [x,gy]$, $gy \in [y,gx]$. The two closed segments $y.gy$ and $gy.g^2y$ thus intersect only at gy. Observe that the smallest convex subgroup Λ_y containing $d(y,gy)$ is the union $\bigcup_{n\in\mathbb{Z}}[nd(y,gy),(n+1)d(y,gy)]$. By induction on n we can define a map from Λ_y to $\mathcal{T}$ whose image is the reunion of segments $A_y = \bigcup_{\mathbb{Z}} g^n y.g^{n+1}y$ and which conjugates the action of g to the translation of length $d(y,gy)$ on Λ_y. In this case g is called a *hyperbolic isometry.*

Let us introduce the following notion. A *partial axis* for a hyperbolic isometry g is a segment $A \subset \mathcal{T}$ which is invariant by g and such that there exists an isometry from A onto an interval $I \subset \Lambda$ which conjugates the restriction $g|A$ to a translation on I. An example of a partial axis is the segment A_y defined above. *An axis for g* is a partial axis A which is *maximal* for the inclusion: any partial axis which contains A is equal to A.

Proposition 20. *If an isometry of $\mathcal{T}$ has a partial axis, then it has a unique axis, and this axis contains all partial axis.*

Proof. We first prove the following lemma.

Lemma 21. *Suppose that A_1 and A_2 are two partial axis for an isometry g. Then $A_1 \cup A_2$ is contained in a partial axis.*

Proof. Denote by δ_i the translation distance of g on A_i. Two cases appear according to $A_1 \cap A_2$ is empty or not.

—Suppose that $A_1 \cap A_2 = \emptyset$. This situation occurs for instance when $\mathcal{T}$ is the group $\Lambda = \mathbb{Z} \times \mathbb{Z}$ endowed with the lexicographic order, g is the translation by $(0,1)$, and A_1, A_2 are two vertical distinct segments $a_1 \times \mathbb{Z}$, $a_2 \times \mathbb{Z}$.

Choose arbitrary points $a_1 \in A_1$, $a_2 \in A_2$. Suppose first that some closed segment $k_1.a_2$ contained in $a_1.a_2$ has the property that its intersection with A_1 is reduced to k_1. In that case we shall reach a contradiction. By Proposition 19, the segment $a_2.ga_2$ is equal to the union of $a_2.k_1$, $k_1.gk_1$ and $gk_1.gk_2$; it follows that g moves the point a_2 by a length $d(a_2,ga_2) = 2d(a_2,k_1) + \delta_1$. Suppose now that the segment $a_2.k_1$ contains a point $a_2' \in A_2$ distinct from a_2 : in particular $d(a_2',k_1) < d(a_2,k_1)$. Since $d(a_2',ga_2') = 2d(a_2',k_1) + \delta_1 < d(a_2,ga_2)$, the restriction of g to A_2 cannot be conjugated a translation. This contradiction implies that $a_2.k_1 \cap A_2 = \{a_2\}$. But then for any point $a_2' \neq a_2$ on A_2, the segment $k_1.a_2'$ is the union of $k_1.a_2$ and the segment $a_2.a_2'$. In particular g moves the point a_2' at a distance $d(a_2',ga_2') = 2d(k_1,a_2') + \delta_1$ which is strictly greater than $d(a_2,ga_2)$: this is again a contradiction. We deduce that the intersections $a_1.a_2 \cap A_1$ and $a_1.a_2 \cap A_2$ are *ends* of A_1 and A_2 respectively. This means the following.

For $j = 1,2$, pick an isometry i_j from an interval $I_j \subset \Lambda$ to A_j which conjugates $g|A_j$ to a translation on I_j. Write $\alpha_j = i_j^{-1}(a_j) \in I_j$. Then after possibly exchanging the role of A_1 and A_2, one has: $i_1^{-1}(a_1.a_2 \cap A_1) = \{\lambda \in I_1 | \lambda \geq \alpha_1\}$ and $i_2^{-1}(a_1.a_2 \cap A_2) = \{\lambda \in I_2 | \lambda \leq \alpha_2\}$. In particular $A_j' = A_j - (A_j \cap a_1.a_2) \cup \{a_i\}$, $j = 1,2$ is a segment with endpoint a_j. The set $A_1 \cup A_2$ is thus the union of three segments A_1', $a_1.a_2$ and A_2' each intersecting the next at its endpoints: it is therefore a segment. This segment is invariant by g and g acts on it as a translation (since it does so on A_1 and on A_2). Therefore $A_1 \cup A_2 \cup a_1.a_2$ is a partial axis for g.

—Suppose that $A_1 \cap A_2 \neq \emptyset$. Denote as before $i_1 : I_1 \to A_1$ the isometry between an interval $I_1 \subset \Lambda$ and A_1. Then $i_1^{-1}(A_1 \cap A_2)$ is an interval J contained in I_1. The axioms of a Λ-tree imply that one of the three following possibilities does occur. (1): J has one endpoint, (2): J has two endpoints, or (3): each end of J is mapped by i_1 to an end of A_1 or A_2 (an *upper end* in a segment is a subsegment of the form $\{a | \, a \geq a_0\}$ for some a_0; one defines in an analogous way the notion of *lower end*.)

Since $A_1 \cap A_2$ is invariant under g the interval J is invariant by a non-trivial translation of Λ: this rules out (1) and (2). When both ends of J are mapped to ends of A_1 (resp. A_2), then $J = A_1$ and therefore $A_1 \subset A_2$ (resp. $A_2 \subset A_1$). When one end of J is mapped to an end of A_1, and the other to an end of A_2, then $A_1 \cup A_2$ is a segment. In that case $A_1 \cap A_2$ is a partial axis for g. □

We notice also the following consequence of the proof. Suppose that A_1 and A_2 are two partial axis for g with $A_1 \cap A_2 \neq \emptyset$. Then any isometry $i_1 : I_1 \to A_1$ which conjugates g to a translation on I_1 extends in a unique way to a isometry from an interval which contains I_1 to $A_1 \cup A_2$.

We continue the proof of Proposition 20. Let $A_0 \subset \mathcal{T}$ be a partial axis for g. Let $i_0 : I_0 \to A_0$ be the isometry between an interval $I_0 \subset \Lambda$ and A_0 which conjugates g and the translation of length δ. For any partial axis A_j which contains A_0 there is a unique interval $I_j \subset \Lambda$ and a unique isometry $i_j : I_j \to A_j$ such that $I_0 \subset I_j$ and $i_j|_{I_0} = i_0$. Let $\mathcal{J}$ be the set of all partial axis A_j which contain A_0. Denote by I the union of all intervals I_j as $j \in \mathcal{J}$; and by A the union of all partial axis A_j, for $j \in \mathcal{J}$. Then I is an interval of Λ since the interval which parameterizes $A_1 \cup A_2$ contains $I_1 \cup I_2$ thanks to the previous observation. Also the uniqueness of the map i_j implies that the restrictions of the isometries i_1 and i_2 to $I_1 \cap I_2$, are equal. Therefore the maps i_j, $j \in \mathcal{J}$ can be glued together to define an isometry $i : I \to \mathcal{T}$, the image of which is equal to A. It is clear that A is invariant by g and that the isometry i conjugates the translation by δ on I to the restriction of g to A. This shows that A is a partial axis. Since any partial axis for g is contained in a partial axis element of $\mathcal{J}$, we conclude that A is an axis. □

Length of an isometry. Let g be an isometry of a Λ-tree. The *length of* g is by definition the infimum of $d(x, gx)$ over all $x \in \mathcal{T}$. We denote it by l_g. When g is an elliptic isometry the infimum is attained and $l_g = 0$ by definition. When g is a phantom inversion l_g may or may not be equal to 0 and the infimum may or may not be attained.

Suppose g is an hyperbolic isometry which translates by a distance δ along its axis A_g; for all $x \in A_g$ $d(x, gx) = \delta$. When $x \notin A_g$, then the segment $x.gx$ intersects a partial axis A_x for g along a closed segment. Therefore there is some closed segment with endpoints x and a point $k \in A_x$ such that the intersection of $x.k$ and A_x is reduced to k. It follows that $d(x, g.x) = \delta_x + 2d(x, k)$, where δ_x is the length of g along the partial axis A_x. Since each partial axis for g is contained in A_g, we infer $\delta_x = \delta$. In particular for any $x \notin A_g$, $d(x, gx) > \delta$. We summarize what we proved in the following proposition.

Proposition 22. *Let g be a hyperbolic isometry of $\mathcal{T}$. Then $l_g = \inf\{d(x, gx), x \in \mathcal{T}\}$ is attained and $l_g > 0$. The axis A_g of g coincides with the set of points x such that $l_g = d(x, gx)$. Furthermore, for any point $x \notin A_g$ there exists $k \in A_g$ such that $x.k \cap A_g = \{k\}$ and $d(x, gx) = l_g + 2d(x, k)$.*

3.3. **Groups acting on Λ-trees.**

Length functions. Let G be a group and let $: G \times \mathcal{T} \to \mathcal{T}$, $(g, x) \mapsto gx$ be an action of G by isometries on a Λ-tree $\mathcal{T}$. *The length function* of this action is the function $G \to \Lambda^+$,

$g \mapsto l_g$. Since l_g only depends on the conjugacy class of g we can view this function as a function $\mathcal{C} \to \Lambda^+$, where $\mathcal{C}$ is the set of conjugacy classes of G.

Proposition 23. *Let G be a finitely generated group which acts by isometries on a Λ-tree and without phantom inversion. Then G has a fixed point if and only each of its elements has a fixed point (that is if the length-function of G is 0 : $l_g = 0$ for all $g \in G$).*

Proof. We argue by induction on the number of generators of G. Suppose that G is generated by n elements $g_1, ..., g_n$ and that $l_g = 0$ for all g. Then by assumption, the set of the fixed points of the group generated by $\langle g_1, ..., g_{n-1} \rangle$ is non-empty: it is a non-empty convex set C. To prove the proposition, we show that the fixed point set of g_n intersects C. We begin with the following observation.

Lemma 24. *Let g, h be isometries of $\mathcal{T}$ and let $x, y \in \mathcal{T}$ be fixed points of g and h respectively. Then either the segment $x.y$ contains a common fixed point of g and h or hg is an hyperbolic isometry.*

Proof. Let $x, y \in \mathcal{T}$ with $gx = x$ and $hy = y$. The intersection $x.y \cap x.gy$ is a closed segment $x.x'$. Since g is an isometry any point in $x.x'$ is fixed by g and the intersection $x'.y \cap x'.gy$ is equal to x'. The same property is satisfied by the intersection $x.y \cap hx.y$. In this way, we find a segment $x'.y'$ contained in $x.y$ where

(i) $gx' = x'$ and $hy' = y'$;
(ii) $x'.y' \cap x'.gy' = \{x'\}$; $x'.y' \cap hx'.y' = \{y'\}$.

If the segment $x'.y'$ is degenerate then $x' = y'$ is fixed by g and h. If it is non-degenerate, consider the union $x'.y' \cup y'.hx'$: it is the segment $x'.hx'$. One deduces from (ii) that $x'.y'$ intersects its image by hg only at hx'. This implies that the non-degenerate segment $x'.hx'$ is a fundamental domain for hg; in particular hg is an hyperbolic isometry. □

Let $x \in \mathcal{T}$ fixed by $g_1, ...$ and g_{n-1}; let $y \in \mathcal{T}$ fixed by g_n. After applying the Lemma to these x and y, we obtain either a point fixed by the entire group G or an hyperbolic element of the form $g_n g_i$ for some $i \leq n-1$. □

Minimal action. Let $G \times \mathcal{T} \to \mathcal{T}$ be an isometric action without phantom inversion of a group G on a Λ-tree $\mathcal{T}$. The action is said to be *minimal* if any Λ-tree $\mathcal{T}' \subset \mathcal{T}$ which is invariant by G is equal to $\mathcal{T}$. There exists an invariant subtree contained in $\mathcal{T}$ on which the action of G is minimal.

Small actions. An action $G \times \mathcal{T} \to \mathcal{T}$ is *small* when for any non-degenerate closed segment $x.y \subset \mathcal{T}$, the subgroup of G which fixes $x.y$ pointwise is "small" in the sense that it is virtually abelian. Recall that a group is said *virtually abelian* when it contains an abelian subgroup with finite index. For instance any free action is small.

Remark. We will be mainly interested in groups which can be embedded into $\mathrm{SL}(2, \mathbb{C})$ as discrete subgroups. Such a group is either virtually abelian or contains a subgroup isomorphic to a free group with two generators. In particular, if a group is not virtually abelian its commutator subgroup is not virtually abelian either. This motivates the assumption made in the following result.

Proposition 25. *Let $G \times \mathcal{T} \to \mathcal{T}$ be a minimal action of a group G on a Λ-tree $\mathcal{T}$ such that any subgroup of G which stabilizes a non-degenerate segment is small. Suppose that g and h are hyperbolic elements of G such that $\langle g, h \rangle$ and its commutator subgroup are not small. Then,*

(1) *if the axis A_g and A_h intersect their intersection $A_g \cap A_h$ is a closed segment, and*

(2) *if the axis A_g and A_h are disjoint, there is a closed non-degenerate segment $k.l \subset \mathcal{T}$ with $k.l \cap A_g = \{k\}$, $k.l \cap A_h = \{l\}$.*

Proof. Suppose that $A_g \cap A_h \neq \emptyset$. Since A_g is an axis, there exists an interval $I \subset \Lambda$ and an isometry $\iota : I \to A_g$ which conjugates g to the translation by l_g on I. Then $\iota^{-1}(A_g \cap A_h)$ is a convex subset of I, and is therefore equal to a sub-interval $J \subset I$. Denote by $\{0\} = \Lambda_0 \subset \Lambda_1 ... \subset \Lambda_r = \Lambda$ the sequence of convex subgroups of Λ. Up to translation, one can assume that J is contained in Λ_s but not in Λ_{s+1}. Pick $\xi \in J$, and consider the interval $[\xi, \infty[\cap J \subset \Lambda_s$ with $[\xi, \infty[= \{\lambda \in \Lambda_s | \, \xi \leq \lambda \leq \infty\}$. We will show that this interval is closed. The same reasoning applied to the interval $]\infty, \xi] \cap J$ will imply (1).

Suppose by contradiction that the interval $[\xi, \infty[\cap J$ is not a closed interval of Λ_s. We may then assume that the segment $\iota([\xi, \infty[\cap J)$ is an end of A_g and that it is mapped into itself by all positive powers of g.

— Suppose $\iota([\xi, \infty[\cap J)$ is also an end of A_h, stabilized by all positive powers of h. Since g and h act by translation on $\iota([\xi, \infty[\cap J)$ any element in the commutator group of $\langle g, h\rangle$ acts as the identity on an end of $\iota([\xi, \infty[\cap J)$. Since the action of G on $\mathcal{T}$ is small the commutator subgroup of $< g, h >$ is virtually abelian. This contradicts our assumption.

— Suppose now that $\iota([\xi, \infty[\cap J)$ is not an end of A_h. One can find a point $y \in A_h$ such that the segment $x.y$ contains $\iota([\xi, \infty[\cap J)$. Then $x.y$ contains the upper end of A_g. By invariance, $gx.gy$ also contains the upper end of A_g. Since $x.gy = x.gx \cup gx.gy$, the intersection $x.y \cap x.gy$ contains the upper end of A_g. As $\mathcal{T}$ is a Λ-tree, the intersection $x.y \cap x.gy$ is a closed segment $x.z$. If $z \in A_g$, then $gz \in A_g$, and so $x.gz$ is contained in $x.y \cap x.gy$. But z is between x and gz, hence $x.gz$ is strictly contained in $x.y \cap x.gy$. We conclude that $z \notin A_g$.

On the other hand, one has $z \in x.y$ hence $gz \in x.gy$. Since g is an isometry $d(gz, gy) = d(z, y)$, $d(x, y) = d(gx, gy) = d(x, y) - d(x, gx)$, therefore, $d(z, y) = d(z, gy) - d(x, gx)$. We infer $gz \in z.gy$, and it follows that $d(z, gz)$ is equal to $d(x, gx)$ the length of g, hence $z \in A_g$. This gives a contradiction and proves that $\iota([\xi, \infty[\cap J)$ is a closed segment. This concludes the proof of (1).

To prove (2) choose points a and b in $\mathcal{T}$ with $a \in A_g$ and $b \in A_h$. Let $k \in A_h$ be the point provided by Proposition 22 such that the intersection of the segment $k.b$ and A_g is equal to $\{k\}$. In a similar way let $l \in A_h$ be the point such that the intersection of $a.l$ and A_h is reduced to $\{l\}$. Then the segment $k.l$ satisfies (2). □

The segment $k.l$ defined in Proposition 25 (2) is the shortest among all segments $a.b$ connecting a point $a \in A_g$ and a point $b \in A_h$. Indeed, the axioms of a Λ-tree easily imply that $a.b$ is the union of the three segments $a.k$, $k.l$ and $l.b$, with $a.k \cap k.l = \{k\}$ and $k.l \cap l.b = \{l\}$. We will call the length of $k.l$ the *distance* between the two axis A_g and A_h and we will denote it by $d(A_g, A_h)$.

Lemma 26. *For any hyperbolic isometries a, b on a Λ-tree, the distance between the two axis A_a and A_b is determined by the lengths of the elements a, b and ab. More precisely, one has*

$$d(A_a, A_b) = \frac{1}{2} \max(0, l_{ab} - l_a - l_b). \tag{1}$$

Proof. the proof is the same as for $\mathbb{R}$-trees. We are reduced to the case when the axis are disjoint. Then let $k.l$ be the shortest segment with $k \in A_a$ and $b \in A_b$ provided by Proposition 25 (2). A fundamental domain for ab on its axis is then the union of the segments $a^{-1}k.k$, $k.l$ $l.bl$ and $bl.bk$. Therefore the length of ab is $l_a + l_b + 2d(k, l)$ proving (1). □

3.4. **Conjugated actions.**

We will now generalize to arbitrary Λ-trees a fundamental result first proved by Culler and Morgan in the case of $\mathbb{R}$-trees (see [9, Theorem 3.7]). Our proof in the general case closely follows their one, and relies in an essential way on Proposition 25.

Theorem 27. [9] *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be Λ-trees and $\rho_1 : G \times \mathcal{T}_1 \to \mathcal{T}_1$ and $\rho_2 : G \times \mathcal{T}_2 \to \mathcal{T}_2$ be small and minimal actions of a group G on $\mathcal{T}_1$ and on $\mathcal{T}_2$. Then there is an isometry between $\mathcal{T}_1$ and $\mathcal{T}_2$ which conjugates ρ_1 and ρ_2 if and only if the length-functions of these actions are equal.*

Being equivariantly isometric the existence of an isometry $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ such that $\phi(\rho_1(g)x) = \rho_2(g)\phi(x)$ for all $g \in G$ and all $x \in \mathcal{T}_1$.

Proof. We follow the main steps of the proof in [9]. One ingredient in their proof is the classification (up to isometry) of groups acting on trees in terms of their based length-functions. This classification is due to Chiswell [4, 5], see also [23, p. 22]. In what follows, we do not rely on this classification although our argument are very close in essence.

— One can find two elements $g, h \in G$ which form *a good pair of isometries of* $\mathcal{T}_1$. This means they satisfy the following three properties:

(1) the axis A^1_g and A^1_h have non-empty intersection ;
(2) the intersection of $A^1_g \cap A^1_h$ is a closed segment and g and h move in the same direction along this segment;
(3) the length of $A^1_g \cap A^1_h$ is shorter than l_g and l_h.

To see this, pick two hyperbolic isometries a and b in G, and denote by A^1_a and A^1_b their respective axis in $\mathcal{T}_1$. If $A^1_a \cap A^1_b = \emptyset$, then one checks using Proposition 25 (2) that the axis of the isometries a and ab have non-empty intersection. The intersection of these axis in then a closed segment by Proposition 25 (1). Up to replacing a by a^{-1} if necessary, both isometries a and ab translate in the same direction. After taking sufficiently large positive powers these isometries, (3) is satisfied.

— Since the length-functions of ρ_1 and ρ_2 are equal, g and h form also a good pair of isometries of $\mathcal{T}_2$.

—Let $p_i \in \mathcal{T}_i$ be the unique point common to the three axis A^i_g, A^i_h and $A^i_{gh^{-1}}$ (it is the "upper" endpoint of the closed segment $A^i_g \cap A^i_h$ if the isometries move in the positive direction). Then there is a formula for the displacement distance of p_i under an element $k \in G$: $d(p_i, kp_i)$ is the maximum of the distances between one of the axis A^i_g, A^i_h, $A^i_{gh^{-1}}$ and the image by k of one of these axis. To see this, consider the segment $p_i.kp_i$. Then for at least one of the axis $C \in \{A^i_g, A^i_h, A^i_{gh^{-1}}\}$, the intersection $C \cap p_i.kp_i$ is equal to $\{p_i\}$. A similar result holds for the intersection with the other three axis.

— It follows that the *displacement function* (or *Chiswell length-function*) of p_i in $\mathcal{T}_i$, that is the function from G to Λ, defined by $k \to d(p_i, kp_i)$ is determined by the length-function of the action $\rho_i : G \times \mathcal{T}_i \to \mathcal{T}_i$. In particular the displacement function of p_1 in $\mathcal{T}_1$ is equal to that of p_2 in $\mathcal{T}_2$.

— Let us now explain the argument of Chiswell for constructing at this point an explicit isometry from $\mathcal{T}_1$ to $\mathcal{T}_2$. Pick any $x_1 \in \mathcal{T}_1$. By minimality, there exists $g \in G$ such that $x_1 \in p_1.gp_1$. Since $d(p_1, gp_1) = d(p_2, gp_2)$, there is a unique point denoted $x_2 \in \mathcal{T}_2$ such that x_2 lies in the segment $p_2.gp_2$ and satisfies $d(p_2, x_2) = d(p_1, x_1)$. Let us check that this definition does not depend on the choice of g. Let g' be any other element of G such that $x_1 \in p_1.g'p_1$. Since the displacement functions of G in $\mathcal{T}_1$ and in $\mathcal{T}_2$ are the same, one has $d(p_1, gp_1) = d(p_2, gp_2)$, $d(p_1, g'p_1) = d(p_2, g'p_2)$ and $d(gp_1, g'p_1) = d(gp_2, g'p_2)$. Therefore the length of the common part of the segments $p_1.gp_1$ and $p_1.g'p_1$ is equal to that of the common part of $p_2.gp_2$ and $p_2.g'p_2$. This implies that the definition of the point x_2 is whether one uses g' or g. Set $\Psi(x_1) = x_2$. One easily checks that Ψ is an isometry. By definition $\Phi(gp_1) = gp_2$, thus Ψ is G-equivariant in restriction to the orbit Gp_1. Since Ψ is an isometry, Ψ is also G-equivariant on $\mathcal{T}_1$. This proves Proposition 27. $\square$

4. **The Bass-Serre tree associated to a valuation of F/k.**

4.1. **The Λ-tree of the lattices in a two dimensional F-vector space.**

Let F/k be a field extension. Let v be a valuation of F/k, $\mathfrak{o}_v$ its valuation ring, and Λ its value group. Denote by V the n-dimensional F-vector space F^n. A *lattice* or $\mathfrak{o}_v$*-lattice* in V is a $\mathfrak{o}_v$-module $L \subset V$ of the form $L = \mathfrak{o}_v e_1 \oplus \cdots \oplus \mathfrak{o}_v e_n$ for some basis $e_1, \cdots e_n$ of V. When we take as basis the canonical basis of $V = F^n$, we obtain the *standard lattice.* By definition, any lattice is a free $\mathfrak{o}_v$-lattice of rank n.

We say that two lattices L and L' in V are *equivalent* when they differ by an homothety of V: for some $\alpha \in F$, $L' = \alpha L$. Denote by $[L]$ the equivalence class of L.

Our aim is now to show that when $n = 2$, the set $\mathcal{T}_v$ of equivalence classes of lattices in V has the structure of a Λ-tree. We define first a Λ-distance on $\mathcal{T}_v$. Let L, L' be two lattices in V. Up to replacing L' by an homothetic lattice we can suppose that $L' \subset L$ and that L' has a basis

$$(e'_1, e'_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (e_1, e_2)$$

where (e_1, e_2) is a basis of L and where the coefficients a, b, c and d are in $\mathfrak{o}_v$. Up to permuting the basis vectors if necessary, we may also impose the condition $v(a) = \min(v(a), v(b), v(c), v(d))$ (with the convention $v(0) = +\infty$). Then a divides in the ring $\mathfrak{o}_v$ all others coefficients of the matrix so that after operations on the rows and colums, we obtain a new basis for L'

$$(e'_1, e'_2) = \begin{pmatrix} a & 0 \\ 0 & d - \frac{b}{a}c \end{pmatrix} (e_1, e_2). \tag{2}$$

In that case, the representative $L'_0 = \dfrac{1}{a} L'$ of the equivalence class $[L']$ satisfies $L/L'_0 \simeq \mathfrak{o}_v / \beta \mathfrak{o}_v$ with $\beta = \dfrac{da - bc}{b^2}$.

We introduce the following terminology: a lattice M' contained in a lattice M such that $M/M' \simeq \mathfrak{o}_v / \gamma \mathfrak{o}_v$ is said *cocyclic* in M. We claim that there exists a unique lattice in the equivalence class $[L']$ which is cocyclic in L. We already proved the existence of such a cocyclic lattice L'_0. Suppose now that L'_1 is another lattice cocyclic in L. Then $L'_1 = \alpha L'_0$ for some $\alpha \in F^*$. Assume by contradiction that $v(\alpha) \neq 0$. We can assume $v(\alpha) > 0$, so that $L'_1 \subset L'_0 \subset L$. It follows that $L/L'_1 \simeq \mathfrak{o}_v/\alpha\mathfrak{o}_v \oplus \mathfrak{o}_v/\alpha\beta\mathfrak{o}_v$. This contradicts the fact that L'_1 cannot be cocyclic in L since $v(\alpha) > 0$ and $v(\beta) \geq 0$. Whence $v(\alpha) = 0$, α is a unit in $\mathfrak{o}_v$, and we conclude $L'_1 = L'_0$ as required.

Note that different choices of the basis of L might possibly affect β but only by multiplying it with a unit, therefore $v(\beta)$ is well-defined. For any two equivalence classes $[L], [L'] \in \mathcal{T}_v$ given by two basis satisfying relation (2), we set:

$$d([L], [L']) = v(ad - bc) - 2\min(v(a), v(b), v(c), v(d)). \tag{3}$$

The preceding discussion shows this expression does not depend on the particular representatives of the classes $[L]$ and $[L']$.

Proposition 28. *The function $d : \mathcal{T}_v \times \mathcal{T}_v \to \Lambda$ is a Λ-distance for which $\mathcal{T}_v$ is a Λ-tree.*

Proof. We first prove that d is a Λ-distance. We keep the same notations as above. Suppose $d([L],[L'])=0$ for two lattices L, L', with L' cocyclic in L. Then $L/L' \simeq \mathfrak{o}_v/\beta\mathfrak{o}_v$ with $v(\beta)=0$, hence $L=L'$. The symmetry axiom is easy to check. To check the triangular inequality, consider three equivalence classes x, x' and x'' in $\mathcal{T}_v$. Let L' be a representative of x' and (e,f) be a basis of L', such that a representative of x is the lattice L generated by e and αf, for some $\alpha \in \mathfrak{o}_v$. Similarly take (u,v) a basis of L' and a representative of x'' generated by the two vectors u and βv, for some $\beta \in \mathfrak{o}_v$. We can write $(u,v)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}(e,f)$ for some invertible matrix with coefficients in $\mathfrak{o}_v$, hence

$$(u,\beta v)=\begin{pmatrix} a & \frac{b}{\alpha} \\ \beta c & \frac{\beta d}{\alpha}\end{pmatrix}(e,\alpha f).$$

A particular representative of $[L'']$ contained in L has basis $(\alpha u, \alpha\beta v)=M(e,\alpha f)$ with $M=\begin{pmatrix} \alpha a & b \\ \alpha\beta c & \beta d\end{pmatrix}$. Since all coefficients of M lie in $\mathfrak{o}_v$ we can apply (3), and we infer $d([L],[L''])\leq v(\det(M))=v(ad-bc)+v(\alpha)+v(\beta)=v(\alpha)+v(\beta)$. This proves the triangular inequality, and show that $\mathcal{T}_v$ is a Λ-metric space.

Suppose that equality holds in the triangular inequality for three distinct equivalence classes: $d(x,x'')=d(x,x')+d(x',x'')$. Keeping the same notations as above, the three lattices x, x' and x'' can be represented respectively by $L=\mathfrak{o}_v e\oplus\mathfrak{o}_v\alpha f$, $L'=\mathfrak{o}_v\alpha u\oplus\mathfrak{o}_v\alpha v$ and $L''=\mathfrak{o}_v\alpha u\oplus\mathfrak{o}_v\alpha\beta v$ with $v(\alpha), v(\beta)>0$. Now if equality holds in the triangular inequality then the minimum of the valuations of the coefficients of the matrix M must be 0. This implies $v(b)=0$ hence (u,v) can be chosen to be $u=f$, $v=e$.

We conclude that $d(x,x'')=d(x,x')+d(x',x'')$ if and only if there is a basis (e_1,e_2) of L such that L', L'' are generated respectively by $(e_1,\alpha e_2)$ and $(e_1,\alpha\beta e_2)$ with $d(x,x')=v(\alpha)$ and $d(x',x'')=v(\beta)$.

We now check that $\mathcal{T}_v$ satisfies the three axioms of a Λ-tree.

— We first construct a segment connecting any two points $x,x'\in\mathcal{T}_v$. Let L', L be any representatives of x and x' such that $L'\subset L$ and cocyclic in L. There is a basis (e,f) of L such that L' is the lattice $\mathfrak{o}_v e\oplus\mathfrak{o}_v\beta f$, with $\beta\in\mathfrak{o}_v$ and $d(x,x')=v(\beta)$. For any z in the interval $[0,v(\beta)]\subset\Lambda_v$ choose a $\gamma_z\in\mathfrak{o}_v$ with $v(\gamma_z)=z$. Consider the lattice $L_z=\mathfrak{o}_v e\oplus\mathfrak{o}_v\gamma_z f$. By construction one has: $d([L_z],[L_{z'}])=|v(\gamma'_z)-v(\gamma_z)|$. So the map $z\to[L_z]$ from $[0,v(\beta)]$ to $\mathcal{T}_v$ is an isometry onto a segment which connects x and x'. The fact that this segment is unique follows from the characterization of the equality case in in the triangular inequality. Observe that the segment $x.x'$ coincides with the set of equivalence classes of lattices L'' such that $L'\subset L''\subset L$.

— Let $x_1.z$ and $x_2.z$ be any two closed segments having common endpoint z. Consider the representatives L_1, L_2 and L_0 of x_1, x_2 and z such that $L_1\subset L_0$ and $L_2\subset L_0$ and both lattices L_1, L_2 are cocyclic in L_0. Then $x_i.z=\{[L]|\, L_i\subset L\subset L_0\}$ for $i=1,2$ therefore $x_1.z\cap x_2.z=\{[L]|\, L_1+L_2\subset L\subset L_0\}$. The sum L_1+L_2 is an $\mathfrak{o}_v$-module contained in L_0 which is torsion free and finitely generated: therefore L_1+L_2 is a lattice. So the intersection of $x_1.z$ and $x_2.z$ is the segment with endpoints $[L_0]$ and $[L_1+L_2]$.

— Suppose that $x_1.z$ and $x_2.z$ are any two closed segments intersecting only at z. We need to show that the union of $x_1.z$ and $z.x_2$ is again a segment. Keeping previous notations, we know that L_1+L_2 represents z, i.e. is equivalent to L_0. But since L_1+L_2 contains L_1 and is contained in L_0, it must be cocyclic in L_0, hence $L_1+L_2=L_0$. We can write $L_1=\mathfrak{o}_v e_1\oplus\mathfrak{o}_v\beta_1 f_1$ and $L_2=\mathfrak{o}_v e_2\oplus\mathfrak{o}_v\beta_2 f_2$ where (e_1,f_1) and (e_2,f_2) are basis of L_0. We have $v(\beta_i)>0$ since the segments are non-degenerate. Let $\mathfrak{m}_v=v^{-1}\{>0\}$ be the maximal ideal of $\mathfrak{o}_v$. Since $\mathfrak{o}_v e_1+\mathfrak{o}_v e_2+\mathfrak{m}L_0=L_0$, we must have $\mathfrak{o}_v e_1+\mathfrak{o}_v e_2=L_0$.

Hence the vectors e_1 and e_2 form a basis of the lattice L_0. This allows to write L_1 and L_2 as $L_1 = \mathfrak{o}_v e_1 \oplus \mathfrak{o}_v \beta_1 e_2$ and $L_2 = \mathfrak{o}_v e_2 \oplus \mathfrak{o}_v \beta_2 e_1$. From the previous description of the segments in $\mathcal{T}_v$ we deduce that $[L_0]$ belongs to the segment $[L_1].[L_2]$. Therefore the union of $x_1.z$ and $z.x_2$ is the segment $[L_1.L_2]$. □

4.2. **The action of** $\mathrm{SL}(2,\mathrm{F})$**.**

The group GL(2,F) naturally acts on the set of lattices preserving equivalence classes. It acts therefore on the Λ-tree $\mathcal{T}_v$ and this action is isometric. The action of GL(2,F) is transitive on lattices and therefore also on $\mathcal{T}_v$: any point in $\mathcal{T}_v$ is equivalent to the class x_0 of the standard $\mathfrak{o}_v$-lattice in F^2. Pick $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ any element of $\mathrm{SL}(2,\mathrm{F})$. Since the determinant of g has valuation 0, relation (3) gives: $d(x_0, g.x_0) = -2\min(v(a), v(b), v(c), v(d))$. It follows that $d(x, g.x)$ is divisible by 2 in Λ for any point $x \in \mathcal{T}_v$. In particular g is not a phantom inversion. Whence $\mathrm{SL}(2,\mathrm{F})$ acts on $\mathcal{T}_v$ by hyperbolic or elliptic isometries.

Elliptic isometries of $\mathcal{T}_v$**.** Let $g \in \mathrm{SL}(2,\mathrm{F})$ be an elliptic isometry. Let $x = [L] \in \mathcal{T}_v$ be a point fixed by g: $gL = \alpha L$ for some $\alpha \in k^*$. Suppose that $v(\alpha) \neq 0$. After possibly replacing g by g^{-1} we can assume that $v(\alpha) \geq 0$; then $gL \subset L$. Since g has determinant 1 $L/gL = \{0\}$. However, since $v(\alpha) > 0$, $L/\alpha L$ is not zero, since its dimension as a vector space over the field $\mathfrak{o}_v/\mathfrak{m}_v$ is equal to 2. Therefore $v(\alpha) = 0$. This means that $gL = L$: the expression of g on any basis of L has coefficients in $\mathfrak{o}_v$. Note that L is the image of the standard lattice by some element of $\mathrm{GL}(2,\mathrm{F})$. We have thus proved that the elements of $\mathrm{SL}(2,\mathrm{F})$ acting by elliptic transformations on $\mathcal{T}_v$ are those which are conjugated in $\mathrm{GL}(2,\mathrm{F})$ to elements of $\mathrm{SL}(2,\mathfrak{o}_v)$. In particular their trace lies in $\mathfrak{o}_v$. Conversely any element of $\mathrm{SL}(2,\mathrm{F})$ whose trace belongs to $\mathfrak{o}_v$ is conjugated in $\mathrm{GL}(2,\mathrm{F})$ to an element in $\mathrm{SL}(2,\mathfrak{o}_v)$ (consider the matrix of this element on the basis (e, ge) for a vector e which is not an eigenvector of g).

The following is now a direct consequence of Proposition 23.

Proposition 29. *Let $G \subset \mathrm{SL}(2,\mathrm{F})$ be a finitely generated subgroup such that the trace of any $g \in G$ belongs to $\mathfrak{o}_v$ (i.e. $v(\mathrm{tr(g)}) \geq 0$). Then G is conjugated in* GL(2,F) *to a subgroup of* $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v})$.

Hyperbolic isometries of $\mathcal{T}_v$**.** Take $g \in \mathrm{SL}(2,\mathrm{F})$. For any $x \in \mathcal{T}_v$ let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix representation of g in a basis of a representative of x. By (3), we have $d(x, gx) = -2\min(v(a), v(b), v(c), v(d))$. In particular $d(x, g.x) \geq -2\min(v(a), v(d)) \geq -2v(a+d) = -2v(\mathrm{tr(g)})$. Pick a vector e which is not an eigenvector of g. The $\mathfrak{o}_v$-lattice generated by e and $g.e$ satisfies: $d([L], [gL]) = -2v(\mathrm{tr(g)})$. If $v(\mathrm{tr(g)}) < 0$, the minimum displacement of g is thus positive: thus g is an hyperbolic isometry with translation distance $-2v(\mathrm{tr(g)})$ and axis consisting of all the equivalence classes of lattices of the form $\mathfrak{o}_v e \oplus \mathfrak{o}_v ge$.

Segment-stabilizers. Consider the subgroup $G_{x.x'}$ of $\mathrm{SL}(2,\mathrm{F})$ which fixes a non-degenerate segment $s = x.x'$. Let $[L]$ and $[L']$ be respective representatives of x and x' with L' cocyclic in L and pick $(e, f) \in L$ such that $L' = \mathfrak{o}_v e \oplus \mathfrak{o}_v.\beta f$. The matrix expression on the basis (e, f) of the stabilizer of x is the group $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v})$ and $G_{x.x'}$ is isomorphic to the subgroup G_s of $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v})$ which fixes L'. Therefore G_s consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathfrak{o}_\mathrm{v})$ with $v(c) \geq v(\beta)$. The group G_s is thus identified with the preimage of the triangular group $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ under the canonical homomorphism $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v}) \to \mathrm{SL}(2,\mathfrak{o}_\mathrm{v}/\beta\mathfrak{o}_\mathrm{v})$.

We have thus proved.

Proposition 30. *Let $G_{x.x'} \subset \mathrm{SL}(2,\mathrm{F})$ be the stabilizer of a non-degenerate segment $x.x'$ of length $v(\beta)$, with $\beta \in F$. Then $G_{x.x'}$ is conjugated in $\mathrm{GL}(2,\mathrm{F})$ to a subgroup G_s of $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v})$ such that the commutator subgroup $[G_s, G_s]$ maps to the identity under the homomorphism $\mathrm{SL}(2,\mathfrak{o}_\mathrm{v}) \to \mathrm{SL}(2,\mathfrak{o}_\mathrm{v}/\beta\mathfrak{o}_\mathrm{v})$.*

4.3. Applications to the character variety.

Let X_0 be an irreducible component of the character variety $X(G)$. Recall that to any sequence of points (x_i) in X_0 tending to infinity is associated a valuation on $\mathbb{Q}(X_0)/\mathbb{Q}$ (cf. section 2.3). Our aim is to attach to any such valuation an action of G on a Λ-tree.

It is technically easier to work with the space of representations than with the character variety. In doing so, we shall use the following classical result.

Proposition 31. [37, Chap. 6, §4], [3, Chap. 6] *Let K/k be a field extension and v be a valuation of K/k. Let K' be an extension of K. Then there is a valuation v' on K'/k extending v. Furthermore one can choose v' such that its value group Λ' is a finite extension of the value group Λ of v: the index of Λ in Λ' is finite.*

Proof. In the sequel, we shall only make use of the case K'/K has finite transcendental degree. We hence give a proof only under this assumption. The extension K'/K is the composition of a purely transcendental extension and a finite extension. By induction, it suffices to consider the case of an extension of transcendental degree 1. The case of a purely transcendental extension $K' = K(x)$ can be solved in an explicit way. Define for a non-zero polynomial $P(x) = \sum a_j x^j$. Clearly this defines a valuation on $K[x]$ that extends to $K(x)$ and has the same value group as v.

Suppose now K'/K is a finite extension. Consider the set $\mathcal{R}$ of all rings $R \subset K'$ that contain $\mathfrak{o}_v = \{v \geq 0\} \subset K$ and such that $R.\mathfrak{m}_v \neq R$ with $\mathfrak{m}_v = \{v > 0\}$. The set $\mathcal{R}$ is partially ordered by the inclusion and one easily checks that any totally ordered subset (R_α) has an upper bound. Hence by Zorn's Lemma $\mathcal{R}$ admits a maximal element R' which satisfies $R'.\mathfrak{m}_v \neq R'$.

Claim 32. *The ring R' has the following property: if $x \in K'$ is $\neq 0$, then either x or $\dfrac{1}{x}$ is in R'.*

Granting this claim we finish the proof. Denote by $\mathfrak{m}'$ the subset of R' consisting of 0 and all inverses of elements of $K' - R'$. Since R' satisfies the claim $\mathfrak{m}'$ is a maximal ideal (see for instance [37, Theorem 1, Ch. VI]). Therefore R' is a place of K', whence the valuation ring of a valuation v'. Let us prove that $K \cap R' = \mathfrak{o}_v$. Suppose by contradiction that $x \in K - \mathfrak{o}_v$ lies in R'. Then we have $x^{-1} \in \mathfrak{m}_v$, so $1 = x.x^{-1}$ lies in $R'.\mathfrak{m}_v$, which is absurd. In the same way we get $K \cap \mathfrak{m}' = \mathfrak{m}_v$ and v' is an extension of v. Notice that the extension v' is not unique (cf. [34] for a description of the possible extensions in the degree 1 case). Suppose that Λ'/Λ contains m elements represented by $v(x_1), \cdots, v(x_m)$. Then the x_j's must be linearly independent over K; in particular, m is less than the degree of the algebraic extension K'/K. Therefore the index of Λ in Λ' is less than the degree of K'/K. □

Proof of the claim. By the maximality property of R' it suffices to show that either $R'[x].\mathfrak{m}_v \neq R'[x]$ or $R'[x^{-1}].\mathfrak{m}_v \neq R'[x^{-1}]$. Assume by contradiction that both are not satisfied. Then there are polynomial relations $1 = a_0 + \cdots + a_n x^n$, $1 = b_0 + \cdots + b_m (x^{-1})^m$, with coefficients a_i and b_j in the ideal $\mathfrak{m}_v$. We may assume that $m \leq n$ and that the degrees m and n are the smallest possible. The first equation gives: $1 - b_0 = a_0(1 - b_0) + \cdots + a_n(1 - b_0)x^n$ and the second $(1 - b_0)a_n x^n = b_1 a_n x^{n-1} + \cdots + b_m a_n x^{n-m}$. After adding these equations and simplifying we get a polynomial equation of degree $\leq n-1$ satisfied by x. This is a contradiction. □

We are now ready for the main result of this section.

Theorem 33. *Let $\rho_i : G \to \mathrm{SL}(2,\mathbb{C})$ be a sequence of representations which does not contain a sequence which converges up to conjugacy. Then there is a Λ-tree $\mathcal{T}$ and a minimal action by isometries $G \times \mathcal{T} \to \mathcal{T}$ such that (ρ_i) converges to $\mathcal{T}$ in the following sense: for any conjugacy classes g, h in G with $\delta_{\mathcal{T}}(h)$ is > 0, one has:*

$$\lim_{i\to\infty} \frac{\log|(tr(\rho_i(g))|+2)}{\log(|tr(\rho_i(h))|+2)} = \frac{\delta_{\mathcal{T}}(g)}{\delta_{\mathcal{T}}(h)}. \tag{4}$$

Proof. Denote by $x_i = t(\rho_i) \in X(G)$ the character of ρ_i. We may assume that (x_i) is contained in the same irreducible component X_0 of $X(G)$. By Proposition 5 and since (ρ_i) does not contain subsequences which converge up to conjugacy, the sequence of characters (x_i) is unbounded. We may assume that (x_i) is a valuating sequence and defines a valuation v on $\mathbb{Q}(X_0)/\mathbb{Q}$. Recall that we defined $\mathcal{I}_g(\rho) = \mathrm{tr}(\rho(\mathrm{g}))$. By Proposition 16, if $v(\mathcal{I}_h) < 0$ then

$$\lim_{i\to\infty} \frac{\log(|\mathcal{I}_g(x_i)|+2)}{\log|(\mathcal{I}_h(x_i)|+2)} = \frac{v(\mathcal{I}_g)}{v(\mathcal{I}_h)}. \tag{5}$$

Let $\mathcal{R}_0$ be a component of $\mathcal{R}(G)$ mapped by t onto a Zariski open subset of X_0. By Lemma 31, we may pick a lift v' of v to a valuation on $\mathbb{Q}(\mathcal{R}_0)/\mathbb{Q}$. Denote by $\mathcal{T}$ the Bass-Serre tree of $\mathrm{SL}(2,\mathbb{Q}(\mathcal{R}_0))$ associated to v'. We may identify G with a subgroup of $\mathrm{SL}(2,\mathbb{Q}(\mathcal{R}_0))$ via the tautological representation. Therefore G acts on the Λ-tree $\mathcal{T}$. Since (x_i) is unbounded this action has no global fixed point. Up to replacing $\mathcal{T}$ by a subtree we may assume that this action is minimal.

Now note that for any $g \in G$ we have $-2\min(0, v(\mathrm{tr}(g))) = -2\min(0, v'(\mathrm{tr}(g))) = \delta_{\mathcal{T}}(g) > 0$ (see §4.2). We conclude by comparing (4) and (5) using $\mathcal{I}_g(t(\rho)) = \mathrm{tr}(\rho(g))$. □

To the sequence (ρ_i), we have thus associated an action of G on a Λ-tree. This action — through the length-function — gives a precise information on the growing rate of traces of group elements. If one wants to describe only the "top order terms", one possibility is to look at the embedding of the space of characters into $\mathbb{P}^{\mathcal{F}}$ as in Chapter 1.

Consider as set of functions $\mathcal{F} \subset \mathbb{Q}[X]$ the set $\{\mathcal{I}_g\}$ where g ranges over all conjugacy classes of G. Let $\bar{X}^{\mathcal{F}}$ be the Morgan-Shalen compactification of X as explained in § 1.3, and denote by $B(X) = \bar{X}^{\mathcal{F}} - X$ the frontier. Then we may interpret points of $B(X)$ as length-functions of an action of G on a Λ-tree where the ordered group Λ is archimedean.

Proposition 34. [17, Th.II.4.3] *Any point in the frontier $B(X)$ is the projectivized length-function of an action of G on a Λ-tree with Λ archimedean; this action is minimal and without phantom inversion.*

Proof. Note first that the frontier $B(X)$ is the union of the frontiers of the (finitely many) irreducible components of $X(G)$. Let X_0 be any such irreducible component and let ξ be a point in the frontier of X_0. By Proposition 16, ξ has homogeneous coordinates $(-\min(0, v(\mathrm{tr}(g)))) \in \mathbb{P}^{\mathcal{F}}$ where v is a valuation on $\mathbb{Q}(X_0)/\mathbb{Q}$ such that for any $g \in G$, $v(\mathcal{I}_g)$ is either positive or in the smallest isolated subgroup Λ_1 of the value group Λ.

As in the proof of the preceding Theorem and using the tautological representation, we may identify G with a subgroup of $\mathrm{SL}(2,\mathbb{Q}(\mathcal{R}_0))$ where $\mathcal{R}_0$ is a component of $\mathcal{R}(G)$ which dominates X_0. Pick any valuation v' on $\mathbb{Q}(\mathcal{R}_0)/\mathbb{Q}$ extending v and let $\bar{v}$ be the quotient valuation obtained after post-composing v' with the quotient homomorphism $\Lambda \to \Lambda/\Lambda_1$. Denote by $\mathcal{T}_{\bar{v}}$ the Bass-Serre tree associated to $\bar{v}$. Since for any element

$g \in G$ $\bar{v}(\mathrm{tr}(g)) \geq 0$, g has a fixed point on $\mathcal{T}_{\bar{v}}$. By Proposition 29 the entire group G has a fixed point, which is to say that the tautological representation of G is conjugated to a representation $\rho : G \to \mathrm{SL}\ (2, \mathfrak{o}_{\bar{v}})$. In particular, the matrix of $\rho(g)$ in the standard basis has coefficients $a_g, b_g, c_g, d_g \in \mathfrak{o}_{\bar{v}}$, and $v'(a_g), v'(b_g), v'(c_g), v'(d_g)$ are either positive or lie in Λ_1.

Let now $\mathcal{T}_v$ denote the Bass-Serre tree of $\mathrm{SL}(2, \mathbb{Q}(\mathcal{R}_0))$ associated this time to the valuation v'. Denote by x_0 the equivalence class of the standard lattice. Since $d(x_0, \rho(g)(x_0)) = -2\min(v'(a_g), v'(b_g), v'(c_g), v'(d_g))$, the distance between any two points of the orbit Gx_0 is in Λ_1. Observe now that the union of all segments $x_0.\rho(g)(x_0)$ is a Λ_1-tree $\mathcal{T}$ with Λ_1 archimedean. The group G acts on $\mathcal{T}$ without phantom inversion and its length-function is projectively the point ξ. □

4.4. **The limit tree of a sequence of discrete and faithfull representations.**

Recall that a group is said *small* if it contains an abelian subgroup of finite index.

A representation $\rho : G \to \mathrm{SL}(2, \mathbb{C})$ is *discrete* if its image is a discrete subgroup, and *faithfull* if it is injective. Denote $\mathcal{DF}(G) \subset \mathcal{R}(G)$ the set of all the discrete and faithfull representations.

Proposition 35. [7] *Let G be a group which is not small. Then $\mathcal{DF}(G)$ is a closed subset of $\mathcal{R}(G)$.*

Theorem 33 admits the following refinement when (ρ_i) is a sequence of discrete and faithfull representations.

Proposition 36. *Let (ρ_i) be a sequence of representations in $\mathcal{DF}(G)$ which does not contain any subsequence which converges up to conjugacy. Let $\mathcal{T}$ be any Λ-tree provided by Theorem 33. Then any subgroup of G which stabilizes a non-trivial segment is small.*

Proof. We keep the notations of the proof of Theorem 33. Recall that v is the valuation associated to a valuating sequence (x_i') which approximates sufficiently closely the sequence of characters $t(\rho_i)$ so that by Proposition 5, $v(f) \geq 0$ if and only if the sequence $\mathrm{tr}(\rho_i(f))$ is bounded. Let $x.y$ be a non-trivial segment of $\mathcal{T}$ and denote $G_{x.y}$ the subgroup of G which fixes $x.y$. Since $G_{x.y}$ fixes x, it is conjugated to a subgroup G_s of $\mathrm{SL}(2, \mathfrak{o}_v)$, hence the traces $\mathrm{tr}(\rho_i(g))$ are bounded for any $g \in G_s$. By Proposition 5 the sequence of representations $\rho_i|G_s$ contains a sequence which converges to a representation $\rho_\infty : G_s \to \mathrm{SL}(2, \mathbb{C})$. Since G_s stabilizes the segment $x.y$, its commutator subgroup $[G_s, G_s]$ is mapped to the identity by the homomorphism $\mathrm{SL}(2, \mathfrak{o}_v) \to \mathrm{SL}(2, \mathfrak{o}_v/\beta\mathfrak{o}_v)$ (Proposition 30). Therefore for any $g \in [G_s, G_s]$, $\mathrm{tr}(\rho_\infty(g)) = 2$. Suppose by contradiction that G_s is not virtually abelian. By Proposition 35 $\rho_\infty|G_s$ is then a discrete and faithfull representation. Since $\mathrm{tr}(\rho_\infty(g)) = 2$ $\rho_\infty([G_s, G_s])$ must be parabolic subgroup and so it fixes a unique point in $\mathbb{C}P^1$. This point must also be fixed by $\rho_\infty(G_s)$. But any discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ having a fixed point is virtually abelian. A contradiction. □

5. **Geometric actions of groups on Λ-trees**

Trying to understand unbounded sequences of points in $\mathcal{R}(G)$, Theorem 33 led us to the study of isometric group actions on trees. In general, this tree is a Λ-tree with Λ non-archimedean but the tree associated to the quotient rank 1 valuation already carries a lot of informations (Proposition 34). In this section, we address the following two questions: which groups G do have a non-trivial action on a Λ-tree (on a $\mathbb{R}$-tree) such that all segment stabilizers are small? When such an action exists can it be described in a geometric way?

We begin with the case of fundamental groups of n-dimensional manifolds and will after focus on *surface groups*, i.e. of groups isomorphic to the fundamental group of a closed surface.

5.1. Λ-measured laminations.

Let M be a manifold of dimension n. A (codimension 1) *lamination* of M is a closed subset $L \subset M$ such that there is a cover of M by open sets V_i which satisfy:

(1) for each i, $(V_i, V_i \cap L)$ has a product structure, i.e. there is a compact set $F_i \subset]0,1[$ and an homeomorphism ϕ_i between the pairs $(V_i, V_i \cap L)$ and $(U_i \times]0,1[, U_i \times F_i)$;

(2) if two open sets U_j and U_k have non-empty intersection, then over their intersection, the homeomorphism $\phi_k \circ \phi_j^{-1} : \phi_j(L \cap U_i \cap U_j) \to \phi_k(L \cap U_i \cap U_j)$ preserves the product structure.

The open sets V_j in this definition are called *flow-boxes for* L. Consider a flow-box V_j for L. The set Y of the connected components $]0,1[-F_j$ is ordered by the order of the interval $]0,1[$. Let Λ- be a countable totally ordered abelian group. A Λ-*measure* on $]0,1[$ supported on F_j is a monotonic bijection i from Y to an interval of Λ. This allows to assign to any interval $[a,b] \subset]0,1[$ with endpoints disjoint from F_j a number in Λ^+, namely the absolute value $|i(a) - i(b)|$. This defines a finitely additive measure. Also the measure of an interval $[a,b]$ is 0 if and only if $[a,b]$ is disjoint from F_j.

A Λ-*measure transverse to* L is the data, for each flow-box $V_j = U_j \times]0,1[$, of a Λ-measure μ_j on $]0,1[$ such that the obvious compatibility relations are satisfied for flow-boxes which intersect. A Λ-*measured lamination* is the data (L,μ) of a lamination L and a Λ-measure μ transverse to L .

A continuous path $c : [0,1] \to M$ in M is *transverse to* L if $c(0)$ and $c(1)$ are disjoint from L and if c can be decomposed as a product $c_1.c_2.\cdots.c_n$ of paths c_j such that for each j, c_j is contained in a flow-box U_i and the projection $c_j \cap L \to F_j$ is strictly monotone. A Λ-measure transverse to L induces a finitely additive measure on c with total mass denoted $\mu(c)$. If c and c' are two paths homotopic by an homotopy c_t such that c_t remains transverse to L, the total mass $\mu(c_t)$ remains constant.

The dual tree to a Λ-measured lamination

Denote by $\tilde{M}$ the universal cover of M. The preimage of a Λ-measured lamination in $\tilde{M}$ is a Λ-measured lamination.

Proposition 37. *Let $\mathcal{L} = (L,\mu)$ be a Λ-measured lamination. Let $\tilde{L}$ denote the preimage of L in $\tilde{M}$. Suppose that its preimage in $\tilde{M}$ satisfies the following axioms:*

(1) *the leaves of $\tilde{L}$ are closed subsets of $\tilde{M}$;*

(2) *given any two connected components x and y of $\tilde{M} - \tilde{L}$, there exists a path $c : [0,1] \to \tilde{M}$ with endpoints contained in x and y respectively which is transverse to $\tilde{L}$ and which intersects each leaf of $\tilde{L}$ at most once.*

Then the set $\mathcal{T}_{\mathcal{L}}$ of the connected components of $\tilde{M} - \tilde{L}$ is a Λ-tree.

Proof: First, the set $\mathcal{T}_{\mathcal{L}}$ is a Λ-metric space. For complementary components x and y of $\tilde{M} - \tilde{L}$, define $d(x,y)$ to be the mass of a path provided by (2). This does not depend on the choice of the path, by (1) and the invariance property of a transverse Λ-measure. To check the triangular inequality consider the induced lamination on a triangle in $\tilde{M}$ with vertices x, y and z in components of $\mathcal{T}_{\mathcal{L}}$ such that the sides intersect exactly once the leaves of $\tilde{L}$. Each leaf of the induced foliation which has one endpoint on the side from x to z has its other endpoint either on the side from x to y or on the side from y to z. The triangular inequality is a consequence of this.

We define segments in $\mathcal{T}_{\mathcal{L}}$. Let c be a path in $\tilde{M}$ joining two connected components x and y of $\mathcal{T}_{\mathcal{L}}$ which satisfies (2). Then $w \mapsto d(x, w)$ is an isometry from the set of components of $\mathcal{T}_{\mathcal{L}}$ that c intersects to the interval $[0, d(x, y)]$. When c is contained in a flow-box, this follows from the definition of a Λ-measured lamination and the general case follows from this one, since c can be written as a composition $c_1 \cdots c_k$ where each path c_j is contained in a flow-box. Therefore any two distinct points can be joined by a segment. We call special those segments obtained on that way. We show that any segment in $\mathcal{T}_{\mathcal{L}}$ is special. Suppose that there is another segment S in $\mathcal{T}_{\mathcal{L}}$ with endpoints x and y. Let w be a point on S; since S is a segment, $d(x, w) + d(w, y) = d(x, y)$. Consider the two paths provided by (2) which connect respectively x to w and w to y. We can suppose that the endpoints of these paths in the component w are the same: their composition is a path $c' \subset \tilde{M}$. Let Δ be a disk which realizes an homotopy fixing endpoints between c and c'. Since L has empty interior, Δ can be chosen to be transverse to L; then $L \cap \Delta$ is a lamination of Δ. Each leaf of this lamination which has one endpoint in c has its other endpoint on the special segment because c satisfies (2). Since $\mu(c') = d(x, y)$, any leaf starting from c' ends on c. In particular, w is on the special segment.

We check now that $\mathcal{T}_{\mathcal{L}}$ satisfies the first axiom of a Λ-tree: the intersection of two segments $x.y$ and $x.z$ is a segment issued from x. Let x, y and $z \in \mathcal{T}_{\mathcal{L}}$. Let Δ be the triangle in $\tilde{M}$ whose sides are paths c, c' and c'' in $\tilde{M}$ whose endpoints succesively in x, y, z and which intersect each leaf of L atmost once. This disc can be made transverse to L. The induced lamination of Δ has the property that each leaf with an endpoint on one side has its other endpoint on another side. The set of the leaves from the side c to the side c'' is ordered by the inclusion of the disc containing x and bounded by the leaf on Δ. Some leaf is maximal for this order among the leaves from c to c'. The component of $\tilde{M} - \tilde{L}$ bounded by this leaf determines a vertex of $\mathcal{T}_{\mathcal{L}}$ such that $x.y \cap x.z = x.z$.

The second axiom can be proven in the same way. □

Remark. When $\Lambda \subset \mathbb{R}$, a Λ-measure with support a closed subset $F \subset]0, 1[$ extends in a unique way to a σ-additive measure on $]0, 1[$ with support F. In that situation $\mathcal{L} = (L, \mu)$ is a Λ-measured lamination; it has *a transverse measure*, i.e. each transversal I carries a Radon measure, supported on $L \cap I$ and these measures are invariant under the holonomy pseudo-group (cf. [18]). When a lamination is the support of a transverse measure, it has the following structure [18, Theorem 3.2]: it is the union of finitely components and each component is either a family of parallel compact leaves or an exceptional minimal set.

When the ordered group Λ is not contained in $\mathbb{R}$, there are no similar descriptions. In particular, a Λ-measured lamination can contain Reeb components and this prevents in general Property (2) in Proposition 37 to be satisfied and also the existence of a dual tree in general.

5.2. Construction of laminations by transversality.

The kind of geometric models we will obtain for describing a given action of a fundamental group $\pi_1(M)$ on an abstract Λ-tree is the action on the dual tree of a Λ-measured lamination on M. We will explain in the next section how such a description exists when M is a closed surface. The first step for finding this geometric model, that is the construction of the Λ-measured lamination, is achieved by "transversality" following a method that was first introduced by John Stallings in the context of simplicial trees.

Definition. Let $\mathcal{L} = (L, \mu)$ be a Λ-measured lamination contained in M and let $\mathcal{T}$ be a Λ-tree. A *transverse map* $\phi : \tilde{M} \setminus \tilde{L} \to \mathcal{T}$ is a locally constant map such that any point $x \in L$ is contained in a flow-box $V \simeq U \times]0, 1[$ for $\tilde{L}$ on which the restriction $\phi|(V \setminus \tilde{L})$ is the composition of the projection $(U \times]0, 1[) \setminus \tilde{L} \to]0, 1[\setminus F$ and map $]0, 1[\setminus F \to \mathcal{T}$ which induces a monotone bijection between the components of $]0, 1[\setminus F$ and a segment in $\mathcal{T}$.

Proposition 38. *Let M be a closed n-manifold. Let $\pi_1(M) \times \mathcal{T} \to \mathcal{T}$ be an action of $\pi_1(M)$ on a Λ-tree $\mathcal{T}$. Then there exists a Λ-measured lamination $\mathcal{L} = (L, \mu)$ contained in M and a transverse map from the set of connected components of $\tilde{M} \setminus \tilde{L}$ to $\mathcal{T}$ which is $\pi_1(M)$-equivariant.*

Proof. We select a triangulation τ of M and denote by $\tilde{\tau}$ the lift of the triangulation to $\tilde{M}$. We construct the support of the measured lamination L by defining its intersection with the i-skeleton of τ inductively.

Over the 0-skeleton we choose any equivariant map ϕ_0 from $\tilde{\tau}_0$ to $\mathcal{T}$. In order to extend this map over the 1-skeleton, we use the following result.

Lemma 39. *Let Λ be a countable ordered group and let $[a, b] \subset \Lambda$ be an interval. There is a Λ-measure μ on $[0, 1]$ with the following properties:*

1. *the support of μ is a closed subset $F \subset]0, 1[$ with empty interior;*
2. *the function $t \mapsto a + \int_0^t d\mu$ is a monotone bijection from the connected components of $]0, 1[\setminus F$ to $[a, b]$.*

Furthermore if μ_1 and μ_2 are two Λ-measures on $]0, 1[$ satisfying properties (1) *and* (2)*, then there is an orientation-preserving homeomorphism $h : [0, 1] \to [0, 1]$ that carries μ_1 to μ_2.*

Let τ_1 be an edge of the 1-skeleton of τ; let $\tilde{\tau}_1$ be one of its lifts to the universal cover, and $\tilde{a}$, $\tilde{b}$ the endpoints of $\tilde{\tau}_1$. Set $a = \phi_0(\tilde{a})$, $b = \phi_0(\tilde{b})$. The edge $\tilde{\tau}_1$ can be identified with $[0, 1]$. An application of Lemma 39 gives a Λ-measure with support contained in $]0, 1[$ and a bijection from the complementary components of the support of this measure. This support will be the intersection of $\tilde{L}$ with $\tilde{\tau}_1$ and we define ϕ_1 will be the bijection. The measures and ϕ_1 can clearly be made equivariant.

In order to extend ϕ_1 over the 2-skeleton, we consider a 2-simplex τ_2; let $\tilde{\tau}_2$ be one of its lifts to $\tilde{M}$. The boundary of $\tilde{\tau}_2$ is the union of 3 edges e_1, e_2 and e_3 of $\tilde{\tau}_1$. Each edge e_j contains a closed subset F_j and ϕ_1 identifies $e_i \setminus F_j$ with a segment $I_j \subset \mathcal{T}$. By the axioms of a Λ-tree, there exists a unique point $x \in \mathcal{T}$ which is common to all segments $\phi_1(e_j)$. Let $c_j \subset I_j$ be the connected component of $e_j \setminus F_j$ which is mapped by ϕ_1 to v. The uniqueness part of Lemma 39 provides for each j an homeomorphism between the intervals components of $e_j \setminus v_j$ which are mapped to the same segment of $\mathcal{T}$ which identifies the traces of $\cup F_j$ on these intervals. Using these homeomorphisms one can construct a lamination L of $\tilde{\tau}_2$ which intersects $\partial\tau_2$ along $\cup F_j$ and a map ϕ_2 from the components of $\tilde{\tau}_2 \setminus \tilde{L}$ which extends ϕ_1. These laminations and maps can be chosen invariant. Qu'on promet When M is a surface, this construction ends the proof of Lemma 38. For the higher dimensional case we refer to [16, Theorem II.3]. □

Remark. Suppose that the Λ-measured lamination $\mathcal{L} = (L, \mu)$ constructed in the proof above satisfies the properties of Proposition 37. Then $\mathcal{L}$ has a dual tree and ϕ induces a map $\mathcal{T}_{\mathcal{L}} \to \mathcal{T}$ which is *a morphism of trees* [16, p. 175]. We don't recall the definition of "morphism" here; we just indicate that in our case it is an equivalent reformulation of the fact that $\phi : \tilde{M} \setminus \tilde{L}$ is a transverse map. By the construction of $\mathcal{L}$, $\mathcal{T}_{\mathcal{L}}$ is a union of segments on which ϕ is an isometry.

5.3. Actions of surface groups.

We consider in this section a closed surface S of negative Euler characteristic. For simplicity we assume that S is closed.

We denote by $\mathcal{DF}(\pi_1(S))$ the set of discrete and faithfull representations of $\pi_1(S)$ into SL(2,$\mathbb{C}$) and by $\mathcal{DF}(\pi_1(S))_{\mathbb{R}}$ those with image contained in SL(2,$\mathbb{R}$). In [16] the Λ-trees which are limits of a diverging sequence of discrete and faithfull representations (ρ_i) belonging to $\mathcal{DF}(\pi_1(S))_{\mathbb{R}}$ are characterized.

Definition 40. *Let $\pi_1(S)\times\mathcal{T}\to\mathcal{T}$ be an action of $\pi_1(S)$ on a Λ-tree $\mathcal{T}$. This action is geometric if there is a Λ-measured lamination $\mathcal{L}=(L,\mu)$ on S which has a dual tree $\mathcal{T}_{\mathcal{L}}$ and a $\pi_1(S)$-equivariant isometry between $\mathcal{T}$ and $\mathcal{T}_{\mathcal{L}}$.*

Theorem 41. [16] *Let (ρ_i) be a sequence in $\mathcal{DF}(\pi_1(S))_{\mathbb{R}}$ which converges to a minimal action of $\pi_1(S)$ on a Λ-tree $\mathcal{T}$ in the sense of Theorem 33. Then $\pi_1(S)\times\mathcal{T}\to\mathcal{T}$ is geometric.*

We sketch the proof of this theorem and begin with the following complement to Proposition 38.

Proposition 42. *Let $\pi_1(S)\times\mathcal{T}\to\mathcal{T}$ be a minimal action of $\pi_1(S)$ by isometries on a Λ-tree. Then there is a Λ-measured lamination $\mathcal{L}$ on S which has a dual tree $\mathcal{T}_{\mathcal{L}}$ and a morphism $\mathcal{T}_{\mathcal{L}}\to\mathcal{T}$ which is $\pi_1(S)$-equivariant.*

Proof. In the case of surfaces the axioms for a lamination that guarantee the existence of a dual tree can be formulated differently. Let us identify the universal cover $\tilde{S}$ with the unit disk after choosing an arbitrary metric of constant curvature -1. Denote by $(S^1\times S^1\setminus \text{diagonal})/\mathbb{Z}_2$ the set of pairs of distinct points of S^1. Let L be a codimension 1 lamination of S and let $\tilde{L}$ its preimage in $\tilde{S}$. Suppose that L satisfies the following set of axioms.

(1) there is a finite cover of S by flow-boxes for L such that each leaf of $\tilde{L}$ intersects the lift of a flow-box in a connected set;
(2) each leaf of $\tilde{L}$ is proper;
(3) each leaf has two distincts ends defining a point in $(S^1\times S^1\setminus \text{diagonal})/\mathbb{Z}_2$;
(4) the map $\tilde{L}\to(S^1\times S^1\setminus \text{diagonal})/\mathbb{Z}_2$ which assigns to a point of $\tilde{L}$ the ends of the leaf of $\tilde{L}$ which passes through it is continuous.

Then it is not difficult to show that L satisfies also the hypothesis of Proposition 37 (cf. [16, Theo. I.4.2]). Therefore if L is the support of a Λ-measured lamination $\mathcal{L}$, then $\mathcal{L}$ has a dual Λ-tree.

In order to construct the lamination $\mathcal{L}=(L,\mu)$, consider the lamination $\mathcal{L}'=(L',\mu')$ provided by Proposition 38. This lamination might contain closed leaves that are homotopic to 0 on S. One proves that the union of those leaves consist of finitely many sublaminations of L' which are the union of parallel leaves [16, Lemma III.1.4]. By removing those sublaminations and modifying accordingly the transverse map, one obtains a Λ-measured lamination $\mathcal{L}=(L,\mu)$ and a transverse map $\tilde{S}\setminus\tilde{L}\to\mathcal{T}$. At this stage, one can prove that each half-leaf of $\tilde{L}$ has one endpoint in S^1; however this does not suffice for verifying (2) and in particular the lamination might contain *Reeb components*. A new modification of $\mathcal{L}$ is required which simplifies the annuli bounded by leaves of $\mathcal{L}$ to guarantee that the lamination has a dual tree (cf. [16]). By Remark 5.2 there is an equivariant morphism $\mathcal{T}_{\mathcal{L}}\to\mathcal{T}$. □

Take back the sketch of the proof of Theorem 41. By Proposition 36 a first property of $\mathcal{T}$ is that any subgroup of $\pi_1(S)$ which stabilizes a non-trivial segment of $\mathcal{T}$ is virtually abelian: therefore non-trivial segments stabilizers are cyclic. A second one is the following. Say first that $g\in\pi_(S)$ is hyperbolic when for some $\rho\in\mathcal{DF}(\pi_1(S))_{\mathbb{R}}$, $\rho(g)$ acts on $\mathbb{H}$ leaving invariant a geodesic, called its *axis*. This property is independant on the representation $\rho\in\mathcal{DF}(\pi_1(S))_{\mathbb{R}}$. One denotes by $l_\rho(g)$ the translation distance of the hyperbolic element $\rho(g)$. Let g and h be hyperbolic elements of $\pi_1(S)$ whose axis intersect in $\mathbb{H}$. This property also does not depend on ρ. By the triangular inequality the translation distance of the elementsgh satisfies for any $\rho \in DF(\pi_1(S))$: $l_\rho(gh) \leq l_\rho(g) + l_\rho(h)$. Therefore since $\mathcal{T}$ is the limit of the sequence (ρ_i) the translation distances in $\mathcal{T}$ satisfy $\delta_{\mathcal{T}}(gh) \leq \delta_{\mathcal{T}}(g)+\delta_{\mathcal{T}}(h)$. A geometric interpretation of this inequality is that any partial axis (or the fixed points sets) of g and h in $\mathcal{T}$ have non-empty intersection.

Theorem 41 follows therefore from the following:

Theorem 43 ([16], [30]). *Let $\pi_1(S) \times \mathcal{T} \to \mathcal{T}$ be an action of $\pi_1(S)$ on a Λ-tree which satisfies*

(1) *the action is minimal and without phantom inversions,*
(2) *the edge stabilizers are cyclic and*
(3) *if g and h are hyperbolic elements whose axis intersect in $\mathbb{H}$, then any partial axis of g and h in $\mathcal{T}$ intersect.*

Then the action $\pi_1(S) \times \mathcal{T} \to \mathcal{T}$ is geometric.

The proof given in [16] of this theorem starts with the lamination $\mathcal{L}$ obtained in Proposition 42 and with the $\pi_1(S)$-equivariant morphism $\iota : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}$. If ι is not an embedding, there is a vertex v and two distinct segments issued from v such that $\iota(e_1) \cap \iota(e_2)$ is a segment which is not reduced to v. The vertex v of $\mathcal{T}_{\mathcal{L}}$ corresponds to a connected component C of $\tilde{S} \setminus \tilde{L}$; for each segment e_i there is a leaf $\tilde{l}_i$ of $\tilde{L}$ in the boundary $\tilde{C}$. Since $e_1 \neq e_2$, $\tilde{l}_1 \neq \tilde{l}_2$. One distinguishes two cases according to the behaviour of the projections l_i of $\tilde{l}_i$ on S: first when both projections are compact and second when both are non-compact. Using the equivariance of the map $\tilde{i}$ one sees that these two cases cover all possible situations. These two cases contradict respectively (1) and (2) (cf. [16, Chapter IV]). □

In [16] the question wether the conclusion of Theorem 43 was still true assuming only condition (1) was left open. It was solved when Λ is archimedean by Richard Skora who proved:

Theorem 44. [31] *Let $\pi_1(S) \times \mathcal{T} \to \mathcal{T}$ be a minimal action of $\pi_1(S)$ by isometries on an $\mathbb{R}$-tree without phantom inversion and such that the edge stabilizers are cyclic. Then this action is geometric.*

Furthermore there is an action on a Λ-tree which satisfies (1) but which is not geometric.

The proof by Skora uses also the $\mathbb{R}$-measured lamination $\mathcal{L}$ and the morphism $\iota : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}$. Assuming by contradiction again that ι is not an isometry, then two germs of segments e_1 and e_2 in $\mathcal{T}_{\mathcal{L}}$ issued from some vertex v are identified; e_1 (resp. e_2) corresponds to a leaf $\tilde{l}_1$ (resp. $\tilde{l}_2$) in the boundary of the component v of $\tilde{S} \setminus \tilde{L}$. Since $\mathcal{L}$ is $\mathbb{R}$-measured, the structure theorem [18, Theorem 3.2] says that L is the union of finitely many disjoint sublaminations which either are formed by parallel closed leaves or are exceptional minimal [18, Theorem 3.2]. The case when each leaf l_i is closed contradicts that the edge stabilizers are cyclic, as in the previous proof. The case when each leaf l_i is contained in a minimal exceptional is handled differently: Skora uses interval exchanges to produce a subinterval of $\iota(e_1) \cap \iota(e_2)$ which has a non-cyclic edge stabilizer [31] (see also [23]).

5.4. **Actions of 3-manifolds groups.**

Let M be a compact 3-manifold. One says that M is *boundary-incompressible* when any closed curve embedded in ∂M which is homotopic to 0 in M bounds a 2-disc properly embedded in M. One says that M is *acylindrical* when any properly embedded annulus or any embedded 2-torus can be homotoped relatively to its boundary into ∂M.

Theorem 45. *Let M be a compact boundary-incompressible and acylindrical 3-manifold. Then the space of discrete and faithfull representations $\mathcal{DF}(\pi_1(M))$ is compact.*

This theorem was first proven by Thurston in [32]: it is one main step in his proof of the Hyperbolization Theorem for non-Haken 3-manifolds. Morgan and Shalen gave a entirely different proof in [19] using trees. Their proof can be outlined as follows. Set $G = \pi_1(M)$. Suppose by contradiction that $\mathcal{DF}(G)$ is not compact. Then by Proposition 36, there exists an $\mathbb{R}$-tree $\mathcal{T}$ and a minimal non-trival action of G on $\mathcal{T}$ by isometries such that the subgroups of G which fix a non-degenerate segment of $\mathcal{T}$ are small. The proof reduces then to show that such an action does not exist when M satisfies the hypothesis of the theorem.

By Proposition 38 there is a a measured lamination $\mathcal{L} = (L, \mu)$ in M and a transverse map $\tilde{M} \setminus \tilde{L} \to \mathcal{T}$. The construction gave that $\mathcal{L}$ is *carried with positive weights* by a *branched surface* [11]. A *branched surface* is a 2-complex B embedded in M with a local model (cf. Picture). Particular neighborhoods of this 2-complex in M have a natural decomposition in intervals. A measured lamination $\mathcal{L} = (L, \mu)$ is *carried by B* if it is contained in a neighborhood of this type in such a way that L is transverse to the intervals.) A difficult part of the proof is to show that one can choose the dual lamination to be carried by a branched surface which is *incompressible* (cf. [11]). This implies that the fundamental group of each leaf l of L maps injectively into $\pi_1(M) = G$. By equivariance of the transverse map, $\pi_1(l)$ stabilizes a non-degenerate segment of $\mathcal{T}$. The property of the segment stabilizers implies that $\pi_1(l)$ is virtually abelian. Therefore the fundamental group of any leaf of L is cyclic or trivial.

On another side, the incompressibility of B implies that no closed surface carried by B is homeomorphic to a disk or a 2-sphere. Under these circonstances, Theorem 5.1 of [18] implies that any surface carried by B has zero Euler characteristic. By approximating $\mathcal{L}$, one constructs compact surfaces carried by B. Such a surface is a union of annuli or tori. This contradicts the acylindricity of M.

The hypothesis on M in Theorem 45 are of a topological nature, but they can also be formulated equivalently in a group-theoretic way. The property that M is boundary-incompressible and irreducible is equivalent to the fact that G is not isomorphic to a free product. The acylindricity of M is equivalent to the non-existence of a non-trivial splitting of G as an amalgamated product over a subgroup isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}^2$. After giving their proof based on 3-dimensional arguments, Morgan and Shalen conjectured the following vast generalization which is now a theorem of Rips.

Theorem 46. *Let G be a finitely presented group which can not be written as an amalgamated free product over a virtually abelian group. Then there are no non-trivial actions of G on an $\mathbb{R}$-tree such that the stabilizers of segments are virtually abelian.*

We refer to [2] for a generalization and to [25] for a survey of the proof.

References

[1] Mladen Bestvina, *Degenerations of the hyperbolic space* Duke Mathematical Journal, vol. 56 (1988), no. 1, 143-161

[2] Mladen Bestvina and Mark Feighn, *Stable actions of groups on real trees*, Inventiones Mathematicae, vol. 121 (1995), no. 2, 287-321.

[3] Nicolas Bourbaki, *Algèbre Commutative*, Chapitres 5-7, Masson, 1985.

[4] Ian Chiswell, *Abstract length functions in groups*, Proc. Cambridge Phil. Soc. 80 (1976), 451-463.

[5] Ian Chiswell, *Introduction to Λ-trees*, Singapore, World Scientific 2001.

[6] Ian Chiswell, *Non standard analysis and the Morgan-Shalen compactification*, Quart. J. Math. Oxford Ser. (2) 42 (1991), no. 167, 257–270.

[7] Vicky Chuckrow, *On Schottky groups with applications to Kleinian groups*, Ann. of Math. 88 (1968), 47-61.

[8] Marc Culler, Cameron Gordon, John Luecke and Peter Shalen, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987), 237–300.

[9] Marc Culler and John Morgan, *Groups actions on $\mathbb{R}$-trees*, Proc. Lond. Math. Soc 55 (1987), 571–604.

[10] Marc Culler and Peter Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. 117 (1983), 109–146.
[11] William Floyd and Ulrich Oertel, *Incompressible surfaces via branched surfaces*, Topology 23 (1984), 117–125.
[12] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, 1977.
[13] Roger Lyndon, *Length functions in groups*, Math. Scand. 12 (1963), 209–234.
[14] J. Morgan, Group actions on trees and the compactification of the space of classes of SO(n,1)-representations, Topology 25, (1986) 1- 33.
[15] John Morgan, *Λ-trees and their applications*, Bull. Amer. Math. Soc., 26 (1992), 87-112.
[16] John Morgan and Jean-Pierre Otal, *Relative growth rates of closed geodesics on closed surfaces*, Comment. Math. Helvitici 68 (1993), 171–208.
[17] John Morgan and Peter Shalen, *Degenerations of hyperbolic structures, I: Valuations, trees and surfaces*, Ann. of Math. 120 (1984), 401–476.
[18] John Morgan and Peter Shalen, *Degenerations of hyperbolic structures II : Measured laminations in 3-manifolds*, Ann. of Math. 127 (1988), 403–456.
[19] John Morgan and Peter Shalen, *Degenerations of hyperbolic structures III: Actions of 3-manifold groups on trees and Thurston's compactness theorem*, Ann. of Math. 127 (1988), 457–519.
[20] John Morgan and Peter Shalen, *Free actions of surface groups on* $\mathbb{R}$*-trees*. Topology 30 (1991), no. 2, 143–154.
[21] David Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 34, Springer-Verlag, New York, 1965.
[22] David Mumford, *Algebraic Geometry I: Complex Projective Varieties*, Grundlehren der Math. Wiss. 221, Springer-Verlag, 1976.
[23] Jean-Pierre Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque no. 235, Soc. Math. France, 1996.
[24] Frédéric Paulin, *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Inv. Math. 94 (1988), 53-80.
[25] Frédéric Paulin, *Actions de groupes sur les arbres*, Séminaire Bourbaki 1995–96, exp. 808, Astérisque no. 241, Soc. Math. France, 1997.
[26] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Math. 149, Springer-Verlag, New York,1994.
[27] Jean-Pierre Serre, *Arbres, Amalgames, SL2*, Astérisque no. 46, Soc. Math. France, 1977.
[28] Peter Shalen, *Dendrology of Groups: An Introduction* in *Essays in Group Theory*, ed. S.M. Gersten, Mathematical Sciences Research Institute Publications, Vol. 8, Springer-Verlag, New York.
[29] Peter Shalen, *Dendrology and its applications* in Group theory from a geometrical viewpoint (Trieste, 1990), 543–616, World Sci. Publ., River Edge, NJ, 1991.
[30] Richard Skora, *Geometric actions of surface groups on Λ-trees*, Comment. Math. Helvetici 65 (1990) 519–533.
[31] Richard Skora, *Splittings of surfaces*, Journal of the American Math. Society, 9, no. 2 (1996), 605–616.
[32] W. P. Thurston, *Hyperbolic structures on 3-manifolds I: Deformations of acylindrical manifolds*, Annals of Math. 124 (1986), 203–246.
[33] Michel Vaquié, *Valuations and local uniformization*, Singularity theory and its applications, 477-527, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo (2006).
[34] Michel Vaquié, *Extension d'une valuation*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3439-3481.
[35] Oscar Zariski, *Fundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc. vol. 53 (1943).
[36] Oscar Zariski, *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. 50, (1944), 683–691.
[37] Oscar Zariski and Pierre Samuel, *Commutative algebra*, vol. 2, Graduate Texts in Mathematics no 29, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

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