Equidistribution of hyperbolic centers using Arakelov geometry

Charles Favre
favre@math.polytechnique.fr

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Number theory and dynamics

- Additive number theory: proof of Szemeredi's theorem by Furstenberg
- Diophantine approximation problems: proof of the Oppenheim conjecture by Margulis; proof of Khintchine theorem by Sullivan
- Arithmetic dynamics: set of conjectures promoted by Silverman concerning rational maps $f : X_K \dashrightarrow X_K$ defined over a number field (Dynamical Lehmer's conjecture, Manin-Mumford, Mordell-Lang, Uniform Boundedness Conjecture, etc.)

The parameter space of quadratic polynomials

Fix $c \in \mathbb{C}$, and define

$$P_c(z) := z^2 + c$$

Interested in the behaviour of

$$z, P_c(z), P_c^2(z) = P_c(P_c(z)), \ldots, P_c^n(z), \ldots$$

- $K(c) := \{z \in \mathbb{C}, |P_c^n(z)| = O(1)\}$ (the filled-in Julia set is compact)
- Dichotomy: either a Cantor set, or a connected set
- $\mathcal{M} := \{c \in \mathbb{C}, K(c) \text{ is connected }\}$ (the Mandelbrot set)

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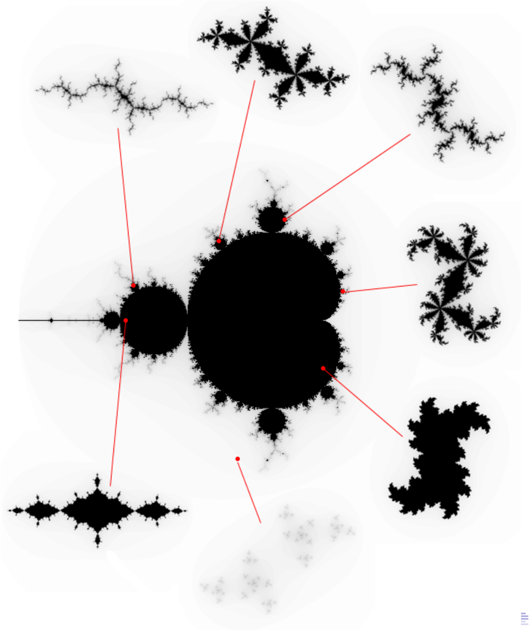
The Mandelbrot set and some Julia sets



Special points in the Mandelbrot set

$$\begin{aligned}\mathrm{Per}(n) &= \{c,\ \text{critical point has exact period } n\}\\ &= \{c,\ P_c^n(0)=0, P_c^k(0)\neq 0 \text{ for } 1\leq k<n\}\end{aligned}$$

$$\begin{aligned}P_c^1(0) &= c\\ P_c^2(0) &= c^2+c=c(c+1)\\ P_c^3(0) &= (c^2+c)^2+c=c(1+c+2c^2+c^3)\\ P_c^4(0) &= (c^2+c)(1+2c^2+3c^3+3c^4+3c^5+c^6)\end{aligned}$$

Theorem (Douady-Hubbard)

$$\sum_{m|n} \#Per(m) = 2^{n-1}$$

whence $\#Per(n)\sim 2^{n-1}$

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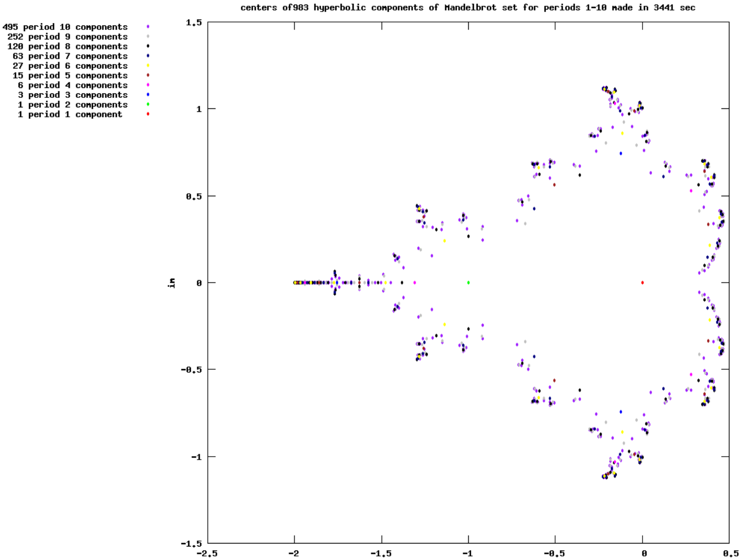
Hyperbolic centers



Hyperbolic centers

Theorem (Levine)

$$\mu_n := \frac{1}{2^{n-1}} \sum_{c \in \mathrm{Per}(n)} \delta_c \longrightarrow \mu_{\mathcal{M}} := \textit{ harmonic measure of } \mathcal{M}$$

Basic tool: harmonic analysis

- $\mu_{\mathcal{M}} = \Delta g_{\mathcal{M}} = \frac{i}{\pi} \partial\bar{\partial} g_{\mathcal{M}}$
- $\mu_n = \Delta g_n$

Aim: prove that $g_n \to g_{\mathcal{M}}$ in L^1_{loc}.

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Analytic proof

- $\Delta \log |w - c| = \delta_c$
- One may take

$$g_n(c) = \frac{1}{2^{n-1}} \log |P_c^n(0)|$$

- Construction of $g_{\mathcal{M}}$:
  - Green function: $g_c(z) := \lim_n \frac{1}{2^{n-1}} \log \max\{1, |P_c^n(z)|\} \geq 0$
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$$g_{\mathcal{M}}(c) := g_c(0) = \lim_n \frac{1}{2^{n-1}} \log \max\{1, |P_c^n(0)|\}$$

1. $g_n \leq g_{\mathcal{M}}$ everywhere
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Adelic proof (Baker-H'sia)

- Construction of a suitable height function such that $\mathrm{Per}(n) \subset \{h_{\mathcal{M}} = 0\}$
- Use equidistribution of points of small height (Autissier)

The latter result goes back to Bilu and Szpiro-Ullmo-Zhang in their work on the Bogomolov conjecture.

Adelic proof (Baker-H'sia)

Fix v a place in $\mathbb{Q}$, and $c \in \mathbb{C}_v$.

$$\begin{aligned} g_{c,v}(z) &:= \lim_n \frac{1}{2^{n-1}} \log \max\{1, |P_c^n(z)|_v\} \\ g_{\mathcal{M},v}(c) &= g_{c,v}(0) \\ g_{\mathcal{M},\infty}(c) &= g_{\mathcal{M}}(c) \text{ and } g_{\mathcal{M},p}(c) = \log^+ |c|_p \end{aligned}$$

$$h_{\mathcal{M}}(c) := \frac{1}{\deg(c)} \sum_{c' \sim c} \sum_{v \in M_{\mathbb{Q}}} g_{\mathcal{M},v}(c')$$

$h_{\mathcal{M}}$ differs from the standard height by a bounded function

Adelic proof (Baker-H'sia)

Theorem (Autissier)

For any sequence of disjoint finite sets $Z_n \subset \bar{\mathbb{Q}}$ that are invariant under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and such that $h_{\mathcal{M}}|_{Z_n} = 0$ then

$$\frac{1}{\# Z_n} \sum_{p \in Z_n} \delta_p \to \mu_{\mathcal{M}}$$

Apply this to $Z_n = \mathrm{Per}(n)$

Analytic vs Global method

Theorem (F.- Rivera-Letelier, Okuyama)

For any C^1 function φ,

$$\left| \frac{1}{2^{n-1}} \sum_{c \in \mathrm{Per}(n)} \varphi(c) - \int \varphi \mu_{\mathcal{M}} \right| \leq C \, \frac{\sqrt{n}}{2^{n/2}} |\varphi|_{C^1}$$

Theorem (Buff-Gauthier)

$$\frac{1}{2^{n-1}} \sum_{c \in \mathrm{Per}(n,\lambda)} \delta_c \longrightarrow \mu_{\mathcal{M}} := \text{ harmonic measure of } \partial\mathcal{M}$$

where

$$\mathrm{Per}(n,\lambda) = \{c,\ P_c^n(z) = z,\ (P_c^n)'(z) = \lambda\}$$

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The parameter space of polynomials of degree $d = 3$

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\begin{aligned}
P_{c,a}(z) &= \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3 \\
\mathrm{Crit}(P_{c,a}) &= \{P'_{c,a} = 0\} = \{c_1 := c, c_0 := 0\} \\
\mathrm{Per}(n_0, n_1) &:= \{(c,a) \in \mathbb{C}^2,\ P^{n_i}_{c,a}(c_i) = c_i \text{ for } i = 0,1\}
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Theorem (F.-Gauthier)

If $n_0^{(k)} \neq n_1^{(k)}$ and $\min n_i^{(k)} \to \infty$ then

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\frac{1}{3^{n_0^{(k)} + n_1^{(k)}}} \sum_{p \in \mathrm{Per}(n_0^{(k)}, n_1^{(k)})} \delta_p \longrightarrow \mu_{\mathcal{M}_3}
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where $\mu_{\mathcal{M}_3}$ is the equilibrium measure of the connectedness locus of cubic polynomials.

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Analytic method

- The Green function is well-defined:

$$g_{c,a}(z) := \lim_n \frac{1}{3^n} \log \max\{1, |P^n_{c,a}(z)|\}$$

- $g_0 = g_{c,a}(c_0)$, $g_1 = g_{c,a}(c_1)$.
- Connectedness locus is $\{g_0 = g_1 = 0\}$ and is compact
- Equilibrium measure: $\mu_{\mathcal{M}} := (dd^c)^2 G(c,a)$ with

$$G = \max\{g_0, g_1\}$$

Warning: the analytic method only applies when $n_0^{(k)} \gg n_1^{(k)} \to \infty$ (Dujardin -F.)

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Strategy

- Construction of a natural height where $\mathrm{Per}(n_0, n_1) \subset \{h_{\mathcal{M}_3} = 0\}$
- Application of Yuan's theorem of equidistribution of points of small heights

Difficulties:

1. Height should be defined at finite places in a special way (semi-positive adelic metric)
2. Points should be generic

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Difficulties:

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Construction of the height

The construction is similar to the quadratic case.

- $g_{c,a,v}(z) = \lim_n \frac{1}{3^n} \log \max\{1, |P^n_{c,a}(z)|_v\}$
- $G_v = \max\{g_{c,a,v}(c_0), g_{c,a,v}(c_1)\}$
- For $p \geq 5$ then $G_v = \log \max\{1, |c|, |a|\}$

$$h_{\mathcal{M}_3}(c,a) := \frac{1}{\deg(c,a)} \sum_{(c',a')\sim(c,a)} \sum_{v\in M_{\mathbb{Q}}} g_{\mathcal{M},v}(c',a')$$

It differs from the standard height by a bounded factor.
$\mathrm{Per}(n_0, n_1) \subset \{h_{\mathcal{M}_3} = 0\}$

Yuan's theorem

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Theorem

- *The line bundle:* $\mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{Q}}$;
- *Metrization:* $|\sigma|_v := e^{-G_v}$ *on* $\mathbb{A}^2$ *(with* $\operatorname{div}(\sigma)$ *the hyperplane at infinity)*

The associated height function is $h_{\mathcal{M}_3}$.
Suppose F_n *is a sequence of finite subsets of* $\mathbb{P}^2(\bar{\mathbb{Q}})$ *that are defined over* $\mathbb{Q}$ *such that*

- $h_{\mathcal{M}_3}(F_n) \to 0$;
- *For any subvariety* $Z \subsetneq \mathbb{P}^2$, $\frac{\#(F_n \cap Z)}{\#F_n} \to 0$.

Then

$$\frac{1}{\#F_n} \sum_{p \in F_n} \delta_p \to \mu_{\mathcal{M}_3} \text{ in } \mathbb{A}^2(\mathbb{C})$$

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Genericity

Theorem

Fix a sequence $n_0^{(k)} \neq n_1^{(k)}$ and $\min n_i^{(k)} \to \infty$, and pick any curve $Z \subset \mathbb{A}^2$. Then

$$\lim_{k\to\infty} \frac{\#\,\mathrm{Per}(n_0^{(k)}, n_1^{(k)}) \cap Z}{\#\,\mathrm{Per}(n_0^{(k)}, n_1^{(k)})} = 0$$

Proof.

- $\mathrm{Per}_\varepsilon(n) = \{(c,a),\ P^n_{c,a}(c_\varepsilon) = c_\varepsilon\}$ has degree 3^n;
- Lower bound
  $\#\,\mathrm{Per}(n_0, n_1) = \#\mathrm{Per}_0(n_0) \cap \mathrm{Per}_1(n_1) = 3^{n_0+n_1}$
- Upper bound
  $\mathrm{Per}(n_0, n_1) \cap Z \subset (\mathrm{Per}_0(n_0) \cap Z) \cup (\mathrm{Per}_1(n_1) \cap Z)$
  $\#\mathrm{Per}(n_0, n_1) \cap Z \leq \deg(Z) 3^{\max n_0, n_1}$

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Transversality problems

Theorem (Adam Epstein)

Pick $n_0 \neq n_1$. Then $\mathrm{Per}_0(n_0)$ and $\mathrm{Per}_1(n_1)$ are smooth at any of their intersection points, and intersect transversally there.

Method inspired by Teichmüller theory. Relies on purely analytical tools (contraction properties of suitable operators in a complex Banach algebra).

Characterization of special subvarieties

Special points:

- $h_{\mathcal{M}_3}(c, a) = 0$
- both critical points have a finite orbit

Question

Describe irreducible curves in $\mathbb{A}^2$ for which the set of special points is infinite.

Chambert-Loir answered this for the standard height function (Bogomolov conjecture for semi-abelian varieties).

Characterization of special subvarieties

Conjecture (Baker-DeMarco)

Let $V \subset \mathbb{A}^2$ be an irreducible curve containing infinitely many (c, a) such that both critical points of $P_{c,a}$ have a finite orbit.
Then

- *either one of the two critical points has finite orbit for all $v \in V$;*
- *or there exists a critical dynamically defined relation, i.e a closed subvariety $Z \subset V \times (\mathbb{A}^1)^2$ invariant by the map $(v, z, w) \mapsto (v, P_v(z), P_v(w))$ and containing (v, c_0, c_1) for all $v \in V$.*

Beyond equidistribution: characterization of special subvarieties

Example

$P_{c,a}(c) = 0$ defines a special curve $\{6a^3 = c^3\}$

Example

The family $P_t(z) = z^3 - 3tz^2 + (2t^3 + t)$ is special.

- $c_0 = 0$, $c_1 = 2t$
- $h_t(z) = -z + 2t$ satisfies $h_t \circ f_t = f_t \circ h_t$
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Characterization of special subvarieties

Theorem (Baker-DeMarco)

In the space of cubic polynomials $P_{a,b} = z^3 + az + b$*. Consider the curve*

$$\mathrm{Per}(\lambda) = \{P_{a,b} \text{ admits a fixed point with multiplier } \lambda\}$$

Then $\mathrm{Per}(\lambda)$ *contains infinitely many points for which both critical points are periodic iff* $\lambda = 0$*.*