Algebraic dynamics of polynomial maps: the dynamical Mordell-Lang conjecture

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The dynamical Mordell-Lang conjecture

- $f : X \to X$ regular dominant map of an algebraic variety defined over $\mathbb{C}$;
- $V \subset X$ a subvariety, and $x \in X$ a point;

Conjecture (Denis, Bell-Ghioca-Tucker)

The set of hitting times $\{n \in \mathbb{N},\ f^n(x) \in V\}$ is a finite union of arithmetic sequences.

An arithmetic sequence is a set $\{an + b, n \in \mathbb{N}\}$ for some integers a, b (possibly zero)

Origin of the conjecture

Theorem (Skolem-Mahler-Lech's theorem)

Suppose $u_n \in \mathbb{C}$ is defined by a recurrence relation $u_{n+k+1} = a_k u_{n+k} + \cdots + a_0 u_n$, $a_i \in \mathbb{C}$. Then the set $\{n \in \mathbb{N},\ u_n = 0\}$ is a finite union of arithmetic sequences.

Conjecture $\Rightarrow$ Theorem.

Take $X = \mathbb{A}^{k+1}_{\mathbb{C}}$, f linear, $x = (u_0, \cdots, u_k)$, and V a hyperplane. □

Theorem (Falting-Vojta)

Let G be a (semi)-abelian variety over $\mathbb{C}$, let V be a subvariety, and let Γ be a finitely generated subgroup of $G(\mathbb{C})$. Then $V(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.

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The polynomial case

- $f : \mathbb{A}^2 \to \mathbb{A}^2$ polynomial dominant map;
- V an irreducible curve, and $x \in X$ a point.

Conjecture

When $\{n \in \mathbb{N},\ f^n(x) \in V\}$ is infinite, then either x or V is pre-periodic.

Theorem (J. Xie)

For any polynomial map $f : \mathbb{A}^2_{\bar{\mathbb{Q}}} \to \mathbb{A}^2_{\bar{\mathbb{Q}}}$ the previous conjecture is true.

Automorphisms

Theorem (Bell-Ghioca-Tucker)

For any polynomial automorphism $f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ *the set* $\{n \in \mathbb{N},\ f^n(x) \in V\}$ *is infinite iff* x *or* V *is periodic.*

- Their method applies to any étale maps in any dimension.
- Elaboration of the original method of Skolem based on *p*-adic methods.

First reduction

- Use a specialization argument to reduce to the case f, x, V have coefficients in $\mathbb{Q}$.
- Pick a large prime number p not dividing denominators in the coef. of f, x, V, and such that f mod p remains an automorphism.

Work in $\mathbb{Q}_p$: completion of $\mathbb{Q}$ w.r.t the p-adic norm $|p| = \frac{1}{p}$.

$$\mathbb{Z}_p := \{\, x \in \mathbb{Q}_p,\ |x|_p \leq 1\} = \text{closure of } \mathbb{Z}\ .$$

Skeleton of the argument

1. Replace f by f^N to get $\bar{f}(\bar{x}) = \bar{x}$ in $\mathbb{A}^2_{\mathbb{F}_p}$;

- The map f is then an analytic automorphism of the open ball $B(x, 1)$;

2. Extend the map $n \mapsto f^n(x)$ to an analytic map $\Phi : \mathbb{Z}_p \to \mathbb{A}^2_{\mathbb{Q}_p}$ s.t. $\Phi(n) = f^n(x)$ for all n;

- For an equation $V = \{h = 0\}$ we have

$$\{n \in \mathbb{N},\, f^n(x) \in V\} \subset \{t \in \mathbb{Z}_p,\, h \circ \Phi(t) = 0\}$$

which is finite or equal to $\mathbb{Z}_p$.

p-adic parameterization

Theorem (Poonen)

Let $f(x) = \sum_I a_I x^I$, $|a_I| \to 0$, $a_I \in \mathbb{Z}_p$ be an analytic automorphism of the closed unit polydisk $\overline{B(0,1)}^d$ such that

$$f \equiv id \mod p^c \text{ with } c > \frac{1}{p-1} .$$

Then there exists an analytic map Φ on $\mathbb{Z}_p \times \overline{B(0,1)}^d$ s.t. $\Phi(n, x) = f^n(x)$ for all n.

- Any point belongs to a one dimensional disk on which f is conjugated to a translation by 1.
- In the complex domain, an analog statement holds in 1d, but not in 2d!
- One line proof but the margin is too small!!!

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Ingredients in Xie's approach

Theorem (J. Xie)

Pick any polynomial map $f : \mathbb{A}^2_{\bar{\mathbb{Q}}} \to \mathbb{A}^2_{\bar{\mathbb{Q}}}$*, any irred. curve* V *and any point* x*. If the set* $\{n \in \mathbb{N},\ f^n(x) \in V\}$ *is infinite, then either* x *or* V *are pre-periodic.*

1. A local analog of DML for special maps.
2. Arithmetical arguments : Siegel's theorem, height argument.
3. Affine geometry: existence of good compactifications, a special device to construct auxiliary polynomials.

A simplified situation

- f, x and V are defined over $\mathbb{Q}$;
- f is alg. stable in $\mathbb{P}^2$ and $\deg(f^n) \to \infty$;
- H_∞ is contracted to a point, say q_∞;
- the invariant valuation in $\mathcal{V}'_3$ is *not divisorial*, e.g. $e(f) < \lambda(f)$;
- the curve V is a line.

Assumption: x is not preperiodic and the set $\mathcal{N} := \{n \in \mathbb{N},\ f^n(x) \in V\}$ is infinite.

Aim: V is preperiodic.

The line contains the super-attracting point I

Step 1: $f^n(x) \to q_\infty \in \mathbb{P}^2_{\mathbb{C}}$ is impossible

- Blow-up at q_∞: f maps again the whole divisor to a fixed point $q_\infty^{<1>}$ that attracts x. Repeat the process until $q_\infty^{<n>}$ is not in the closure of V.
- same argument works when $\mathbb{C}$ is replaced by some $\mathbb{P}^2_{\mathbb{C}_p}$ for some prime p.

The line contains the super-attracting point II

Step 2: the point x is preperiodic

- Uniform upper bound for $|f^n(x)|_p$ for all $n \in \mathcal{N}$ and all place p.
- Height of $f^n(x)$ is bounded for all $n \in \mathcal{N}$.

Remark

This ends the proof when f is a Hénon automorphism. When f is birational, Xie proves that V not periodic implies $f^n(V) \ni q_\infty$ for some n.

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The line does not contain the super-attracting point

- Build a sequence of irrreducible pre-images

$$V_{-k+1} \xrightarrow{f} V_{-k} \xrightarrow{f^k} V$$

with $\mathcal{N}_k := \{n \in \mathbb{N}, f^n(x) \in V_{-k}\}$ infinite.

- Siegel's theorem: V_{-k} has at most two places at infinity

Simplification: V_{-k} has a single place for all k.

$$\nu_{-k}(P) := \mathrm{ord}_\infty(P|_{V_{-k}}) \in \mathbb{Z} \cup \{+\infty\} \text{ associated to } V_{-k} \ .$$

Auxiliary polynomial

Assumption: the map f has at least three points of indeterminacy in a good compactification

Theorem

There exists $P \in \mathbb{C}[x, y]$ s.t. $\nu_{-k}(P) > 0$ for all $k \geq 0$.

Consequence

The function $P|_{V_{-k}}$ is identically zero for all k and V is pre-periodic.

Auxiliary polynomial : existence

- Choose a resolution $f^M : X \to \mathbb{P}^2$
- $X = \mathbb{A}^2 \sqcup \left(\cup_1^s E_i \cup F\right)$ with F (reducible but) connected, and $f^{-M}\{-\deg\} \subset \{\mathrm{ord}_{E_i}\}$
- The curve V_{-k} does not intersect F

Aim: build an ample divisor supported on F so that $X \setminus F$ is affine.

- roughly: start with $\frac{1}{\lambda(f)^M}(f^M)^* H_\infty \in \mathrm{NS}_{\mathbb{R}}(X)$;
- modify it to get zero value on the E_i's.
- Need to contract a couple of E_i's.

$\to$ Look at Xie's paper for detail !!!

The difficulties in the general case

- The curve might have more than one place at infinity ($\leq$ 2 by Siegel's theorem).
- The case the invariant valuation is divisorial is substantially harder.
- Remove the assumption on the existence of sufficiently many indeterminacy points: need to construct suitable height and prove a height bound.
- Need to treat the case $e = \lambda(f)^2$ separately.