New trends in holomorphic dynamics
II: Moduli space

Salt Lake City Workshop

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CNRS

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Brief recap

$f(z) = \frac{P(z)}{Q(z)} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $d = \max\{\deg(P), \deg(Q)\} \geq 2$

- $F(f) = \{z, \{f^n\}_n$ is normal near $z\}$: open, tame dynamics
- $J(f) = \hat{\mathbb{C}} \setminus F(f)$: compact, chaotic dynamics

Study how $J(f)$ behaves when f is varying

- Define the space of rational maps

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The quadratic case

$f_c(z) = z^2 + c$ with $c \in \mathbb{C}$.

https://www.math.univ-toulouse.fr/~Cheritat/Applets

Theorem

- *Either $f^n(0)$ is bounded, and $K(f_c), J(f_c)$ are connected.*
- *Or $f^n(0)$ is unbounded, and $K(f_c) = J(f_c)$ is a Cantor set.*

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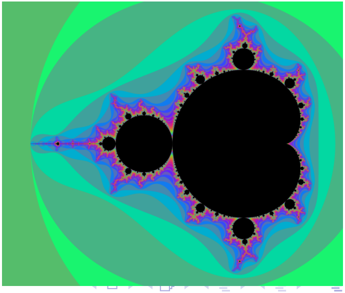
Mandelbrot set

$\mathcal{M} = \{c \in \mathbb{C}, f_c^n(0) \text{ is bounded}\}$



The space of rational maps of degree d

$$f_{a,b}(z) = \frac{P_a(z)}{Q_b(z)} = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_d}$$

- f is determined by $[a:b] \in \mathbb{P}^{2d+1}_{\mathbb{C}}$;
- $P^{-1}(0) \cap Q^{-1}(0) = \emptyset$ is equivalent to $\mathrm{Res}(P_a, Q_b) = 0$

Observation

Rat_d *is an affine subvariety of dimension* $2d+1$*, Zariski open in* $\mathbb{P}^{2d+1}_{\mathbb{C}}$.

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Action of $\mathrm{SL}(2,\mathbb{C})$ on Rat_d by conjugation: $f \sim \phi \circ f \circ \phi^{-1}$

Theorem (Silverman)

- *The ring* $R := \mathbb{C}[\mathrm{Rat}_d]^{\mathrm{SL}(2,\mathbb{C})}$ *is finitely generated, and* $M_d := \mathrm{Spec}(R)$ *is a connected affine algebraic variety of dimension* $2d-2$.
- *The map* $\mathrm{Rat}_d \to M_d$ *induces a bijection*

$$\mathrm{Rat}_d \,/\, \mathrm{SL}(2,\mathbb{C}) \xrightarrow{\simeq} M_d.$$

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Interesting subfamilies of M_d and variants

(Levy): M_d is irreducible and rational

- $M_2 \simeq \mathbb{A}^2_{\mathbb{C}}$ (Milnor)
- Polynomial case: $P(z) = z^d + a_2 z^{d-2} + \cdots + a_d$, hence $MPol_d = \mathbb{C}^{d-1}/G$ with G finite;
- Marked points:

$$(f, p_1, \cdots, p_N) \sim (\phi \circ f \circ \phi^{-1}, \phi(p_1), \cdots, \phi(p_N))$$

marked periodic points, marked critical points

Up to a finite cover, may suppose that M_d is smooth...

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Stability theory: preview

$\{f_\lambda\}$ holomorphic family of rational maps of degree $d \geq 2$

$\lambda \in \Lambda$ complex manifold (e.g., finite cover of M_d, or of $Mpoly_d$)

$F\colon \Lambda \times \hat{\mathbb{C}} \to \Lambda \times \hat{\mathbb{C}}$, $F(\lambda, z) = (\lambda, f_\lambda(z))$ holomorphic

Slogan

Decompose $\Lambda = \mathrm{Stab} \sqcup \mathrm{Bif}$ *where*

- Stab *is open, the dynamics is stable under perturbation;*
- Bif *is closed, the dynamics is unstable under perturbation.*

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The stable locus I

Definition

$\mathrm{Stab}_c =$
$\{\lambda_0, \{\lambda \mapsto f_\lambda^n(c)\}_n$ *is normal at* λ_0 *for any critical point* $c\}$

- In $MPoly_2$, $\mathrm{Stab}_c = \mathrm{Int}(\mathcal{M}) \sqcup \mathbb{C} \setminus \mathcal{M}$;
- If $f \in \mathrm{Rat}_d$ is hyperbolic (all critical points converge to attracting cycles), then $f \in \mathrm{Stab}_c$.

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Definition

$\mathrm{Stab}_p = \{\lambda,$ *the type of periodic orbit remains locally the same*$\}$

Proposition

if $\lambda_0 \in \mathrm{Stab}_p$*, then there exists a map* $h\colon \mathrm{Per}(f_{\lambda_0}) \times U \to \hat{\mathbb{C}}$ *such that*

- $h(\cdot, \lambda)\colon \mathrm{Per}(f_{\lambda_0}) \to \mathrm{Per}(f_\lambda)$ *is bijective;*
- $\lambda \mapsto h(p, \lambda)$ *is holomorphic*

$\longrightarrow$ get a holomorphic motion of $J(f_{\lambda_0})$

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Mané-Sad-Sullivan theory

Theorem

The following conditions are equivalent:

1. $\lambda \in \mathsf{Stab}_c$
2. $\lambda \in \mathsf{Stab}_p$
3. *there exists a holomorphic motion of $J(f_\lambda)$ compatible with the dynamics*
4. *the Julia set moves continuously in the Hausdorff topology*

Stab *is open and dense in Λ.*

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Fatou conjecture

Recall f is hyperbolic iff all critical points converge to some attracting orbit.

Conjecture

The set of hyperbolic maps coincides with the set of stable maps in M_d

Theorem (Douady-Hubbard)

If $\mathcal{M}$ is locally connected, then the set of hyperbolic quadratic polynomials is dense in $MPoly_2$.

McMullen, Yoccoz, Avila, Kahn, Lyubich,...

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2. $g_{\mathcal{M}}(c) = g_c(c)$
3. $g_{\mathcal{M}}(c) = \log|\Phi(c)|$ with $\Phi(c) := \lim_n (f_c^n(c))^{1/2^n}$

$\longrightarrow$ Potential theoretic approach to stability: DeMarco, Berteloot

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References

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