New trends in holomorphic dynamics
I: Fatou-Julia theory

Salt Lake City Workshop

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What is holomorphic dynamics?

Let X be any complex manifold ($\mathbb{C}$, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}/\Lambda$, $\mathbb{C}^d$, $\mathbb{P}^d_{\mathbb{C}}$, etc.)

Q1 Pick $f\colon X \to X$ holomorphic.
Describe the orbits $\{f^{\circ n}(z)\}_{n\in\mathbb{N}}$ for all $z \in X$.

Q2 Suppose $\{f_t\}_{t\in\Lambda}$ is a family of holomorphic maps.
Describe the changes in the dynamics of f_t in terms of t.

Focus on $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$f(z) = \frac{P(z)}{Q(z)} \text{ with } P, Q \in \mathbb{C}[z],\ P^{-1}(0) \cap Q^{-1}(0) = \emptyset,$$

$$d := \max\{\deg(P), \deg(Q)\} \geq 2$$

Generalizations: meromorphic maps, groups, correspondences, ...

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Original developments (1910 –)

- Normal families
- Fatou, Julia, Montel

QC revolution (1980 –)

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- Fatou set: $F_f := \{z, \{f^n\}_n \text{ normal family near } z\}$ (tame dynamics)
- Julia set: $J_f = \hat{\mathbb{C}} \setminus F_f$ (chaotic dynamics)

Observation

The Fatou set (resp. Julia set) is open (resp. closed) and totally invariant.

Theorem

The Julia set is always non-empty (uncountable and perfect)

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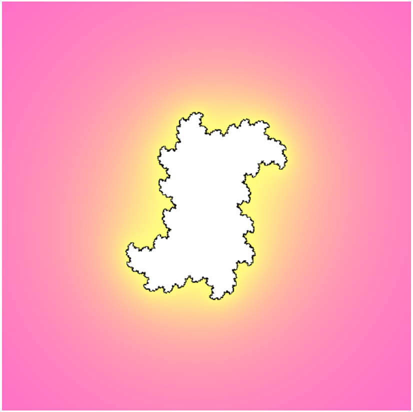
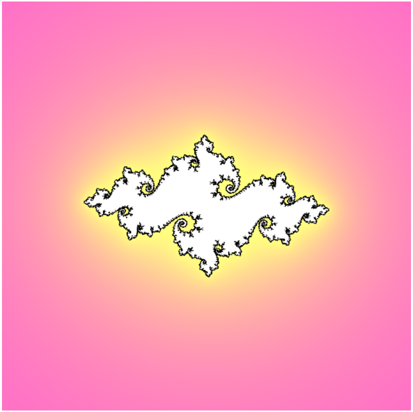
Examples of Julia sets

- $f(z) = z^{\pm d}$, $J(f) = S^1 = \{|z| = 1\}$, $f^{-1}\{0, \infty\} = \{0, \infty\}$;
  $f(z) = z^{\pm d} + \epsilon$, $J(f)$ is a quasi-circle.





Examples of Julia sets

- Lattès maps: $\pi : \mathbb{C}/\Lambda \to \hat{\mathbb{C}}$, $f_L(\pi(z)) = \pi(az)$ with $|a|^2 > 1$, $a\Lambda \subset \Lambda$, $J(f_L) = \hat{\mathbb{C}}$;

Observation

$a = 2$, π is $2:1$, then

$$f(z) = \frac{4z(1-z)(1-t^2z)}{(1-t^2z^2)^2}$$

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Polynomial Julia sets

$$f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z];\ f^{-1}\{\infty\} = \{\infty\}$$

- For $|z| \geq R \gg 1$, then $|f(z)| \geq \frac{1}{2}|z|^d$, and $|f^n(z)| \sim |z|^{d^n} \to \infty$
- Filled-in Julia set $K(f) = \{z, |f^n(z)| = O(1)\}$.

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$J(f) = \partial K(f)$.

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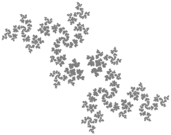
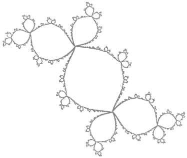
Examples of Julia sets (pictures)



$c = -0.25 + 0.74\,i$



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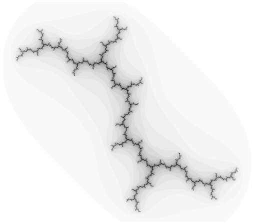
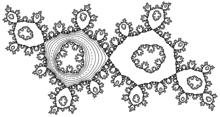
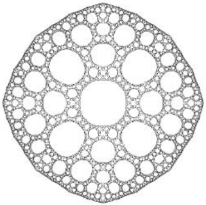


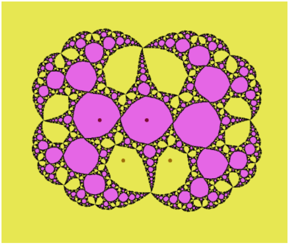
$c = i$



$e^{2i\pi t}\ 2(z-4)/(1-4z)$ with

$t = .6151732$





$z^2 - 0,06/z^2$

Fatou-Julia and periodic orbits

Theorem

Suppose $f(z) = z$, and write $\lambda := f'(z)$.

1. *If $|\lambda| < 1$, then $z \in F(f)$ (attracting);*
2. *if $|\lambda| > 1$, then $z \in J(f)$ (repelling);*
3. *if λ is a root of unity then $z \in J(f)$ (parabolic);*
4. *$\lambda = e^{2i\pi\theta}$, θ badly approximable by rationals (Siegel, Brjuno), then $z \in F(f)$.*

$$\theta = \frac{p_n}{q_n} \text{ and } \sum_n \frac{\log q_{n+1}}{q_n} < \infty$$

Remark

When $f(z) = z^2 + c$ and θ is well-approximable then $z \in J(f)$ (Yoccoz). Open in general (Perez-Marco, Cheraghi, ...)

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Fatou components

Theorem

Let U be a fixed Fatou component. One of the following possibilities occur:

1. *U contains an attracting fixed point p and $f^n|_U \to p$;*
2. *∂U contains a parabolic fixed point p, and $f^n|_U \to p$;*
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Sullivan's theorem

Theorem

Any Fatou component is eventually mapped to a periodic component.

Remark

- *Not true if f is transcendental (Baker, Rippon-Stellard, Benini-Fagella-Evdoridou, Martí-Pete-Rempe-Waterman),*
- *not true in higher dimensions (Astorg-Buff-Dujardin-Peters-Raissy, Berger-Biebler).*

Dynamics on the Julia set

Slogan

The dynamics $f: J(f) \to J(f)$ is chaotic!

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1. $\cup_{n\geq 0} f^{-n}(z)$ *is dense in* $J(f)$ *for all* $z \in J(f)$*;*
2. *the set* $\{z \in J(f), \overline{\{f^n(z)\}_n} = J(f)\}$ *is dense;*
3. *repelling periodic orbits are dense in* $J(f)$*;*
4. $z \in J(f)$, $U \ni z$, *then* $f^n(U) \supset J(f)$ *for some* n*.*

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f *admits a unique measure of maximal entropy* $\log d$*, which is ergodic, and represents the distribution of the repelling periodic orbits.*

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References

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- Carleson-Gamelin: *Complex dynamics*
- Hubbard: *Dynamics in one complex variable*

Pictures:

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- Robert Devaney
- Arnaud Chéritat