Cours 3

lunedì 1 marzo 2021 13:58

Refresh:

$\mathbb{K} = \mathbb{K}^{alg}$, $\mathrm{char}(\mathbb{K}) = 0$, $f \in \mathbb{K}(T)$, $d = \deg(f) \geq 2$

Terminology: a parabolic cycle for f is a periodic orbit $\{x_0, \dots, x_{n-1}\}$ with multiplier $(f^n)'(x_0)$ a root of unity.

Theorem A: The number of parabolic cycles is bounded by $2d-2$

($\Rightarrow \mathrm{Card}(\mathrm{Fix}(f^n)) = d^n + O(1)$)

• Reduction to the case $\mathbb{K} = \mathbb{C}$:

. Suppose that Th. A is true when $\mathbb{K} = \mathbb{C}$

$f = \frac{P}{Q}$, $P = \sum_{i=0}^{d} a_i T^i$, $Q = \sum_{j=0}^{d} b_j T^j$.

Introduce $\mathbb{K}' = \mathbb{Q}(a_i, b_j)^{alg}$ (the algebraic closure of a number field)

$\mathbb{K}'$ is finitely generated over $\mathbb{Q}^{alg}$. $\quad (\mathrm{tr\,deg}(\mathbb{C}, \mathbb{Q}^{alg}) = +\infty)$

Since $\mathrm{tr\,deg}(\mathbb{K}'/\mathbb{Q}^{alg}) < +\infty$, $\mathbb{K}'$ can be embedded in $\mathbb{C}$

One concludes observing that all periodic points of f lie in $\mathbb{P}^1(\mathbb{K}')$

1) Complex parabolic fixed points

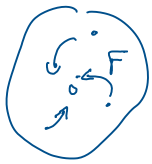
$f \in \mathbb{C}(T)$, $d \geq 2$ $\quad f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ holomorphic

Riemann sphere

Goal: analyze the dynamics of a holomorphic germ having a parabolic fixed point: $F : (\mathbb{C}, 0) \circlearrowleft$ $\quad F(0) = 0$

$\lambda = F'(0)$ its multiplier

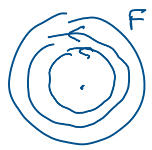
• $\lambda = 0$ $\quad 0$ is superattracting: Böttcher $F \sim z^p$ $\quad p \geq 2$



• $0 < |\lambda| < 1$ $\quad 0$ is attracting: Königs $F \sim \lambda z$

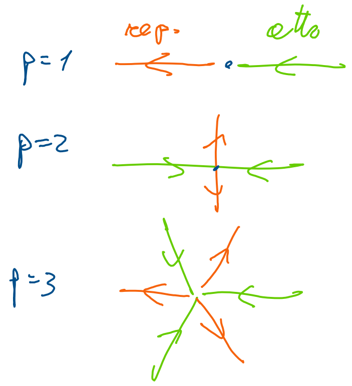
• $|\lambda| > 1$, 0 is repelling: $F \sim \lambda z$ (apply Königs to F^{-1})

- $|\lambda|>1$, 0 is repelling: $F\sim\lambda z$ (apply Königs to F^{-1})
- $|\lambda|=1$, 0 is neutral. $\to$ $\exists N,\ \lambda^N=1$ (parabolic)
  $\to$ $\forall n,\ \lambda^n\neq 1$ (irrational)



We will focus on the parabolic case.
Irrational case: hard in general. easy situation: if $F\sim\lambda z$
in general this is not the case.

Setting: $F:(\mathbb{C},0)\circlearrowleft$ holomorphic germ, $\lambda=F'(0)=1$ (ok up to iterates)
Expand $F(z)=z-z^{p+1}+O(z^{p+2})$ where $p\geq 1$
"First order analysis"
$v\in\mathbb{C}^*$ attracting direction if $v^{p+1}=v$;
repelling " " $v^{p+1}=-v$

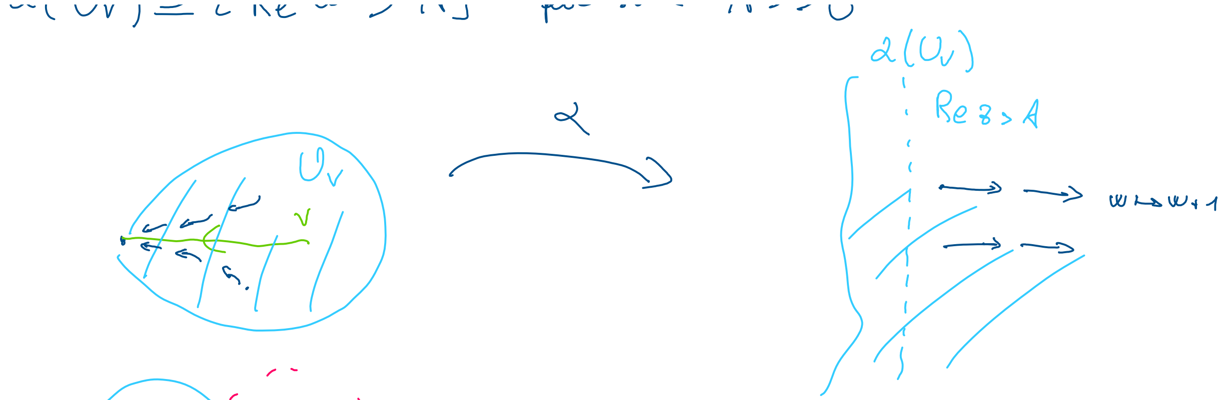


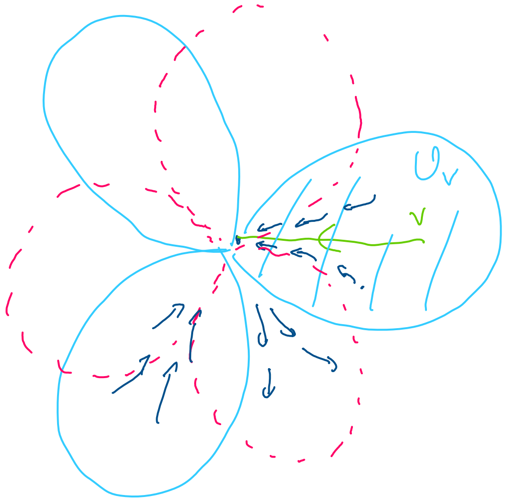
Suppose $F(z)=z-z^{p+1}$. If v is an attracting direction
$F(tv)=v(t-t^{p+1})$ $\forall t\in\mathbb{R}$, if $0<t\ll 1$, $t-t^{p+1}<t$ and $F^n(tv)\to 0$
(If v is repelling, apply the previous argument to F^{-1})

Theorem (Leau-Fatou flower theorem).
$F:(\mathbb{C},0)\circlearrowleft$, $F'(0)=1$, $F\not\equiv id$. For each attracting direction v, there exists an open set U_v and a univalent holomorphic map $\alpha: U_v\to\mathbb{C}$ such that:

1) $U_v\supseteq\,]0,\varepsilon]v$ for some $0<\varepsilon\ll 1$ $(0\in\partial U_v)$;
   $F(\overline{U_v})\subseteq U_v\cup\{0\}$, $F^n(z)\to 0$, $\dfrac{F^n(z)}{|F^n(z)|}\to v$ $\forall z\in U$.
2) $\alpha(F(z))=\alpha(z)+1$
   $\alpha(U_v)\supseteq\{\mathrm{Re}\,w>A\}$ for some $A\gg 0$

$\alpha(U_v)$





← all petals together.

Goal: apply Leau-Fatou to prove thm A.

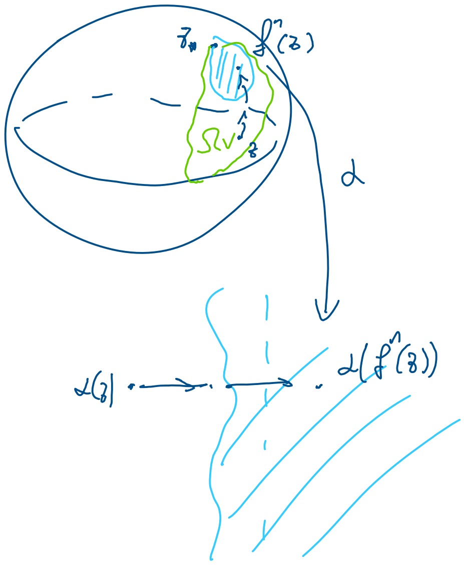
Take $f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ $f \in \mathbb{C}(T)$, $d \geq 2$

Take $z_* \in \mathrm{Fix}(f)$ $f'(z_*) = 1$

We want to extend α as much as possible.

Set Ω_v = connected component of the set of points z, $f^n(z) \in U_v$ for some n, containing U_v.

We set $\alpha : \Omega_v \to \mathbb{C}$, $\alpha(z) = \alpha(f^n(z)) - n$ for $n >> 0$ so that $f^n(z) \in U_v$.



Lemma: α is a holomorphic surjective map from Ω_v to $\mathbb{C}$, and possesses at least one critical point: $\exists z_0 \in \Omega_v$, $\alpha'(z_0) = 0$.

Corollary. For any attracting direction v, Ω_v contains at least a critical point for f.

Proof: take $z_0 \in \Omega_v$ as in the lemma. $0 = \alpha'(z_0) = \alpha'(f(z_0)) f'(z_0)$

if $f'(z_0) = 0$ we are done, if not, $\alpha'(f(z_0)) = 0$.

Apply recursively.. If we never have $f'(f^k(z_0)) = 0$, then we get a sequence $z_n = f^n(z_0) \to z_*$ so that $\alpha'(z_n) = 0$, in contradiction with α being univalent.

sequence $z_n = f^n(z_0) \to z_*$ so that $\alpha(z_n) = 0$, in contradiction with α being univalent.

Proof of Thm A: $f \in \mathbb{C}(T)$ $d \geq 2$. $\{z_1, \dots, z_k\}$ periodic cycle.
$(f^k)'(z_1)$ is a root of unity $\Rightarrow \exists\, \ell$ multiple of k, $f^\ell(z_1) = z_1$, $(f^\ell)'(z_1) = 1$
$\Rightarrow \exists\, c$ s.t. $(f^\ell)'(c) = 0$ and $(f^{n\ell})(c) \xrightarrow{n \to \infty} z_1$.
In particular, there exists c' so that $f'(c') = 0$, and $d(f^n(c), \{z_1, \dots, z_k\}) \underset{n \to +\infty}{\longrightarrow} 0$
"$f^h(c)$ for some $h < \ell$"
Then $\mathrm{Card}\,(\text{parabolic cycles}) \leq \mathrm{Card}\,(\mathrm{Crit}(f)) \leq 2d - 2$ $\square$

Sketch of proof of lemma

$\alpha : \Omega_V \to \mathbb{C}$, $\alpha(z) = \alpha(f^n(z)) - n$.

- $f : \Omega_V \to \Omega_V$ is proper.

($z_n \in \Omega_V$, $f(z_n) \to w \in \Omega_V$, extract $z_n \to z_\infty$, $z_\infty \in \Omega_V$ because open).

- Ω_V connected, $f(\Omega_V)$ is open (because f is open) and closed in Ω_V (because f is proper) $\Rightarrow f(\Omega_V) = \Omega_V$.

- $\alpha(\Omega_V) \supseteq \alpha(U_V) \supseteq \{\mathrm{Re} \geq A\}$.

Take $w \in \mathbb{C}$. $\exists\, n \in \mathbb{N}$ so that $\mathrm{Re}(w + n) > A$.
$\Rightarrow w + n = \alpha(z_0)$ for some $z_0 \in U$. $\Rightarrow \exists\, z_n \in \Omega_V$, $f^n(z_n) = z_0$,
and $\alpha(z_n) = \alpha(f^n(z_n)) - n = \alpha(z_0) - n = w + n - n = w$.
Hence α is surjective.

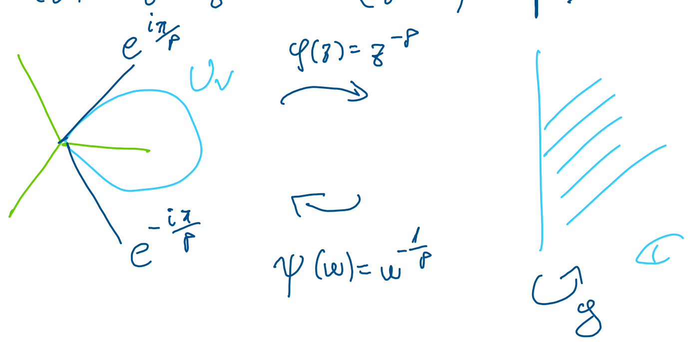
- By contradiction, suppose that $\alpha : \Omega_V \to \mathbb{C}$ has no critical point.

Then α is a covering map. We show that Ω_V is biholomorphic to $\mathbb{C}$.
This is not possible, since f admits ∞-many periodic points, and they do not belong to Ω_V. By Picard theorem, there is no non-constant holomorphic map from $\mathbb{C}$ to $\mathbb{P}^1(\mathbb{C}) \setminus \{3 \text{ points}\}$.
To show that α is a biholomorphism: set $\tilde{U}_V = \alpha^{-1}(\{\mathrm{Re}\, w > A\})$
We know that $\alpha : \tilde{U}_V \xrightarrow{\cong} \{\mathrm{Re}\, w > A\}$ is a biholomorphism (for $A \gg 0$)
Set $U^{(1)} = $ c.c. of $f^{-1}(\tilde{U}_V)$ cont. $\tilde{U}_V$ ...

We know that $\alpha : U_V \xrightarrow{\sim} \{\mathrm{Re}\, w > A\}$ is a biholomorphism (for $A >> 0$)
Set $U^{(1)}$ = c.c. of $f^{-1}(\tilde{U}_V)$ containing $\tilde{U}_V$. Using the Fatou equation and f proper, get $\alpha : U^{(1)} \xrightarrow{\sim} \{\mathrm{Re}\, w > A-1\}$. Continue by induction
use $\nexists$ critical point to show that α is a finite cover, and $U^{(1)}$ connected, $\{\mathrm{Re}\, w > A-1\}$ simply connected to get an isomorphism. $\square$

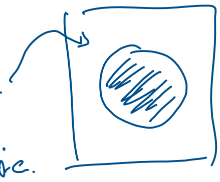
Proof of Leau-Fatou's thm.

$F(z) = z - z^{p+1} + O(z^{p+2})$ $p \geq 1$. $\mathbb{R}_+$ = positive real axis.



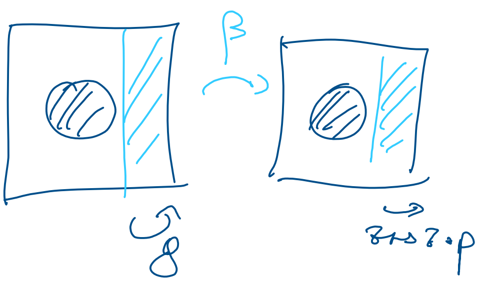
Step 0 (first reduction)
We may conjugate analytically F to a map $z \mapsto z - z^{p+1} + a z^{2p+1} + \text{h.o.t.}$
If $f(z) = z - z^{p+1} + b z^{q} + \text{h.o.t}$, with $p+2 \leq q \leq 2p$, $b \neq 0$, look for
$\chi(z) = z + \alpha z^{q-p}$ so that $F \circ \chi = \chi \circ F_i$, with
$F_i(z) = z - z^{p+1} + O(z^{q+1})$. Solution: take $\alpha = \frac{b}{q-2p-1}$

Step 1 $g(w) = \varphi \circ F \circ \psi(w)$ g well defined on the complement of a disc.

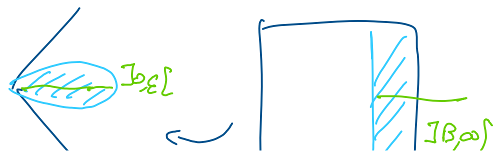


$F \circ \psi(w) = w^{-\frac{1}{p}} - \left(w^{-\frac{1}{p}}\right)^{p+1} + O\left(w^{-\frac{1}{p}}\right)^{2p+1} + \text{h.o.t.}$
$= w^{-\frac{1}{p}}\left(1 - w^{-1} + a w^{-2} + O\left(w^{-2-\frac{1}{p}}\right)\right)$
$g(w) = w\left(1 + p w^{-1} + b w^{-2} + O\left(w^{-2-\frac{1}{p}}\right)\right) = w + p + \frac{b}{w} + O\left(w^{-1-\frac{1}{p}}\right)$ $|w| \to \infty$.

Step 2: linearisation of g in $\{\mathrm{Re}\, w > B\}$.
$\exists \beta$ biholomorphism from $\{\mathrm{Re}\, w > B\}$ onto its image $\supseteq \{\mathrm{Re}\, w > A\}$ and $\beta(g(w)) = \beta(w) + p$.

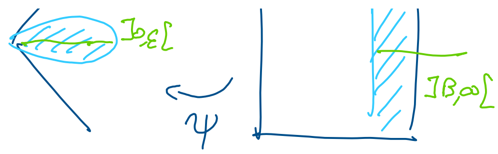


$U_V = \varphi^{-1}(\mathrm{Re}\, w > B) = \psi(\mathrm{Re}\, w > B)$



$U_V = \varphi(\mathrm{Re}\, w > B) = \psi(\mathrm{Re}\, w > B)$

Define $\alpha(z) = \beta \circ \varphi(z)$, and we are done



Construction of β:

If $B >> 0$ and $\mathrm{Re}(w) \geq B$, then $\left|\mathrm{Re}\, g(w) - R(w) - p\right| \leq \left|\frac{b}{w} + O(w^{-1-\frac{1}{p}})\right| \leq \frac{1}{10}$

$\mathrm{Re}(g(w)) \geq \mathrm{Re}(w) + \underbrace{p - \frac{1}{10}}_{\geq \frac{1}{2}} \geq B + \frac{1}{2}$. By induction, $\mathrm{Re}(g^n(w)) \geq \mathrm{Re}(w) + \frac{n}{2}$.

Set : $h_n(w) = g^n(w) - pn - \frac{b}{p}\log n$. We want to show that $h_n \to \beta$.

$|h_{n+1}(w) - h_n(w)| = O\left(\frac{1}{n}\right)$.

$$\left|g^{n+1}(w) - p(n+1) - \frac{b}{p}\log(n+1) - g^n(w) + pn + \frac{b}{p}\log n\right| = \left|g^{n+1}(w) - g^n(w) - p\right| + O\left(\frac{1}{n}\right)$$

$\Rightarrow |h_n(w) - w| = O(\log n)$

$\cdot\ |h_{n+1}(w) - h_n(w)| \overset{(\star)}{=} O\left(\frac{\log n}{n^2}\right) + O\left(\frac{1}{n^{1+\frac{1}{p}}}\right) \Rightarrow h_n \to \beta$ uniformly on $\{\mathrm{Re}\, w > B\}$

β is holomorphic, and it is easy to show that $\beta \circ g \overset{(*)}{=} \beta + p$.

Moreover it is also a biholomorphism with its image (by Hurwitz, the limit is either constant or injective, and it is not constant since it satisfies (*)).

($\star$) $h_n(w) = g^n(w) - np - \frac{b}{p}\log n$.

$$|h_{n+1} - h_n| = \left|g^{n+1}(w) - g^n(w) - p - \frac{b}{p}\log\left(1 + \frac{1}{n}\right)\right| \leq \left|\frac{b}{g^n(w)} - \frac{b}{p}\log\left(1+\frac{1}{n}\right) + O\left(\frac{1}{n^{1+\frac{1}{p}}}\right)\right|$$

$$= \left|\frac{b}{h_n(w) + np + \frac{b}{p}\log n} - \frac{b}{p}\log\left(1+\frac{1}{n}\right)\right| + O\left(\frac{1}{n^{1+\frac{1}{p}}}\right) =$$

$$= \underbrace{\left|\frac{b}{w + np + O(\log n)} - \frac{b}{pn}\right|}_{= O\left(\frac{\log n}{n^2}\right)} + O\left(\frac{1}{n^{1+\frac{1}{p}}}\right)$$

□