Lecture 9 February 4, 2019

§2.3. Weierstrass theory

germ of a holomorphic function at $0 \in \mathbb{C}^n$

Given $f: U \to \mathbb{C}$ holomorphic map $U \ni 0$ open, and $g: V \to \mathbb{C}$ holomorphic map $V \ni 0$ open.

We say that $f \sim_0 g$ if $f = g$ in a common neighborhood of 0.

Definition: A germ of hol. function at $0 \in \mathbb{C}^n$ is an equivalence class of equivalence relation $\sim_0$.

Observation: $f \sim_0 g \Leftrightarrow$ their power series expansions at 0 coincide.

$$\Leftrightarrow \quad \frac{\partial^{|I|} f}{\partial z^I}(0) = \frac{\partial^{|I|} g}{\partial z^I}(0) \quad \forall I.$$

Now we can put together all the germs in one space.

Notation:

$$\mathcal{O}_{(\mathbb{C}^n,0)} = \{\text{holomorphic germs at } 0 \text{ in } \mathbb{C}^n\}$$

$$= \left\{ \sum_I a_I z^I : \begin{array}{l} z=(z_1,\dots,z_n) \\ a_I \in \mathbb{C} \end{array} \right\}$$

$\exists \rho > 0$ s.t. $\sum_I |a_I| \rho^{|I|} < \infty$.

Other notations: $\mathcal{O}(\mathbb{C}^n, 0) = \mathcal{O}_n = {}_0\mathcal{O}_n$

Observation: If M is a $\mathbb{C}$-manifold, define analogously:

$$\mathcal{O}_{(M,p)} = \{\text{hol. germs of } M \text{ at } p\}.$$

Fact: $\mathcal{O}_{(\mathbb{C}^n,0)}$ is a ring $\begin{array}{l} f+g \\ f\cdot g \end{array}$

- has a unit ($f \equiv 1$)
- it is an integral domain ($fg = 0 \Rightarrow f=0$ or $g=0$)
- it is a $\mathbb{C}$-algebra ($\lambda \in \mathbb{C} \to \lambda f \in \mathcal{O}_{(\mathbb{C}^n,0)}$, $f \in \mathcal{O}_{(\mathbb{C}^n,0)}$)

There is a canonical evaluation morphism:

$$\begin{aligned} ev_0 : \mathcal{O}_{(\mathbb{C}^n,0)} &\longrightarrow \mathbb{C} \\ f &\longmapsto f(0) \end{aligned}$$

This is a morphism of $\mathbb{C}$-algebras.

Let

$\mathfrak{m}_{(\mathbb{C}^n,0)} = \{f \mid f(0)=0\}$ is a maximal ideal.

$$\mathcal{O}_{(\mathbb{C}^n,0)} \setminus \mathfrak{m}_{(\mathbb{C}^n,0)} = \{f \in \mathcal{O}_{(\mathbb{C}^n,0)} \text{ s.t. } f(0) \neq 0\}$$
$$= \mathcal{O}^{\times}_{(\mathbb{C}^n,0)} = \{\text{units in } \mathcal{O}_{(\mathbb{C}^n,0)}\}$$

$\Rightarrow \mathfrak{m}_{(\mathbb{C}^n,0)}$ is the unique maximal ideal of $\mathcal{O}_{(\mathbb{C}^n,0)}$.

Thus, $\mathcal{O}_{(\mathbb{C}^n,0)}$ is a local ring.

$$\begin{tikzcd} \mathcal{O}_{(\mathbb{C}^n,0)} \arrow[r] \arrow[rr, bend right, "ev_0"'] & \mathcal{O}_{(\mathbb{C}^n,0)}/\mathfrak{m}_{(\mathbb{C}^n,0)} \arrow[r, "\cong"] & \mathbb{C} \end{tikzcd}$$

Observation: $(n=1)$

$$\mathcal{O}_{(\mathbb{C},0)} = \{ \sum a_n z^n \mid \sum |a_n| \rho^n < +\infty \text{ for some } \rho > 0 \}$$

$$0 \neq f(z) = \sum a_n z^n = \underset{\substack{\# \\ 0}}{a} \, z^{\underset{k \geq 0}{k}} \underbrace{(1 + \dots)}_{\text{unit.}}$$

If $\mathfrak{a}$ is any ideal of $\mathcal{O}_{(\mathbb{C},0)}$

$\mathfrak{a} = (z^k)$ for some $k \geq 0$

$= \mathfrak{m}^k_{(\mathbb{C},0)}$

$\Rightarrow \mathcal{O}_{(\mathbb{C},0)}$ is a PID (principal ideal domain) + Local domain

In particular, it is Noetherian.

"Weierstraß theory allows one to make induction on the dimension and to view holomorphic function as a polynomial in one variable z_n with coefficients that are holomorphic in the other variables $z_1, z_2, \dots, z_{n-1}$".

Notation: $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$

$= (z', w)$

where $z' = (z_1, \dots, z_{n-1})$ and $w = z_n$.

Definition: $f \in \mathcal{O}_{(\mathbb{C}^n,0)}$ is distinguished in z_n with degree $p \geq 0$, if

$$f(0, z_n) = z^p \times (\text{unit})$$

"big Oh" notation

$$= \underset{\neq 0}{a} z_n^p + \mathcal{O}(z_n^{p+1})$$

- A Weierstraß polynomial of degree $p \geq 1$ is a holomorphic function of the form

$$f(z) = w^p + a_1(z')w^{p-1} + \ldots + a_p(z')$$

where $a_i \in \mathcal{O}_{(\mathbb{C}^{n-1},0)}$ and $a_i(0) = 0$ (i.e. $a_i \in \mathfrak{m}_{(\mathbb{C}^{n-1},0)}$).

Remark: In Hörmander's book, distinguished is called normalized

In Gunning's book, distinguished is called regular.

Theorem: (Weierstraß division theorem)
Given $f \in \mathcal{O}_{(\mathbb{C}^n,0)}$ distinguished in z_n of degree p. Then:
$\exists \Delta \ni 0$ open neighborhood, $\exists C > 0$,
for all $g \in \mathcal{O}(\Delta)$, $\sup_\Delta |g| < +\infty$
there exists a Weierstraß polynomial r of degree $\leq p-1$, and $q \in \mathcal{O}(\Delta)$
such that

- $g = qf + r$ $\quad (\star)$

$\max\left(\sup_\Delta |q|, \sup_\Delta |r|\right) \leq C \cdot \sup_\Delta |g|$

Moreover, this decomposition is unique.

Corollary (Weierstraß Preparation Theorem)
$f \in \mathcal{O}_{(\mathbb{C}^n,0)}$ is distinguished in z_n of degree p.

$\exists h \in \mathcal{O}^\times_{(\mathbb{C}^n,0)}$ and $W \in \mathcal{O}_{(\mathbb{C}^n,0)}$ Weierstraß polynomial of degree p.
Such that $f = h \cdot W$.
Furthermore, this decomposition is unique.

Proof of Corollary: Apply Weierstraß division theorem to $g = z_n^p$

$\Rightarrow \quad z_n^p = q \cdot f + r \qquad\qquad z = (z'; w)$

$$r = \sum_{j=0}^{p-1} a_j(z') w^j$$

Look at $(0, w)$ (points of this form)

$$\underbrace{w^p - r(0,w)}_{\substack{\text{monic poly.}\\ \text{of degree } p}} = q(0,w) \cdot \underbrace{f(0,w)}_{(a w^p + \mathcal{O}(w^{p+1}))}$$

$a \neq 0$

$\Rightarrow q(0) \neq 0$ and $r(0,w) \equiv 0$.

$\Rightarrow q \in \mathcal{O}^{\times}_{(\mathbb{C}^n, 0)}$ and so

$$f = \underbrace{q^{-1}}_{\text{unit}} \underbrace{(z_n^p - r)}_{\substack{\text{Weierstraß poly.}\\ \text{of degree } p.}}$$

Uniqueness is left as an exercise (it is a consequence of the uniqueness of the Weierstraß division theorem)

Theorem (Weierstraß Division Theorem)
$f \in \mathcal{O}_{(\mathbb{C}^n,0)}$ distinguished in z_n of degree p. $\exists \Delta \ni 0$ open neighborhood $\exists C > 0$, for all $g \in \mathcal{O}(\Delta)$, $\sup_\Delta |g| < +\infty$ $\exists r \in \mathcal{O}_{(\mathbb{C}^{n-1},0)}[z_n]$ of degree $\leq p-1$ $\exists q \in \mathcal{O}(\Delta)$ such that

- $g = qf + r$

$\max(\sup_\Delta |q|, \sup_\Delta |r|) \leq C \cdot \sup_\Delta |g|$

Obs: $g(z', w) = \sum_{j=0}^{\infty} a_j(z') w^j$

where $a_j \in \mathcal{O}_{(\mathbb{C}^{n-1},0)}$. We split the sum:

$$g(z', w) = \underbrace{\sum_{j=0}^{p-1} a_j(z') w^j}_{R_g} + \underbrace{\left(\sum_{j=p}^{\infty} a_j(z') w^{j-p}\right)}_{Q_g} w^p$$

If $\Delta \ni 0$ is a polydisk, $g \in \mathcal{O}(\Delta)$, $\sup_\Delta |g| < \infty$

$\Longrightarrow R_g, Q_g \in \mathcal{O}(\Delta)$

$\Delta = \{|z'| < \rho,\ |w| < R\},\ \rho, R > 0.$

Cauchy estimate applied to

$w \mapsto g(z', w)$ for a fixed $|z'| < \rho$.

$$|a_j(z')| \leq \sup_\Delta |g| \cdot \frac{1}{R^j}$$

<u>Lemma</u>: $\sup\limits_\Delta |R_g| \leq p \cdot \sup\limits_\Delta |g|$

$$\sup_\Delta |Q_g| \leq \frac{(p+1)}{R^p} \sup_\Delta |g|$$

<u>Proof</u>:

$$|a_j(z') \cdot w^j| \leq \sup_\Delta |g| \cdot \frac{1}{R^j} \cdot R^j = \sup_\Delta |g|$$

$$\Rightarrow \sup_\Delta |R_g| \leq p \sup_\Delta |g|.$$

$$|w|^p \cdot |Q_g(z', w)| \leq |g| + |R_g(z', w)|$$
$$\leq (p+1) \sup_\Delta |g|$$

$|Q_g| \leq \frac{p+1}{R^p} \sup\limits_\Delta |g|$ true on $|w| = R$

hence everywhere by the maximum principle.

• $f = f_1 + w^p f_2$
$\qquad \overset{\shortparallel}{R_f} \qquad \overset{\shortparallel}{Q_f}$

f distinguished $\Longrightarrow f_2(0) = 1.$

$\Rightarrow g = qf + r \iff g = q(f_1 + w^p f_2) + r$

$\iff (q f_2)(f_1 f_2^{-1} + w^p) + r$

• Define $h(z', w) = f_1 f_2^{-1}(z', w) = \text{unit} \cdot$ (poly in w with degree $\leq p-1$ and coeff in $\mathcal{O}_{(\mathbb{C}^{n-1}, 0)}$)

Note $f_1(0, w) \equiv 0$

so $h(0, w) \equiv 0$.

Aim: Solve $g = (w^p + h) \cdot \tilde{q} + \tilde{r}$

Idea: successive approximation.

$g = w^p \cdot q_1 + r_1$ with $q_1 = Q_g$, $r_1 = R_g$

$g - h(z', w) q_1 = w^p q_2 + r_2$

with $q_2 = Q_{g-hq}$ and $r_2 = R_{g-hq_1}$

⋮ (continue inductively)

$g - h q_n = w^p \cdot q_{n+1} + r_{n+1}$

where $q_{n+1} \in Q_{g-hq_n}$ and $r_{n+1} \in R_{g-hq_n}$

After subtracting two levels:

$$-h(q_n - q_{n-1}) = w^p(\underbrace{q_{n+1} - q_n}_{\substack{\shortparallel \\ Q_{(-h(q_n-q_{n-1}))}}}) + (r_{n+1} - r_n)$$

by linearity of the operator Q.

$$\sup_{\Delta} |q_{n+1} - q_n| \le \frac{\sup_{\Delta}|h|}{R^p}(p+1) \sup_{\Delta} |q_n - q_{n-1}|$$

Since $h(0,w) \equiv 0$, by choosing $\rho \ll 1$, we can ensure that

$$\sup_{\substack{|z'|<\rho \\ |w|<R}} |h| \le \frac{1}{2} \frac{R^p}{p+1}$$

$$\Rightarrow \sup_{\Delta} |q_{n+1} - q_n| \le \frac{1}{2} \sup_{\Delta} |q_n - q_{n-1}|$$

$$q_n \xrightarrow[\sup]{} \tilde{q}$$

$$\sup |\tilde{q}| \le C \sup |q_1| \le C' \sup_{\Delta} |g|$$

$$r_n \xrightarrow[\Delta]{\sup} \tilde{r}$$

$$g - h\tilde{q} = w^p \tilde{q} + \tilde{r} \quad \Rightarrow \sup_\Delta |\tilde{r}| \leq C'' \sup_\Delta |g|$$

so in the limit, we set

$$g - h\tilde{q} = w^p \tilde{q} + \tilde{r}$$

$$g = (w^p + h)\tilde{q} + \tilde{r}. \qquad \square$$

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uniqueness: $(w^p + h)q + r = 0$

$r \in \mathcal{O}_{(\mathbb{C}^{n-1}, 0)}[w]$ with $\deg \leq p-1$.

$$-hq = w^p q + r$$

where $q = Q(-hq)$

$$\Rightarrow \sup_\Delta |q| \leq \underbrace{\frac{(p+1)}{R^p} \sup_\Delta |h|}_{\leq \frac{1}{2}} \cdot \sup_\Delta |q|$$

$$\Rightarrow \sup_\Delta |q| = 0 \Rightarrow q = 0.$$

and so we also get $r = 0$.

Lecture 10 Thursday, February 6

$\mathcal{O}_n = \{$ hol. germs at 0 in $\mathbb{C}^n\}$ for $n \geq 1$

$$= \{ \sum_I a_I z^I \mid \sum |a_I| \rho^I < \infty \text{ for some } \rho > 0 \}$$

Recall that for R an integral domain:
R is Noetherian if one of the following equivalent conditions is satisfied:

- $I_n \subseteq I_{n+1}$ ideals $\Rightarrow \exists n_0$ s.t. $I_n = I_{n_0}$ $\forall n \geq n_0$.
- any ideal is finitely generated.

Theorem: (Hilbert Basis Theorem)
R Noetherian $\Rightarrow R[T]$ is Noetherian.

Observation: $R[T_1, T_2, \dots, T_N]/\mathcal{I}$ is Noetherian for any ideal $\mathcal{I}$.

- divisibility in R: $f \mid g$ if $g = f \times h$ for some $h \in R$.
- g is irreducible if $f \mid g \Rightarrow f \in R^\times$ or $\frac{g}{f} \in R^\times$
  (Here $R^\times$ = units in R).

R is factorial if any $f \in R$ admits a unique decomposition as a product of irreducible elements.

In other words, given any two different factorizations,

$$f = f_1 f_2 \cdots f_r \quad (f_i = \text{irreducible})$$
$$= \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_s \quad (\tilde{f}_i = \text{irreducible})$$

$\Rightarrow$ $r = s$ and $\exists$ permutation $\pi : \{1, 2, \ldots, r\} \circlearrowleft$ such that

$$f_i / \tilde{f}_{\pi(i)} \in R^\times \text{ for all } i.$$

Examples of factorial rings: Eucliden domains, PID, etc.

<u>Thm 4</u>: $\mathcal{O}_n$ is both Noetherian + factorial.

<u>Lemma</u>: R Noetherian, $n \geq 0$

Any submodule of R^n is finitely generated.

<u>Proof</u>: $n = 1$, $I \subseteq R$, I = ideal

$\Rightarrow$ I is finitely generated (as R is Noetherian).

<u>Induction on n</u>: Given $M \leq R^n$ a submodule, consider projection $\pi : R^{n+1} \longrightarrow R$

$$(a_1, a_2, \ldots, a_{n+1}) \mapsto a_1$$

$\pi(M)$ is an ideal of R, so we get that $\pi(M)$ is finitely-generated.

$\pi(M) = \langle \pi(m_1), \ldots, \pi(m_r) \rangle$ for some elements $m_1, m_2, \ldots, m_r \in M$.

Now, given $m \in M$, note that

$$m - \sum r_i m_i \in \underbrace{\{0\} \times R^n}_{= \ker(\pi)}$$

So by induction, $M \cap \ker(\pi) \subseteq R^n$ is finitely-generated, and we can combine the generators of $M \cap \ker(\pi)$ together with $m_1, \ldots, m_r$ to get a generating set for M. using induction

$\square$

Proof that $\mathcal{O}_n$ is Noetherian

$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$

$z = (z', w)$ where $z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$ and $w = z_n$

$f \in \mathcal{O}_n$ is distinguished in z_n of order $p \geq 0$

$f(0, w) = a w^p + \mathcal{O}(w^{p+1})$ with $a \neq 0$.

Lemma: $0 \neq f \in \mathcal{O}_n$. There exists an open and dense set $\mathcal{U} \subseteq GL_n(\mathbb{C})$ such that: for every $A \in \mathcal{U}$, we have: $f(A \cdot z)$ is distinguished in z_n.

Proof: Expand f into power series:

$$f(z) = \sum a_I z^I = f_k + f_{k+1} + \dots$$

$$f_j = \sum_{|I|=j} a_I z^I \longleftarrow \text{homogeneous poly. of degree } j.$$

Suppose $f_k \neq 0$.

$V = \{v \in \mathbb{C}^n \mid f_k(v) = 0\}$ closed nowhere else.

$\mathcal{U} = \{A \in GL_n(\mathbb{C}) : A \cdot e_n \notin V\}$

$\mathcal{U}$ is open and dense.

$A \in \mathcal{U}$, $\quad z \to f(A \cdot z) =: f_A(z)$

$$t \longmapsto f(A \cdot te_n) = f_A(0, t)$$

$$= f_k(tA \cdot e_n) + f_{k+1}(tA \cdot e_n) + \dots$$

$$= t^k \underbrace{f_k(A \cdot e_n)}_{\neq 0} + \mathcal{O}(t^{k+1})$$

$\square$

Proof that $\mathcal{O}_n$ is Noetherian

We argue by induction on n.

$n=1$: $\mathcal{O}_1$ is PID (so in particular Noetherian).

Inductive step: $n-1 \longrightarrow n$

Pick an ideal $\mathcal{I} \subseteq \mathcal{O}_n$.

Take $f \neq 0 \in \mathcal{I}$. May assume that f is distinguished in z_n of degree $p \geq 0$.

Weierstrass division theorem:

$\forall g \in \mathcal{I}, \quad g = qf + r, \quad r \in \mathcal{O}_{n-1}[w]$

$\deg_w(r) \leq p-1$

$\mathcal{O}_{n-1} = \{ \sum a_I (z')^I \}$

$$r = \sum_{j=0}^{p-1} r_j(z') \cdot w^j$$

$M = \{ r(g) : g \in \mathcal{I} \}$ is a submodule of $\mathcal{O}_{n-1}^{\oplus p}$

so it is finitely-generated by induction + Lemma.

Finally, $\mathcal{O} = \langle f \rangle + M$ is also finitely-generated. $\square$

Proof of factoriality of $\mathcal{O}_n$

We want to prove:

(1) Suppose $(f_i) \in \mathcal{O}_n$ is a sequence such that $f_{i+1} | f_i$. Then $f_i / f_{i+1} \in \mathcal{O}_n^\times$ for all $i >> 0$.

(2) If f is irreducible and $f | (g \cdot h)$ then $f | g$ or $f | h$.

We proceed by induction on n.

$\underline{n=1}$: clear $\mathcal{O} = (z)^k \Rightarrow f = \text{unit} \times z^k$

Inductive step: Gauß Lemma

$\mathcal{O}_{n-1}$ is factorial $\Rightarrow \mathcal{O}_{n-1}[w]$ is factorial.

Observation: $\mathcal{O}_n^\times = \{ f \in \mathcal{O}_n \mid f(0) \neq 0 \}$.

$\mathcal{O}_{n-1}[w]^\times = \mathcal{O}_{n-1}^\times \subsetneq \mathcal{O}_n^\times \cap \mathcal{O}_{n-1}[w]$.

(for $n \geq 2$)

We prove (2): f irreducible in $\mathcal{O}_n$ and $f | gh$. We may assume that f is distinguished in z_n of order $p \geq 1$ and that $f \notin \mathcal{O}_n^\times$.

Weierstraß preparation $\Rightarrow f = \underset{\mathcal{O}_n}{\text{unit in}} \cdot \underset{\text{polynomial}}{\text{Weierstraß}}$.

We may assume that

$f(z', w)$ is a Weierstrass polynomial

$$\Rightarrow f(z', w) = w^p + \sum_{i=0}^{p-1} a_i(z') w^i, \quad a_i(0) = 0.$$

Weierstrass division $\Rightarrow g = q f + g_0$

$$h = \bar{q} f + h_0$$

where $g_0, h_0 \in \mathcal{O}_{n-1}[w]$.

$\Rightarrow$ $f \mid g_0 h_0$ in $\mathcal{O}_n$ (because $f \mid gh$).

We need two lemmas:

Lemma A: f Weierstraß poly, $\varphi \in \mathcal{O}_{n-1}[w]$.
$f \mid \varphi$ in $\mathcal{O}_n \Longleftrightarrow f \mid \varphi$ in $\mathcal{O}_{n-1}[w]$

Lemma B: f weierstraß poly.
f is irreducible in $\mathcal{O}_n$ iff it is irreducible in $\mathcal{O}_{n-1}[w]$.

Let's use lemmas A and B to finish the proof of factoriality.

$$f \mid g_0 h_0 \text{ in } \mathcal{O}_n \xrightarrow{A} f \mid g_0 h_0 \text{ in } \mathcal{O}_{n-1}[w]$$

$$\xrightarrow{B} f \text{ irreducible in } \mathcal{O}_{n-1}[w]$$

$$\xrightarrow[\text{induction}]{} f \mid g_0 \text{ or } f \mid h_0 \text{ in } \mathcal{O}_{n-1}[w]$$

$$\xrightarrow[\substack{\text{Lemma A} \\ \text{(trivial direction)}}]{} f \mid g_0 \text{ or } f \mid h_0 \text{ in } \mathcal{O}_n. \qquad \square$$

Now we can prove Lemmas A + B.

Note, before we start, that

$$\mathcal{O}_n^\times = \{ f \mid f(0) \neq 0 \}$$

$$(\mathcal{O}_{n-1}[w])^\times = \mathcal{O}_{n-1}^\times$$

<u>Proof of A</u>: ($\Leftarrow$) clear. ($\mathcal{O}_{n-1}[w] \subset \mathcal{O}_n$).

($\Rightarrow$) $f \mid \varphi$ in $\mathcal{O}_n$. Both f, φ are Weierstraß polynomials in variable w, and f is <u>monic</u>.

We do Euclidean division inside $\mathcal{O}_{n-1}[w]$.

$\varphi = qf + r$ where $q \in \mathcal{O}_{n-1}[w]$

$\deg(r) \leq p-1$

But we also have $\varphi = h \cdot f$ $(h \in \mathcal{O}_n)$

By uniqueness of weierstrass division theorem, we get $r = 0$.

Proof of Lemma B

$(\Longleftarrow)$ exercise

$(\Longrightarrow)$ f is reducible in $\mathcal{O}_{n-1}[w]$

φ_1, φ_2 non-invertible in $\mathcal{O}_{n-1}[w]$

$f = \varphi_1 \varphi_2$

$\Rightarrow \underbrace{f(0,w)}_{w^p} = \varphi_1(0,w) \cdot \varphi(0,w)$

$\Rightarrow \varphi_1(0,w) = \overset{0}{\overset{\shortparallel}{a}} \cdot w^k, \quad \varphi_2(0,w) = \overset{0}{\overset{\shortparallel}{b}}\, w^{p-k}$

$\Rightarrow \varphi_1$ & φ_2 are both Weierstrass polynomials of degree k & $p-k$.

If $k = 0$, $\varphi_1 \in \mathcal{O}_{n-1}^{\times} = (\mathcal{O}_{n-1}[w])^{\times}$ which contradicts the hypothesis that φ_1 was non-invertible.

Thus, $K \geq 1$, and similarly $p-K \geq 1 \Rightarrow f$ is reducible in $\mathcal{O}_n$ as $\varphi_1, \varphi_2 \in \mathfrak{m}_{\mathcal{O}_n}$ (maximal ideal) $\square$.

Now we prove 1:

$f_{i+1} | f_i$ for all $i \Rightarrow f_i$ is a Weierstraß polynomial.

$\deg_w(f_i)$ is decreasing by Lemma A.

for $i \gg 0$, $\deg_w(f_i) = \delta$ (stabilizes) $= \deg_w(f_{i+1})$

$$f_i = \underset{\substack{\cap \\ \mathcal{O}_n^\times}}{h} \cdot f_{i+1}, \quad \underbrace{f_i(0,w)}_{w^\delta} = \underbrace{h(0,w)}_{\neq 0} \underset{\substack{\| \\ w^\delta}}{f_{i+1}(0,w)}$$

$\square$

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§2.4. Applications to the geometry of analytic sets

- A germ of analytic set in a $\mathbb{C}$-manifold M at p is an analytic subset Z of an open set $\mathcal{U}$ of p modulo the relation $Z \sim_p Z'$ if:

$Z \cap V = Z' \cap V$ for some open set $V \subset \mathcal{U} \cap \mathcal{U}'$ which contains p_0

$Z \subseteq M$ analytic subset

Any point p determines a germ (Z, p) at p.

Observation: $p \notin Z \Rightarrow (Z,p) = (\emptyset, p)$

(Z,p) = germ of analytic subset at $p \in M$.

$\rightsquigarrow \mathcal{I}(Z,p) = \{ f \in \mathcal{O}_{M,p} \mid f|_Z \equiv 0 \}$

is an ideal of $\mathcal{O}_{M,p}$ $\left(\underset{\mathbb{C}\text{-algebras}}{\overset{\text{isom.}}{\cong}} \mathcal{O}_n \right)$

- If $\mathcal{I} \subseteq \mathcal{O}_{M,p}$ is ideal, since $\mathcal{O}_n$ is Noetherian, we get $\mathcal{I} = \langle f_1, f_2, \ldots, f_r \rangle$. Consider the set $V(\mathcal{I}) := \bigcap_{i=1}^{r} (f_i = 0)$ is analytic.

$(V(\mathcal{I}), p)$ is a germ of analytic set at p which does not depend on choice of generates.

Observation:

$$J \subseteq \mathcal{I}(V(J), p)$$

$$V(\mathcal{I}_{(Z,x)}, x) = (Z, x)$$

This is exactly going to be the analogue of the usual Hilbert's Nullstellensatz.