ANALYSIS IN SEVERAL COMPLEX VARIABLES: ANALYTIC SETS. EXERCICES FOR THE CHAPTER 2

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Exercice 1.

- Show that the unit sphere S^2 can be endowed with a structure of complex manifold.
- Prove that no sphere of odd dimension can be endowed with a structure of complex manifold.

Comment: it is a theorem of Borel and Serre that if S^n can be endowed with a structure of complex manifold, then $n \in \{2, 6\}$, and it is still open whether S^6 can be endowed with a structure of complex manifold, see `https://mathoverflow.net/questions/101888`

Exercice 2 (Hopf manifold). Pick any $\lambda \in \mathbb{C}^*$ such that $0 < |\lambda| < 1$, and consider the linear map $\Lambda(z_1, \cdots, z_n) = (\lambda z_1, \cdots, \lambda z_n)$.

- Prove that Λ acts properly discontinuously on $\mathbb{C}^n \setminus \{0\}$ (ie for any compact set K, the set $\{n \in \mathbb{Z}, \Lambda^n(K) \cap K \neq \emptyset\}$ is finite).
- Prove that the quotient space $S := (\mathbb{C}^n \setminus \{0\})/\langle\Lambda\rangle$ can be endowed with a unique structure of complex manifold such that the canonical projection map $\pi\colon (\mathbb{C}^n \setminus \{0\} \to S$ is holomorphic. Here we identify two points $p, p' \in \mathbb{C}^n \setminus \{0\}$ iff $p' = \Lambda^n(p)$ for some $n \in \mathbb{Z}$.
- Show that S is compact.
- Prove that S is homeomorphic to $S^1 \times S^{2n-1}$.

Comment: it is possible to endow any product $S^{2m-1} \times S^{2n-1}$ with a structure of complex manifold (Calabi-Eckmann manifolds).

Exercice 3. A lattice Λ in $\mathbb{C}^n$ is an additive subgroup which is discrete.

- Prove that the quotient space $T = \mathbb{C}^n/\Lambda$ can be endowed with a unique structure of complex manifold such that the canonical projection map $\pi\colon \mathbb{C}^n \to T$ is holomorphic.
- Prove that for any integer $k \in \mathbb{Z}$, the linear map $\phi_k(z) = kz$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ induces a holomorphic self-map $\Phi_k\colon T \to T$.

Exercice 4. Let Ω be any open subset of $\mathbb{C}^n$ and let $f\colon \Omega \to \mathbb{C}^k$ be any holomorphic function. Prove that for any $l \leq \min\{n, k\}$ the set of point $z \in \Omega$ such that the rank of $dF(z)$ is $\leq l$ is an analytic subset.

- Construct an example with $k = n = 2$ where $\{\text{rank } dF(z) = 0\}$ is a point, and $\{\text{rank } dF(z) \leq 1\}$ is a line.
- Construct an example with $k = n = 2$ where $\{\text{rank } dF(z) = 0\}$ is a line, and $\{\text{rank } dF(z) = 1\}$ is empty.

Exercice 5. Let $f\colon \Omega \to \mathbb{C}$ be any holomorphic function defined on an open subset of $\mathbb{C}^n$.

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- Suppose that $\{f=0\}$ is included inside $\{z_1=z_2=0\}$. Using Hartog's Kugelsatz, prove that $\{f=0\}=\emptyset$.
- Suppose that the holomorphic function $\frac{\partial f}{\partial z_1}$ is not vanishing on any connected component of $\{f=0\}$. (Hard) Prove that the singular locus of $\{f=0\}$ is equal to the set
$$f=\frac{\partial f}{\partial z_1}=\cdots=\frac{\partial f}{\partial z_n}=0\ .$$
- Determine the singular locus of the Whitney's umbrella $\{x^2=y^2z\}\subset\mathbb{C}^3$.
- Determine the singular locus $C := \{y^3+2x^2y-x^4=0\}\subset\mathbb{C}^2$. Show that the real trace of this curve $C\cap\mathbb{R}^2$ is actually smooth. See `https://mathoverflow.net/questions/98366`.

Exercice 6.
- Prove that $\mathcal{O}_{\mathbb{C},0}$ is a principal ideal domain (i.e. every ideal is a power of the maximal ideal).
- Prove that $\mathcal{O}_{\mathbb{C}^n,0}$ is not a principal ideal domain for any $n\geq 2$.

Exercice 7. Let f be a holomorphic map defined in a neighborhood of $0\in\mathbb{C}^n$ with values in $\mathbb{C}$ such that $f(0)=0$ and $df(0)\neq 0$.
- Prove that f is a distinguished polynomial of degree 1 in one of the variables $z_1,\cdots,z_n$.
- Using Weierstrass preparation theorem, prove that $\{f=0\}$ is a submanifold in a neighborhood of 0.

Exercice 8. Suppose that $f\in\mathcal{O}_{\mathbb{C}^n,0}$ is a Weierstrass polynomial in the variables $z=(z',w)$. Prove that if f is irreducible in $\mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ then it is irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$.

Exercice 9. Let Z be any analytic subset of a complex manifold X.

A continuous function $f\colon Z\to\mathbb{C}$ is said to be holomorphic if for any $x\in Z$ there exists a open neighborhood $x\in V\subset X$ and a holomorphic function $F\colon V\to\mathbb{C}$ such that $F|_Z=f$.

For any open subset Ω of V, we let $\mathcal{O}_V(\Omega)$ be the set of all holomorphic functions on Ω.
- Prove that $\mathcal{O}_V$ is a sheaf of local $\mathbb{C}$-algebras (i.e. each stalk $\mathcal{O}_{V,x}$ is a local ring).
- Prove that for each $x\in V$, the stalk $\mathcal{O}_{V,x}$ is isomorphic to $\mathcal{O}_{X,x}/\mathcal{J}_{V,x}$.
- Deduce that $\mathcal{O}_{V,x}$ is Noetherian, and that it is a domain iff the germ (V,x) is irreducible.
- Suppose $Z=\{x^2=y^3\}\subset\mathbb{C}^2$. Using the normalization map $t\in\mathbb{C}\to(t^3,t^2)\in Z$ show that $\mathcal{O}_{Z,0}$ is not isomorphic to $\mathcal{O}_{\mathbb{C},0}$ as a $\mathbb{C}$-algebra (consider the quotient $\mathcal{O}_{Z,0}$ by its maximal ideal).

Exercice 10. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_V$-modules on a complex manifold V. Prove that the support of $\mathcal{F}$ (defined as the set of points for which $\mathcal{F}_x\neq(0)$) is an analytic subset whenever $\mathcal{F}$ is of finite type.

Exercice 11. Let $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3$ be sheaves of $\mathcal{O}_V$-modules on a complex manifold V. Suppose that we have a morphism $\rho\colon\mathcal{F}_1\to\mathcal{F}_2$.
- Prove that $\ker(\rho)$ is a coherent sheaf if $\mathcal{F}_1$ and $\mathcal{F}_2$ are coherent.
- Prove that $\Im(\rho)$ is a coherent sheaf if $\mathcal{F}_1$ and $\mathcal{F}_2$ are coherent.
- Suppose that we have an exact sequence $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$. Prove that if any two of the sheaves are coherent then the third one is also coherent.

Exercice 12. Consider the sheaves $\mathcal{F} = \mathcal{O}_V^{\oplus 3}$ and $\mathcal{G} = \mathcal{O}_V$on $V = \mathbb{C}^3$, and define the morphism

$$\varphi\colon \mathcal{F} \to \mathcal{G},\ (f_1, f_2, f_3) \mapsto z_1 f_1 + z_2 f_2 + z_3 f_3$$

Show that φ is a surjective sheaf morphism, and compute $\ker(\varphi)$ (the difficulty is in computing the stalk over 0).

Exercice 13 (Hilbert syzygies theorem). Let V be any complex manifold of dimension n, and let $\mathcal{F}$ be any coherent analytic sheaf of $\mathcal{O}_V$-modules.

- Using Oka's theorem, prove that there exists an exact sequence of sheaves of $\mathcal{O}_V$-algebras
$$\cdots \mathcal{O}_V^{\nu_n} \xrightarrow{\sigma_n} \mathcal{O}_V^{\nu_{n-1}} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} \mathcal{O}_V^{\nu_1} \xrightarrow{\sigma_1} \mathcal{F} \to 0\ .$$
- Fix a point $x \in V$, and choose local coordinates $(z_1, \cdots, z_n)$ centered at that point. For each $j = 1, \cdots, n$, let $\mathfrak{m}_j \cdot \mathcal{O}_{V,x}$ be the ideal generated by $z_1, \cdots, z_j$, and denote by $\mathcal{S}_k \subset \mathcal{O}_{V,x}^{\nu_k}$ the kernel of σ_k (at x).

  We shall prove that
$$\mathcal{S}_k \cap \mathfrak{m}_j \mathcal{O}_{V,x}^{\nu_k} = \mathfrak{m}_j \cdot \mathcal{S}_k \text{ for all } 1 \le j \le k\ . \tag{1}$$
  - Prove the inclusion $\subset$.
  - Prove the statement for $k = 1$.
  - Proceed by induction on j, and pick
$$F = z_1 G_1 + \cdots + z_{j+1} G_{j+1} \in \mathcal{S}_k \cap \cap \mathfrak{m}_{j+1} \mathcal{O}_{V,x}^{\nu_k}$$
    for some $k \ge j+1$. Prove that $\sigma_k(G_{j+1}) = z_1 G_1' + \cdots + z_j G_j'$ for some other $G_1', \cdots, G_j' \in \mathcal{O}_{V,x}^{\nu_k}$.
  - Apply σ_{k-1} to the previous equality and show there exist $H_1, \cdots, H_j \in \mathcal{O}_{V,x}^{\nu_k}$ such that $G_{j+1} - (z_1 H_1 + \cdots + z_j H_j) \in \mathcal{S}_k$.
  - Conclude the induction step.
- Let $F_1, \cdots, F_r$ be a minimal set of generators of $\mathcal{S}_{n-1}$, and consider the natural morphism $\rho\colon \mathcal{O}_{V,x}^r \to \mathcal{S}_{n-1}$ they induce.

  Let $\mathcal{K}$ be the kernel of ρ. Show that $\mathcal{K} \subset \mathfrak{m}_{V,x} \cdot \mathcal{O}_{V,x}^r$, and use (1) to conclude that $\mathcal{K} = \mathfrak{m}_{V,x} \cdot \mathcal{O}_{V,x}^r$.
- Apply Nakayama's result to prove that $\mathcal{K} = 0$.
- What did we prove about coherent sheaves?

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