Topology of the Lyubich-Minsky Laminations for Quadratic Maps: Deformation and Rigidity (2nd lecture)

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Jussieu, Paris

Nagoya Univ. / LATP CMI
Tomoki KAWAHIRA

The Riemann Surface Laminations Constructions/Examples

II: Lyubich-Minsky C-laminations

Abstruct of Today's Talk

Holo. Dynamics

$f \curvearrowright \mathbb{C}$

holo. dynam.

$\hat{f} \curvearrowright \mathcal{N}_f = \underleftarrow{\lim}(\overline{\mathbb{C}}, f)$

natural extension

Riemann Surf. Lamin.

Topology/Geometry

?

Problem of Rigidity

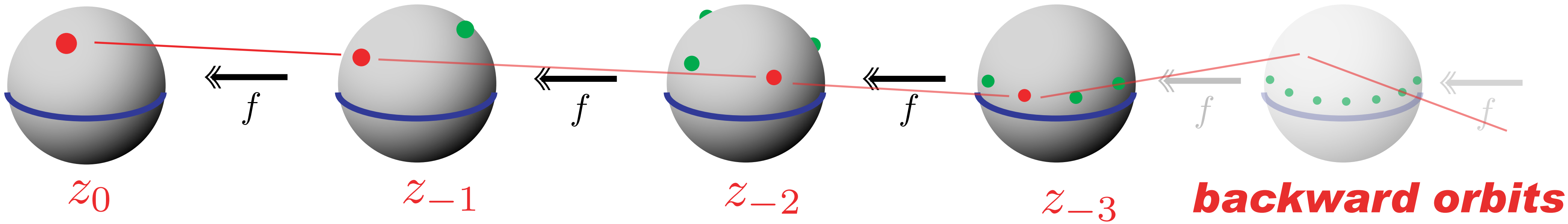
$\hat{f} \curvearrowright \mathcal{R}_f$ **regular part**

$\hat{f} \curvearrowright \mathcal{A}_f$

affine part
= C-lamination

Natural Extension

- Rational map: $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \ \deg f \geq 2$



- *Inverse limit:* $\varprojlim(\overline{\mathbb{C}}, f) = \left\{ \hat{z} = (z_0, z_{-1}, \ldots) : \begin{array}{l} z_0 \in \overline{\mathbb{C}}, \\ f z_{-n} = z_{-n+1} \end{array} \right\}$

$\subset \overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \cdots$ **natural extension**

- Natural lifted action: $\hat{f} \curvearrowright \mathcal{N}_f = \varprojlim(\overline{\mathbb{C}}, f)$

right shift $\hat{f}\hat{z} := (f z_0, f z_{-1}, \ldots) = (f z_0, z_0, \ldots)$

left shift $\hat{f}^{-1}\hat{z} := (z_{-1}, z_{-2}, \ldots)$

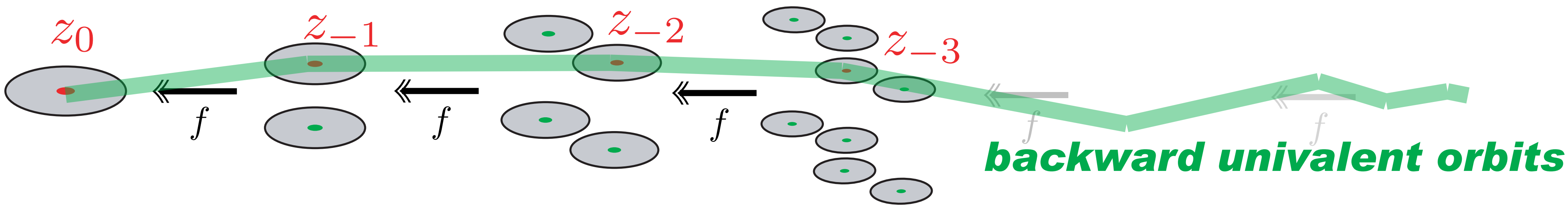
projection $\pi_{-n}(\hat{z}) := z_{-n}$

$$\begin{array}{c} \hat{f} \curvearrowright \mathcal{N}_f \\ \big\downarrow \pi_{-n} \\ f \curvearrowright \overline{\mathbb{C}} \end{array}$$

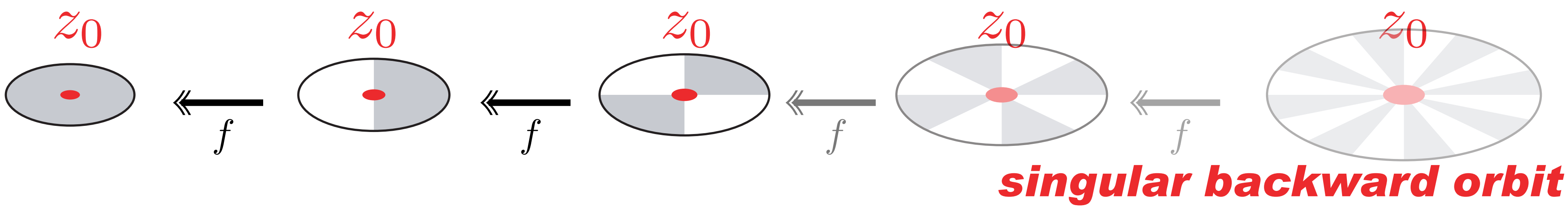
semiconj.

"Regular" Backward Orbits

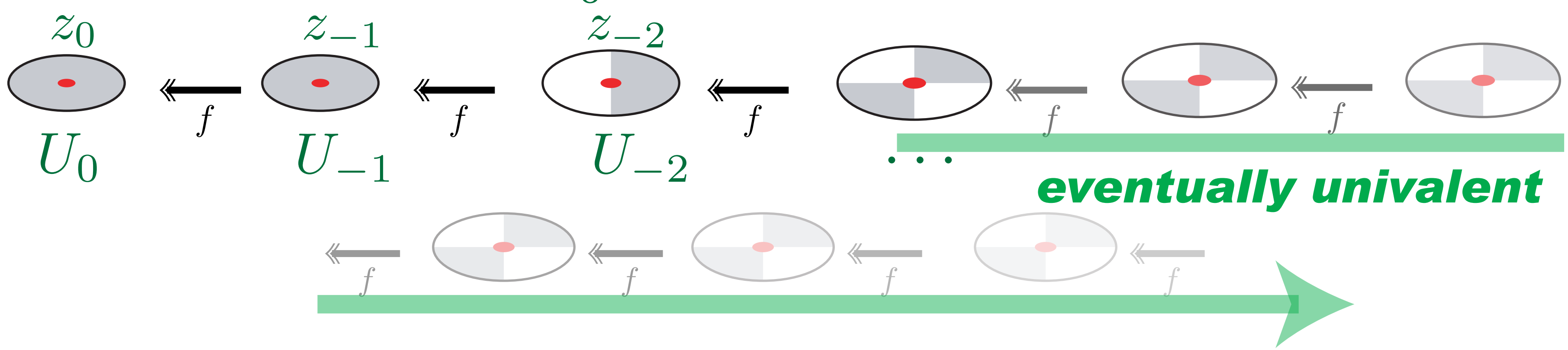
- Ex. $fz = z^2$, $\hat{z} = (z_0, z_{-1}, \ldots)$, $z_0 \neq 0, \infty$



- Ex. $fz = z^2$, $\hat{z} = (z_0, z_{-1}, \ldots)$, $z_0 = 0$ or ∞



- **Definition:** A backward orbit $\hat{z} = (z_0, z_{-1}, \ldots)$ is *regular* if there exists a nbd. U_0 of z_0 st

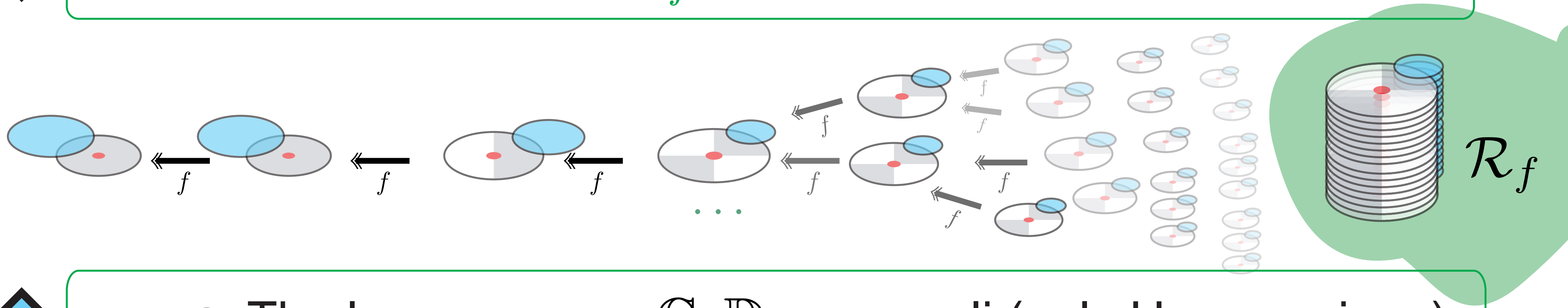


Regular Part

- **Definition:** The set of regular backward orbits in $\mathcal{N}_f = \varprojlim(\overline{\mathbb{C}}, f)$ is called the *regular part* $\mathcal{R}_f$.

- **Ex:** Backward orbits in a repelling cycle: $\mathcal{R}_f$ **regular pt.**
  in an attracting or parabolic cycle: $\mathcal{N}_f - \mathcal{R}_f$ **irregular pt.**

- **Fact 1:** The regular part $\mathcal{R}_f$ is a "rough" Riem. surf. lamin.



- **Fact 2:** The leaves are $\simeq \mathbb{C}, \mathbb{D}$, or annuli (only Herman rings). In particular, any leaf $\simeq \mathbb{C}$ is dense in $\mathcal{N}_f$.

- **Fact 3:** The action $\hat{f} \curvearrowright \mathcal{R}_f$ is a leafwise conformal homeo.

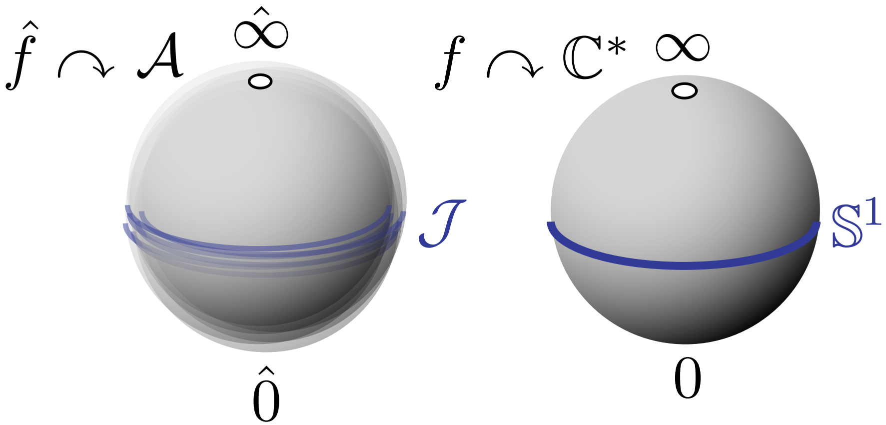
Affine Part: C-lamination

- **Definition:** For future 3D extension (" $\mathbb{C} \longrightarrow \mathbb{H}^3$ ") we denote the union of leaves $\simeq \mathbb{C}$ by $\mathcal{A}_f$, and call it the affine part.

- **Ex:** Backward orbits in a repelling cycle are in $\mathcal{A}_f$.

- **Ex:** When $fz = z^2$, we have

$$\mathcal{R}_f = \mathcal{N}_f - \{\hat{0}, \hat{\infty}\}$$
$$= \varprojlim(\mathbb{C}^*, f) = \mathcal{A}_f$$



- **Proposition(Lyubich-Minsky):**
  If f is critically non-recurrent, then

$$\mathcal{R}_f = \mathcal{A}_f = \mathcal{N}_f - \left\{ \begin{array}{l} \textit{cyclic backward orbits in} \\ \textit{attracting/parabolic cycles} \end{array} \right\}$$

Deformation and Rigidity of L-M Affine Parts (C-laminations)

Deformation of Hyperbolic Maps

- A rational map f is called *Hyperbolic* if all critical points are attracted to attracting cycles.
- Consider a perturbation of Hyperbolic f in the space of rational functions of the same degree >1.
- **Fact:** *For small enough perturbation f_ϵ of f, the dynamics near the Julia sets are quasiconformally the same. (**NOT globally conjugate, because of superattracting cycles!**)*
- ***Stable dynamics implies stable topology:***

Theorem(K): *For small enough perturbation f_ϵ of f, the affine parts $\mathcal{A}_f$ and $\mathcal{A}_{f_\epsilon}$ are quasiconformally homeomorphic. (**No matter how many superattracting cycles are!**)*

Quadratic Maps: Cabrera's Theorem

- Let $f_c z = z^2 + c$ and $f_{c'} z = z^2 + c'$ be hyperbolic quadratic maps. If c and c' are in the same hyperbolic component of the Mandelbrot set, then we have $\mathcal{A}_c \approx \mathcal{A}_{c'}$ (qc homeo) by previous **Theorem (K)**. As we expect, the converse is true:

- ***Topology determines the holomorphic dynamics!***

Theorem(Cabrera): *If there exists an orientation preserving homeo. between* $\mathcal{A}_c$ *and* $\mathcal{A}_{c'}$, c *and* c' *must be in the same hyperbolic component.*

Cor 1: *These laminations are actually qc homeo.*

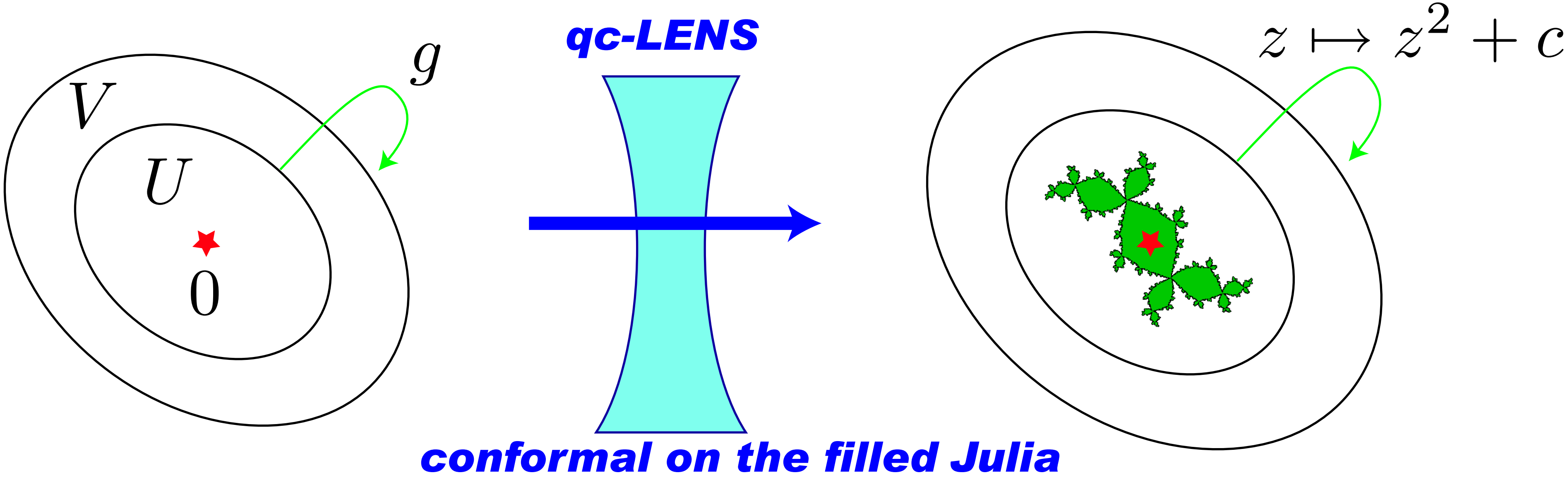
Cor 2: *If they have superattracting cycles,* $c = c'$.

- **Cf. Mostow's Rigidity:** *If two complete finite volume hyperbolic 3-manifolds are homeomorphic, then they are isometrically homeo. (hence are quotients of conformally the same Kleinian groups).*

Topology and Rigidity of LM's C-lamination for infinitely renormalizable quadratic maps

Quadratic-like maps / Straightening

- A *Quadratic-like map* $g : U \to V$ is a proper holo. branched covering of degree 2, like this:

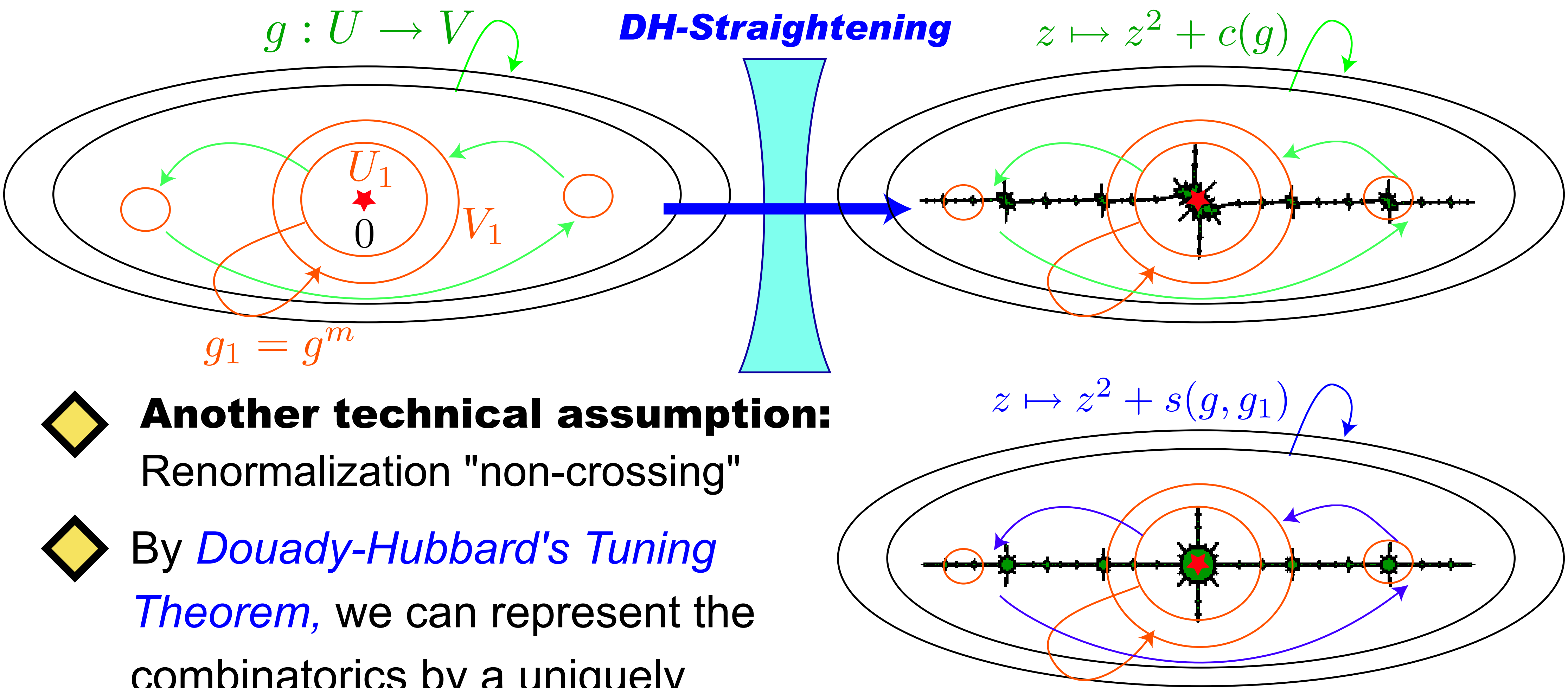


- **A technical assumption:** The critical orbit never goes out by iteration (This implies the connected filled Julia set.) .

- By *Douady-Hubbard's Straightening Map,* we may regard the Q-like map $g : U \to V$ as a deformed image of a quadratic map $z^2 + c$ with uniquely determined $c = c(g)$.

Renormalization / Combinatorics

- A Q-like map $g : U \to V$ is *renormalizable* if there exists "sub-Q-like map" $g_1 = g^m|\, U_1 \to V_1$ like this:

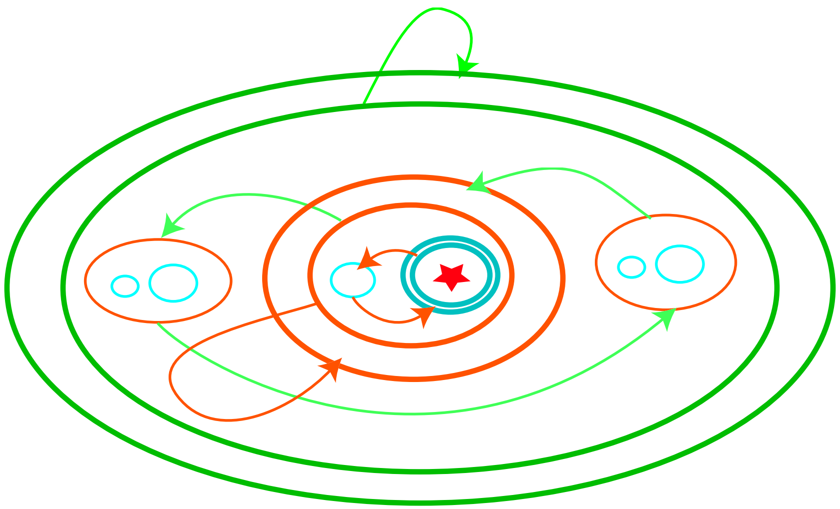


- **Another technical assumption:** Renormalization "non-crossing"
- By *Douady-Hubbard's Tuning Theorem,* we can represent the combinatorics by a uniquely determined superattracting dynamics $z \mapsto z^2 + s(g, g_1)$

Infinite Renormalization

- A quad. map $f_c(z) = z^2 + c$ is *infinitely renormalizable* if there exist "nested-Q-like maps" $\{g_n : U_n \mapsto V_n\}_{n \geq 0}$ like this:

$$\begin{cases} g_0 = f_c : \mathbb{C} \to \mathbb{C} \\ g_{n+1} = g_n^{m_n}|U_{n+1} \end{cases}$$:renormaliazation with $m_n \geq 2$



- The sequence of superattracting parameters $\sigma(c) = (s_0, s_1, \ldots)$ given by $s_n = s(g_n, g_{n+1})$ is called the *combinarotics* of c or $f_c(z) = z^2 + c$.

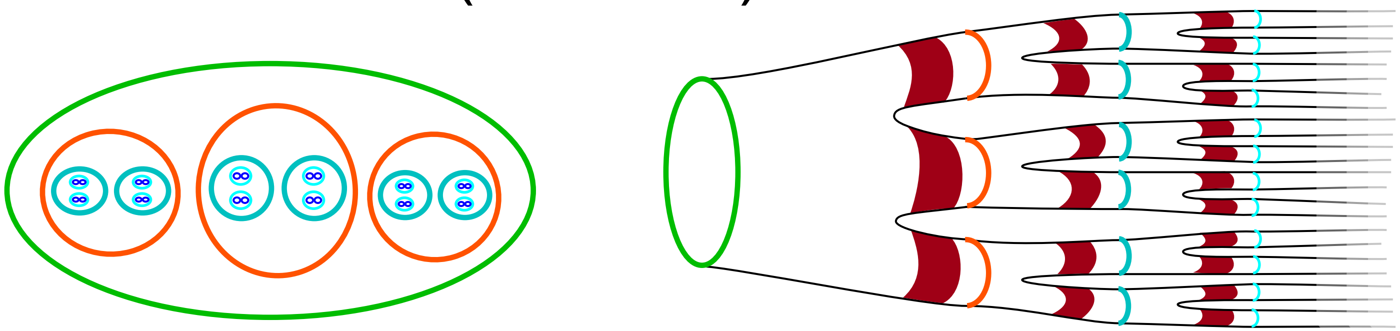
- **Question (Combinatorial Rigidity):**

$$\sigma(c) = \sigma(c') \implies c = c' \ ?$$

(Comb. Rigidity $\Longleftrightarrow$ MLC $\Longrightarrow$ Hyperbolic Density)

Rigidity of Combinatorics

- An infinitely renormalizable $f_c(z) = z^2 + c$ has *a priori bounds* when each level of the renormalization is separated by a definite size (modulus) of annulus:



- **Proposition (Kaimanovich-Lyubich):**

  f_c *has a priori bounds* $\Longrightarrow$ $\mathcal{R}_c = \mathcal{A}_c$ ***(C-lamination)***

- ***Topology of lamin. determines the combinatorics:***

  Theorem(Cabrera-K): *If* f_c *and* $f_{c'}$ *have a priori bounds then:* $\mathcal{A}_c \approx \mathcal{A}_{c'}$ *(homeo.)* $\Longrightarrow$ $\sigma(c) = \sigma(c')$

  Cor: *Plus, MLC at* c $\Longrightarrow$ $c = c'$

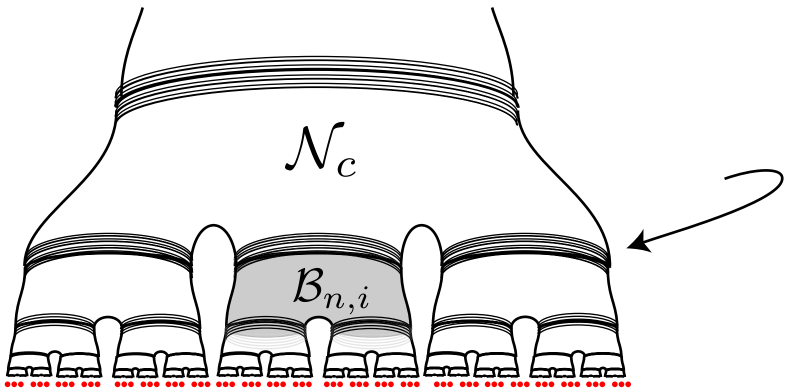
Structure Theorem

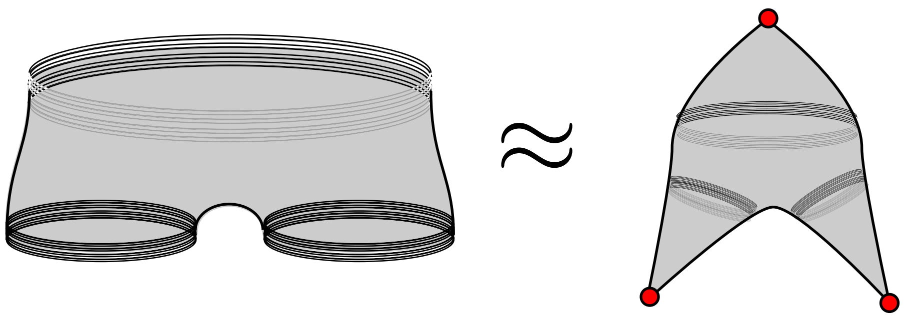
- For the proof, we use the following theorem:

Structure Theorem(C-K): *For infinitely renormalizable* f_c *with combinatorics* $\sigma(c) = (s_0, s_1, \ldots)$ *, its natural extension supports a decomposion by blocks* $\{\mathcal{B}_{n,i}\}$ *and* $\{\mathcal{W}_{n\ j}\}$ *as follows:* $\mathcal{B}_{n,i} \approx \mathcal{A}_{s_n}$,

$$\mathcal{W}_{n,j} \approx \varprojlim(\mathbb{C}, f_{c(g_n)}) \ , \quad \partial\mathcal{W}_{n,j} \approx \varprojlim(\mathbb{S}^1, f_0)$$

$$\mathcal{N}_c = \left(\bigsqcup_{0 \le n \le N} \bigsqcup_{1 \le i \le p_n} \overline{\mathcal{B}_{n,i}} \right) \sqcup \left(\bigsqcup_{1 \le j \le p_{N+1}} \mathcal{W}_{N+1,j} \right)$$ *for any* $N \ge 0$





$\mathcal{A}_{s_n} = \mathcal{N}_{s_n} -$ **{ *irregular pts* }**