SYSTEMS OF PARTICLES INVOLVING PERMUTATIONS AND THEIR CONTINUOUS LIMITS

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Abstract

We discuss several elementary dynamical systems of particles involving permutations. We show that they can approximate nontrivial models of continuum Mechanics and Physics. This includes the simplest model of adhesion dynamics, linked to one-dimensional scalar conservation laws, the Euler equations of inviscid incompressible fluids, some models in Electrodynamics and Geophysics (such as the semigeostrophic equations for atmospheric fronts) and, finally, isothermal gas dynamics equations through the concept of harmonic functions "up to rearrangement".

1 Permutations and sticky particles

1.1 Sticky particles

Let us consider a set of particles, labelled by $a \in \{1, ..., N\}$, of unit mass, moving along the real axis, with position $X(t, a)$ and velocity $X'(t, a)$ at time $t \in [0, T]$, where $T > 0$ is a fixed horizon time. These particles are supposed to freely stream until they collide. Then, they stick and their total momentum is preserved. Typically, as a binary collision occurs at time t^* between particle a and particle $\tilde{a}$, we get

$$X'(t^* + 0, a) = X'(t^* + 0, \tilde{a}) = \frac{1}{2}(X'(t^* - 0, a) + X'(t^* - 0, \tilde{a})).$$

Of course, such collisions dissipate energy. So, this model of 'sticky particles', also known as 'adhesion dynamics' [3], [15], [13]..., is probably the simplest model of particle interaction, combining free streaming, conservation of momentum and maximal dissipation of energy.

1.2 Velocity exchanges

As well known among specialists of rarefied gas dynamics and Boltzmann equation, purely elastic collisions between particles in only one space dimension mean no effective collision at all! Indeed, in one space dimension, an elastic collision between two particles exactly means an exchange of velocity. So, nothing has effectively changed after collision, since these two particles, up to an irrelevant relabelling, continue their way with unchanged velocity. So, we are very far from sticky particles and adhesion dynamics. However, if we introduce a slight delay in the exchange process, we can recover the adhesion dynamics in the vanishing delay limit. More precisely, let us introduce a time step $h > 0$. Assume that all particles freely stream between $t = (n-1)h + 0$ and $t = nh - 0$, for all integer $n > 0$. Namely

$$X(t,a) = X((n-1)h, a) + (t-(n-1)h)X'((n-1)h+0, a), \quad \forall t \in [(n-1)h, nh].$$

Then, they may cross and the order along the real axis of their positions may change accordingly. Although each particle has constant speed, we attach to each particle a 'virtual'velocity subject to change. At time $t = (n-1)h + 0$, the virtual velocity does not differ from the actual velocity. However, at each crossing, particles exchange their virtual velocities, without changing their actual velocity. At the end of the time interval, each particle takes its virtual velocity as its new actual velocity. Mathematically, this amounts to use the following rule. At the end of the time interval, namely at $t = nh$, we look at the positions of the particles $X(nh, a)$, $a = 1, ..., N$. At least in the generic case when all particle positions are distinct from each other, there is a unique permutation $a \to \sigma_n(a)$ such that $X(nh, \sigma_n(a))$ is strictly increasing in a. Then, the velocities are exchanged accordingly. Namely,

$$X'(nh+0, \sigma_n(a)) = X'((n-1)h+0, \sigma_{n-1}(a)).$$

(In other words, velocities are assigned to each particle according to its order along the real axis.) Notice that there is no energy dissipation in this process! However, the process is not reversible in time. As the delay $h > 0$ goes to zero, we get the following convergence theorem :

Theorem 1.1 *Let us denote by $X(t,a)$ and $X_h(t,a)$ the particle positions, for $t \in [0,T]$ and $a = 1, ..., N$, corresponding respectively to the sticky particle dynamics and the velocity exchanging dynamics. Assume that at time $t = 0$,*

$$X_h(0,a) = X(0,a), \quad X_h'(0,a) = X'(0,a), \quad a = 1, ..., n.$$

Then there is a constant C depending only on T and $\max_a |X'(0,a)|$ *such that*

$$\sup_{0\leq t\leq T} \frac{1}{N}\sum_{a=1}^{n} |X_h(t,a) - X(t,a)| \leq C\sqrt{h}.$$

The proof easily follows from two earlier results. First, it has been established in [10] (in the spirit of Dafermos' polygonal approximations of scalar conservation laws [12]) that sticky particles dynamics correspond to general scalar conservation laws in one space dimension with nondecreasing initial data. Next, the velocity exchanging dynamics is nothing more than a discrete version, described in [7], of the "transport-collapse"method for scalar conservation laws [4], [6], or, equivalently, the kinetic approach to scalar conservation laws by Giga and Miyakawa [14]. (A more recent and popular variant is the Perthame-Tadmor "BGK model" for scalar conservation laws [17].) As h goes to zero, the transport-collapse approximation is known to converge to the "entropy solutions" of scalar conservation laws.

1.3 Asymptotic velocity exchanges

From the theory of scalar conservation laws, a necessary and sufficient condition on the initial velocities $V_a = X'_a(0)$ is known to guarantee the following asymptotic behaviour of sticky particles for large time : after a finite time, whatever their initial positions $X_1(0) < X_2(0) < ... < X_N(0)$ are, all particles have merged and form a unique particle with zero speed. This 'single shock condition' is

$$\sum_{\tilde{a}=1}^{a} V_{\tilde{a}} > 0, \quad 1 \leq a < N, \qquad \sum_{a=1}^{N} V_a = 0.$$

It is natural to study the asymptotic behaviour of velocity exchanging particles under this 'single shock' condition. A preliminary step is to find stationary states for the time iteration scheme, with rescaled delay $h = 1$. This exactly amounts to look for a set of positions $X_1 \leq X_2 \leq ... \leq X_N$, i.e. a 'discrete shock profile' and a permutation σ such that

$$X_{\sigma(a)} - X_a = V_a.$$

It turns out [5] that, under the 'single shock condition' on the V_as and the additional non-degeneracy assumption that *all* partial sums of the V_a are non zero, this equation is uniquely solvable. More precisely there is a unique schock profile (up to an irrelevant additive constant) and a unique permutation solving this functional equation. In addition, the locations X_a are distinct from each other, and permutation σ is cyclical (which is the discrete analogous of being ergodic for a Lebesgue measure preserving map).

2 Incompressible inviscid fluids

2.1 Harmonic oscillators with exchange of centers

Consider the periodic cube $D = \mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$. Dispose $N = n^d$ particles equally spaced in D at points $Y(1), ..., Y(N) \in D$. Let us call them "ions". Now, consider N other particles, called "electrons", with label $a = 1, ..., N$ and position $X(t, a) \in D$ at time t. A time step $h > 0$ is fixed. In each time interval $nh \leq t < (n+1)h$, there is a one-to-one pairing (i.e. a permutation) $a \to \sigma(a)$ between each electron a and each ion $Y(\sigma(a))$. A spring links each pair so that each electron oscillates around the corresponding ion with frequency proportional to ϵ^{-1} :

$$\epsilon^2 X''(t,a) + X(t,a) = Y(\sigma(a)) \tag{1}$$

During the time interval $nh < t < (n+1)h$, the total energy of the springs

$$E(t) = \frac{1}{2}\sum_{a=1}^{N}(|X'(t,a)|^2 + \epsilon^{-2} d(X(t,a), Y(\sigma(a)))^2), \tag{2}$$

(where $d(.,.)$ is the Euclidean distance on D) is conserved. At each discrete time $t = nh$, the permutation σ is updated and chosen to minimize, among all permutations, the potential energy

$$\frac{1}{2}\sum_{a=1}^{N}\epsilon^{-2} d(X(t,a), Y(\sigma(a)))^2. \tag{3}$$

(Notice that they may be several solutions, in which case we arbitrarily choose one of them.) So, σ is time dependent, piecewise constant, and denoted by $\sigma(t)$. Of course, we assume both positions and velocities of each particle to be continuous at each discrete time nh and we prescribe their values at time 0. This gives a complete description of the dynamical system. Notice that the total energy, defined by (2), is preserved on each interval $nh < t < (n+1)h$, and can only decay at each time nh, by definition of $\sigma(nh)$. So the total energy is a non-increasing function of time. The possible dissipation is due to the fact that $\sigma(t)$ is updated only at $t = nh$, and not continuously in time (in which case, the system would be formally conservative). It has been shown in [8] that this model of harmonic oscillators with exchange of centers describe incompressible inviscid fluids in the limit as ϵ, h, N^{-1} approach zero. More precisely

Theorem 2.1 *Let* (v, p) *a smooth solution to the Euler equations*

$$(\partial_t + v.\nabla)v + \nabla p = 0, \quad \nabla.v = 0,$$

where $v(t, x)$ *and* $p(t, x)$ *are respectively the velocity and the pressure at* $t \in [0, T]$*,* $x \in D$*. Assume the following ordering to be true*

$$N^{-1} \leq O(\epsilon^{8d}), \quad h \leq O(\epsilon^4)$$

and $X(0,a) = Y(a) + o(\epsilon)$. *Then, if*

$$X'(t,a) = v(0, X(t,a)) + o(1),$$

holds true at $t = 0$, *it holds true also for all* $t \in [0,T]$.

2.2 A caricature of Coulomb interaction

At least formally, in the limit $h = 0, N = \infty$, as $\epsilon > 0$ is fixed, the density field ρ and the velocity field v of the particles are subject to the "Euler-Monge-Ampère" system

$$(\partial_t + v.\nabla)v + \nabla p = 0,$$

$$\partial_t \rho + \nabla.(\rho v) = 0,$$

$$det(\delta_{ij} - \epsilon^2 \partial^2_{x_i\ x_j} p) = \rho.$$

As $\epsilon > 0$ goes to zero, we can expand the determinant and get

$$1 - \epsilon^2 \Delta p = \rho + O(\epsilon^4)$$

(with no error if $d = 1$). Neglecting the $O(\epsilon^4)$ term in the last equation leads to the (pressureless) Euler-Poisson system which describes the Coulomb interaction between a continuum of electrons and a uniform background of ions. So the model of harmonic oscillators with exchange of centers can be seen as a caricature of Coulomb interaction.

3 The semi-geostrophic equations

Hoskins' frontogenesis model and the related semi-geostrophic equations in atmospheric sciences [11], see also [2]) can be discretized in a way involving permutations. Following [1], at least in two space dimensions, we have a set of N particles moving in the periodic square $D = \mathbf{T}^2$ with positions $X(t,a) \in D$, for $a = 1, ..., N$, $t \in [0,T]$, according to the dynamical system

$$i.X'(t,a) + X(t,a) = Y(\sigma(t,a)) \qquad (4)$$

where i is the rotation matrix of angle $\pi/2$ in the plane, and $Y(a), \sigma(t,a)$ are defined exactly in the same way as in the previous section. This (formally) Hamiltonian system differs from the one discussed in the previous section exactly as vortex point dynamics (as described in [16] for instance) differs from classical Coulomb dynamics.

4 Jointly vibrating strings

4.1 The joint potential energy

Let us consider a collection of vibrating strings parameterized by

$$s \in [0, S] \rightarrow X(t, s, a) \in \mathbf{R}^d,$$

and labelled by $a \in A = [0, 1]$. The kinetic energy of each vibrating string at time t is evaluated individually

$$\int_0^S \frac{1}{2} |\partial_t X(t, s, a)|^2 ds$$

and then integrated in $a \in A$. The potential energy of X is not evaluated *individually* for each vibrating string by the usual formula

$$\int_0^S \frac{1}{2} |\partial_s X(t, s, a)|^2 ds,$$

but *jointly* by rearranging the labels of the strings for each fixed value of (t, s), in order to get the lowest possible energy. In other words, we consider all possible collections of fictitious vibrating strings

$$s \in [0, S] \rightarrow Y(t, s, a)$$

having the spatial density as X has, i.e. satisfying

$$\int_A \delta(x - Y(t, s, a)) da = \rho(t, s, x), \tag{5}$$

where ρ is the nonnegative measure defined by

$$\rho(t, s, x) = \int_A \delta(x - X(t, s, a)) da \tag{6}$$

for all continuous test function h and for all $s \in [0, S]$. Then, we define the joint potential energy to be the infimum of

$$\int_A \int_0^S \frac{1}{2} |\partial_s Y(t, s, a)|^2 ds da,$$

with respect to all Y satisfying (5). Then, it can be shown (see [9] for more details) that, assuming ρ to be a smooth positive function of t, s, x, an optimal choice for Y is obtained by solving the following ode in s (t and a being fixed)

$$\partial_s Y(t, s, a) = (\nabla \phi)(t, s, Y(t, s, a)), \quad Y(t, s = 0, a) = X(t, 0, a), \tag{7}$$

where $\phi(t,s,x)$ solves the elliptic equation in x (t and s just being parameters)

$$-\nabla.(\rho\nabla\phi)=\partial_s\rho. \tag{8}$$

(N.B. In order to have a precise statement, we need appropriate boundary conditions. The easiest way, adopted in [9], is to consider that the strings vibrate in the peridic cube $D=\mathbf{T}^d=(\mathbf{R}/\mathbf{Z})^d$. Then, the elliptic equation can be uniquely solved in ϕ with prescribed zero mean on D.)

4.2 The Dynamical equations

Dynamical equations for systems of vibrating string are obtained by varying the difference between the potential energy and the kinetic energy both integrated in time. In the case of an individual string, we would get the simple one-dimensional wave equation

$$\partial_{tt}X=\partial_{ss}X.$$

Here, because the potential energy is computed after optimizing in labels a, the dynamical equations for the jointly vibrating strings are far more complicated. They are obtained by varying

$$\int\frac{1}{2}(\eta|\partial_t X|^2+|\partial_s Y|^2)dtdsda-\int(q(t,s,X(t,s,a))-q(t,s,Y(t,s,a)))dtdsda, \tag{9}$$

where $\eta=-1$ and $q(t,s,x)$ is the Lagrange multiplier for constraint (5). We get

$$\eta\partial_{tt}X(t,s,a)=(-\nabla q)(t,s,X(t,s,a)), \tag{10}$$

$$\partial_{ss}Y(t,s,a)=(+\nabla q)(t,s,Y(t,s,a)). \tag{11}$$

Because of (7), $\nabla\phi$, denoted by e, must satisfy

$$\partial_s e+(e.\nabla)e=\nabla q, \tag{12}$$

which leads, after integrating in x, to the following expression for q (up to an irrelevant additive constant)

$$q=\partial_s\phi+\frac{1}{2}|\nabla\phi|^2.$$

Summarizing the equations we obtained for the jointly vibrating strings, we have

$$\eta\partial_{tt}X(t,s,a)=(-\nabla q)(t,s,X(t,s,a)),\quad \rho(t,s,x)=\int\delta(x-X(t,s,a))da, \tag{13}$$

$$\partial_s\rho+\nabla.(\rho\nabla\phi)=0,\quad \partial_s\phi+\frac{1}{2}|\nabla\phi|^2=q. \tag{14}$$

The second order ode (13) can be expressed as well in terms of the "phase density" $f(t,s,x,\xi) \geq 0$ associated to X and defined by

$$f(t,s,x,\xi) = \int \delta(x - X(t,s,a))\delta(\xi - \partial_t X(t,s,a))da, \tag{15}$$

for $\xi \in \mathbf{R}^d$. We get for f

$$\partial_t f + \xi.\nabla_x f - \eta \nabla q.\nabla_\xi f = 0, \quad \int f(t,s,x,d\xi) = \rho(t,s,x). \tag{16}$$

So, the jointly vibrating strings equations can be written either as the combination of (13), (14), in terms of (X,ρ,ϕ), or as the combination of (14), (16), in terms of (f,ρ,ϕ). Special solutions of the latter formulation can be found by assuming f and q to be time independent. Indeed, in such a case,

$$f(s,x,\xi) = F(s, \frac{|\xi|^2}{2} - q(s,x)), \tag{17}$$

provides a solution for (16), for any choice of $F(s,r)$ as a smooth nonnegative function with sufficient decay at $r = +\infty$. To solve the jointly vibrating string equations, we now need

$$\rho(s,x) = \int f(s,x,\xi)d\xi = R(s,q(s,x)), \tag{18}$$

where (using polar coordinates)

$$R(s,\sigma) = C_d \int_0^\infty F(s,\tau - \sigma)\tau^{d/2-1}d\tau, \tag{19}$$

C_d being the $d-1$ Hausdorff measure of the unit sphere in $\mathbf{R}^d$. In particular, as $F(s,r) = (2\pi)^{-d/2}\exp(-r)$, we get

$$\rho(s,x) = \exp(q(s,x)), \tag{20}$$

and

$$f(s,x,\xi) = \rho(s,x)(2\pi)^{-d/2}\exp(-\frac{|\xi|^2}{2}). \tag{21}$$

From this calculation, it is not hard to deduce

Theorem 4.1 *Let us consider a smooth solution $(f,\rho > 0,\phi)$ to the jointly vibrating string equations (16), (14). Assume that ρ does not depend on t and is not uniform in (s,x). Then f has the "thermalized" form*

$$f(s,x,\xi) = \rho(s,x)(2\pi\Theta)^{-d/2}\exp(-\frac{|\xi|^2}{2\Theta})$$

for some positive constant $\Theta > 0$, *if and only if* $(\rho, u = \nabla\phi)$ *satisfies*

$$\partial_s \rho + \nabla.(\rho u) = 0, \tag{22}$$

$$\partial_s u + (u.\nabla)u + \frac{\nabla p}{\rho} = 0, \tag{23}$$

$$p = \eta\rho\Theta. \tag{24}$$

Equations (22),(23) describe an isothermal gas, with density $\rho(s,x)$, velocity $u(t,x)$ with an unphysical negative temperature $\eta\Theta$, since $\eta = -1$. This yields a way to recover the physical isothermal gas dynamics equations just by using $\eta = +1$, instead of $\eta = -1$ in definition (9) for the Action principe! This amounts to consider no longer jointly vibrating strings but rather "harmonic functions up to rearrangement" in the following sense :

Definition 4.2 *A collection of functions* $(t,x,a) \to X(t,s,a)$ *is harmonic up to rearrangement if it minimizes*

$$\int |\partial_t X(t,s,a)|^2 dtdsda + \inf_Y \int |\partial_s Y(t,s,a)|^2 dtdsda, \tag{25}$$

with respect to local perturbations, where the infimum in Y *is performed over all possible rearrangement* Y *of* X *with respect to* $a \in A$, *i.e. all* Y *such that*

$$\int_A \delta(x - Y(t,s,a))da = \int_A \delta(x - X(t,s,a))da.$$

Then it can be established than smooth solutions of the isothermal gas dynamics equations generate non trivial examples of harmonic maps up to rearrangement $X(t,s,a)$ [9].

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