HOMOGENEOUS HYDROSTATIC FLOWS WITH CONVEX VELOCITY PROFILES

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Abstract

We consider the Euler equations of an incompressible homogeneous fluid in a thin two-dimensional layer $-\infty < x < +\infty$, $0 < z < \epsilon$, with slip boundary conditions at $z = 0$, $z = \epsilon$ and periodic boundary conditions in x. After rescaling the vertical variable and letting ϵ go to zero, we get the following hydrostatic limit of the Euler equations

$$\partial_t u + u\partial_x u + w\partial_z u + \partial_x p = 0, \tag{1}$$

$$\partial_x u + \partial_z w = 0, \quad \partial_z p = 0, \tag{2}$$

supplemented by slip boundary conditions at $z = 0$ and $z = 1$ and periodic boundary conditions in x. We show that the corresponding initial-value problem is locally, but generally not globally, solvable in the class of smooth solutions with strictly convex horizontal velocity profiles, with constant slopes at $z = 0$ and $z = 1$.

Résumé

On considère les équations d'Euler d'un fluide incompressible homogène se mouvant dans une couche mince $-\infty < x < +\infty$, $0 < z < \epsilon$, glissant sur les bords $z = 0$, $z = \epsilon$ et périodique en x. Par changement d'échelle verticale et passage à la limite $\epsilon \to 0$, on arrive aux équations hydrostatiques

$$\partial_t u + u\partial_x u + w\partial_z u + \partial_x p = 0,$$

$$\partial_x u + \partial_z w = 0, \quad \partial_z p = 0,$$

avec conditions de glissement en $z = 0$ et $z = 1$ et périodicité en x. Nous montrons qu'on peut résoudre en temps petit, mais pas globalement en général, le problème de Cauchy dans la classe des solutions régulières dont les profils de vitesse horizontale sont strictement convexes avec pentes constantes en $z = 0$ et $z = 1$.

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1 Introduction.

The Boussinesq equations of a three-dimensional incompressible inviscid fluid in hydrostatic balance write :

$$\partial_t u + (u.\nabla_x)u + w\partial_z u + \nabla_x p = 0. \tag{3}$$

$$\nabla_x.u + \partial_z w = 0, \quad \partial_z p + \rho = 0, \tag{4}$$

$$\partial_t \rho + (u.\nabla_x)\rho + w\partial_z \rho = 0. \tag{5}$$

In these equations, $(x,z) = (x_1, x_2, z)$ stands for the space variable, $x \in \mathbb{R}^2$, $0 < z < 1$ being the vertical coordinate, $\nabla_x = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient, $(u,w) = (u_1, u_2, w)$ stands for the velocity field, $p(t,x,z)$ and $\rho(t,x,z)$ are the pressure and the density fields. Typical boundary conditions are $w(t,x,z) = 0$ at $z = 0$ and $z = 1$ (slip boundary conditions) and spatial periodicity in x. A discussion of these equations can be found in [6] from the Hamiltonian and non-linear stability point of view. Notice that, when the Coriolis force is added, the so-called primitive equations widely used in oceanography and meteorology [10] (see also [2] and [8]) are then recovered. There has been apparently little interest in the literature for the somewhat degenerate case when the density is uniform, which corresponds to a homogeneous fluid. Then the density can be entirely removed from the equations (since a constant density can be absorbed in the pressure term) and we obtain the simpler system :

$$\partial_t u + (u.\nabla_x)u + w\partial_z u + \nabla_x p = 0. \tag{6}$$

$$\nabla_x.u + \partial_z w = 0, \quad \partial_z p = 0. \tag{7}$$

The resulting equations, that we call homogeneous hydrostatic equations (HHE), exactly correspond to the hydrostatic model mentioned in [9] (ch. 4.6) and may play an important role in the understanding of the 3D Euler equations from which they can be formally obtained after rescaling the vertical variable z as follows :

$$u(t,x,z) \to u(t,x,z/\epsilon), \quad w(t,x,z) \to \epsilon w(t,x,z/\epsilon), \tag{8}$$

$$p(t,x,z) \to p(t,x,z/\epsilon),$$

and letting ϵ go to zero.

In this paper, we mainly address the two-dimensional version (1), (2) of the HHEs, when the horizontal variable is one-dimensional, and introduce a particular class of solutions, called class C solutions and defined as follows. We assume $((u,w)(t,x,z),p(t,x))$ to be smooth functions, say C^1 in $t>0$, $x\in\mathbb{R}$, and C^2 in z, 1-periodic in x, for which the horizontal velocity profile $z\to u(t,x,z)$ has zero mean

$$\int_0^1 u(t,x,z)dz=0 \tag{9}$$

and is strictly *convex*

$$\partial_{zz}u(t,x,z)>0. \tag{10}$$

In addition, we assume the normal derivative of u to be constant along both vertical boundaries, which leads, after normalization, to :

$$\partial_z u(t,x,0)=k,\ \ \partial_z u(t,x,1)=k+1, \tag{11}$$

for some real constant k.

The main assumption is (10) and is clearly reminiscent of the famous Rayleigh stability criterion for stationary solutions of the Euler and Navier-Stokes equations [1]. As a matter of fact, uniform strictly convex profiles

$$u(t,x,z)=U(z),\ \ w(t,x,z)=0,\ \ p(t,x)=0, \tag{12}$$

are trivial examples of global class C solutions.

To solve the initial-value problem, we only need to know the initial horizontal profile $u(0,x,z)$, that we assume to be C^1 in x, C^2 in z, and compatible with class C conditions (9), (10), (11). Our main result is :

Theorem 1.1 *The initial-value problem for (1), (2) is locally solvable in class C, defined by (9), (10), (11). However, for a dense open set of initial conditions $u(0,x,z)$, class C solutions cannot be global.*

2 Main steps of the proof.

The first step is a semi-Lagrangian reformulation of the HHE's, where we use a Lagrangian label $a\in[0,1]$ instead of the vertical coordinate z. In this section, the horizontal variable x is not supposed to be one-dimensional. We define a Lagrangian foliation to be a family of sheets $z=Z(t,x,a)$, labeled by $a\in[0,1]$, where Z is a smooth function such that :

$$0\le Z(t,x,a)\le 1,\ \ Z(t,x,0)=0,\ \ Z(t,x,1)=1, \tag{13}$$

$$\partial_a Z(t,x,a) > 0, \tag{14}$$

$$\partial_t Z(t,x,a) + u(t,x,Z(t,x,a)).\nabla_x Z(t,x,a) = w(t,x,Z(t,x,a)). \tag{15}$$

If $Z(0,x,a)$ is given and compatible with (13), (14), it is always possible to get a Lagrangian foliation on a short interval of time if (u,w) is smooth. (For larger times, it is in general no longer possible to maintain Z smooth.) Then, simple algebraic manipulations translate HHE's into

$$\partial_t c + \nabla_x.(cv) = 0, \tag{16}$$

$$\partial_t v + (v.\nabla_x)v + \nabla_x p = 0, \tag{17}$$

$$\int_0^1 c(t,x,a)da = 1, \tag{18}$$

where the new unknowns (c,v) are defined by :

$$c(t,x,a) = \partial_a Z(t,x,a) > 0, \tag{19}$$

$$v(t,x,a) = u(t,x,Z(t,x,a)) \tag{20}$$

and p is unchanged. This new system of "semi-Lagrangian equations" (SLE), (16), (17) and (18), which involves differential operators only in the horizontal variables x, is equivalent to the original HHE's (6), (7), as long as there is no folding of the Lagrangian sheets generated by the flow from the surface $z = Z(0,x,a)$ at time 0. Then, we show the variational nature of the solutions of the SLE's, when the velocity field v is curl-free, and show that their lifespan is finite for a large set of initial conditions.
Next, we address the initial-value problem for the HHE's in one horizontal space variable. Then, $\omega = \partial_z u$ is solution to the transport equation

$$\partial_t \omega + \partial_x(u\omega) + \partial_z(w\omega) = 0. \tag{21}$$

For class C solutions, the level lines of $\partial_z u$ must be smooth graphs, because of (10). By (21), they are also transported by the velocity field (u,w). It is therefore natural to use the semi-Lagrangian formulation associated with them. In other words, we write :

$$\partial_z u(t,x,z) = k + \int_0^1 H(z - Z(t,x,a))da, \tag{22}$$

where H is the Heaviside function and Z satisfies (13), (14), (15). The strict convexity of the velocity profiles is directly linked to the strict ordering of the curves $z = Z(t,x,a)$ with respect to a. Indeed

$$\partial_{zz}u(t,x,Z(t,x,a)) = \frac{1}{\partial_a Z(t,x,a)}. \tag{23}$$

So, for this special choice of foliation, a class C solution to the HHE's cannot be global unless the corresponding semi-Lagrangian solution (c,v) is also global.
Next, we get for Z a self-consistent system of first-order conservation laws

$$\partial_t Z + \partial_x(\Psi(Z)) = 0, \tag{24}$$

where the flux Ψ is the non-linear operator defined by

$$\Psi(Z)(a) = \frac{k}{2}Z(a)(1-Z(a)) \tag{25}$$

$$+\frac{1}{2}\int_0^1 [(Z(a)-Z(b))^2 H(a-b) - Z(a)(1-Z(b))^2]db.$$

This system turns out to be symmetric and we obtain, through standard energy estimates and maximum principles, the differential inequality

$$\frac{d}{dt}N \le CN^3(1+N), \tag{26}$$

where C is a purely numerical constant and

$$N(t) = \max(|\partial_a Z(t,.)|_\infty, |\partial_{xxa}Z(t,.)|_2, |\partial_{xxx}Z(t,.)|_2), \tag{27}$$

where $|.|_p$ denotes the L^p norm, with respect to $x \in \mathbb{R}/\mathbb{Z}$, $a \in [0,1]$. This is enough to establish that the initial-value problem is locally solvable in class C.
Finally, we deduce from the lifespan estimate obtained for the SLE's that class C solutions cannot be global, except for a set of initial conditions $Z(0,x,a)$ which is closed with empty interior in the class

$$\partial_a Z(0,x,a) > 0, \quad Z(0,x,0) = 0, \quad Z(0,x,1) = 1, \tag{28}$$

equipped with the C^1 topology.

3 A semi-Lagrangian formulation of the homogeneous hydrostatic equations.

In this section, we obtain the semi-Lagrangian formulation (16), (17), (18) of the HHE's. Let (u, w, p) be a solution of the HHE's. We assume (u, w, p) to be smooth, 1-periodic in x_1 and x_2. For each $a \in [0, 1]$, we solve the non-linear first order equation (15), where initial conditions $Z(0, x, a)$ are supposed to be smooth and satisfy (28). (A typical, but not necessary, choice is $Z(0, x, a) = a$.) Observe that, in equation (15), a is just a parameter. This problem has, at least for a short interval of time $[0, T]$, a smooth solution $Z(t, x, a)$. From the maximum principle, we deduce (13) (because of the slip boundary condition $w(t, x, 0) = w(t, x, 1) = 0$) and (14). (Indeed, for $a > a'$, $Z(0, x, a) > Z(0, x, a')$ and the ordering $Z(t, x, a) > Z(t, x, a')$ is preserved as t grows.) We define $c(t, x, a)$ by (19) and get

$$c(t, x, a) > 0, \quad \int_0^1 c(t, x, a) da = 1. \tag{29}$$

Then, $v(t, x, a)$ is defined by (20) and we prove :

Proposition 3.1 *Let (u, w, p) be a smooth solution of the HHE's (6), (7). We assume (u, w, p) to be 1-periodic in x_1 and x_2. Then, at least for a short interval of time, the fields $(Z, v)(t, x, a)$ defined by (15) and (20), are smooth and $(c = \partial_a Z, v)$ are solutions to (16), (17), (18).*

Proof.

We have already defined Z and v. Let us introduce the Lagrangian coordinate X corresponding to v and defined by

$$\partial_t X(t, x, a) = v(t, X(t, x, a), a), \quad X(0, x, a) = x. \tag{30}$$

Differentiating (15) with respect to a yields :

$$\partial_t c(t, x, a) + \partial_z u(t, x, Z) c(t, x, a).\nabla_x Z + u(t, x, Z).\nabla_x c = \partial_z w(t, x, Z) c,$$

where Z stands for $Z(t, x, a)$, for notational simplicity. So, we get :

$$\partial_t c(t, x, a) + \nabla_x.(c u(t, x, Z)) - (\nabla_x.u)(t, x, Z) c = \partial_z w(t, x, Z) c,$$

which leads to (16) by using definition (20) and the fact that (u, w) is divergence-free. Now, let us compute

$$\partial_{tt} X(t, x, a) = (\partial_t v + v.\nabla_x v)(t, X(t, x, a))$$

and show that

$$\partial_{tt}X(t,x,a) = (\partial_t u + u.\nabla_x u + w\partial_z u)(t, X(t,x,a), Z(t, X(t,x,a), a)). \quad (31)$$

By definition (20),

$$\partial_{tt}X(t,x,a) = \partial_t[u(t, X, Z(t, X, a))]$$

(where X stands for $X(t,x,a)$)

$$= (\partial_t u)(t, X, Z(t, X, a)) + (u.\nabla_x u)(t, X, Z)$$

$$+(\partial_z u)(t, X, Z(t, X, a))\partial_t[(Z(t, X, a)].$$

Since (15) implies

$$\partial_t[Z(t, X, a)] = w(t, X, Z(t, X, a)),$$

the proof of (31) is now complete. From (31), we directly obtain (17) from (6). This concludes the proof of Proposition 3.1.
Let us point out that the correspondence between the hydrostatic and the semi-Lagrangian equations is only local in time. Indeed, generally after a finite time, equation (15) will generate shocks, namely discontinuous gradients in x for $Z(t,x,a)$. This corresponds to the folding of the Lagrangian sheets $z = Z(t,x,a)$, which does not rule out the continuation of a smooth solution to the HHE's. Anyway, it should be emphasized that the mapping between (u,w) and (c,w) is one-to-one *only modulo* the choice of $Z(0,x,a)$, subject to (28). There is a definite arbitrariness in this choice, which will be used subsequently.

4 A lifespan estimate for the semi-Lagrangian equations.

In this section, we show that the SLE's have a variational nature (related to [3] and [4]) and we consider smooth solutions (c,v,p) with curl-free velocity fields :

$$(\partial_i v_j - \partial_j v_i)(t,x,a) = 0, \;\; i,j = 1,...,d. \quad (32)$$

Of course, (32) is always satisfied when $d = 1$.

Theorem 4.1 .

Let $((c,v)(t,x,a),p(t,x))$ *be a smooth solution,* $\mathbb{Z}^d$ *periodic in* x*, of the semi-Lagrangian equations (16), (17), (18), satisfying (32), for* $0 \le t \le T$*. Then, for each* $a \in [0,1]$*, the mean velocity and the mean concentration are time independent :*

$$\int v(t,x,a)dx = \int v(0,x,a)dx = V(a), \tag{33}$$

$$\int c(t,x,a)dxda = \int c(0,x,a)dxda = M(a) \tag{34}$$

and

$$|\ ||v|| - ||V||\ | \le \frac{2D}{T} + 4(\frac{2D||v||}{T})^{1/2}, \tag{35}$$

where $D^2 = d/12$,

$$||v||^2 = \int |v(t,x,a)|^2 c(t,x,a)dxda \tag{36}$$

is time independent and

$$||V||^2 = \int V(a)^2 M(a)da = \int |\int v(t,x',a)dx'|^2 c(t,x,a)dxda. \tag{37}$$

In particular, the solution cannot be global unless the initial data satisfy

$$||v|| = ||V||. \tag{38}$$

Notice that, in the particular case $c(0,x,a) = M(a)$, condition (38) implies $v(0,x,a) = V(a)$, by Cauchy-Schwarz' inequality. The proof of Theorem 4.1 is based on the following least action principle :

Proposition 4.2 *Let us call admissible solution any smooth pair*

$$(t,x,a) \in [0,T] \times \mathbb{R}^d \times [0,1] \to (c'(t,x,a) > 0, v'(t,x,a) \in \mathbb{R}^d),$$

$\mathbb{Z}^d$ *periodic in* x*, satisfying (16), (18), such that* $c' = c$ *at times* $t = 0$ *and* $t = T$*. Then*

$$\int |v(t,x,a) - V(a)|^2 c(t,x,a)dxdadt \le \int |v'(t,x,a) - V(a)|^2 c'(t,x,a)dxdadt. \tag{39}$$

Notice that (c, v) is a *global* minimizer of the action among all admissible solutions. This is a very peculiar situation, since most of the so-called least action principles in classical Mechanics and Physics involve either *local* minima or saddle points. From the variational principle, we first get the following estimate for v.

Proposition 4.3 .
For each smooth function $\zeta(t)$ compactly supported in $0 < t < T$, and each vector $h \in \mathbb{R}^d$,

$$\int |v(t, x + h\zeta(t), a) - v(t, x, a)|^2 c(t, x, a) dx da dt \leq 4|h|^2 \int \zeta'(t)^2 dt. \quad (40)$$

It is remarkable that this estimate does not depend at all on the initial conditions ! Then, we construct a suitable admissible solution and prove :

Proposition 4.4 .
There is an admissible solution (c', v') such that

$$\int v'(t, x, a) c'(t, x, a) dx dt = \int v(t, x, a) c(t, x, a) dx dt, \quad (41)$$

for all $a \in [0, 1]$ and

$$\int |v'(t, x, a)|^2 c'(t, x, a) dx da dt \leq (2D + T||V_T||)^2 / T, \quad (42)$$

where $D^2 = d/12$,

$$||V_T||^2 = \int |V_T(a)|^2 M(a) da, \quad (43)$$

$$V_T(a) = \frac{1}{M(a)T} \int v(t, x, a) c(t, x, a) dx dt. \quad (44)$$

Before proving these two last propositions, let us first establish the least action principle of Proposition 4.2 and deduce Theorem 4.1 from them.

4.1 Proof of Theorem 4.1.

We first observe that (34) and

$$\int |v(t, x, a)|^2 c(t, x, a) dx da = \int |v(0, x, a)|^2 c(0, x, a) dx da \quad (45)$$

directly follow from the semi-Lagrangian equations and the the x periodicity of v, c and p. Next, using assumption (32), we can rewrite equation (17) as

$$\partial_t v + \nabla_x(\frac{1}{2}|v|^2 + p) = 0 \tag{46}$$

and (33) follows from the spatial periodicity of (v, p). Because of the curl-free condition (32), there is a smooth potential $\phi(t, x, a)$, periodic in x, such that

$$v(t, x, a) = V(a) + \nabla_x \phi(t, x, a), \tag{47}$$

and (46) can be integrated in x, which leads to :

$$\partial_t \phi + \frac{1}{2}|v|^2 + p = 0, \tag{48}$$

after adding a suitable function of (t, a) to ϕ, which does not affect (47). Since (c', m') is an admissible solution, we have :

$$\int ((c - c')\partial_t \phi + (cv - c'v').\nabla_x \phi) dxdadt = 0,$$

which, combined with (48), leads to :

$$\int (-(c - c')\frac{1}{2}|v|^2 + (cv - c'v').(v - V)) dxdadt = 0. \tag{49}$$

(Notice that there is no term involving p because $\int (c - c')(t, x, a)da = 0$, since (c', v') is admissible.) Thus

$$\int (c|v - V|^2 + c|v - v'|^2 - c'|v' - V|^2) dxdadt = 0,$$

which concludes, by definition (39), the proof of Proposition 4.2.

Let us now deduce the lifespan estimate (35) from Propositions 4.3 and 4.4. We have

$$||v||^2 T = \int |v(t, x, a)|^2 c(t, x, a) dxdadt$$

(by definition (36))

$$= \int |v(t, x, a) - V(a)|^2 c(t, x, a) dxdadt$$

$$+ \int (2v(t, x, a).V(a) + |V(a)|^2) c(t, x, a) dxdadt$$

$$= \int |v(t,x,a) - V(a)|^2 c(t,x,a) dx da dt$$

$$+ \int (2v'(t,x,a).V(a) + |V(a)|^2) c'(t,x,a) dx da dt$$

(since (c', v') is admissible and satisfies (41))

$$\leq \int |v'(t,x,a) - V(a)|^2 c'(t,x,a) dx da dt$$

$$+ \int (2v'(t,x,a).V(a) + |V(a)|^2) c'(t,x,a) dx da dt$$

(by Proposition 4.2)

$$= \int |v'(t,x,a)|^2 c'(t,x,a) dx dx' da dt$$

$$\leq (2D + T||V_T||)^2 / T$$

(by Proposition 4.4). Thus, we have already obtained

$$||v|| \leq \frac{2D}{T} + ||V_T||. \tag{50}$$

Next, we observe

$$TM(a)(V_T(a) - V(a)) = \int (v(t,x,a) - v(t,x',a)) c(t,x,a) dx dx' dt$$

(by definitions (33) and (44)). Thus

$$||V_T - V||^2 = \int |V_T(a) - V(a)|^2 M(a) da$$

$$= \int |T^{-1} \int (v(t,x,a) - v(t,x',a)) c(t,x,a) dx dx' dt|^2 M(a)^{-1} da$$

$$\leq T^{-1} \int |v(t,x,a) - v(t,x',a)|^2 c(t,x,a) dx dx' da dt$$

(by Cauchy-Schwarz' inequality and (34))

$$\leq T^{-1}(8\tau||v||^2 + \int_\tau^{T-\tau} dt |v(t,x,a) - v(t,x',a)|^2 c(t,x,a) dx dx' da),$$

for all $0 < \tau < T/2$. According to Proposition 4.3), the right-hand side can be bounded by

$$\leq T^{-1}(8\tau||v||^2 + \frac{32D^2}{\tau})$$

(by choosing $\zeta(t) = 1$ for $\tau \leq tT - \tau$, with minimal total variation). After optimizing in τ, we have obtained

$$||V_T - V||^2 \leq 32DT^{-1}||v||,$$

if $||v||T \geq 2D$. This, combined with (50), leads to

$$|||v|| - ||V||| \leq 2DT^{-1} + 4T^{-1/2}(2D||v||)^{1/2},$$

in all cases, which completes the proof of Theorem 4.1.

4.2 Proof of Proposition 4.4.

In this subsection, we use the Lagrangian coordinates defined by

$$X(t, x, a) = x + \int_0^t v(s, X(s, x, a), a)ds, \quad 0 \leq t \leq T$$

and set

$$h(x, a) = X(T, x, a).$$

Because of (16), we have, for all periodic function f,

$$\int f(x)c(t, x, a)dx = \int f(X(t, x, a))c(0, x, a)dx$$

which implies, according to definition (44),

$$V_T(a) = \frac{1}{M(a)T} \int (h(x, a) - x)c(0, x, a)dx. \tag{51}$$

Indeed :

$$\int (h(x, a) - x)c(0, x, a)dx = \int (X(T, x, a) - x)c(0, x, a)dx$$

$$= \int (\int_0^T v(t, X(t, x, a), a)dt)c(0, x, a)dx = \int v(t, x, a)c(t, x, a)dxdt.$$

Let us now start the construction of an admissible solution. For each (x, a), $y \in [-1/2, 1/2]^d$ and $0 < r < 1/2$, we introduce the piecewise linear trajectory $t \to \xi(t, x, a, y)$, linking x at time 0 and $h(x, a)$ at time T and passing through $x+y$ at time $t = rT$ and $h(x, a)+y$ at time $(1-r)T$. More precisely,

$$\xi(t, x, a, y) = x + \frac{ty}{rT}$$

for $0 \leq t \leq rT$,

$$\xi(t,x,a,y) = x + y + (h(x,a) - x)\frac{t - rT}{(1-2r)T}$$

for $rT \leq t \leq (1-r)T$ and

$$\xi(t,x,a,y) = h(x,a) - \frac{(t-T)y}{rT}$$

for $(1-r)T \leq t \leq T$. Then, we set, for every 1-periodic continuous function f on $\mathbb{R}^d$,

$$\int f(x)c'(t,x,a)dx = \int f(\xi(t,x,a,y))c(0,x,a)dxdy,$$

where the integrals are performed with $x \in [0,1]^d$ and $y \in [-1/2, 1/2]^d$,

$$\int f(x)(c'v')(t,x,a)dx = \int \partial_t\xi(t,x,a,y)f(\xi(t,x,a,y))c(0,x,a)dxdy,$$

which define $c'(t,x,a) > 0$ and $v'(t,x,a) \in \mathbb{R}^d$ as smooth functions, both $\mathbb{Z}^d$-periodic in x. Notice that, by taking $f = 1$ in the definition of $c'v'$ and integrating in $t \in [0,T]$, we immediately get

$$\int (c'v')(t,x,a)dxdt = \int \partial_t\xi(t,x,a,y)c(0,x,a)dxdydt$$

$$= \int (h(x,a) - x)c(0,x,a)dx = TM(a)V_T(a),$$

by (51), which already proves (41). For $0 \leq t \leq rT$, we get, more explicitly,

$$c'(t,x,a) = \int c(0, x - \frac{yt}{rT}, a)dy \tag{52}$$

$$c'(t,x,a)v'(t,x,a) = \int \frac{y}{rT}c(0, x - \frac{yt}{rT}, a)dy \tag{53}$$

and

$$\int |v'(t,x,a)|^2 c'(t,x,a)dxda \leq (rT)^{-2}D^2, \tag{54}$$

where $D^2 = \int |y|^2 dy = d/12$. For $(1-r)T \leq t \leq T$, we get analogous formulae for c' and v', and (54) is still valid. For $rT \leq t \leq (1-r)T$, we have

$$c'(t,x,a) = \int c(0,x',a)dx' = M(a) \tag{55}$$

$$v'(t,x,a) = M(a)^{-1}\int c(0,x',a)\frac{h(x',a)-x}{(1-2r)T}dx' = \frac{V_T(a)}{1-2r}, \qquad (56)$$

by definition (51), and

$$\int |v'(t,x,a)|^2 c'(t,x,a)dxda = ((1-2r)T)^{-2}\int |V_T(a)|^2 M(a)da. \qquad (57)$$

Let us check that (c',v') is an admissible solution. Indeed, (16) and $c=c'$ for $t=0$ and $t=T$ are automatically enforced by our definition, and (18) can be easily checked by showing

$$\int f(x)c'(t,x,a)dxda = \int f(x)dx,$$

for each continuous $\mathbb{Z}^d$ periodic function f and each $t\in[0,1]$. (Indeed, if we call $I(t)$ this integral, we get for $0\le t\le rT$

$$I(t) = \int f(x+\frac{yt}{rT})dxdy$$

$$= \int f(x')dx'dy$$

-after the change of variable $x' = x+\frac{yt}{rT}$ for each fixed y, using that f is $\mathbb{Z}^d$ periodic-

$$= \int f(x)dx,$$

and a similar result for $(1-r)T\le t\le T$. If $rT\le t\le(1-r)T$, then

$$I(t) = \int f(y+(t/T-r)\frac{h(x,a)-x}{1-2r})dxdy$$

$$= \int f(y')dxdy'$$

-after the change of variable

$$y' = y+(t/T-r)\frac{h(x,a)-x}{1-2r}$$

for each fixed x-, which completes the proof of (18).) Finally, because of (54), (57), we get

$$\int |v'(t,x,a)|^2 c'(t,x,a)dxda \le \frac{2D^2}{rT} + \frac{1}{(1-2r)T}\int |V_T(a)|^2 M(a)da,$$

which leads to (42), after optimizing in r this inequality. Thus, the proof of Proposition 4.4 is now complete.

Proof of Proposition 4.3.

Given $h \in \mathbb{R}^d$ and ζ, we set, for every $\mathbb{Z}^d$ periodic continuous function f on $\mathbb{R}^d$,

$$\int f(x)c'(t,x,a)dx = \int f(x+\zeta(t)h)c(t,x,a)dx,$$

$$\int f(x)(c'v')(t,x,a)dx = \int (\zeta'(t)h + v(t,x,a))f(x+\zeta(t)h)c(t,x,a)dx.$$

This makes (c', v') an admissible solution. Thus, we can use (49) and get

$$\int (-c|v|^2 + c|v_h|^2 + 2vc.(v-V) - 2(v+\zeta'(t)h)c.(v_h - V))dxdadt = 0,$$

where we denote $v(t, x+\zeta(t)h, a)$ by $v_h(t,x,a)$. Thus

$$\int ((|v-v_h|^2)c + 2\zeta'(t)hc.(v-v_h))dxdadt = 0.$$

(after noticing that

$$\int c(v-V)dxda$$

is time independent because of (17) multiplied by c and integrated in a). So, by Cauchy-Schwarz' inequality

$$\int |v-v_h|^2 cdxdadt \leq 4|h|^2 \int |\zeta'(t)|^2 cdxdadt$$

$$= 4|h|^2 \int |\zeta'(t)|^2 dt,$$

which concludes the proof of Proposition 4.3.

5 Local existence of class C solutions.

5.1 Reduction to a system of symmetric conservation laws.

From now on, we consider the HHE's with only one horizontal space variable (1), (2). with period 1, $x \in \mathbb{R}/\mathbb{Z}$, and we denote $\nabla_x = \partial_x$. Let us first observe that we can assume (9). Indeed, from (2), integrated in $z \in [0,1]$, and the slip boundary condition $w(t,x,z) = 0$ for $z=0$ and $z=1$, we first deduce :

$$\partial_x \int_0^1 u(t,x,z)dz = 0.$$

Then,

$$\partial_t \int_0^1 u(t,x,z)dz = -\partial_x(p(t,x) + \int_0^1 u^2(t,x,z)dz)$$

(by integrating (1) in z and using (2)). Since p and u are 1-periodic in x, after integrating in $x \in [0,1]$, we conclude that $\int_0^1 u(t,x,z)dz$ is a constant. So, by Galilean invariance, we can assume (9). Now, we observe that

$$\int_0^1 w(t,x,z)dx = 0. \tag{58}$$

Indeed, this is true for $z = 0$, because of the slip boundary conditions, and

$$\partial_z \int_0^1 w(t,x,z)dx = -\int_0^1 \partial_x u(t,x,z)dx = 0,$$

because of (2) and by periodicity of u in x. Since the velocity field (u,w) is a two-dimensional divergence-free vector field, and because of (9), (58), we can now introduce a stream-function ψ, 1-periodic in x, such that

$$u(t,x,z) = \partial_z\psi(t,x,z) \quad w(t,x,z) = -\partial_x\psi(t,x,z), \tag{59}$$

$$\psi(t,x,0) = \psi(t,x,1) = 0. \tag{60}$$

Thus, the z derivative of $u(t,x,z)$, denoted by $\omega(t,x,z)$, entirely determines ψ, since

$$\partial_{zz}\psi = \omega. \tag{61}$$

By differentiating (1) with respect to z, we get (21). Thus, we have obtained a completely equivalent formulation of the HHE's with (59), (60), (61) and (21). This, of course, very similar to the well-known "$\omega - \psi$" formulation of the 2D Euler equations.

Proposition 5.1 *Let (u,w,p) be a smooth, 1-periodic in x, class C solution of the HHE's (which means (1), (2), (9), (10), (11)). Then (22) holds true, where $Z(t,x,a)$ satisfies (13), (14) and is a smooth solution of the (infinite-dimensional) system of conservation laws (24) with flux (25). In addition $c = \partial_a Z$ satisfies*

$$\partial_t c + \partial_x(cv) = 0, \tag{62}$$

with $v(t,x,a) = \frac{k}{2}(1 - 2Z(t,x,a))$

$$+\frac{1}{2}\int_0^1 [2(Z(t,x,a) - Z(t,x,b))H(a-b) - (1 - Z(t,x,b))^2]db. \tag{63}$$

Proof

We use the semi-Lagrangian variables (Z, v) defined by (15) and (20). They depend on the choice of $Z(0, x, a)$ which will be made later on. Notice first that (15) can be written in conservation form :

$$\partial_t Z(t, x, a) + \partial_x(\psi(t, x, Z(t, x, a))) = 0, \tag{64}$$

because of (59). From (21), we get :

$$(\partial_t + v(t, x, a)\partial_x)(\omega(t, x, Z(t, x, a))) = 0. \tag{65}$$

Let us now choose $z = Z(0, x, a)$ as the level set $\{\partial_z u(0, x, z) = k + a\}$ for each $0 \leq a \leq 1$, which is compatible with (28), for initial data in class C. Then, we get

$$\omega(0, x, Z(0, x, a)) = k + a,$$

which leads to

$$\omega(t, x, Z(t, x, a))) = k + a, \tag{66}$$

because of (65). Thus, (22) is preserved. Because of (60), (61), the stream-function ψ is now given by

$$\psi(t, x, z) = \frac{k}{2} z(1 - z) \tag{67}$$

$$+\frac{1}{2}\int_0^1 [((z - Z(t, x, b))_+)^2 - z(1 - Z(t, x, b))^2]db$$

and we can rewrite (64) as (24), with (25). Differentiating (24) with respect to a leads to (62), (63). This concludes the proof of Proposition 5.1.

5.2 A priori estimates.

In this subsection, we get a priori estimates for the solutions of (24). We first observe that the flux operator (25) is (formally) symmetric (with respect to the $L^2([0, 1])$ inner product, in the a variable). Indeed, one can check that Ψ is the (formal) gradient of

$$\Phi(Z) = \frac{1}{12}\int_0^1 \int_0^1 [(3Z(a)^2 - 2Z(a)^3)k \tag{68}$$

$$+2(Z(a) - Z(b))^3 H(a - b) + (Z(a) + Z(b))^3 - 3Z(a)^2 Z(b)^2 - 6Z(b)^2]dbda.$$

So, formally, (24),(25) is an infinite-dimensional symmetric system of first order conservation laws, with values in $L^2([0,1])$. Unfortunately, Kato's local existence and uniqueness result [7] cannot be applied because Ψ is not a nice operator in $L^2([0,1])$ (even if we introduce a cut-off to remove the quadratic non-linearity). Notice, however, that if a becomes a discrete variable (ranging from $a = 1$ to M) and da becomes the (normalized) counting measure, then (24), (25) becomes a standard $M \times M$ symmetric system of conservation laws, for which the existence of local smooth solutions (in the Sobolev space $H^2(\mathbb{R}/\mathbb{Z})$) is a direct application of Kato's theorem. This remark will be used later to complete the proof of Theorem 1.1 by a standard approximation argument with $M \to \infty$. However, in the discrete case, the existence time may depend on M and we need a priori estimates to control it. This is possible because, already at the continuous level, we can obtain a priori energy estimates. In order to control the a dependence of $Z(t,x,a)$, we use the fact that $c(t,x,a) = \partial_a Z(t,x,a)$ is solution to

$$\partial_t + \partial_x(cv) = 0, \tag{69}$$

where

$$v(t,x,a) = u(t,x,Z(t,x,a)) = \frac{k}{2}(1 - 2Z(t,x,a)) \tag{70}$$

$$+ \int_0^1 [(Z(t,x,a) - Z(t,x,b))H(a-b) - \frac{1}{2}(1 - Z(t,x,b))^2]db.$$

(This can be obtained as in the previous sections or directly from (24) and (25) after differentiating in a.) Let us now perform a priori energy estimates on both (24), (25) and (69), (70). We denote by $|.|_p$ the norm in $L^p([0,1])$, we use subscripts for the partial derivatives and denote by C *purely* numerical constants.

Proposition 5.2 *Let Z be a smooth solution of (24), (25), and c such that*

$$Z(t,x,a) = \int_0^a c(t,x,b)db.$$

Let

$$N(t) = \max(|c(t)|_\infty, |c_{xx}(t)|_2, |Z_{xxx}(t)|_2).$$

Then

$$\frac{d}{dt}\ln|c|_\infty \leq CN(1+N), \tag{71}$$

$$\frac{d}{dt}\ln|c_{xx}|_2 \leq CN^2(1+N), \tag{72}$$

$$\frac{d}{dt}\ln|v_{xxx}|_2 \leq CN^2(1+N). \tag{73}$$

To prove Proposition 5.2, we use the following lemmas :

Lemma 5.3 *Let $c(x,a)$ and $Z(x,a) = \int_0^a c(t,x,b)db$, smooth enough and 1-periodic in x. Then we have the following estimates :*

$$|Z|_\infty \leq |c|_\infty \leq N, \tag{74}$$

(where N is defined by (27)),

$$|Z_{xx}|_2 \leq |Z_{xxx}|_2 \leq N, \tag{75}$$

$$|Z_x|_\infty \leq |c_{xx}|_2 \leq N, \tag{76}$$

$$|Z_{xx}|_4^2 \leq |Z_{xxx}|_2|c_{xx}|_2 \leq N^2. \tag{77}$$

Lemma 5.4 *Let $v(t,x,a)$ be defined by (70). Then we have the following estimates :*

$$|v_x|_\infty \leq C(1+|c|_\infty)|c_{xx}|_2 \leq CN(1+N), \tag{78}$$

$$|v_{xxx}|_2 \leq C|Z_{xxx}|_2(1+|c|_\infty+|c_{xx}|_2) \leq CN(1+N), \tag{79}$$

$$|c_x v_{xx}|_2^2 \leq C|c_{xx}|_2^4(1+|c|_\infty)^2 \leq CN^4(1+N^2). \tag{80}$$

Proof of Lemma 5.3.

The two first estimates are obvious. The third one follows from :

$$Z_x(t,x,a) = \int_0^a c_x(t,x,b)db = \int_0^a (\int_y^x c_{xx}(t,\tilde{x},b)d\tilde{x})db$$

(for some $y = y(t,x,b) \in [0,1]$ chosen such that $c_x(t,y,b) = 0$, which is possible because $\int_0^1 c_x(t,y,b)dy = 0$ follows from the assumption that $c(t,x,a)$ is 1-periodic in x), the last one is obtained in a similar way :

$$\int Z_{xx}(t,x,a)^4 dxda = \int (\int_y^x Z_{xxx}(t,\tilde{x},a)d\tilde{x})^2 (\int_0^a c_{xx}(t,x,b)db)^2 dadx$$

$$\leq \int Z_{xxx}(t,\tilde{x},a)^2 c_{xx}(t,x,b)^2 d\tilde{x}dbdxda$$

$$= |Z_{xxx}|_2^2 |c_{xx}|_2^2,$$

which completes the proof of Lemma 5.3.

Proof of Lemma 5.4.

We have

$$v_x(t,x,a) = -kZ_x(t,x,a)$$

$$+\int (H(a-b)(Z_x(t,x,a) - Z_x(t,x,b)) + (1-Z(t,x,b))Z_x(t,x,b))db.$$

Thus, (78) follows from (76) and (74). Next,

$$v_{xx}(t,x,a) = -kZ_{xx}(a)$$

$$+\int (H(a-b)(Z_{xx}(a) - Z_{xx}(b)) + (1-Z(b))Z_{xx}(b) - Z_x(b)^2)db,$$

(where we denote $Z(t,x,a)$ by $Z(a)$ for simplicity)

$$v_{xxx}(t,x,a) = -kZ_{xxx}(a) + \int (H(a-b)(Z_{xxx}(a) - Z_{xxx}(b))$$

$$+(1-Z(b))Z_{xxx}(b) - 3Z_x(b)Z_{xx}(b))db$$

implies (79) because of (76), (74), (75), and :

$$\int c_x(t,x,a)^2 v_{xx}(t,x,a)^2 dxda$$

$$\leq C\int (\int_y^x c_{xx}(t,\tilde{x},a)d\tilde{x})^2((1+|Z|_\infty)\int |c_{xx}(t,x,b)|db$$

$$+|Z_x|_\infty^2)^2 dxda$$

$$\leq C\int c_{xx}(t,\tilde{x},a)^2((1+|c|_\infty)^2 c_{xx}(t,x,b)^2 + |c_{xx}|_2^4)d\tilde{x}dxdadb$$

$$\leq C|c_{xx}|_2^4(1+|c|_\infty)^2,$$

which completes the proof of Lemma 5.4.

Let us now prove Proposition 5.2. From (69), we immediately get

$$\frac{d}{dt}\ln|c|_\infty \leq |v_x|_\infty$$

and (71) follows from (78). From (69) again, differentiated twice in x, multiplied by c_{xx} and integrated by part, we get

$$\frac{1}{2}\frac{d}{dt}|c_{xx}|_2^2 = -\int [\frac{5}{2}c_{xx}^2 v_x + 3c_x c_{xx} v_{xx} + c c_{xx} v_{xxx}]dxda$$

$$\leq C(|c_{xx}|_2^2|v_x|_\infty + |c_{xx}|_2|c_x v_{xx}|_2 + |c|_\infty|c_{xx}|_2|v_{xxx}|_2)$$
$$\leq CN^3(1+N),$$

because of Lemma 5.4, which gives (72). Now, let us get an energy estimate for Z_{xxx}. It is only now that we are going to use the symmetry of (24). After differentiating (24) three times in x, multiplying by Z_{xxx} and integrating by part, we get :

$$-\frac{d}{dt}\int Z_{xxx}^2(t,x,a)dxda = \frac{k}{2}\int[Z(a)(1-Z(a))]_{xxxx}Z_{xxx}(a)dadx \qquad (81)$$

$$+\int[(Z(a)-Z(b))^2_{xxxx}H(a-b)-(Z(a)(1-Z(b))^2)_{xxxx}]Z_{xxx}(a)dbdadx$$

$= I + J$, where I denotes the sum of the terms involving four space derivatives. First, we easily bound J by $CN^3(1+N)$ (using Lemma 5.3, (77) in particular). Next, we have $I = I_0 + I_1 + I_2 + I_3$, with

$$I_0 = k\int Z_{xxxx}(a)(1-2Z(a))Z_{xxx}(a)dadx,$$

$$I_1 = \int(Z_{xxxx}(a)-Z_{xxxx}(b))(Z(a)-Z(b))H(a-b)Z_{xxx}(a)dbdadx,$$

$$I_2 = -\int Z_{xxxx}(b)(1-Z(b))Z(a)Z_{xxx}(a)dbdadx.$$

$$I_3 = -\frac{1}{2}\int Z_{xxxx}(a)(1-Z(b))^2 Z_{xxx}(a)dbdadx.$$

We easily deals with I_0 and I_3 which can be integrated by part and then bounded by $CN^3(1+N)$. Let us split $I_1 = I_{11} + I_{12}$, with

$$I_{11} = -\int Z_{xxxx}(b)(Z(a)-Z(b))H(a-b)Z_{xxx}(a)dbdadx.$$

$$I_{12} = \int Z_{xxxx}(a)(Z(a)-Z(b))H(a-b)Z_{xxx}(a)dbdadx,$$

We first get

$$I_{12} = -\int\frac{1}{2}(Z(a)-Z(b))_x Z_{xxx}(a)^2 dbdadx,$$

which can be bounded by CN^3. Then,

$$I_{11} = \int Z_{xxx}(b)(Z(a)-Z(b))_x H(a-b)Z_{xxx}(a)dbdadx$$

$$+\int Z_{xxx}(b)(Z(a)-Z(b))_x H(a-b) Z_{xxxx}(a) dbdadx$$

(by integration by part). By symmetry a/b, the second term is equal to

$$\int Z_{xxx}(a)(Z(b)-Z(a))_x (1-H(a-b)) Z_{xxxx}(b) dbdadx$$

that is :

$$-I_{11}+\int Z_{xxx}(a)(Z(b)-Z(a))_x Z_{xxxx}(b) dbdadx.$$

We deduce

$$I_{11}=I_{111}+I_{112},$$

where

$$I_{111}=\frac{1}{2}\int Z_{xxx}(a)(Z(b)-Z(a))_x Z_{xxxx}(b) dbdadx,$$

$$I_{112}=\frac{1}{2}\int Z_{xxx}(b)(Z(a)-Z(b))_x H(a-b) Z_{xxx}(a) dbdadx.$$

The second term is bounded by CN^3. Let us now consider $I_2=I_{21}+I_{22}$, where

$$I_{21}=\int Z_{xxxx}(b) Z(a) Z_{xxx}(a) dbdadx.$$

$$I_{22}=\int Z_{xxxx}(b)(-Z(b)) Z(a) Z_{xxx}(a) dbdadx.$$

The first term exactly balances I_{111}. Indeed

$$\int Z_{xxxx}(b) Z(a) Z_{xxx}(a) dbdadx = -\int Z_{xxx}(b) Z(a) Z_{xxxx}(a) dbdadx$$

$$= -\int Z_{xxx}(a) Z(b) Z_{xxxx}(b) dbdadx.$$

$$= \frac{1}{2}\int Z_{xxxx}(b)(Z(a)-Z(b)) Z_{xxx}(a) dbdadx = -I_{11}.$$

Next, by integration by part,

$$I_{22}=\int Z_{xxx}(b)[Z_x(b)Z(a)+Z(b)Z_x(a)] Z_{xxx}(a) dbdadx$$

$$\int Z_{xxx}(b) Z(b) Z(a) Z_{xxxx}(a) dbdadx.$$

Thus, by symmetry,

$$I_{22}=\int Z_{xxx}(b) Z_x(b) Z(a) Z_{xxx}(a) dbdadx,$$

which is bounded by CN^4. So, we have now $|I|$ bounded by CN^3 meanwhile $|J|$ is bounded by $CN^3(1+N)$, which concludes the proof of Proposition 5.2.
We can now complete the proof of the existence part of Theorem 1.1 in a standard fashion by passing to the limit in the discrete system obtained from (22), when a is a discrete variable, ranging from $a=1$ to $a=M$ and the (normalized) counting measure is used instead of the Lebesgue measure da. This situation is analogous to the so-called vortex patches for the 2D Euler equations [5]. The discrete system is a finite-dimensional system of symmetric conservation laws, for which the existence of local smooth solutions $(Z(t,x,a), a = 1,...,M)$ is straightforward. Then, energy estimates can be obtained in the same way as for Proposition 5.2, after noticing that the discrete analogous of Lemmas 5.3, 5.4 hold true with constants that do not depend on M. It follows that the existence time is uniform in M, provided that the discrete analogous of norm (27) is bounded at time $t=0$, uniformly in M. Then, standard compactness arguments can be used to pass to the limit $M \to +\infty$ and get local existence of smooth solutions for the continuous problem.

6 Generic class C solutions cannot be global.

In this section, we prove that class C solutions cannot be global time for most initial conditions. Let us recall that there is a one-to-one correspondence between initial conditions in class C and smooth functions $Z(0,x,a)$ satisfying (28), through :

$$\partial_z u(0,x,z) = k + \int_0^1 H(z - Z(0,x,a))da. \tag{82}$$

Theorem 6.1 *Let us consider initial conditions $Z(0,x,a)$ in class (28) equipped with the C^1 topology. Then the subset for which there is no global class C solutions to the HHE's contains a dense open set.*

Proof.

Let us consider a global class C solution of the HHE's with initial condition $Z(x,a)$. This defines a global solution (c,v) of the semi-Lagrangian equations through definitions (19) and (63). According to Theorem 4.1, this implies that the initial values $(c(0,x,a), v(0,x,a))$ satisfy condition (38).

This exactly means $F(Z) = 0$, where

$$F(Z) = \int v(x,a)^2 c(x,a)dxda - \int (\int v(x',a)^2 dx')c(x,a)dxda, \qquad (83)$$

with

$$c(x,a) = \partial_a Z(x,a), \qquad (84)$$

$$v(x,a) = \frac{k}{2} + (a-k)Z(x,a) - \int_0^a Z(x,b)db - \int_0^1 (1-Z(x,b))^2 db. \qquad (85)$$

Therefore the set of initial conditions $Z(x,a)$ for which the solution of the HHE's is global is contained in the zero level set N of F. It is now enough to show that N is a closed set with empty interior. We see that $F(Z)$ is a smooth function of Z with respect to the C^1 topology and therefore N is closed. Let us now consider Z in the interior of N and get a contradiction. The second order differential $F''(Z)$ must vanish. After straightforward calculations, we find that :

$$F''(Z)(z,z) = \int (c(x,a)\eta(x,a)^2 + 2(\eta\gamma v)(x,a))dxda \qquad (86)$$

$$-\int (< c(a) >< \eta(a) >^2 + 2 < \eta v(a) >< \gamma(a) >)da,$$

where integrals with respect to x are denoted by brackets $< . >$, and

$$\gamma(x,a) = \partial_a z(x,a), \qquad (87)$$

$$\eta(x,a) = (a-k)z(x,a) - \int_0^a z(x,b)db + 2\int_0^1 (1-Z(x,b))z(x,b)db. \qquad (88)$$

Thus $F''(Z)(z,z)$ must vanish for all smooth variations $z(x,a)$ vanishing at $a = 0$, $a = 1$, and 1 periodic in x. By taking high frequency variations z_n of the form

$$z_n(x,a) = \phi(x)\psi(a)\sin(n\pi x)\sin(n\pi a), \qquad (89)$$

where ϕ and ψ are smooth test functions, and n a large even integer, we find, in the limit $n \to +\infty$, after simple calculations (involving integration by parts and straightforward limits of oscillatory integrals)

$$\lim F''(Z)(z_n, z_n) = \frac{1}{4}\int \phi(x)^2\psi(a)^2 v(x,a)dxda. \qquad (90)$$

This cannot be identically equal to zero unless $v(x,a) = 0$, which is impossible in class C. Therefore the interior of N is empty, which concludes the proof of Theorem 6.1.

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