STICKY PARTICLES AND SCALAR CONSERVATION LAWS

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Abstract

One dimensional scalar conservation laws with nondecreasing initial conditions and general fluxes are shown to be the appropriate equations to describe large systems of free particles on the real line, which stick under collision with conservation of mass and momentum.

Introduction

There has been a recent interest for the one dimensional model of pressureless gases with sticky particles. This model can be described at a discrete level by a finite collection of particles that get stuck together right after they collide with conservation of mass and momentum. At a continuous level, the gas can be described by a density and a velocity fields $\rho(t,x)$, $u(t,x)$ that satisfy the mass and momentum conservation laws

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1}$$

$$\partial_t \rho u + \partial_x(\rho u^2) = 0. \tag{2}$$

This system can be formally obtained from the usual Euler equations for ideal compressible fluids by letting the pressure go to zero, or from the Boltzmann equation by letting the temperature go to zero. This model of adhesion dynamics is connected to the sticky particle model of Zeldovich [18],[16], which also includes gravitational interactions and has interesting statistical properties. (See [6] as the most complete and recent reference.)

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For smooth solutions and positive densities, (2) is equivalent to the inviscid Burgers equation

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = 0, \tag{3}$$

which has been studied from a statistical point of view in [15],[14]. However, the reduction to the inviscid Burgers equation is not correct for general data. Bouchut [2] pointed out some mathematical difficulties to get a rigorous derivation of the sticky particle model (1), (2). First, at the continuous level, ρ must be considered as a nonnegative measure, with possible singular parts, and not as a function. Second, the velocity field must be well defined ρ almost everywhere. Third, the system of conservation laws has to be supplemented by some entropy conditions, and the most obvious condition, say

$$\partial_t(\rho U(u)) + \partial_x(\rho u U(u)) \leq 0, \tag{4}$$

for all smooth convex function U, turns out to be insufficient to guarantee uniqueness for prescribed initial values, as observed in [2]. After [2] and independently, Grenier [7] (see also [8] and [4]) and E, Rykov and Sinai [6] proved the global existence of measure solutions to system (1), (2),(4), obtained as limits of the discrete solutions when the number of particles goes to $+\infty$ with bounded total mass and initial velocities. Similar results can also be deduced from the theories independently developed by Bouchut James [3] and Poupaud Rascle [11] for linear transport equations. In the present paper, we show that the continuous model can be fully described, in an alternative and more straightforward way, by *scalar* conservation laws, with non decreasing initial conditions, general flux functions and the usual Kruzhkov entropy condition (for which, we refer to [13]). These scalar conservation laws are *not* the inviscid Burgers equation (3) and their flux functions depend on the initial data $\rho(0,.)$ and $u(0,.)$. We also relate the continuous model to the Hamilton-Jacobi equation and provide two kinetic formulations. In addition, we indicate how the pressureless system (1),(2), without entropy condition (4), can be *directly* recovered from scalar conservation laws, by using standard BV calculus [17]. However, the direct recovery of (4) from Kruzhkov's entropy condition seems unclear except in the particular case of piecewise smooth solutions, for which we give an elementary argument.

1. Sticky particles and scalar conservation laws

Let us consider n particles on the real line. We describe the $i-$th particle by its weight, position and velocity $P_i(t) = (m_i(t), x_i(t), v_i(t))$ at time $t \geq 0$. We say that this set of particles has a sticky particle dynamics with initial conditions (m_i, x_i, v_i), with $x_1 < ... < x_n$, if

- $m_i(t) = m_i$ for all $t \geq 0$,
- $x_i(0) = x_i$ and $v_i(0) = v_i$,
- the speed of P_i is constant as long as it meets no new particles,
- the speed changes only when shocks occur : if at t_0 there exists j such that $x_j(t_0) = x_i(t_0)$ and $x_j(t) \neq x_i(t)$ for all $t < t_0$ then

$$v_i(t_0+) = \frac{\sum_{j/x_j(t_0)=x_i(t_0)} m_j v_j(t_0-)}{\sum_{j/x_j(t_0)=x_i(t_0)} m_j}. \tag{5}$$

Notice that only a finite number of shocks can occur because particles remain stuck together after a shock. Moreover, we observe, as the total momentum is conserved through the shocks, that the speed of the particles is defined, for positive t, by

$$m_i x_i'(t) = \sum_{j/x_j(t)=x_i(t)} m_j v_j, \tag{6}$$

for $i = 1, ..., m$, where x_i' denotes the right time derivative of $x_i(t)$. In other words, the particles having the same position at time t move together at the same speed and their total momentum is the sum of their initial momentum. Notice that a discrete and exact algorithm can be straightforwardly designed to compute the solutions (by properly ordering the possible collision times). This scheme is strongly reminiscent of Dafermos' [5] (see also [9]) polygonal approximation methods for scalar conservation laws, where each particle corresponds to a jump of an entropy solution of a scalar conservation law with a piecewise linear continuous flux function. Thus, we can expect that the continuous limit of the sticky particle dynamics is properly described by a scalar conservation law. Indeed, we show

Theorem 1..1 *Let n sticky particles, with initial weight m_i, position x_i and velocity $u^0(x_i)$, where u^0 is a fixed given bounded continuous function on $\mathbb{R}$, and the total mass is normalized $\sum_j m_j = 1$. Assume that the x_i belong to a fixed compact interval $[-R, +R]$ and*

$$\sum_j m_j \delta(x - x_j) \to \rho^0(x)$$

weakly on $x \in \mathbb{R}$, when $n \to +\infty$. Then, for each $t \geq 0$

$$\sum_j m_j(t)\delta(x - x_j(t)) \to \partial_x M(t, x)$$

weakly on $x \in \mathbb{R}$, where M is the unique entropy solution of the scalar conservation

$$\partial_t M + \partial_x(A(M)) = 0, \tag{7}$$

with initial condition $M(0, .) = M^0$ given by

$$M^0(x) = \rho^0(] - \infty, x]) \tag{8}$$

and flux function $A(m) = \int_0^m a(m')dm'$, where

$$a(m) = u^0(x), \tag{9}$$

for all $M^0(x-) \leq m < M^0(x+)$ and $x \in \mathbb{R}$.

We use the following elementary lemma on probability measures :

Lemma 1..1 *Let (ρ_n) be a sequence of probability measures on $\mathbb{R}$ supported on a fixed compact subset of $\mathbb{R}$ and let M_n be the corresponding distribution function*

$$M_n(x) = \rho_n(] - \infty, x]).$$

Then the following statement are equivalent. i) $\rho_n \to \rho$ weakly ii) the L^1 norm of $M_n - M$ on $\mathbb{R}$ goes to zero, where

$$M(x) = \rho(] - \infty, x]);$$

iii) for all nondecreasing C^1 function B on $[0, 1]$ such that $B(0) = 0$, $B(1) = 1$,

$$\partial_x(B(M_n)) \to \partial_x(B(M))$$

weakly.

Proof of Lemma 1..1

Let us first assume ii). Because $M_n - M \to 0$ in $L^1(\mathbb{R})$, $\rho_n \to \rho$ holds true in the sense of distributions since $\rho_n = \partial_x M_n$, $\rho = \partial_x M$. Because ρ_n and ρ are nonnegative distributions, this means that $\rho_n \to \rho$ weakly in the sense of measures. If B is a nondecreasing C^1 function on $[0,1]$ with $B(0) = 0$, $B(1) = 1$, then we can apply the same reasoning to $B(M_n)$ (instead of M_n) and deduce that $\partial_x B(M_n)$ is a probability measure and weakly converges to $\partial_x B(M)$ when $n \to +\infty$. Thus iii) holds true. Since i) trivially follows from iii), let us now prove that i) implies ii). We use a classical result of measure theory [12] asserting that, if $\rho_n \to \rho$ weakly and is supported on a fixed compact set, then $\int f(x)d\rho_n(x) \to \int f(x)d\rho(x)$ is true for all bounded ρ Riemann integrable (that is continuous on a set of full ρ measure) function f. For Lebesgue almost every fixed $y \in \mathbb{R}$, $\rho(\{y\}) = 0$ and therefore $f(x) = H(x-y)$ is ρ Riemann integrable in x. Thus $M_n(y) = \rho_n(]-\infty, y]) \to \rho(]-\infty, y]) = M(y)$ for Lebesgue almost every $y \in \mathbb{R}$. Since $M_n = M$ outside of a fixed compact subset of $\mathbb{R}$, the Lebesgue dominated convergence theorem implies that

$$\int_{\mathbb{R}} |M_n(y) - M(y)|dy \to 0,$$

which proves i) and concludes the proof of Lemma 1..1.

Proof of Theorem 1..1

To prove Theorem 1..1, we first observe that *already at the discrete level*, the sticky particle dynamics is described by a scalar conservation, with appropriate initial values and flux function. Then, we rely on standard results on conservation laws [13] to conclude. Let us be more precise. We denote, for each $t \geq 0$,

$$\rho_n(t,x) = \sum_j m_j \delta(x - x_j(t)),$$

and

$$M_n(t,x) = \rho_n(]-\infty, x]) = \sum_j m_j H(x - x_j(t)), \tag{10}$$

where H is the Heaviside function (with $H(0) = 1$). Now, we denote $M_n^0(x) = M_n(0,x)$, then set

$$a_n(m) = v_i, \tag{11}$$

for $M_n^0(x_i-) \leq m < M_n^0(x_i+)$ and $i = 1, ..., n$, and finally define the Lipschitz continuous, piecewise linear flux function

$$A_n(m) = \int_0^m a_n(m')dm'. \tag{12}$$

Then, we observe, in the spirit of [5], [9]

Proposition 1..2 *The function $M_n(t,x)$ defined by (10) is the unique entropy solution of the scalar conservation*

$$\partial_t M_n + \partial_x(A_n(M_n)) = 0, \tag{13}$$

with $M_n^0(x)$ as initial conditions.

Let us first assume this Proposition and end the proof of Theorem (1..1). By assumption and Lemma 1..1, we know that

$$\int_{\mathbb{R}} |M_n^0(x) - M^0(x)|dx \to 0,$$

and

$$\partial_x B(M_n^0) \to \partial_x B(M^0),$$

weakly, for all nondecreasing C^1 function B on $[0,1]$ such that $B(0) = 0$, $B(1) = 1$. Since u^0 is continuous, it follows that

$$< \partial_x(B(M_n^0)), u^0 > \to < \partial_x(B(M^0)), u^0 >,$$

where $< ., . >$ denote measure brackets. This is still true for all C^1 function B. By definition (11), this exactly means

$$\int_0^1 a_n(m)B'(m)dm \to \int_0^1 a(m)B'(m)dm.$$

Since the a_n are uniformly bounded by the sup norm of u^0, we deduce that $a_n \to a$ for the weak-* topology of L^∞. Now, if we consider any C^1 function U on $[0,1]$ and define

$$A_{n,U}(m) = \int_0^m a_n(w)U'(w)dw, \quad A_U(m) = \int_0^m a(w)U'(w)dw,$$

then $A_{n,U}$ uniformly converges to A_U on $[0,1]$. Next, standard results on scalar conservation laws [13] show that, for all $T > 0$,

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} |M_n(t,x) - M(t,x)|dx \to 0,$$

where M is the unique solution of (7). (Indeed, the $M_n - M$ are compact in $C([0,T], L^1(\mathbb{R}))$ and we can pass to the limit in all entropy inequalities

$$\partial_t(U(M_n)) + \partial_x(A_{n,U}(M_n)) \leq 0,$$

for all C^1 convex function U.) Using again Lemma 1..1, we deduce, for all $t \geq 0$,

$$\rho_n(t,x) \to \partial_x M(t,x),$$

weakly, which is enough to prove Theorem (1..1).

Let us now prove Proposition (1..2). As M_n is piecewise constant, it is sufficient to check that the shocks at $x_j(t)$ $(1 \leq i \leq n)$, which are of finite number, satisfy Rankine Hugoniot and entropy conditions of (7) at $x_j(t)$.

But

$$M_n(x_i(t)+) - M_n(x_i(t)-) = \sum_{j|x_j(t)=x_i(t)} m_j$$

and

$$A_n(M(x_i(t)+)) - A_n(M(x_i(t)-)) = \sum_{j|x_j(t)=x_i(t)} m_j v_j,$$

therefore, using (5),

$$v_i(t)[M_n(x_i(t)+) - M_n(x_i(t)-)] = [A_n(M(x_i(t)+)) - A_n(M(x_i(t)-))]$$

which is exactly the Rankine Hugoniot for (7).

M_n satisfies entropy conditions provided

$$\frac{A_n(\theta) - A_n(M(x_i(t)-))}{\theta - M(x_i(t)-)} \geq v_i(t). \tag{14}$$

Notice that as A_n is piecewise linear, it is sufficient to check (14) for θ of the form $\sum_{j \leq i_0} m_j$. But (14) is exactly the discrete version of the Generalized Variational Principle studied in [6] and is a consequence of the following very simple barycentric lemma (used in [4] and [7])

Lemma 1..2 *Let $t \geq 0$ and let us assume that $x_i(t) = x_j(t)$ for $i_0 \leq i, j \leq i_1$ and that $x_i(t) = x_{i_0}(t)$ implies $i_0 \leq i \leq i_1$. Then for $i_0 \leq k \leq i_1$,*

$$\frac{\sum\limits_{i_0 \leq j \leq k} m_j v_j(0)}{\sum\limits_{i_0 \leq j \leq k} m_j} \geq \frac{\sum\limits_{i_0 \leq j \leq i_1} m_j v_j(0)}{\sum\limits_{i_0 \leq j \leq i_1} m_j} \tag{15}$$

To end the proof of Proposition (1..2), just notice that

$$\frac{A_n(\theta) - A_n(M(x_i(t)-))}{\theta - M(x_i(t))-)} = \frac{\sum\limits_{i_0 \leq j \leq k} m_j v_j(0)}{\sum\limits_{i_0 \leq j \leq k} m_j}$$

when $\theta = \sum_{j \leq k} m_j$, and that

$$v_i(t) = \frac{\sum\limits_{i_0 \leq j \leq i_1} m_j v_j(0)}{\sum\limits_{i_0 \leq j \leq i_1} m_j}.$$

Now to prove Lemma (1..2), just observe that when a group of particles hits another group of particles, the averaged velocity of the *left* group of particles decreases. But the group of particles $i_0 \leq j \leq k$ only meets particles on its *right* by definition of i_0 (before time t). Therefore its averaged velocity which equals $(\sum_{i_0 \leq j \leq k} m_j v_j(0))(\sum_{i_0 \leq j \leq k} m_j)^{-1}$ at $t = 0$ is greater than its averaged at time $t+$, which is also the mean velocity $\bar{v}$ of the set $i_0 \leq j \leq i_1$ by definition of i_0 and i_1. But by conservation of momentum through the shocks, $\bar{v}$ equals $(\sum_{i_0 \leq j \leq i_1} m_j v_j(0))(\sum_{i_0 \leq j \leq i_1} m_j)^{-1}$, which ends the proof of the Lemma.

2. Recovery of the pressureless gas equations without entropy conditions.

Theorem 2..1 *Let $M(t,x)$ be a weak solution of (7),*

$$\partial_x M \geq 0, \tag{16}$$

$M(t, -\infty) = 0$, $M(t, +\infty) = 1$. *Let*

$$\rho(t,x) = \partial_x M(t,x) \quad \textit{and} \quad q(t,x) = \partial_x(A(M(t,x))). \tag{17}$$

Then q is uniformly continuous with respect to ρ, and there exists $u(t,x)$ with $q = u\rho$. Moreover, (ρ, u) satisfy (1) and (2).

Proof

M is a bounded BV function in space time. (Indeed $\partial_x M(t,x)$ is a bounded measure on each finite interval $[0,T]$ of total mass T, since $\partial_x M \geq 0$, and $\partial_t M(t,x)$ therefore is a bounded measure because of (7)). Using BV calculus [17], we can define a Borel function $\underline{a}(t,x)$ valued in $[\inf u^0, \sup u^0]$, uniquely defined for $\partial_x M$ almost every (t,x), such that

$$\nabla(A(M(t,x)) = \underline{a}(t,x)\nabla M(t,x) \tag{18}$$

holds in the sense of vector valued measures, where $\nabla = (\partial_t, \partial_x)$. Let us define

$$\rho(t,x) = \partial_x M(t,x), \tag{19}$$

$$q(t,x) = \partial_x(A(M(t,x))) = \underline{a}(t,x)\partial_x M(t,x) \tag{20}$$

and the Radon-Nikodym derivative

$$u(t,x) = \underline{a}(t,x), \tag{21}$$

uniquely defined for ρ almost every (t,x). Then (1) immediately follows from (the weak form of) (7) differentiated (in the distributional sense) with respect to x. To get (2), we write

$$\partial_t q = \partial^2_{tx}(A(M))$$

(by (20) and (7))

$$= \partial_x(\underline{a}\partial_t M) = -\partial_x(\underline{a}^2\partial_x M)$$

(by (18), twice, and (7) once again)

$$= -\partial_x(\rho u^2)$$

(by (21). Thus both conservations of mass and momentum are easily derived from the weak form of (7).

Remarks on entropy conditions

There does not seem to be any easy way to recover (4) from the usual Kruzhkov's entropy condition for scalar conservation laws. However, let us recall that (4) is not sufficient to assure the uniqueness of the solution of (1,2), as noticed in [2]. If we consider for instance the evolution of two particles (when ρ is the sum of two Dirac masses), (4) merely says that some kinetic energy is lost during a shock, but does not say that the particles remain stuck after they collide, for instance free transport of a finite number of particles satisfies (4), which is far from the desired behaviour.

In fact the entropy condition (4) is completely embedded in our formalism. More precisely, $\partial_x M \geq 0$ can be seen as a kind of entropy condition : as the flux A is independent on time, the monotonicity of M implies that particles can not cross, and therefore that, at a discrete level, particles have a sticky behaviour. If M is a sum of Dirac masses, this implies uniqueness, and (4). Now, the monotonicity of M is preserved by time evolution if M satisfies the usual Kruzhkov's entropy conditions associated to (7). Therefore, at the discrete level, the Kruzhkov's entropy conditions on (7) are stronger than (4) (which is not sufficient to enforce uniqueness of (1, 2)).

Therefore Kruzhkov's entropy conditions appear to be the natural entropy conditions for the sticky particle dynamics. The problem of finding the right entropy conditions to complete (1, 2) is however open.

Notice moreover that at the discrete level, the sticky particle dynamics satisfies, $i < j$,

$$\frac{v_j(t) - v_i(t)}{x_i(t) - x_j(t)} \leq \frac{1}{t} \tag{22}$$

(see [4], [7]). In fact (22) is a natural entropy condition, which leads when ρ is a sum of a finite number of Dirac masses to uniqueness of the solution of (1,2). Its counterpart at the continuous level would be

$$\partial_x u(t,x) \leq \frac{1}{t}.$$

3. Link with the Hamilton-Jacobi equation.

It is known that the *potential*

$$\Psi(t,x) = \int_{-\infty}^{x} M(t,y)dy. \tag{23}$$

is a viscosity solution (in the sense of Crandall Lions) of the Hamilton-Jacobi equation

$$\partial_t \Psi + A(\partial_x \Psi) = 0 \tag{24}$$

if and only if M is an entropy solution to (7). Since the initial condition $\Psi^0 = \Psi(0,.)$ is convex, the (second) Hopf formula [1] asserts that the unique viscosity solution with $\Psi^0(x)$ as initial conditions is given by

$$\Psi(t,x) = \sup_{0\le m\le 1}\{xm - \Phi^0(m) - tA(m)\}, \tag{25}$$

where $\Phi^0 = \Phi(0,.)$ and $\Phi = \Psi^*$ denotes the Legendre-Fenchel transform

$$\Phi(t,m) = \sup_{x\in\mathbb{R}}\{xm - \Psi(t,x)\}. \tag{26}$$

(For each $t \ge 0$, the reciprocal function of $x \in \mathbb{R} \to M(t,x) \in [0,1]$ is the partial derivative of $\Phi(t,m)$ with respect to m.) In short, we have

$$\Psi(t,.) = (\Psi^{0*} + tA)^*, \tag{27}$$

where * denotes the Legendre-Fenchel transform, which geometrically means that, at time $t \ge 0$, $\Phi(t,m)$ is the convex hull of $\Phi^0(m) + tA(m)$ on the interval $0 \le m \le 1$.

This correspondence with the Hamilton-Jacobi equations is true both at the continuous and discrete levels. Let us examine the formulae in the discrete case. We have

$$\Phi_n^0(m) = x_1 m + \sum_{i=1}^{n-1}(m - \sum_{j=1}^{i} m_j)_+(x_{i+1} - x_i). \tag{28}$$

We can express A_n in a similar way as

$$A_n(m) = v_1 m + \sum_{i=1}^{n-1}(m - \sum_{j=1}^{i} m_j)_+(v_{i+1} - v_i). \tag{29}$$

So,

$$\Phi_n(t,m) = x_1(t)m + \sum_{i=1}^{n-1}(m - \sum_{j=1}^{i} m_j)_+(x_{i+1}(t) - x_i(t)), \tag{30}$$

where $x_i(t)$ denotes the position of particle i at time t, is nothing but the convex hull on the interval $m \in [0,1]$ of

$$(x_1 + tv_1)m + \sum_{i=1}^{n-1}(m - \sum_{j=1}^{i} m_j)_+(x_{i+1} + tv_{i+1} - x_i - tv_i)$$

(which would correspond to a collisionless particle dynamics). Notice that this formula yields an efficient algorithm to compute the *exact* positions of n sticky particles at a given positive time T without following the detailed interactions occurring when $t \in [0,T]$. Indeed, convex hull can be computed in $O(n \log n)$ elementary operations.

4. Two kinetic formulations.

In the one-dimensional case, we can use the kinetic formulation of scalar conservation laws as in [10]. We define a kinetic density $F(t,x,m)$ by setting

$$F(t,x,m) = H(M(t,x) - m), \quad \forall (x,m) \in \mathbb{R} \times [0,1]. \tag{31}$$

where H denotes the Heaviside function. Following [10], we know that F satisfies the kinetic equation

$$\partial_t F + a(m)\partial_x F = \partial_m \mu, \tag{32}$$

where $a(m) = A'(m)$ and $\mu = \mu(t,x,m)$ is some nonnegative bounded measure. However, there is a more natural and conventional kinetic formulation, which was already considered in [2], [6], in the usual phase space (x,v),where the density $f(t,x,v) \geq 0$ is subject to

$$\partial_t f + v\partial_x f + \partial_v^2 \nu = 0, \tag{33}$$

for some nonnegative measure $\nu(t,x,v)$, and

$$f(t,x,v) = \rho(t,x)\delta(v - u(t,x)) \tag{34}$$

(which means that the phase density is *monokinetic*). Indeed, (1),(2) and (4), are equivalent to (33), (34), integrated in v against appropriate test functions U (respectively $U(v) = 1$, $U(v) = v$, $U =$ any convex function). The relationship between the (t,x,m) and the (t,x,v) kinetic formulations is not entirely clear. Notice, in particular, that the entropy condition (4) does not enforce uniqueness, as noticed in [2],[6], so that the second kinetic formulation is not as sharp as the first one, which is strictly equivalent to the Kruzhkov formulation of scalar conservation laws (as shown in [10]) and, therefore guarantees uniqueness.

5. Recovery of the entropy conditions for piecewise smooth entropy solutions to scalar conservation laws.

In this last section, we show that for simple piecewise smooth entropy solutions to scalar conservations laws we recover entropy condition (4).

Theorem 5..1 *Let $M(t,x)$ be an entropy solution of (7) and Ω be an open rectangle in of $\mathbb{R}_+ \times \mathbb{R}$ such that M is of form*

$$M(t,x) = M^l(t,x)H(c(t)-x) + M^r(t,x)H(x-c(t))$$

for $(t,x) \in \Omega$, where both M^l and M^r are local classical solutions of (7) and $t \to c(t)$ is smooth. Let $f(t,x,v)$ be the nonnegative measure defined by

$$f(t,x,v) = \rho(t,x)\delta(v - u(t,x)), \tag{35}$$

where $u = a$ and

$$\rho(t,x) = \partial_x M(t,x), \quad \rho(t,x)a(t,x) = \partial_x(A(M(t,x))).$$

Then, f is a solution to (33). In particular, (1), (2) and (4) are satisfied by ρ and u.

Let us use the shorter notations $M^l(t)$, $M^r(t)$, instead of $M^l(t,c(t))$ and $M^r(t,c(t))$. We have $M^l(t) \leq M^r(t)$, since $M(t,x)$ is nondecreasing in x, for each t. The curve $t \to c(t)$ satisfies the Rankine-Hugoniot condition

$$c'(t) = \frac{A(M^r) - A(M^l)}{M^r - M^l}$$

and Lax entropy condition

$$A'(M^l) \geq c'(t) \geq A'(M^r).$$

The measure f is defined by (35), where

$$\rho(t,x) = M^l_x(t,x)H(c(t)-x) + M^r_x(t,x)H(x-c(t)) + (M^r(t) - M^l(t))\delta(x-c(t))$$

$$\rho a = A'(M^l)M^l_x H(c-x) + A'(M^r)M^r_x H(x-c) + (A(M^r) - A(M^l))\delta(x-c)$$

(where we use subscripts t and x instead of ∂_t et ∂_x). In order to get (33),(35), it is enough to show the distributional inequality

$$T_t + S_x \leq 0,$$

where

$$T = \rho U(a), \quad S = \rho\psi(a),$$

with $\psi(v) = vU(v)$, when U is convex, and the distributional equality

$$T_t + S_x = 0,$$

when U is affine. (This inequality precisely expresses the nonnegativeness of measure ν in (33).) The computation of S_x is easy. First, notice that

$$S = M_x^l \psi(A'(M^l)) H(c-x) + M_x^r \psi(A'(M^r)) + \psi(c')(M^r - M^l)\delta(x-c),$$

since, at $x = c$,

$$a = \frac{A(M^r) - A(M^l)}{M^r - M^l} = c'.$$

Thus

$$S_x = (M_x^l \psi(A'(M^l)))_x H(x-c) + (M_x^r \psi(A'(M^r)))_x H(c-x)$$

$$+(M_x^r \psi(A'(M^r)) - M_x^l \psi(A'(M^l)))\delta(x-c) + \psi(c')(M^r - M^l)\delta'(x-c).$$

The computation of T_t is longer. We have

$$T = M_x^l U(A'(M^l)) H(c-x) + M_x^r U(A'(M^r)) H(x-c) + U(c')(M^r - M^l)\delta(x-c).$$

Thus

$$T_t = (M_x^l U(A'(M^l)))_t H(x-c) + (M_x^r U(A'(M^r)))_t H(c-x)$$

$$-(M_x^r U(A'(M^r)) - M_x^l U(A'(M^l)))c'\delta(x-c)$$

$$+\frac{d}{dt}(U(c')(M^r - M^l))\delta(x-c) - c'U(c')(M^r - M^l)\delta'(x-c).$$

Since $M^l(t) = M^l(t, c(t))$, we get

$$M^{l\prime}(t) = M_t^l(t,c) + c'M_x^l(t,c) = M_x^l(t,c)(c' - A'(M^l(t)))$$

(indeed, $M^l(t,x)$ is a smooth local solution of (7)) and, similarly,

$$M^{r\prime} = (c' - A'(M^r))M_x^r.$$

Next, we find

$$(M^r - M^l)c'' = (M^r - M^l)\frac{d}{dt}[\frac{A(M^r) - A(M^l)}{M^r - M^l}] = (c' - A'(M^l))^2 M_x^l - (c' - A'(M^r))^2 M_x^r,$$

then

$$\frac{d}{dt}[U(c')(M^r - M^l)] = U'(c')[(c' - A'(M^l))^2 M_x^l - (c' - A'(M^r))^2 M_x^r]$$

$$+U(c')[-(c' - A'(M^l))M_x^l + (c' - A'(M^r))M_x^r].$$

By using again that both $M^l(t,x)$ and $M^r(t,x)$ are classical solutions of (7), we get

$$T_t + S_x = \delta(x-c)(M_x^l[U'(c')(c'-A'(M^l))^2 + (U(A'(M^l)) - U(c'))(c'-A'(M^l))]$$

$$+M_x^r[-U'(c')(c' - A'(M^r))^2 - (U(A'(M^r)) - U(c'))(c' - A'(M^r))]).$$

But, this term vanishes when U is affine and is non positive when U is convex, since, by Lax' entropy condition,

$$A'(M^l) \geq c' \geq A'(M^r),$$

which completes the proof. Notice that we have only used Lax entropy condition which is not as precise as Oleinik's entropy condition (which is equivalent to the Kruzhkov entropy condition for piecewise smooth solution of scalar conservation laws). This is not contradictory, since, as already mentioned, (4) is not a satisfactory condition to enforce uniqueness.

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