DERIVATION OF PARTICLE, STRING AND MEMBRANE MOTIONS FROM THE BORN-INFELD ELECTROMAGNETISM

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Abstract. We derive classical particle, string and membrane motion equations from a rigorous asymptotic analysis of the Born-Infeld nonlinear electromagnetic theory. We first add to the Born-Infeld equations the corresponding energy-momentum conservation laws and write the resulting system as a non-conservative symmetric 10×10 system of first-order PDEs. Then, we show that four rescaled versions of the system have smooth solutions existing in the (finite) time interval where the corresponding limit problems have smooth solutions. Our analysis is based on a continuation principle previously formulated by the second author for (singular) limit problems.

1. Introduction

The Born-Infeld (BI) equations were originally introduced in [1] as a nonlinear correction to the standard linear Maxwell equations for electromagnetism. They form a 6×6 system of conservation laws, together with two solenoidal constraints on the magnetic field and electric displacement. This system has many remarkable physical and mathematical features. Introduced in 1934, the BI model was designed to cure the classical divergence of the electrostatic field generated by point charges, by introducing an absolute limit to it (just like the speed of light is an absolute limit for the particle velocity in special relativity). The value of the absolute field was fixed by Born and Infeld according to physical considerations. As a result, for moderate electromagnetic fields, the discrepancy between the BI model and the classical Maxwell equations is noticable only at subatomic scales (10^{-15} meters). However, for very large values of the field, the BI model gets very different from the Maxwell model and, as will be rigorously established in this paper, rather describes the evolution of point particles along straight lines, or vibrating strings or vibrating membranes, depending on the considered scales. Although the BI model was rapidly given up due to the emergence of quantum Electrodynamics (QED) in the 40', there has been a lot of recent interest for it. In high energy Physics, D-branes can be modelled according to a generalization of the BI model [14, 5]. In differential geometry, the BI equations are closely related to the study of extremal surfaces in the Minkowski space. From the PDEs viewpoint, the initial value problem (IVP) has been recently investigated by Lindblad (in the "scalar case" of extremal surfaces [12]) and by Chae and Huh [3]. They show the existence of global smooth solutions, for small initial data (in a regime sufficiently closed to the Maxwell limit), using Klainerman's null forms and energy estimates. In mathematical physics, QED has recently been revisited by Kiessling who used a quantization technique well-suited to nonlinear PDEs, involving a relativistic version of the Fisher information [9].

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In [2], the first author exploited the fact that the energy density and the Poynting vector satisfy certain additional conservation laws and lifted the Born-Infeld model to a 10×10 system of conservation laws, by using the energy density and the Poynting vector as new unknown variables. The resulting ABI (augmented Born-Infeld) system provides a set of equations coupling the electromagnetic field and a virtual fluid having the electromagnetic energy as mass and the Poynting vector as momentum. It was pointed out that the ABI system has some remarkable structural properties like existence of a strictly convex entropy, Galilean invariance of fluid mechanics, and full linear degeneracy.

Moreover, three asymptotic regimes of the ABI system are studied in [2], using Dafermos' relative entropy method [4] to analyse the resulting (singular) limit problems. With such analysis, the linear Maxwell equations are derived for low fields, some pressureless MHD equations, describing vibrating strings, for high fields, and pressureless gas equations for very high fields. Unfortunately, these results postulate the existence of global weak solutions for the ABI system (although they do not require a priori bounds on them). This is a major weakness, since the global existence of weak solutions to the IVP remains an outstanding open problem for essentially all multidimensional system of non-linear hyperbolic conservation laws.

The goal of this paper is to use the framework of smooth solutions and energy estimates to get definite asymptotic results, using a non-conservative form of the ABI system. It will be shown that IVPs of the rescaled ABI systems have smooth solutions existing in the (finite) time interval where the corresponding reduced problems have smooth solutions. The analysis is based on a continuation principle previously formulated in [17] for (singular) limit problems, and combines formal asymptotic expansions with error estimates of energy type for symmetrizable hyperbolic systems. We also consider a new high field regime, involving vibrating membranes, which was disregarded in [2].

The use of the continuation principle makes our analysis quite different from the classical one due to Klainerman and Majda [10, 11, 13]. With the latter, one shows the existence in a scaling-independent time interval, which may be properly contained in the time interval where the corresponding reduced problems have smooth solutions. The difference of the two approaches makes significant sense when the reduced problems have global smooth solutions. Such an example is given in Subsection 4.3 for low fields. See also Section 3.

As a by-product and first step of our analysis, we observe that the non-conservation form of the ABI equations constitutes a *symmetric* (not only *symmetrizable*) hyperbolic system. Thus, the local well-posedness of the ABI system becomes obvious. Note that the symmetry does not follow directly from the existence of a strictly convex entropy proved in [2], since the latter involves the solenoidal constraints and thus the entropy is not that in the usual sense. About this point see also Serre [15]. Moreover, we show that the solenoidal constraints are compatible with the symmetric hyperbolic systems and point out a few possibly important structural properties thereof. In addition, the non-conservative ABI system, remarkably enough, is well defined for all states in $\mathbb{R}^{10}$, in sharp contrast with the conservative version, which requires the density field h to be nonnegative. Indeed, the non-conservative system involves the inverse density field τ, which substitutes for h^{-1} and can take any real values, negative, positive or null. As a consequence, the previously mentioned asymptotic results trivially follows from the symmetry of the non-conservative ABI system, at least for short time intervals. So, an important technical issue of this paper is to extend these time intervals according to the existence time interval of the solutions to the limit equations, by using the method discussed above.

The paper is organized as follows. In Section 2, we introduce the non-conservative augmented Born-Infeld equations and point out some of its structural properties. In Section 3, we introduce three high field limits of the ABI equations and show that they respectively describe particle, string and membrane motions. In Section 4, a crude asymptotic analysis is performed just by using the symmetric structure of the non-conservative ABI system. In Section 5, an abstract theorem is established for the rescaled ABI systems. This theorem is applied to four concrete asymptotic regimes in Section 6 to get sharper results. The paper ends with an appendix, which contains the continuation principle for (singular) limit problems of symmetrizable hyperbolic systems.

Notation. Let $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$ (the d-dimensional torus). L^2 is the space of square integrable (vector- or matrix-valued) functions on Ω and its norm is denoted by $\|\cdot\|$. In case A depends on another variable t as well as on $x \in \Omega$, we write $\|A(t)\|$ or $\|A(\cdot,t)\|$ to recall that the norm is taken with respect to x while t is viewed as a parameter. Similar notation will be adopted for the function spaces introduced below. For a nonnegative integer k, the Sobolev space $H^k = H^k(\Omega)$ is defined as the space of functions whose distributional derivatives of order $\leq k$ are all in L^2. We use $\|\cdot\|_k$ to denote the norm of H^k. Furthermore, $C\big(J, H^k\big)$ denotes the space of continuous functions on the interval J with values in H^k. Finally, partial time derivatives will be frequently denoted by u_t, instead of $\partial_t u$.

2. The Born-Infeld system and its non-conservative augmented version

Let B and D be time-dependent vector fields in $\mathbb{R}^3$. The Born-Infeld (BI) equations read (see, e.g., [2])

$$B_t + \nabla \times (\frac{B \times V + D}{h}) = 0,\ D_t + \nabla \times (\frac{D \times V - B}{h}) = 0,\ \operatorname{div} B = \operatorname{div} D = 0, \tag{2.1}$$

where

$$h = \sqrt{1 + |B|^2 + |D|^2 + |D \times B|^2}, \qquad V = D \times B, \tag{2.2}$$

and $|\cdot|$ stands for the Euclidean norm. Immediately notice that the classical (homogeneous) Maxwell equations

$$B_t + \nabla \times D = 0,\ D_t - \nabla \times B = 0,\ \operatorname{div} B = \operatorname{div} D = 0, \tag{2.3}$$

can be seen as the limit of the BI equations for weak fields $B, D << 1$.

In [2], the first author exploited the fact that, for smooth solutions of the BI system, the energy density h and the Poynting vector V satisfy additional conservation laws—the first two lines in (2.4) below, and used h and V as new unknown variables to augment the BI model as a 10×10 system of conservation laws. Set

$$v = V/h, \qquad b = B/h, \qquad d = D/h.$$

The ABI (augmented Born-Infeld) system can be written

$$
\begin{aligned}
h_t + \operatorname{div}(hv) &= 0,\\
(hv)_t + \operatorname{div}(hv\otimes v - hb\otimes b - hd\otimes d) &= \nabla h^{-1},\\
(hb)_t + \operatorname{div}(hb\otimes v - hv\otimes b) + \nabla\times d &= 0,\\
(hd)_t + \operatorname{div}(hd\otimes v - hv\otimes d) - \nabla\times b &= 0
\end{aligned} \tag{2.4}
$$

along with

$$
\operatorname{div}(hb) = \operatorname{div}(hd) = 0. \tag{2.5}
$$

See [2] for further discussions about this ABI system.

Here we only consider smooth solutions to the ABI system. Therefore, we may focus on the non-conservative form of (2.4) with (2.5). Set $\tau = h^{-1}$. By using the identity

$$
\operatorname{div}(d\otimes b) = (b\cdot\nabla + \operatorname{div} b)d,
$$

we can easily verify that smooth solutions to (2.4) with (2.5) satisfy

$$
\begin{aligned}
&\tau_t + v\cdot\nabla\tau - \tau\operatorname{div} v = 0,\\
&v_t + v\cdot\nabla v - b\cdot\nabla b - d\cdot\nabla d - \tau\nabla\tau = 0,\\
&b_t + v\cdot\nabla b - b\cdot\nabla v + \tau\nabla\times d = 0,\\
&d_t + v\cdot\nabla d - d\cdot\nabla v - \tau\nabla\times b = 0.
\end{aligned} \tag{2.6}
$$

This is a symmetric hyperbolic system

$$
W_t + \sum_{j=1}^{3} A_j(W)W_{x_j} = 0 \tag{2.7}
$$

for $W = (\tau, v^T, b^T, d^T)^T$, with homogeneous quadratic nonlinearities. Here the superscript "T" denotes the transpose operation and the coefficient matrix is

$$
A_j(W) = v_j I_{10} + \begin{pmatrix} 0 & -\tau e_j^T & 0_3^T & 0_3^T \\ -\tau e_j & 0_{3\times3} & -b_j I_3 & -d_j I_3 \\ 0_3 & -b_j I_3 & 0_{3\times3} & \tau e_j\times \\ 0_3 & -d_j I_3 & -\tau e_j\times & 0_{3\times3} \end{pmatrix} \tag{2.8}
$$

with v_j the j-th component of v, I_k the unit matrix of order k, e_j the j-th column of I_3, 0_3 the origin of $\mathbb{R}^3$, and $0_{3\times3}$ the origin of $\mathbb{R}^{3\times3}$. Notice that (2.7) makes sense for all states

$$
W = (\tau, v^T, b^T, d^T)^T \in \mathbb{R}^{10}, \tag{2.9}
$$

not only for $\tau > 0$ and even if the solenoidal constraints (2.5) do not hold.

It is remarkable that the coefficient matrix $A_j(W)$ *linearly* depends on W and the symmetry is independent of the solenoidal constraints, which are needed in [2] to show the existence of a strictly convex entropy function.

Also notice that, in the original BI equations, B, D, h and V are linked together by the algebraic relations (2.2). This means that the original BI equations exactly correspond to the non-conservative formulation (2.6), with the further restriction that $W = (\tau, v^T, b^T, d^T)^T$ must be valued in the "BI manifold" defined by:

$$
\tau > 0\ , \quad \tau^2 + b^2 + d^2 + v^2 = 1\ , \quad \tau v = d\times b\ . \tag{2.10}
$$

Of course, as for the original ABI system, the BI manifold is an invariant set for system (2.6).

The equivalence of (2.6) and (2.4) with (2.5) is illustrated here.

Proposition 2.1. *If initial data for (2.6) satisfy the constraints in (2.5), then the corresponding smooth solutions to (2.6) satisfy (2.4) as well as (2.5).*

Proof. It suffices to verify (2.5). To do this, we notice that the smooth solutions to (2.6) satisfy $h_t + \operatorname{div}(hv) = 0$ and

$$(hb)_t + \nabla \times (hb \times v) + \nabla \times d + \operatorname{div}(hb)v = 0,$$

where $h = \tau^{-1}$. Thus, we have

$$(\operatorname{div}(hb))_t + v \cdot \nabla(\operatorname{div}(hb)) + \operatorname{div}(hb) \operatorname{div} v = 0.$$

Since $\operatorname{div}(hb) = 0$ initially, we have $\operatorname{div}(hb) = 0$ for all t. Similarly, we can show $\operatorname{div}(hd) = 0$. This completes the proof. □

Furthermore, we point out the following important property of the symmetric hyperbolic system (2.6) or (2.7) with (2.8).

Proposition 2.2. *Let $W = (\tau, v^T, b^T, d^T)^T$ be a smooth solution to (2.6) with $\tau > 0$. Then*

$$\partial_t(hI_{10}) + \sum_j \partial_{x_j}(hA_j(W)) = \operatorname{div}(hb)C_1 + \operatorname{div}(hd)C_2$$

holds with C_1 and C_2 constant symmetric matrices and $h = \tau^{-1}$. In particular, if intial data for (2.6) satisfy the solenoidal constraints in (2.5), then

$$\partial_t(hI_{10}) + \sum_j \partial_{x_j}(hA_j(W)) = 0.$$

Proof. From the explicit expression of the coefficient matrix given in (2.8) it follows that

$$\begin{aligned}
&\partial_t(hI_{10}) + \sum_j \partial_{x_j}(hA_j(W)) \\
=&(h_t + \operatorname{div}(hv))I_{10} + \sum_j \partial_{x_j} \begin{pmatrix} 0 & -e_j^T & 0_3^T & 0_3^T \\ -e_j & 0_{3\times3} & -hb_jI_3 & -hd_jI_3 \\ 0_3 & -hb_jI_3 & 0_{3\times3} & e_j\times \\ 0_3 & -hd_jI_3 & -e_j\times & 0_{3\times3} \end{pmatrix} \\
=&-\begin{pmatrix} 0 & 0_3^T & 0_3^T & 0_3^T \\ 0_3 & 0_{3\times3} & \operatorname{div}(hb)I_3 & \operatorname{div}(hd)I_3 \\ 0_3 & \operatorname{div}(hb)I_3 & 0_{3\times3} & 0_{3\times3} \\ 0_3 & \operatorname{div}(hd)I_3 & 0_{3\times3} & 0_{3\times3} \end{pmatrix}.
\end{aligned}$$

This completes the proof. □

3. High field limits: particle, string and membrane motions

Observe that, for the non-conservative system (2.6), the states $W = (0, v^T, b^T, d^T)^T$, for which $\tau = 0$, are not singular, while, for the conservative ABI system (2.4), they correspond to fields (B, D) of infinite intensity. (Indeed $\tau = h^{-1}$.) (Let us recall that, for classical gas dynamics equations, the state $\tau = 0$ is always singular.)

Due to the special structure of (2.6), the corresponding "reduced" states $(v^T, b^T, d^T)^T$ solve the following "reduced" system:

$$\begin{aligned} &v_t + v\cdot\nabla v - b\cdot\nabla b - d\cdot\nabla d = 0,\\ &b_t + v\cdot\nabla b - b\cdot\nabla v = 0,\\ &d_t + v\cdot\nabla d - d\cdot\nabla v = 0. \end{aligned} \tag{3.1}$$

A further reduction is obtained as $d = 0$, which leads to:

$$\begin{aligned} &v_t + v\cdot\nabla v - b\cdot\nabla b = 0,\\ &b_t + v\cdot\nabla b - b\cdot\nabla v = 0. \end{aligned} \tag{3.2}$$

Finally, $\tau = 0$, $b = d = 0$, reduce (2.6) to a single equation:

$$v_t + v\cdot\nabla v = 0. \tag{3.3}$$

Notice the parallel reduction of the BI manifold (2.10) to the following reduced manifolds

$$b^2 + d^2 + v^2 = 1\ ,\quad d\times b = 0\ ,\quad v\cdot b = v\cdot d = 0\ , \tag{3.4}$$

$$b^2 + v^2 = 1\ ,\quad v\cdot b = 0\ , \tag{3.5}$$

$$v^2 = 1\ , \tag{3.6}$$

respectively associated to (3.1), (3.2) and (3.3). These three sets of "high field" equations have a simple physical and geometrical interpretation. Indeed, system (3.3) describes a continuum of particles moving along straight lines with constant speed, as well known. If, in addition, (3.6) holds true, these particles can be interpreted as massless particles with unit velocities. System (3.2) is more subtle and describe collections of vibrating strings, as will be shown in a moment. Condition (3.5) guarantees that these strings are genuine relativistic strings. Notice that (3.2) can also interpreted as a shallow water MHD equation (without gravity terms), following [6]. Similarly, system (3.1) describes vibrating membranes. These statements follow from the following observation:

Proposition 3.1. *Let $(s, r, u) \in \mathbb{R}^3 \to X(t, s, r, u)$ be a smooth family of diffeomorphisms of $\mathbb{R}^3$, depending on $t \in [-T, T]$. Let λ, μ be two nonnegative real constants. Assume that*

$$\partial_{tt} X = \lambda \partial_{ss} X + \mu \partial_{rr} X \tag{3.7}$$

holds true. Implicitely define

$$b(t, X(t, s, r, u)) = \partial_s X(t, s, r, u), \tag{3.8}$$

$$d(t, X(t, s, r, u)) = \partial_r X(t, s, r, u), \tag{3.9}$$

$$v(t, X(t, s, r, u)) = \partial_t X(t, s, r, u). \tag{3.10}$$

Then (b, d, v) respectively is a smooth solution to system (3.1), if $\lambda = \mu = 1$, system (3.2) if $\lambda = 1$, $\mu = 0$, and equation (3.3) if $\lambda = \mu = 0$.

Proof. The proof is a straightforward application of the chain rule:
differentiate (3.8,3.9,3.10) with respect to t, use equation (3.7) and get the desired equations (3.1,3.2,3.3).

Geometrical interpretation. According to (3.7), in the case $\lambda = \mu = 0$, each trajectory $t \to X(t, r, s, u)$ is a straight line, as (r, s, u) varies in $\mathbb{R}^3$.
In the case $\lambda = 1$ and $\mu = 0$, each surface $(t, s) \to X(t, r, s, u)$ solves the wave equation

$$\partial_{tt} X = \partial_{ss} X,$$

and describes a vibrating string, as (r, u) varies in $\mathbb{R}^2$. Notice that the algebraic constraint (3.5) reads

$$\partial_t X^2 + \partial_s X^2 = 1, \quad \partial_t X \cdot \partial_s X = 0,$$

which, together with the wave equation, means that these strings are genuinely relativistic (i.e. they are extremal surfaces in the Minkowski space). Similarly, in the case $\lambda = \mu = 1$, each $(t, s, r) \to X(t, r, s, u)$ describes a vibrating membrane as u varies along the real line.

Global smooth solutions to the string system (3.2). From Proposition 3.1, we also see that the high field equation (3.2) have non-trivial global smooth solutions for smooth initial conditions sufficiently close to suitable trivial solutions. More precisely, take any constant vector $B_0 \neq 0$. This provides a trivial solution $b = B_0$, $v = 0$ and a corresponding family of diffeomorphisms is $X(t, s, r, u) = sB_0 + rD_0 + uB_0 \times D_0$, where $D_0 \neq 0$ is arbitrarily chosen such that $B_0 \cdot D_0 = 0$. Then, for any initial condition (b, v) chosen sufficiently close to this trivial solution, equation (3.2) has a global smooth solutions. Indeed, it is enough to i) introduce an initial diffeomorphism $X(0, s, r, u)$ implicitely defined by

$$\partial_s X(0, s, r, u) = b(0, X(0, s, r, u)),$$

ii) for each fixed (s, u) solve the wave equation (3.7) with $\mu = 0$ and

$$\partial_t X(0, s, r, u) = v(0, X(0, s, r, u)),$$

iii) ensure that, for all t, $(s, r, u) \to X(t, s, r, u)$ still is a diffeomorphism of $\mathbb{R}^3$, by chosing initial data $(b, v)(0, x)$ sufficiently close to $(B_0, 0)$,
iv) globally define a solution (b, v) to (3.2) by:

$$\partial_s X(t, s, r, u) = b(t, X(t, s, r, u)), \quad \partial_t X(t, s, r, u) = v(t, X(t, s, r, u)).$$

So, we see that the high field limit equations formally derived from the ABI system do have global smooth solutions in the neighborhood of some trivial solutions. This is also true for the original BI equations, as shown by Chae and Huh [3]. We conjecture the same property for the ABI system. (Concerning the existence of global classical solutions for the gas dynamics equations, we refer to [7].)

4. A crude asymptotic analysis

According to well-known results on symmetric system of first-order PDEs [13], there is a positive continuous function θ attached to system (2.6) such that, for all initial condition W_0 belonging to the (homogeneous) Sobolev space H^s, where $s > 3/2 + 1$ is fixed (say $s = 3$), there is a unique strong solution $t \to W(t)$ to (2.6) such that $W(0) = W_0$, defined at least in the time interval $[-T, T]$ where $T = \theta(||W_0||_s)$. In addition, this solution depends continuously on W_0 in the space $C^0([-T, T], H^{s'})$ for all $s' < s$. Thus, we get without effort the following asymptotic result:

Theorem 4.1. *Let $\epsilon \in]0, 1]$. Let $W_{0\epsilon} = (\tau_{0\epsilon}, v_{0\epsilon}, b_{0\epsilon}, d_{0\epsilon})$ be a smooth initial condition, depending on ϵ, uniformly bounded in H^s for some $s > 3/2 + 1$, distant from $W_0 = (0, v_0, b_0, d_0)$ by ϵ in $H^{s'}$ norm for some $s' < s$. Then, there is a time interval $[-T, T]$, where $T > 0$ does not depend on ϵ, such that:*
Systems (2.6) and (3.1) both have a unique strong solution $W^\epsilon = (\tau^\epsilon, v^\epsilon, b^\epsilon, d^\epsilon)$ and (v, b, d), with respective initial condition $W_{0\epsilon}$ and (v_0, b_0, d_0) on $[-T, T]$. Moreover, their distance in $C^0([-T, T], H^s)$ norm is of order ϵ.

Of course, in the special case when $d_0 = 0$, the limit equation (3.1) reduces to the "string equation" (3.2). Similarly, as $b_0 = d_0$, (3.1) reduces to (3.3). Let us point out that this result is obtained effortless, because of the remarkable structure of the non-conservative augmented version of the BI equations. A direct asymptotic analysis of the *original* BI equations (2.1) would have been considerably more difficult.

Need for refined asymptotic results. The main weakness of Theorem 4.1 is that the uniform existence time T is not at all optimal. Indeed, T depends on the H^s norm of the initial conditions, which is very far from being sharp. As a matter of fact, in many situations the optimal existence times T^* for the limit systems (3.1,3.2,3.3) can be explicitely computed. Therefore, we want a sharper existence time of form $T = T^* + O(\epsilon)$. This goal will be achieved in the next section through more refined arguments.

5. Refined asymptotic analysis

Consider IVPs of the ABI system (2.6) (or (2.7) with (2.8)) with initial data $W_{0\epsilon}$, which depends on ϵ in a certain topological space. Suppose an approximate smooth solution $W_\epsilon = W_\epsilon(x, t)$ has been constructed (see the next section) and is well defined for $(x, t) \in \Omega \times [0, T_*]$ with a certain $T_* > 0$. Here Ω stands for the 3-dimensional torus (For simplicity, we consider periodic initial data only). Define the *residual* of W_ϵ as

$$R = R(W_\epsilon) := W_{\epsilon t} + \sum_{j=1}^{3} A_j(W_\epsilon) W_{\epsilon x_j}. \tag{5.1}$$

This section is devoted to the proof of the following general result.

Theorem 5.1. *Let $s \geq 3$ be an integer. Suppose $W_{0\epsilon} \in H^s$ for each ϵ different from a certain singular point (say 0), $W_\epsilon(\cdot, t) \in H^{s+1}$, and*

$$\delta := \|W_{0\epsilon} - W_\epsilon(\cdot, 0)\|_s^2 + \int_0^{T_*} \|R(\cdot, t)\|_s^2 dt \longrightarrow 0$$

as ϵ approaches to the singular point 0. Moreover, suppose there is a 10×10*-matrix* L_ϵ*, which is bounded and invertible for* $\epsilon \neq 0$*, such that* $|L_\epsilon^{-1}|\sqrt{\delta} \longrightarrow 0$ *as* $\epsilon \to 0$*,* $\sup_\epsilon \|L_\epsilon^{-1} W_{0\epsilon}\|_s < \infty$*, and* $\sup_{t,\epsilon} \|L_\epsilon^{-1} W_\epsilon(\cdot, t)\|_{s+1} < \infty$*.*

Then there is a neighborhood of $\epsilon = 0$ *such that, for all* ϵ *in the neighborhood, the ABI system (2.6) with initial data* $W_{0\epsilon}$ *has a unique classical solution*

$$W^\epsilon \in C([0, T_*], H^s).$$

Moreover, the error estimate

$$\|W^\epsilon(\cdot, t) - W_\epsilon(\cdot, t)\|_s \leq K\sqrt{\delta}, \qquad \forall t \in [0, T_*], \tag{5.2}$$

holds with K *a constant independent of* ϵ*.*

Proof. Set $U_\epsilon = L_\epsilon^{-1} W_\epsilon$ for $\epsilon \neq 0$. Then (5.1) becomes

$$U_{\epsilon t} + \sum_{j=1}^{3} A_j(U_\epsilon, \epsilon) U_{\epsilon x_j} = L_\epsilon^{-1} R \tag{5.3}$$

with

$$A_j(U, \epsilon) = L_\epsilon^{-1} A_j(L_\epsilon U) L_\epsilon.$$

Accordingly, we consider the following IVP

$$\begin{aligned} U_t + \sum_j A_j(U, \epsilon) U_{x_j} &= 0, \\ U(x, 0) = \bar{U}(x, \epsilon) &:= L_\epsilon^{-1} W_{0\epsilon}. \end{aligned} \tag{5.4}$$

This is a symmetrizable hyperbolic system with $A_0(U, \epsilon) := L_\epsilon^T L_\epsilon$ as its symmetrizer.

Since $\sup_\epsilon \|\bar{U}(\cdot, \epsilon)\|_s < \infty$ and $\sup_{t,\epsilon} \|U_\epsilon(\cdot, t)\|_{s+1} < \infty$ with $s \geq 3$, we deduce from the Sobolev embedding theorem that both $\bar{U}$ and U_ϵ take values in a bounded subset of the state space $\mathbb{R}^{10}$. Namely, there is an open set G such that

$$\bigcup_{x,t,\epsilon} \left\{ \bar{U}(x, \epsilon), U_\epsilon(x, t) \right\} \subset G \subset\subset \mathbb{R}^{10}.$$

Thus, we can choose G_1 so that

$$G \subset\subset G_1 \subset\subset \mathbb{R}^{10}.$$

For each fixed $\epsilon (\neq 0)$, since $\bar{U}(x, \epsilon) \in G \subset\subset G_1$ for all $x \in \Omega$ and $\bar{U}(\cdot, \epsilon) \in H^s$ with $s \geq 3$, it follows from the local-in-time existence theory [13] for IVPs of symmetrizable hyperbolic systems that there is a maximal time $T_\epsilon = T_\epsilon(G_1) > 0$ so that the rescaled problem (5.4) has a unique classical solution

$$U^\epsilon \in C([0, T_\epsilon), H^s) \qquad \text{and} \qquad U^\epsilon(x, t) \in G_1 \quad \forall (x, t) \in \Omega \times [0, T_\epsilon).$$

Thus we only need to show $T_\epsilon > T_*$ and the error estimate in (5.2). Moreover, it suffices to prove the estimate (5.2) for $t \in [0, \min\{T_*, T_\epsilon\})$, thanks to the continuation principle (Lemma 9.1 in [17], see also the Appendix of this paper) and $|L_\epsilon^{-1}|\sqrt{\delta} \to 0$ as $\epsilon \to 0$.

Now we turn to derive the error estimate (5.2) for $t \in [0, \min\{T_*, T_\epsilon\})$. Notice that, in that time interval, both U^ϵ and U_ϵ are regular. We compute from (5.3) and (5.4) that

$E = W_\epsilon - W^\epsilon = L_\epsilon(U_\epsilon - U^\epsilon)$ satisfies

$$E_t + \sum_j A_j(W^\epsilon)E_{x_j} = R - \sum_j A_j(E)W_{\epsilon x_j}.$$

Here we have used the linearity of $A_j = A_j(W)$ with respect to W. Differentiating this equation with ∂^α for a multi-index α satisfying $|\alpha| \le s$ and setting $E_\alpha = \partial^\alpha E$, we get

$$E_{\alpha t} + \sum_j A_j(W^\epsilon)E_{\alpha x_j} = R_\alpha + F^\alpha. \tag{5.5}$$

Here

$$F^\alpha = \sum_j [A_j(W^\epsilon), \partial^\alpha]E_{x_j} - \big\{\sum_j A_j(E)W_{\epsilon x_j}\big\}_\alpha.$$

Recall that $A_j(W^\epsilon)(j = 1, 2, \cdots, d)$ are all symmetric. Multiplying the equation (5.5) with E_α^T from the left, we get

$$(|E_\alpha|^2)_t + \sum_j \big\{E_\alpha^T A_j(W^\epsilon)E_\alpha\big\}_{x_j} = 2\mathrm{Re}E_\alpha^T(R_\alpha + F^\alpha) + E_\alpha^T\Big\{\sum_j \frac{\partial A_j(W^\epsilon)}{\partial x_j}\Big\}E_\alpha. \tag{5.6}$$

For the right-hand side of (5.5), we treat as follows.

$$\begin{aligned} &2\mathrm{Re}E_\alpha^T(R_\alpha + F^\alpha) \le 2|E_\alpha|^2 + |R_\alpha|^2 + |F^\alpha|^2, \\ &\sum_j \frac{\partial A_j(W^\epsilon)}{\partial x_j} \le C\sum_j |W^\epsilon_{x_j}| \le C\|W^\epsilon\|_s, \end{aligned} \tag{5.7}$$

where C is a generic constant and the well-known Sobolev inequality has been used. Moreover, we apply the Moser-type calculus inequalities in Sobolev spaces [13] to F^α and obtain

$$\begin{aligned} \|F^\alpha\| \le& C\|W_{\epsilon x_j}\|_s\|E\|_{|\alpha|} + C\|W^\epsilon\|_s\|E_{x_j}\|_{|\alpha|-1} \\ \le& C\|W_\epsilon\|_{s+1}\|E\|_{|\alpha|} + C(\|W_\epsilon\|_s + \|E\|_s)\|E\|_{|\alpha|} \\ \le& C(1 + \|E\|_s)\|E\|_{|\alpha|}. \end{aligned} \tag{5.8}$$

Here the boundedness of $\|W_\epsilon\|_{s+1}$ is used. Integrating (5.6) over $(x,t) \in \Omega \times [0,T]$ with $T < \min\{T_\epsilon, T_*\}$ and using (5.7) and (5.8) yields

$$\|E_\alpha(T)\|^2 \le \|E_\alpha(0)\|^2 + \int_0^T \|R_\alpha(t)\|^2 dt + C\int_0^T (1 + \|E(t)\|_s^2)\|E(t)\|_{|\alpha|}^2 dt. \tag{5.9}$$

Summing up (5.9) for all multi-indices α with $|\alpha| \le s$, we get

$$\begin{aligned} \|E(T)\|_s^2 \le& \|E(0)\|_s^2 + \int_0^T \|R(t)\|_s^2 dt + C\int_0^T (1 + \|E(t)\|_s^2)\|E(t)\|_s^2 dt \\ \le& \delta + C\int_0^T (1 + \|E(t)\|_s^2)\|E(t)\|_s^2 dt. \end{aligned}$$

Applying the Gronwall lemma to the last inequality yields

$$\|E(T)\|_s^2 \le \delta \exp\Big[C\int_0^T (1 + \|E(t)\|_s^2)dt\Big] \equiv \Phi(T). \tag{5.10}$$

Thus, we have

$$\Phi'(t) = C(1 + \|E(t)\|_s^2)\Phi(t) \leq C\Phi(t) + C\Phi^2(t).$$

Applying the nonlinear Gronwall-type inequality in [16] to the last inequality yields

$$\|E(t)\|_s^2 \leq \Phi(t) \leq \exp(CT_*) \tag{5.11}$$

for $t \in [0, \min\{T_\epsilon, T_*\})$ if $\Phi(0) = \delta < \exp(-CT_*)$. From (5.10) and (5.11) it follows that

$$\|E(t)\|_s \leq K\sqrt{\delta}$$

for all $t \in [0, \min\{T_\epsilon, T_*\})$. This completes the proof. □

We conclude this section with a remark.

Remark 5.1. In case $W_\epsilon(x,t)$ is defined globally in time and the conditions of Theorem 5.1 hold for $T_* = \infty$, we actually prove the following existence result for the ABI system (2.6): For any $T < \infty$, there is a neigborhood of $\epsilon = 0$ such that, for all ϵ in the neighborhood, the ABI system (2.6) with initial data $W_{0\epsilon}$ has a unique classical solution

$$W^\epsilon \in C([0,T], H^s).$$

Moreover, the error estimate in (5.2) holds for $t \leq T$ and the constant K depends on T.

6. Asymptotic Regimes

In this section, we apply Theorem 5.1 to four concrete asymptotic regimes. For simplicity, we will always take

$$W_{0\epsilon}(x) = W_\epsilon(x, 0).$$

In addition, we won't make remarks parallel to Remark 5.1.

6.1. **Membrane motion equations.** Let ϵ be a small parameter. If the ABI system (2.6) is solved with initial data of the form

$$\tau(x,0) = O(\epsilon), \qquad v(x,0) = O(1), \qquad b(x,0) = O(1), \qquad d(x,0) = O(1),$$

it is natural to take

$$L_\epsilon = \mathrm{diag}(\epsilon, I_3, I_3, I_3)$$

in Theorem 5.1. This L_ϵ is bounded for $\epsilon << 1$, invertible for $\epsilon \neq 0$ and $|L_\epsilon^{-1}| = \epsilon^{-1}$. This is a high field regime disregarded in [2]. It might be incompatible with the BI regime (2.10), contrary to the other ones to be discussed in the following subsections. The rescaled system (5.4) reads

$$\begin{aligned}
&\tilde{\tau}_t + \tilde{v} \cdot \nabla \tilde{\tau} - \tilde{\tau}\,\mathrm{div}\,\tilde{v} = 0, \\
&\tilde{v}_t + \tilde{v} \cdot \nabla \tilde{v} - \tilde{b} \cdot \nabla \tilde{b} - \tilde{d} \cdot \nabla \tilde{d} - \epsilon^2 \tilde{\tau} \nabla \tilde{\tau} = 0, \\
&\tilde{b}_t + \tilde{v} \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla \tilde{v} + \epsilon \tilde{\tau} \nabla \times \tilde{d} = 0, \\
&\tilde{d}_t + \tilde{v} \cdot \nabla \tilde{d} - \tilde{d} \cdot \nabla \tilde{v} - \epsilon \tilde{\tau} \nabla \times \tilde{b} = 0.
\end{aligned} \tag{6.1}$$

Here $\tilde{\tau}, \tilde{v}, \tilde{b}$ and $\tilde{d}$ denote the components of the scaled variable $U = L_\epsilon^{-1} W = (\tilde{\tau}, \tilde{v}^T, \tilde{b}^T, \tilde{d}^T)^T$.

In order to apply Theorem 5.1 to this regime, we drop the tildes in (6.1) and seek an approximate solution W_ϵ of the form

$$W_\epsilon = L_\epsilon U_\epsilon, \qquad U_\epsilon = U_0(x,t) + \epsilon U_1(x,t)$$

with the first component of U_1 being 0. Plugging this ansatz into (6.1), we see that $U_0 = (\tau_0, v_0, b_0, d_0)$ solves

$$\begin{aligned} &\tau_t + v \cdot \nabla \tau - \tau \operatorname{div} v = 0, \\ &v_t + v \cdot \nabla v - b \cdot \nabla b - d \cdot \nabla d = 0, \\ &b_t + v \cdot \nabla b - b \cdot \nabla v = 0, \\ &d_t + v \cdot \nabla d - d \cdot \nabla v = 0 \end{aligned} \tag{6.2}$$

and $U_1 = (0, v_1, b_1, d_1)$ satisfies

$$\begin{aligned} v_t + v_0 \cdot \nabla v - b_0 \cdot \nabla b - d_0 \cdot \nabla d &= -v \cdot \nabla v_0 + b \cdot \nabla b_0 + d \cdot \nabla d_0, \\ b_t + v_0 \cdot \nabla b - b_0 \cdot \nabla v &= b \cdot \nabla v_0 - v \cdot \nabla b_0 - \tau_0 \nabla \times d_0, \\ d_t + v_0 \cdot \nabla d - d_0 \cdot \nabla v &= d \cdot \nabla v_0 - v \cdot \nabla d_0 + \tau_0 \nabla \times b_0. \end{aligned} \tag{6.3}$$

Notice that the last three equations in (6.2) are just (3.1).

Now we solve (6.2) and (6.3) to obtain U_0 and U_1. Note that the last three equations in (6.2) form a symmetric hyperbolic system of nonlinear equations. By the local existence theory [8, 13] for IVPs of symmetrizable hyperbolic systems, if $(v_0, b_0, d_0)(\cdot, 0) \in H^s$ with $s \geq 3$, then there is $T_* > 0$ such that the corresponding IVP has a unique classical solution $(v_0, b_0, d_0) \in C([0, T_*], H^s)$. With v_0 thus obtained, we obtain $\tau_0 \in C([0, T_*], H^{s-1})$ by solving the first equation in (6.2):

$$\tau_t + v_0 \cdot \nabla \tau - \tau \operatorname{div} v_0 = 0,$$

which is a linear equation, with $\tau(\cdot, 0) \in H^{s-1}$. Similarly, by using the existence theory [8] for IVPs of linear symmetrizable hyperbolic systems, we obtain $U_1 = (0, v_1, b_1, d_1) \in C([0, T_*], H^{s-1})$ by solving (6.3) with appropriate initial data.

With $U_\epsilon = U_0 + \epsilon U_1 \in C([0, T_*], H^{s-1})$ thus obtained, it is easy to see that the residual R, defined in (5.1), satisfies

$$R = \epsilon^2 O(1) \in C([0, T_*], H^{s-2}).$$

Thus, we deduce the following conclusion from Theorem 5.1, together with Corollary 1 of Theorem 2.2 in [13]—a continuation principle.

Corollary 6.1. *Let $s \geq 3$ be an integer. Assume $W_{0\epsilon}(x) = W_\epsilon(x, 0) \in H^{s+2}$. Then there exists $T_* > 0$ and $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, the ABI system (2.6) with initial data $W_{0\epsilon}$ has a unique classical solution*

$$W^\epsilon = (\tau^\epsilon, v^\epsilon, b^\epsilon, d^\epsilon) \in C([0, T_*], H^{s+2}).$$

Moreover, the error estimates

$$\|(\tau^\epsilon - \epsilon\tau_0, v^\epsilon - v_0 - \epsilon v_1, b^\epsilon - b_0 - \epsilon b_1, d^\epsilon - d_0 - \epsilon d_1)(\cdot, t)\|_s \leq K\epsilon^2,$$

for $t \in [0, T_*]$, hold with K a constant independent of ϵ. In particular, we have

$$\|(\tau^\epsilon, v^\epsilon - v_0, b^\epsilon - b_0, d^\epsilon - d_0)(\cdot, t)\|_s = \epsilon O(1).$$

Remark 6.1. Theorem 5.1 claims that $W^\epsilon \in C([0, T_*], H^s)$. Since $W_{0\epsilon} = W_\epsilon(\cdot, 0) \in H^{s+2}$, it follows from Corollary 1 of Theorem 2.2 in [13] that this solution W^ϵ has better regularity, that is,

$$W^\epsilon \in C([0, T_*], H^{s+2}).$$

This argument applies to Corollaries 6.2–6.4 in the following subsections.

6.2. **String motion equations.** Consider the ABI system (2.6) with initial data of the form

$$\tau(x,0)=O(\epsilon), \qquad v(x,0)=O(1), \qquad b(x,0)=O(1), \qquad d(x,0)=O(\epsilon)$$

with $\epsilon << 1$. This is the high field regime considered in [2]. As in the previous subsection, we take

$$L_\epsilon = \mathrm{diag}(\epsilon, I_3, I_3, \epsilon I_3)$$

in Theorem 5.1. This L_ϵ is bounded for $\epsilon << 1$, invertible for $\epsilon \neq 0$ and $|L_\epsilon^{-1}| = \epsilon^{-1}$. Then, the rescaled system (5.4) reads

$$\begin{aligned} &\tau_t + v\cdot\nabla\tau - \tau\,\mathrm{div}\, v = 0,\\ &v_t + v\cdot\nabla v - b\cdot\nabla b - \epsilon^2 d\cdot\nabla d - \epsilon^2\tau\nabla\tau = 0,\\ &b_t + v\cdot\nabla b - b\cdot\nabla v + \epsilon^2\tau\nabla\times d = 0,\\ &d_t + v\cdot\nabla d - d\cdot\nabla v - \tau\nabla\times b = 0. \end{aligned} \tag{6.4}$$

Dropping the ϵ^2 terms, we obtain

$$\begin{aligned} &\tau_t + v\cdot\nabla\tau - \tau\,\mathrm{div}\, v = 0,\\ &v_t + v\cdot\nabla v - b\cdot\nabla b = 0,\\ &b_t + v\cdot\nabla b - b\cdot\nabla v = 0,\\ &d_t + v\cdot\nabla d - d\cdot\nabla v - \tau\nabla\times b = 0. \end{aligned} \tag{6.5}$$

Notice that the second and third equations in (6.5) are just (3.2).

To solve (6.5), we note that the second and third equations in (6.5) form a symmetric hyperbolic system of nonlinear equations. Thus, if $(v,b)(\cdot,0)\in H^s$ with $s\geq 3$, then there is $T_*>0$ such that the corresponding IVP has a unique classical solution $(v_0,b_0)\in C([0,T_*],H^s)$. With v_0 and b_0 thus obtained, we see from the existence theory [8] for linear problems that the decoupled hyperbolic system of linear equations:

$$\begin{aligned} \tau_t + v_0\cdot\nabla\tau - \tau\,\mathrm{div}\, v_0 = 0,\\ d_t + v_0\cdot\nabla d - d\cdot\nabla v_0 - \tau\nabla\times b_0 = 0, \end{aligned}$$

with appropriate initial data, has a unique classical solution $(\tau_0,d_0)\in C([0,T_*],H^{s-1})$.

Now we take $W_\epsilon = L_\epsilon U_\epsilon$ in Theorem 5.1 with U_ϵ obtained above. It is clear that the residual R, defined in (5.1), satisfies

$$\sup_t \|\epsilon^{-2}R(\cdot,t)\|_{s-2} < \infty.$$

Thus we deduce the following conclusion from Theorem 5.1, together with Corollary 1 of Theorem 2.2 in [13].

Corollary 6.2. *Let $s\geq 3$ be an integer. Assume $W_{0\epsilon} = W_\epsilon(\cdot,0)\in H^{s+2}$. Then there is $\epsilon_0>0$ such that, for all $\epsilon\in(0,\epsilon_0]$, the ABI system (2.6) with initial data $W_{0\epsilon}$ has a unique classical solution*

$$W^\epsilon = (\tau^\epsilon, v^\epsilon, b^\epsilon, d^\epsilon)\in C([0,T_*],H^{s+2}).$$

Moreover, the error estimates

$$\|(\tau^\epsilon - \epsilon\tau_0, v^\epsilon - v_0, b^\epsilon - b_0, d^\epsilon - \epsilon d_0)(\cdot,t)\|_s \leq K\epsilon^2, \qquad \forall t\in[0,T_*],$$

hold with K a constant independent of ϵ.

The interested reader can derive Corollary 6.2 directly from Corollary 6.1 with initial data satisfying $(v_1, b_1, d_0)(x, 0) = 0$.

6.3. **Particle motion equations.** Now we consider the ABI system (2.6) with initial data of the form

$$\tau(x,0) = O(\epsilon^2), \qquad v(x,0) = O(1), \qquad b(x,0) = O(\epsilon), \qquad d(x,0) = O(\epsilon)$$

with $\epsilon << 1$. This is the very high field regime considered in [2]. As in the previous subsections, we take

$$L_\epsilon = \mathrm{diag}(\epsilon^2, I_3, \epsilon I_3, \epsilon I_3)$$

in Theorem 5.1. This L_ϵ is bounded for $\epsilon << 1$, invertible for $\epsilon \neq 0$ and $|L_\epsilon^1| = \epsilon^{-2}$. Then, the rescaled system (5.4) reads

$$\begin{aligned} &\tau_t + v \cdot \nabla\tau - \tau \,\mathrm{div}\, v = 0, \\ &v_t + v \cdot \nabla v - \epsilon^2 b \cdot \nabla b - \epsilon^2 d \cdot \nabla d - \epsilon^4 \tau \nabla \tau = 0, \\ &b_t + v \cdot \nabla b - b \cdot \nabla v + \epsilon^2 \tau \nabla \times d = 0, \\ &d_t + v \cdot \nabla d - d \cdot \nabla v - \epsilon^2 \tau \nabla \times b = 0. \end{aligned} \tag{6.6}$$

In order to see what Theorem 5.1 means for this rescaled system, we look for an approximate solution W_ϵ of the form

$$W_\epsilon = L_\epsilon U_\epsilon, \qquad U_\epsilon = (\tau_0, v_0 + \epsilon^2 v_1, b_0, d_0)(x,t).$$

Plugging this ansatz into (6.6), we see that (τ_0, v_0, b_0, d_0) should solve

$$\begin{aligned} &\tau_t + v \cdot \nabla\tau - \tau \,\mathrm{div}\, v = 0, \\ &v_t + v \cdot \nabla v = 0, \\ &b_t + v \cdot \nabla b - b \cdot \nabla v = 0, \\ &d_t + v \cdot \nabla d - d \cdot \nabla v = 0 \end{aligned} \tag{6.7}$$

and v_1 should satisfy

$$v_t + v_0 \cdot \nabla v + v \cdot \nabla v_0 = b_0 \cdot \nabla b_0 + d_0 \cdot \nabla d_0. \tag{6.8}$$

Notice that the second equation in (6.7) is just (3.3).

As in the previous subsections, equations in (6.7) and (6.8) can be solved by using the local-in-time existence theorey [8, 13] for IVPs of quasilinear and linear symmetrizable hyperbolic systems. In particular, if initial data for (6.7) satisfy $v_0(\cdot, 0) \in H^{s+3}$ and $(\tau_0, b_0, d_0)(\cdot, 0) \in H^{s+2}$, then there exists $T_* > 0$ so that the corresponding IVP has a unique classical solution (τ_0, v_0, b_0, d_0) satisfying

$$v_0 \in C([0, T_*], H^{s+3}), \quad (\tau_0, b_0, d_0) \in C([0, T_*], H^{s+2}).$$

Moreover, if $v_1(\cdot, 0) \in H^{s+1}$, then the IVP of (6.8) has a unique classical solution

$$v_1 \in C([0, T_*], H^{s+1}).$$

With W_ϵ thus obtained, it is easy to see that the residual R, defined in (5.1), satisfies

$$R = \epsilon^3 \mathrm{diag}(\epsilon, \epsilon I_3, I_3, I_3) O(1) \in C([0, T_*], H^s).$$

Thus we have $\|R(\cdot,t)\|_s = \epsilon^3 O(1)$ and the following corollary from Theorem 5.1 together with Corollary 1 of Theorem 2.2 in [13].

Corollary 6.3. *Let $s \geq 3$ be an integer. Assume $W_{0\epsilon} = W_\epsilon(\cdot,0) \in H^{s+3}$. Then there is $\epsilon_0 > 0$ such that, for all $\epsilon \in (0,\epsilon_0]$, the ABI system (2.6) with initial data $W_{0\epsilon}$ has a unique classical solution*

$$W^\epsilon = (\tau^\epsilon, v^\epsilon, b^\epsilon, d^\epsilon) \in C([0,T_*], H^{s+3}).$$

Moreover, the error estimates

$$\|(\tau^\epsilon - \epsilon^2\tau_0, v^\epsilon - v_0 - \epsilon^2 v_1, b^\epsilon - \epsilon b_0, d^\epsilon - \epsilon d_0)(\cdot,t)\|_s \leq K\epsilon^3,$$

for $t \in [0,T_*]$, hold with K a constant independent of ϵ.

6.4. **The Maxwell equations.** Finally, we consider the ABI system (2.6) with initial data of the form

$$\tau(x,0) = 1 + O(\epsilon^2), \qquad v(x,0) = O(\epsilon^2), \qquad b(x,0) = O(\epsilon), \qquad d(x,0) = O(\epsilon)$$

with $\epsilon << 1$. This is the low field regime considered in [2]. As in the previous subsections, we take

$$L_\epsilon = \mathrm{diag}(1, \epsilon^2 I_3, \epsilon I_3, \epsilon I_3)$$

in Theorem 5.1. This L_ϵ is bounded for $\epsilon << 1$, invertible for $\epsilon \neq 0$ and $|L_\epsilon^{-1}| = \epsilon^{-2}$. Then, the rescaled system (5.4) reads

$$\begin{aligned}
&\tau_t + \epsilon^2 v \cdot \nabla\tau - \epsilon^2 \tau\,\mathrm{div}\, v = 0,\\
&v_t + \epsilon^2 v \cdot \nabla v - b \cdot \nabla b - d \cdot \nabla d - \epsilon^{-2}\tau\nabla\tau = 0,\\
&b_t + \epsilon^2 v \cdot \nabla b - \epsilon^2 b \cdot \nabla v + \tau\nabla \times d = 0,\\
&d_t + \epsilon^2 v \cdot \nabla d - \epsilon^2 d \cdot \nabla v - \tau\nabla \times b = 0.
\end{aligned} \tag{6.9}$$

In order to apply Theorem 5.1 to this case, we seek an approximate solution W_ϵ of the form

$$W_\epsilon = L_\epsilon U_\epsilon, \qquad U_\epsilon = (1 + \epsilon^2\tau_1, v_0, b_0, d_0).$$

Plugging this ansatz into (6.9), we see that (v_0, b_0, d_0) solves

$$\begin{aligned}
&v_t - b \cdot \nabla b - d \cdot \nabla d - \nabla\tau_1 = 0,\\
&b_t + \nabla \times d = 0,\\
&d_t - \nabla \times b = 0
\end{aligned} \tag{6.10}$$

and τ_1 satisfies

$$\tau_t - \mathrm{div}\, v_0 = 0. \tag{6.11}$$

Having (6.10) and (6.11), we determine U_ϵ as follows. By solving the last two lines in (6.10) (the standard linear Maxwell equations), we obtain b_0 and d_0. Then v_0 and τ_1 solve the inhomogeneous linear hyperbolic system

$$\begin{aligned}
v_t - \nabla\tau &= b_0 \cdot \nabla b_0 + d_0 \cdot \nabla d_0,\\
\tau_t - \mathrm{div}\, v &= 0.
\end{aligned}$$

Note that here solved are only hyperbolic systems of linear equations with constant coefficients, Assume $(b_0, d_0, v_0, \tau_1)(\cdot, 0) \in H^s$. It is easy to see that $W_\epsilon \in C([0,\infty), H^{s-1})$ and the residual R satisfies

$$R = \epsilon^3 \text{diag}(\epsilon, \epsilon I_3, I_3, I_3) O(1) \in C([0,\infty), H^{s-2}).$$

Thus $R(\cdot,t)\|_{s-2} = \epsilon^3 O(1)$. In conclusion, from Theorem 5.1 and Corollary 1 of Theorem 2.2 in [13] we have

Corollary 6.4. *Let $s \geq 3$ be an integer. Assume $W_{0\epsilon} = W_\epsilon(\cdot, 0) \in H^{s+2}$. Then for any $T > 0$ there is $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, the ABI system (2.6) with initial data $W_{0\epsilon}$ has a unique classical solution*

$$W^\epsilon = (\tau^\epsilon, v^\epsilon, b^\epsilon, d^\epsilon) \in C([0,T], H^{s+2}).$$

Moreover, the error estimates

$$\|(\tau^\epsilon - 1 - \epsilon^2\tau_1, v^\epsilon - \epsilon^2 v_0, b^\epsilon - \epsilon b_0, d^\epsilon - \epsilon d_0)(\cdot,t)\|_s \leq K\epsilon^3,$$

for $t \in [0,T]$, hold with K a constant independent of ϵ but dependent on T.

Appendix: A Continuation Principle for Singular Limit Problems

For the convenience of the reader, we present in this appendix the convergence-stability lemma previously formulated[1] by the second author in [17] for IVPs of quasilinear symmetrizable hyperbolic systems depending (singularly) on parameters:

$$U_t + \sum_{j=1}^d A_j(U,\epsilon) U_{x_j} = Q(U,\epsilon) \tag{6.1}$$

for $x \in \Omega = \mathbb{R}^d$ or $\mathbb{T}^d$ (the d-dimensional torus). Here ϵ represents a parameter in a *topological space*, $A_j(U,\epsilon)(j = 1, 2, \cdots, d)$ and $Q(U,\epsilon)$ are (matrix- or vector-valued) smooth functions of $U \in G \subset \mathbb{R}^n$ (state space) for each ϵ (possible) different from a certain singular point, say 0.

For each fixed $\epsilon(\neq 0)$, consider the IVP of (6.1) with initial data $\bar{U}(x,\epsilon)$. Assume $\bar{U}(x,\epsilon) \in G_0 \subset\subset G$ for all $x \in \Omega$ and $\bar{U}(\cdot,\epsilon) \in H^s$ with $s > d/2 + 1$ an integer. Let G_1 be a subset of the state space and satisfy $G_0 \subset\subset G_1$ (see (6.3) below). According to the local-in-time existence theory for IVPs of symmetrizable hyperbolic systems (see Theorem 2.1 in [13]), there exists $T > 0$ so that (6.1) with initial data $\bar{U}(x,\epsilon)$ has a unique classical solution

$$U^\epsilon \in C([0,T], H^s) \qquad \text{and} \qquad U^\epsilon(x,t) \in G_1 \quad \forall (x,t) \in \Omega \times [0,T].$$

Define

$$T_\epsilon = \sup\big\{T > 0 : U^\epsilon \in C([0,T], H^s) \quad \text{and} \quad U^\epsilon(x,t) \in G_1 \quad \forall (x,t) \in \Omega \times [0,T]\big\}. \tag{6.2}$$

Namely, $[0, T_\epsilon)$ is the maximal time interval for the existence of H^s-solutions with values in G_1. Note that $T_\epsilon = T_\epsilon(G_1)$ depends on G_1 and may tend to zero as ϵ approaches to the singular point 0.

In order to show that $\underline{\lim}_{\epsilon\to 0} T_\epsilon > 0$, we make the following

[1] Yong thanks Heinrich Freistühler and Denis Serre for pointing out two errors in an earlier version of this formulation. Thanks also go to Ya-Guang Wang for valuable discussions which improve the presentation.

<u>*Convergence Assumption*</u>: there exists $T_* > 0$ and $U_\epsilon = U_\epsilon(x,t)$ defined for $(x,t) \in \Omega \times [0, T_*]$ and $\epsilon (\neq 0)$, satisfying

$$\bigcup_{x,t,\epsilon} \{U_\epsilon(x,t)\} \subset\subset G, \qquad U_\epsilon(\cdot,t) \in H^s \qquad \text{and} \qquad \sup_{t,\epsilon} \|U_\epsilon(\cdot,t)\|_s < \infty,$$

such that for $t \in [0, \min\{T_*, T_\epsilon\})$,

$$\sup_{x,t} |U^\epsilon(x,t) - U_\epsilon(x,t)| = o(1),$$
$$\sup_{t} \|U^\epsilon(\cdot,t) - U_\epsilon(\cdot,t)\|_s = O(1)$$

as ϵ goes to the singular point.

Under this assumption, we slightly modify the argument in [16] to prove

Lemma 6.5. *Suppose $\bar{U}(x,\epsilon) \in G_0 \subset\subset G$ for all $x \in \Omega$ and $\epsilon(\neq 0)$, $\bar{U}(\cdot,\epsilon) \in H^s$ with $s > d/2+1$ an integer, and the convergence assumption holds. Then, for each G_1 satisfying*

$$G_0 \bigcup_{x,t,\epsilon} \{U_\epsilon(x,t)\} \subset\subset G_1 \subset G, \tag{6.3}$$

there is a neighborhood of the singular point such that

$$T_\epsilon(G_1) > T_*$$

for all ϵ in the neighborhood.

Proof. Otherwise, there is a G_1 satisfying (6.3) and a sequence $\{\epsilon_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and $T_{\epsilon_k} = T_{\epsilon_k}(G_1) \leq T_*$. Thanks to (6.3) and the convergence assumption, there exists $\tilde{G}$, satisfying $\bigcup_{x,t,\epsilon}\{U_\epsilon(x,t)\} \subset\subset \tilde{G} \subset\subset G_1$, and a certain k such that $U^{\epsilon_k}(x,t) \in \tilde{G}$ for all $(x,t) \in \Omega \times [0, T_{\epsilon_k})$. On the other hand, we deduce from

$$\|U^\epsilon(\cdot,t)\|_s \leq \|U^\epsilon(\cdot,t) - U_\epsilon(\cdot,t)\|_s + \|U_\epsilon(\cdot,t)\|_s$$

and the convergence assumption that $\|U^{\epsilon_k}(\cdot,t)\|_s$ is bounded uniformly with respect to $t \in [0, T_{\epsilon_k})$. Now we could apply Theorem 2.1 in [13], beginning at a time t less than T_{ϵ_k} (k is fixed here!), to continue the solution beyond $T_{\epsilon_k}(G_1)$. This contradicts the definition of $T_{\epsilon_k}(G_1)$ in (6.2) and, hence, the proof is complete. □

To our knowledge, such a sharp continuation principle has not appeared explicitly in the published literature other than [17]. Thanks to this lemma, the study of the singular limit problems is reduced to find a $U_\epsilon(x,t)$ such that the convergence assumption holds. In verifying the two error estimates in the convergence assumption, we often take G_1 satisfying $G_1 \subset\subset G$ and being convex. Furthermore, we notice that, in the time interval $[0, \min\{T_*, T_\epsilon\})$, both U^ϵ and U_ϵ are regular and take values in the precompact subset G_1.

Remark 6.2. Similar lemmas can be easily formulated for other evolution differential equations. In fact, such a lemma can be regarded as a part of the local-in-time existence theory of any evolution equations.

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