EVOLUTION EQUATIONS AND GENERALIZED FOURIER INTEGRAL OPERATORS

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Introduction

It is well known that, given a classical selfadjoint pseudodifferential operator of order 1, one can define the strongly continuous group (P_t) of unitary operators, such that $u_t = P_t u_0$ gives the solution of the hyperbolic equation $\frac{\partial}{\partial t} + iAu = 0$ with Cauchy data u_0. Moreover, the operators $P_t = e^{-itA}$ are classical Fourier integral operators associated to the canonical transformations F_t, where $(t, x) \mapsto F_t(x)$ is the flow of the Hamiltonian field $H_a = (\partial a/\partial \xi_j; -\partial a/\partial x_j)$ and a is the principal symbol of A. In particular, we have the following two properties :
— The operators P_t are bounded from the Sobolev space H^s into itself.
— The conjugate $P_{-t}BP_t$ of a classical pseudodifferential operator B (with principal symbol b) is a classical pseudodifferential, whose principal symbol is $b \circ F_t^{-1}$.

It turns out that, for extending this theory to more general evolution equation such as Schrödinger type equations, one has just to modify the properties above. Let us consider for instance the harmonic oscillator $A = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$. The group of unitary operators $P_t = e^{-itA}$ is well known and, in particular, for $t = \pi/2$, P_t is, up to some factor, the Fourier transformation while the canonical transformation (still associated to the Hamiltonian flow of the principal symbol) becomes $F_t(x, \xi) = (\xi, -x)$. One has the corresponding properties :
— P_t maps the Sobolev spaces H^s into weighted L^2 spaces.
— The symbols of B and of its conjugate $\widetilde{B} = P_{-t}BP_t$ are still related by $\widetilde{b} = b \circ F_t^{-1}$ but, if b is an symbol of order m satisfying the standard estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta b\right| \leq C^{\text{st}}(1+|\xi|)^{m-|\alpha|},$$

the symbol $\widetilde{b}$ satisfies the exotic ones

$$\left|\partial_\xi^\alpha \partial_x^\beta \widetilde{b}\right| \leq C^{\text{st}}(1+|x|)^{m-|\beta|}.$$

Such $\widetilde{b}$ can be considered as symbols of (generalized) pseudodifferential symbols if we use the Weyl-Hörmander calculus.

In this theory, many different pseudodifferential calculi are defined, each of which is associated to a "good" Riemannian metric g on the phase space $\mathcal{X} = \mathbb{R}^n \times (\mathbb{R}^n)^*$. Moreover, to any "good" positive function

M on $\mathcal{X}$, one can associate a generalized "Sobolev space" $H(M,g)$. In the example above, the P_t maps the usual Sobolev spaces into unusual ones (the weighted L^2 spaces), and conjugates of usual pseudodifferential operators are unusual ones.

We will use systematically the Weyl-Hörmander calculus and, in order to generalize Fourier integral operators and hyperbolic equations, we have to study two problems.

1. Are given a canonical transformation F (symplectic diffeomorphism) of $\mathcal{X}$ onto itself, an initial calculus (defined by a Riemannian metric g) and a final calculus (defined by $\widetilde{g}$). One can then define (under convenient assumptions) a class $\mathrm{FIO}(F,g,\widetilde{g})$ of operators whose main property is the following : conjugates of g-pseudodifferential operators are $\widetilde{g}$-pseudodifferential ones. These generalized Fourier integral operators have good properties (composition, boundedness in generalized Sobolev spaces) and enjoy a symbolic calculus. This has been developped in [Bo3] and is recalled in section 2.

2. Are given an evolution equation $\frac{\partial}{\partial t}+iAu=0$ and an initial calculus (defined by a metric g_0). Then, one can expect that the propagators P_t exist and belong to $\mathrm{FIO}(F_t,g_0,g_t)$. The calculus at time t depends on t and is actually forced by the Hamiltonian flow. Theorems 3.1 and 3.2 give sufficient conditions (on the symbol a and its Hamiltonian flow F_t) for getting such results. Proofs will be sketched in sections 4 and 5.

Our assumption are exclusively expressed in terms of differential geometry, starting from the symbol a of A. In particular, no selfadjoint extension in L^2 is a priori given and an important part of the task is to deduce from the dynamic assumption on a that A is essentially selfadjoint. One can see easily that these assumptions are grosso modo necessary if one wants to fulfil the program above. However, they are not so easy to check : they require estimates which may be touchy, not only on a but also on its Hamiltonian flow.

1. Weyl-Hörmander calculus of pseudodifferential operators

We refer to [Hö, §§18.5, 18.6] but we will need some results from [B-L], [B-C], [Bo1] and [Bo2].

1.1. **Quantization.** We will denote by $X=(x,\xi)$ a point of the *phase space* $\mathcal{X}=\mathbb{R}^n\times(\mathbb{R}^n)^*$. The *symplectic form* σ on $\mathcal{X}$ is defined by

$$\sigma(X,Y)=\langle\xi,y\rangle-\langle\eta,x\rangle\,;\qquad X=(x,\xi),\ Y=(y,\eta).$$

For $a(x,\xi)$ belonging to the Schwartz space $\mathcal{S}(\mathcal{X})$, the operator $a^w(x,D)$, or a^w for short, is defined by

$$a^w(x,D)u(x)=\iint e^{i(x-y)\cdot\xi}\,a\left(\tfrac{x+y}{2},\xi\right)u(y)\,\tfrac{dy\,d\xi}{(2\pi)^n}. \tag{1}$$

Such an operator maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. If now a belongs to the space $\mathcal{S}'(\mathcal{X})$ of tempered distributions on $\mathcal{X}$, the same formula, taken in the weak sense, defines an operator mapping $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. One says that a is the Weyl symbol of a^w.

The product of composition of two symbols a and b (belonging say to $\mathcal{S}(\mathcal{X})$, but this will be widely extended) is defined by $(a\#b)^w = a^w \circ b^w$ and is given by the formula

$$a\#b(X) = \iint e^{-2i\sigma(X-S,X-T)} a(S)b(T) \tfrac{dS\,dT}{\pi^{2n}}. \tag{2}$$

The following expansion is given here with a remainder of order 3, which is sufficient for our purpose, but it exists at any order.

$$a\#b = ab + \tfrac{1}{2i}\{a\,,\,b\} + \tfrac{1}{2}\left(\tfrac{1}{2i}\sigma(\partial_Y,\partial_Z)\right)^2 a(Y)b(Z)|_{Y=Z=X} + R_3(a,b). \tag{3}$$

Here, $\{a\,,\,b\}$ is the usual Poisson bracket in $\mathcal{X}$. There is an integral formula, more or less similar to (2) and for which we refer to [Bo2], giving the value of $R_3(a,b)$. An important point is that it depends only on the derivatives of order 3 of a and b.

1.2. **Admissible metrics.** A Riemannian metric g on the phase space is identified to a family $Y \mapsto g_Y$ of positive definite quadratic forms on $\mathcal{X}$. For each Y, one can choose symplectic coordinates (depending on Y but still denoted by (x,ξ)) such that g_Y is diagonalized :

$$g_Y(dx,d\xi) = \sum_1^n \frac{dx_j^2}{a_j^2} + \sum_1^n \frac{d\xi_j^2}{\alpha_j^2}. \tag{4}$$

The a_j and α_j depend on the choice of the coordinates, but the products $a_j\alpha_j$ depend just on Y.

Such a metric g is said *admissible* if the following 5 properties are satisfied.

A1. Simplifying assumption. — The products $a_j\alpha_j$ above are equal and their common value is denoted by $\lambda(Y)$. This means that there is a (linear) symplectic transformation mapping the unit ball $B_Y = \{X|\, g_Y(X-Y) \le 1\}$ onto the euclidean ball of radius $\sqrt{\lambda(Y)}$. One has

$$|\sigma(S,T)| \le \lambda(Y) g_Y(S)^{1/2} g_Y(T)^{1/2}.$$

A2. Fundamental assumption. — $\forall Y, \quad \lambda(Y) \ge 1.$
This means that localising in unit balls is not a violation of the uncertainty principle.

A3. Slowness. — There exists $C > 0$ such that

$$g_Y(Y-Z) \le C^{-1} \Longrightarrow \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \le C$$

uniformly with respect to Y, Z, T.

A4. Temperance. — There exist C and N such that

$$\forall Y, \forall Z, \forall T, \quad \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \le C\big(1 + \lambda(Y)^2 g_Y(Y-Z)\big)^N.$$

A5. Geodesic temperance. — The geodesic distance $D(Y,Z)$ *for the Riemannian metric* $\lambda(Y)^2 g_Y(dx,d\xi)$ is equivalent to $\lambda(Y)g_Y(Y-Z)^{1/2}$ in the following sense :

$$\exists C, \exists N, \forall Y, \forall Z$$
$$C^{-1}(1+D(Y,Z))^{1/N} \leq 1+\lambda(Y)g_Y(Y-Z)^{1/2} \leq C(1+D(Y,Z))^N.$$

In view of A4, this property is equivalent to

$$\exists C, \exists N, \forall Y, \forall Z, \forall T, \quad \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \leq C\big(1+D(Y,Z)\big)^N.$$

Remark. The first assumption A1 makes things simpler, for instance it is not necessary to introduce the inverse metric g^σ which in this case is just $\lambda^2 g$, but it is not necessary. On the contrary, the geodesic temperance plays an important rôle : thanks to A5, one has a simple characterization of pseudodifferential operators (see n°1.4), one can define very easily the Fourier integral operators and thus prove in a few lines our theorem 3.2.

It could be possible to define the Fourier integral operators without A5, using localized twisted commutators (as in [B-C, th. 5.5]), but the proofs are much more complicated. Moreover, there is no known example of a metric satisfying A4 and not A5.

1.3. **Weights and symbols.** A positive function M defined on $\mathcal{X}$ is a *g-weight* if it satisfies the following conditions (slowness and temperance), for convenient constants C' andN'.

$$g_Y(Y-Z) \leq C'^{-1} \Longrightarrow \big(M(Y)/M(Z)\big)^{\pm 1} \leq C'$$
$$\big(M(Y)/M(Z)\big)^{\pm 1} \leq C'\big(1+\lambda(Y)^2 g_Y(Y-Z)\big)^{N'}.$$

Modifying the constants if necessary, $\big(1+\lambda(Y)^2 g_Y(Y-Z)\big)$ can be replaced above by $\big(1+D(Y,Z)\big)$.

The *classes of symbols* $S(M,g)$ (for admissible metrics and g-weights) are defined as the set of functions $a \in C^\infty(\mathcal{X})$ such that

$$|\partial_{T_1} \dots \partial_{T_k} a(X)| \leq C_k M(X) \quad \text{for } g_X(T_j) \leq 1. \tag{5}$$

Here, $\partial_T a = \langle T, da\rangle$ denotes the directional derivative along T. The space of operators a^w for $a \in S(M,g)$ (the pseudodifferential operators of weight M) is denoted by $\Psi(M,g)$. The following properties are now well known.

- $\Psi(M,g) \subset \mathcal{L}(\mathcal{S},\mathcal{S})$ et $\Psi(M,g) \subset \mathcal{L}(\mathcal{S}',\mathcal{S}')$.
- $\Psi(1,g) \subset \mathcal{L}(L^2,L^2)$.
- In the expansion (3), for $a \in S(M_1,g)$ and $b \in S(M_2,g)$, one has

$$a\#b \text{ and } ab \in S(M_1M_2,g)$$
$$\{a\,,\,b\}\,,\ (a\#b-ab) \text{ and}(a\#b-b\#a) \in S(M_1M_2\lambda^{-1},g) \tag{6}$$
$$R_3(a,b) \in S(M_1M_2\lambda^{-3},g). \tag{7}$$

Let us recall some complements which are proved in [Bo2].

Proposition 1.1. *The classes de symbols* $\hat{S}\ (M,g)$ [resp. $\hat{\hat{S}}\ (M,g)$, $\hat{\hat{\hat{S}}}\ (M,g)$] *are defined as the spaces of functions satisfying* (5) *for* $k \geq 1$ [resp. $k \geq 2$, $k \geq 3$].
(a) *There exist a weight* M' *dépending on* M *such that* $\hat{\hat{\hat{S}}}\ (M,g) \subset S(M',g)$.
(b) *The properties* (6) *are still valid for* $a \in \hat{S}\ (M_1,g)$ *and* $b \in \hat{S}\ (M_2,g)$.
(c) *The property* (7) *is still valid for* $a \in \hat{\hat{\hat{S}}}\ (M_1,g)$ *and* $b \in \hat{\hat{\hat{S}}}\ (M_2,g)$.

The semi-norms of the spaces $S(M,g)$, $\hat{S}\ (M,g)$, ... are the best constants C_k in (5). The $S(M,g)$ are Fréchet spaces, the $\hat{S}\ (M,g)$, ... are complete but not Hausdorf.

1.4. **Characterization of pseudodifferential operators.**

Theorem 1.2. (a) *Given* $b \in \hat{S}\ (\lambda,g)$ *and* $A \in \Psi(M,g)$*, one has*

$$\operatorname{ad} b^w \cdot A \stackrel{\text{def}}{=} b^w A - A\, b^w \in \Psi(M,g).$$

When $M = 1$*, this operator is thus bounded on* L^2*.*

(b) *Conversely, let* A *be an operator which is bounded on* $L2$ *as well as its iterated commutators*

$$\operatorname{ad} b_1^w \dots \operatorname{ad} b_k^w \cdot A \quad \textit{for } b_j \in \hat{S}\ (\lambda,g).$$

Then A *belongs to* $\Psi(1,g)$*.*

The first part is an immediate consequence of the proposition 1.1 (b). For the converse, we refer to [Bo1] where the geodesic temperance plays a decisive rôle.

Generalized Sobolev spaces $H(M,g)$**.** — We refer to [B-C] for equivalent definitions ; the following properties will be sufficient

- For any g-weight M, there exist $A \in \Psi(M,g)$ and $B \in \Psi(M^{-1},g)$ such that $AB = BA = I$.
- The *Sobolev space* $H(M,g)$ (sometimes denoted $H(M)$ for short), is the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $Au \in L^2$ for any $A \in \Psi(M,g)$. It is sufficient that $Au \in L^2$ for one invertible A as above, and one can choose $\|u\|_{H(M)} = \|Au\|_{L^2}$.
- For any g-weights M and M_1, any $A \in \Psi(M,g)$ maps continuously $H(M_1)$ into $H(M_1/M)$.
- If $A \in \Psi(M,g)$ est bijective from $H(M_1)$ onto $H(M_1/M)$ for some g-weight M_1, then A^{-1} belongs to $\Psi(M^{-1},g)$.

2. Generalized Fourier integral operators

We recall here some of the definitions and results of [Bo3]. We consider only Fourier integral operators P of weught 1 (or of order 0, i.e.

bounded on L^2), Fourier integral operators of weight M being just products PA of such P with $A \in \Psi(M, g)$.

An *admissible triple* $(F, g, \widetilde{g})$ is made of a diffeomorphism F of $\mathcal{X}$ onto itself, and of two Riemannian metrics g and $\widetilde{g}$, such that the four following conditions are satisfied.

B1. — F is a canonical transformation (or symplectomorphism), which means that $F_*\sigma = \sigma$. For any $Y \in \mathcal{X}$, the differential $F'(Y)$ belongs to the symplectic group $\mathrm{Sp}(n)$.

B2. — F is an isometry of $(\mathcal{X}, g)$ onto $(\mathcal{X}, \widetilde{g})$. This means that $\widetilde{g}$ is the direct image F_*g of g, i.e. the Riemannian metric defined by

$$\widetilde{g}_{F(Y)}(T) = g_Y(F'(Y)^{-1} \cdot T).$$

B3. — g and $\widetilde{g}$ are admissible metrics, satisfying conditions A1 to A5.

B4. — One has the following estimates on the derivatives of F, for convenient constants C_k :

$$\widetilde{g}_{F(X)}\big(\partial_{T_1} \dots \partial_{T_k} F(X)\big) \leq C_k \quad \text{for } g_X(T_j) \leq 1. \tag{8}$$

Remark. In most applications, the canonical transformation F and an admissible metric g are given and $\widetilde{g}$ is thus determined by B2. The problem is to know and to prove that $\widetilde{g}$ is also admissible. It is easy to see that A1 and A2 are satisfied and that the slowness A3 is a consequence of B4 for $k = 2$, but the temperance is touchy.

It cannot be expressed simply in terms of F and g because it mixes up the symplectic and Riemannian structures (which are preserved by F) and the affine structure (which is not). One has to compare the values of the quadratic forms $\widetilde{g}_Y$ and $\widetilde{g}_Z$ for *the same vector* T in two points which can be very far, and this requires a good knowledge of the behaviour of F at infinity.

For g and $\widetilde{g}$, the functions defined in the n°1.2 are denoted by λ and $\widetilde{\lambda}$. The quadratic forms g_Y and $\widetilde{g}_{F(Y)}$ being symplectically equivalent, one has $\widetilde{\lambda}(F(Y)) = \lambda(Y)$.

The condition B4 for $k = 1$ is automatically satisfied (with $C_1 = 1$) for F and F^{-1}. A simple computation shows that the conditions B4 (for $k > 1$) are also valid for F^{-1} which imply that the triple $(F^{-1}, \widetilde{g}, g)$ is also admissible.

The condition B4 is actually equivalent to the following properties, which are of course essential for our purpose.

Proposition 2.1. *If m is a g-weight, then $\widetilde{m} = m \circ F^{-1}$ is a $\widetilde{g}$-weight and one has*

$$a \circ F^{-1} \in S(\widetilde{m}, \widetilde{g}) \iff a \in S(m, g)$$

$$a \circ F^{-1} \in \hat{S}\,(\widetilde{m}, \widetilde{g}) \iff a \in \hat{S}\,(m, g)$$

There is no analogous result for $a \in \hat{\hat{S}}\,(m, g)$: an estimate of the second derivatives of $a \circ F^{-1}$ requires an estimate of the first derivatives of a.

Definition 2.2 (Fourier integral operators and twisted commutators)**.** *The space* $\mathrm{FIO}(F, g, \widetilde{g})$ *of Fourier integral operators associated to the admissible triple* $(F, g, \widetilde{g})$ *is the set of operators* P *such that*

$$\widetilde{\mathrm{ad}}(b_1) \dots \widetilde{\mathrm{ad}}(b_k) \cdot P \in \mathcal{L}(L^2) \quad \textit{for } b_j \in \hat{S}\,(\lambda, g), \tag{9}$$

where $\widetilde{\mathrm{ad}}(b) \cdot P$ *is a notation for the twisted commutator :*

$$\widetilde{\mathrm{ad}}(b) \cdot P = (b \circ F^{-1})^w P - P b^w. \tag{10}$$

This definition is of course modelled on the characteristic property of pseudodifferential operators given in theorem 1.2. It implies easily the following properties

- $\mathrm{FIO}(I, g, g) = S(1, g)$
- For $P \in \mathrm{FIO}(F, g, \widetilde{g})$, its adjoint P^* belongs to $\mathrm{FIO}(F^{-1}, \widetilde{g}, g)$.
- For $P \in \mathrm{FIO}(F, g, \widetilde{g})$ and $Q \in \mathrm{FIO}(G, \widetilde{g}, \overline{g})$, where $(F, g, \widetilde{g})$ and $(G, \widetilde{g}, \overline{g})$ are two admissible triples, one has $QP \in \mathrm{FIO}(G \circ F, g, \overline{g})$.

For proving the existence of non trivial Fourier integral operators, a more concrete definition is necessary.

2.1. **Principal symbol of Fourier integral operators.** Let Γ be the graph of F and for each point $(Y, F(Y)) \in \Gamma$, let χ_Y the affine tangent map, defined by $\chi_Y(X) = Y + F'(Y) \cdot (X - Y)$. One can define a fiber bundle $\widetilde{\Gamma} \to \Gamma$ such that its fiber at $(Y, F(Y))$ is made of the metaplectic operators V associated to χ_Y, i.e. such that $a^w V = V (a \circ \chi_Y)^w$ for any symbol a. Such a V is determined by χ_Y up to multiplication by a complex number $\omega \in U(1)$ and the fiber is thus a circle.

We refer to [Bo3] for the definition of the *horizontal sections* $Y \mapsto V_Y$ of $\widetilde{\Gamma}$ as well as for the construction of a refined partition of unity $Y \mapsto \psi_Y$. The following result is the theorem 6.6 of [Bo3]

Theorem 2.3. (i) *For such* V_Y *and* ψ_Y *and for* $p \in S(1, g)$*, the following integral*

$$P_1 = \int p(Y)\, V_Y \circ \psi_Y^w \, \tfrac{dY}{\pi^n} \tag{11}$$

defines an element of $\mathrm{FIO}(F, g, \widetilde{g})$.
(ii) *Conversely, any* $P \in \mathrm{FIO}(F, g, \widetilde{g})$ *can be written* $P = P_1 + R$*, with* P_1 *as above and* R *a regularizing Fourier integral operator, i.e. such that*

$$\forall N, \quad (\widetilde{\lambda}^w)^N \circ R \circ (\lambda^w)^N \in \mathrm{FIO}(F, g, \widetilde{g}) \tag{12}$$

(iii) *The section* $(Y, F(Y)) \mapsto p(Y) V_Y$ *of the line bundle* $\widetilde{\Gamma} \otimes_{U(1)} \mathbb{C}$ *is said to be a principal symbol of* P*. The principal symbol of* P *is unique, up to a symbol* $(Y, F(Y)) \mapsto q(Y) V_Y$ *with* $q \in S(\lambda^{-1}, g)$

A principal symbol for the adjoint P^* is $(F(Y), Y) \mapsto \overline{p(Y)} V_Y^*$. With evident notations, for $Q \in \mathrm{FIO}(G, \widetilde{g}, \overline{g})$, a principal symbol of $Q \circ P$ is the section $(Y, G \circ F(Y)) \mapsto p(Y) q(F(Y))\, W_{F(Y)} \circ V_Y$. Thanks to part (i) of the theorem, there exist almost invertible Fourier integral operators.

3. Evolution equations

Let a be a real valued and C^∞ function on $\mathcal{X}$ symbol, belonging to a class of symbols which will be precised later, let g_0 be an admissible metric and $T > 0$. We make the following assumptions

C1. — The flow F_t of the hamiltonian field of a is global : it is defined for all $t \in \mathbb{R}$ par $\frac{d}{dt} F_t(X) = H_a(F_t(X))\,;\ F_0(X) = X$. Set $g_t = F_{t*} g_0$.

C2. — The metrics g_t satisfy A1,..., A5 for any $t \in [-T, T]$, with uniform constants.

C3. — The triples (F_t, g_0, g_t) satisfy B1,..., B4 for any $t \in [-T, T]$, with uniform constants.

The group law $F_{t+s} = F_t \circ F_s$, imply that the triples (F_t, g_s, g_{s+t}) are admissible when s and $s+t$ belong to $[-T, T]$.

The "function λ" defined in the n°1.2 corresponding to g_t will be doneted by λ_t. One has $\lambda_t = \lambda_0 \circ F_t^{-1}$. For any g_0-weight μ_0, we will denote by μ_* the family of g_t-weight $\mu_t = \mu_0 \circ F_t^{-1};\ t \in [-T, T]$.

Theorem 3.1. *Assume that a belongs uniformly to $\overset{\wedge\wedge\wedge}{S}(\lambda_t^3, g_t)$ (i.e. the k^{th} semi-norm of a in these spaces is bounded by a constant C_k independent on t). Then*
(i) *The operator a^w with domain $\mathcal{S}(\mathbb{R}^n)$ is essentially selfadjoint on L^2. The domain of its closure A is $\{u \in L^2 |\, a^w u \in L^2\}$, which means that weak and strong extension coincide.*
(ii) *A is thus the infinitesimal generator of a one parameter strongly continuous group $P_t = e^{-itA}$. For any g_0-weight μ_0 and for $|t| \le T$, the operator P_t is bounded from $H(\mu_0, g_0)$ onto $H(\mu_t, g_t)$.*

The assumption on a is satisfied when $a \in \hat{S}(\lambda_0^3, g_0)$ but it is not sufficient in general that $a \in \overset{\wedge\wedge\wedge}{S}(\lambda_0^3, g_0)$. For the same reason, it is sufficient to assume $a \in \hat{S}(\lambda_0^2, g_0)$ in the next theorem.

Theorem 3.2. *Assume now that a belongs uniformly to $\overset{\wedge\wedge\wedge}{S}(\lambda_t^2, g_t)$. Then P_t belongs to $\mathrm{FIO}(F, g_0, g_t)$ for $|t| \le T$.*

Remark. The meaning of the condition $a \in \overset{\wedge\wedge\wedge}{S}(\lambda_0^2, g_0)$ depends strongly on the choice of the initial metric g_0. For instance, for the standard metric $dx^2 + \frac{d\xi^2}{1+|\xi|^2}$, terms like $|\xi|^2 \log|\xi|$ or x^3 are allowed. If g_0 is the euclidean metric, any polynomial of total degree 3 (in x and ξ) belongs to $\overset{\wedge\wedge\wedge}{S}(1, g_0)$.

It is clear on these examples that the assumption on the class of a cannot imply the global character of the flow nor the essential selfadjointness of a^w. The dynamic assumption C1 is crucial.

4. Proof of theorem 3.1

Let us write $A = a^w$. If we think of the equation $\frac{d}{dt}u_t + iAu_t = f_t$ as a Schrödinger equation, the associated "Heisenberg equation" is $\frac{d}{dt}B_t = i(B_tA - AB_t)$. It turns out that our dynamic assumptions give immediately approximate solutions of this last equation, which will give a priori estimates.

Let b_0 a symbol for the metric g_0 whose weight will be specified later, and $b_t = b_0 \circ F_t^{-1}$. We have $\frac{\partial}{\partial t}b_t = \{b_t\,,\,a\}$ and thus, according to (3)

$$b_t \# a = ab_t + \tfrac{1}{2i}\{b_t\,,\,a\} + \text{order } 2 + R_3(b_t, a)$$
$$a \# b_t = b_t a + \tfrac{1}{2i}\{a\,,\,b_t\} + \text{order } 2 + R_3(a, b_t)$$

Terms of order 0 and 2 are symmetric in a and b, and thus

$$\tfrac{d}{dt}B_t = i(B_tA - AB_t) + R_t \tag{13}$$

where the symbol of R_t belongs to the same class as $R_3(a, b_t)$. As a consequence of the proposition 1.1 (c), under the assumptions of the theorem 3.1, and for $b_0 \in S(\mu_0, g_0)$ (or $b_0 \in \hat{S}\ (\mu_0, g_0)$), one has $R_t \in \Psi(\mu_t, g_t)$.

We have to define the spaces $L^p([-T, T]; H(\mu_*))$ made of (classes) of measurable functions $u : t \mapsto u_t$ (the weak neasurability with values in $\mathcal{S}'$ is sufficient) such that

$$\|u\|_{L^p(H(\mu_*))} = \left(\int_{-T}^{T} \|u_t\|^p_{H(\mu_t)}\,dt\right)^{1/p} < \infty, \tag{14}$$

with the usual convention for $p = \infty$. This definition is meaningful if we define the norms of the spaces $H(\mu_t, g_t)$ in a coherent way. This can be specified thanks to the following proposition.

Proposition 4.1. *Let μ_0 be a g_0-weight. There exist $\delta > 0$ and for each $\theta \in [-T, T]$ a $b_\theta \in S(\mu_\theta, g_\theta)$ such that, for $|s| \le \delta$, the operators $(b_\theta \circ F_s^{-1})^w$ have an inverse belonging to $S(\mu_{\theta+s}, g_{\theta+s})$.*

Choosing a finite number of points θ, each t can be written $\theta + s$ and we can choose $\|u\|_{H(\mu_t)} = \|(b_\theta \circ F_s^{-1})^w u\|_{L^2}$ in (14). Changing the points θ and the b_θ would replace the norm in $L^p(H(\mu_*))$ by an equivalent one.

We should verify, in the proof of [B-C, th. 6.4], that we can choose $b_\theta \in S(\mu_\theta, g_\theta)$ and $c_\theta \in S(\mu_\theta^{-1}, g_\theta)$ whose semi-norms are independant of θ such that $b_\theta \# c_\theta = 1$. We are thus reduced to prove the result for $\theta = 0$. With evident notations for c_s and C_s, we get

$$\tfrac{d}{ds}B_sC_s = i(B_sC_sA - AB_sC_s) + R_s$$

where R_s belongs to $\Psi(1, g_s)$ (with uniform semi-norms). Setting $e_s = (b_s \# c_s) \circ F_s$, we get an equation $\frac{d}{ds} e_s = r'_s$ with a right hand side bounded uniformly in $S(1, g_0)$. For s small, the semi-norms de $(1 - e_s)$ in $S(1, g_0)$ and thus those of $(1 - b_s \# c_s)$ in $S(1, g_s)$ are small. As a consequence, $B_s C_s$ is invertible in $\mathcal{L}(L^2)$, its inverse belongs to $\Psi(1, g_s)$ (see n°1.4), and B_s itself is invertible.

The functions of class C^∞ are dense in $L^1(H(\mu_*))$ and the dual of this space is $L^\infty(H(\mu_*^{-1}))$. The space $C(H(\mu_*))$ ("continuous" functions with values in a variable space!) is defined as the closure, in $L^\infty(H(\mu_*))$, of the set of continuous functions with value in $\mathcal{S}$.

Proposition 4.2. *Let μ_0 a g_0-weight.*
(a) *There exists C such that, for any $u \in C^1([-T, T], \mathcal{S})$ solution of the equation*

$$\frac{d}{dt} u_t + iAu_t = f_t \tag{15}$$

one has

$$\|u\|_{L^\infty(H(\mu_*))} \leq C \left(\|u_0\|_{H(\mu_0)} + \|f\|_{L^1(H(\mu_*))} \right). \tag{16}$$

(b) *There exist $\overline{\mu}_0 > \mu_0$ such that any solution u of* (15) *which belongs to $L^\infty(H(\overline{\mu}_*))$ belongs to $C(H(\mu_*))$ and satisfy the estimate* (16).

It suffices to prove the result on an interval of size δ centered at 0. Keeping the notations above, we have

$$\begin{aligned} \frac{d}{dt}(B_t u_t) = iB_t A u_t - iAB_t u_t + R_t u_t - iB_t A u_t + B_t f_t \\ = -iA(B_t u_t) + R_t u_t + B_t f_t \end{aligned}$$

and thus

$$\frac{d}{dt} \|u_t\|^2_{H(\mu_t)} = \frac{d}{dt} \|B_t u_t\|^2_{L^2} \leq C \left(\|u_t\|^2_{H(\mu_t)} + \|u_t\|_{H(\mu_t)} \|f_t\|_{H(\mu_t)} \right)$$

which proves the part (a) of the theorem for δ small.

For proving the part (b), we need the following lemma, where μ_0 and $\overline{\mu}_0$ will be g_0-weights, and where $\mathcal{H}^N$ is the classical weighted Sobolev space $\left\{ u \,\middle|\, x^\alpha D^\beta u \in L^2 \,;\, |\alpha + \beta| \leq N \right\}$ for $N \geq 0$, and is the dual of $\mathcal{H}^{-N}$ for $N < 0$.

Lemma 4.3.

$$\begin{aligned} &\forall \mu_0,\ \exists N,\ \exists C,\ \forall t \in [-T, T], \qquad \|u\|_{H(\mu_t)} \leq C \|u\|_{\mathcal{H}^N}. \\ &\forall N,\ \exists \overline{\mu}_0,\ \exists C,\ \forall t \in [-T, T], \qquad \|u\|_{\mathcal{H}^N} \leq C \|u\|_{H(\overline{\mu}_t)}. \end{aligned}$$

For a fixed t, the first estimate says that pseudodifferential operators are bounded from $\mathcal{S}$ into L^2, while the second one is a consequence of the fact that any linear form on $\mathcal{X}$ belongs to a class of symbol for a convenient weight. One has just to make uniform these arguments.

Let us go back to part (b) of the proposition 4.2. We know (proposition (1.1) (a)) that there exist a weight m_0 such that $a \in S(m_0, g_0)$. We have also $a \in S(m_t, g_t)$ because $a \circ F_t^{-1} = a$. We can choose $\overline{\mu}_0$

sufficiently large, such that $H(m_t\mu_t) \supset \mathcal{H}^N \supset H(\overline{\mu}_t/m_t)$. We know then that $du/dt \in L^\infty(\mathcal{H}^N)$ and u is continuous with values in $\mathcal{H}^N$. It is then possible to find a sequence $u^\nu \in C^1(\mathcal{S})$ such that $u_\nu \to u$ in $C^0(\mathcal{H}^N)$. The estimate (16) is valid for u_ν, we have $u^\nu(0) \to u(0)$ in $H(\mu_0)$ while $u^\nu \to u$ and $Au^\nu \to Au$ in $L^\infty(H(\mu_*))$, which ends the proof.

Theorem 4.4. *Let μ_0 be a g_0-weight, let u_0 and f belong to $H(\mu_0, g_0)$ and $L^1([0,T]; H(\mu_*))$ respectively. Then there exists a unique solution $u \in C([0,T]; H(\mu_*))$ of the Cauchy problem*

$$\frac{du(t)}{dt} + iAu(t) = f(t); \qquad u(0) = u_0.$$

We use a classical duality argument. Let $v \in \mathcal{S}(\mathbb{R}^{n+1})$ vanishing near $t = T$ and let $g = \frac{\partial v}{\partial t} + iAv$. From (16) (with the time going from T to 0), we know that one has

$$\|v\|_{L^\infty(H(\mu_*^{-1}))} \leq C\, \|g\|_{L^1(H(\mu_*^{-1}))}$$

and thus that v is uniquely determined by g. The linear form $g \mapsto (u_0 \,|\, v(0)) + \int_0^T (f(t) \,|\, v(t))\, dt$ is defined and continuous on the subspace of $L^1(H(\mu_*^{-1}))$ made of such g. From the Hahn-Banach theorem, we get the existence of $u \in L^\infty(H(\mu_*))$ such that

$$\forall v \in \mathcal{S},\ (u_0 \,|\, v(0)) + \int_0^T (f(t) \,|\, v(t))\, dt = -\int_0^T \bigl(u(t) \bigm| \tfrac{\partial v}{\partial t} + iAv\bigr)\, dt \tag{17}$$

Using functions v vanishing also near $t = 0$, this proves that u, in the sense of distributions, is solution de $\frac{\partial u}{\partial t} + iAu = f$ in $]0, T[\times\mathbb{R}^n$. Let us choose a weight $\underline{\mu}_0 \ll \mu_0$ such that the part (b) of proposition 4.2 apply to this couple of weights. One has $u \in C(H(\underline{\mu}_*)))$ and $u(0)$ is now well defined. Integrating by parts in (17) we get that $u(0) = u_0$. The estimate (16), with μ replaced by $\underline{\mu}$, shows the uniqueness of u.

It remains to prove that $u \in C([0,T]; H(\mu_*))$. Let us introduce a weight $\overline{\mu}_0 \gg \mu_0$ such that the part (b) of proposition 4.2 apply. Let us approximate u_0 and f, in $H(\mu_0, g_0)$ and $L^1([0,T]; H(\mu_*))$ respectively, by regular functions u_0^ν and f^ν. From the analysis above, one gets solutions u^ν belonging to $L^\infty(H(\overline{\mu}_*))$ and thus to $C([0,T]; H(\mu_*))$. Using again (16), the sequence u^ν is a Cauchy sequence in $L^\infty(H(\mu_*))$, its limit u should belong to $C([0,T]; H(\mu_*))$ which ends the proof of theorem 4.4.

Proof of the theorem 3.1 (end). — Taking $\mu_0 = 1$, the last theorem shows that, for any $u_0 \in L^2$, there exists a unique solution $t \mapsto u_t = P_t u_0$, continuous from $[-T, T]$ into L^2, of the equation $\frac{\partial u}{\partial t} + ia^w u = 0$. The group law and the relation $P_t^* = P_t^{-1}$ are valid in the interval in view of the uniqueness. One can thus extend P_t to $\mathbb{R}$

and get a strongly continuous group of unitary operators. Its infinitesimal generator will be denoted by $-i\mathcal{A}$, where $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$, is selfadjoint. Moreover, the P_t, for $|t| \leq T$, are continuous from $H(\mu_0)$ into $H(\mu_t)$.

We know that $\frac{1}{t}(u_0 - P_t u_0)$ converges always towards $-ia^w u_0$ in the sense of distributions, and this limit belongs thus to L^2 when $u_0 \in \mathcal{D}(\mathcal{A})$. This proves that

$$\mathcal{D}(\mathcal{A}) \subset \left\{ u_0 \in L^2 \middle| \, a^w u_0 \in L^2 \right\}. \tag{18}$$

Conversely, assume that u_0 and $a^w u_0$ belong to L^2. For $|t| \leq T$, one has

$$\tfrac{d}{dt} P_t u_0 = -ia^w P_t u_0 = -iP_t(a^w u_0).$$

The right hand side is continuous from $[-T, T]$ into L^2, and $P_t u_0$ has a derivative in L^2. This proves that $\mathcal{D}(\mathcal{A})$ is exactly the right hand side of (18). It is well known that a^w, with that domain, is the adjoint of the closure of a^w defined on $\mathcal{S}$. The selfadjointness of $\mathcal{A}$ shows that the weak and strong extensions coincide, which ends the proof of theorem 3.1.

5. Proof of the theorem 3.2

We assume now that $a \in \hat{\hat{\hat{S}}} \, (\lambda_t^2, g_t)$ and we have to prove that the iterated twisted commutators of P_t are bounded on L^2.

Let $b_0 \in \hat{S} \, (\lambda_0, g_0)$, let $b_t = b_0 \circ F_t^{-1}$ and, using capital letters for the corresponding operators, set

$$K_t = P_{-t} \, \widetilde{\mathrm{ad}}(b) \cdot P_t = P_{-t} B_t P_t - B_0.$$

One has

$$\frac{d}{dt} K_t = P_{-t} \left\{ iAB_t - iB_t A + \{b_t \, , \, a\}^w \right\} P_t = P_{-t} R_t P_t$$

The proposition 1.1 (c) shows that R_t belongs to $\Psi(1, g_t)$ (its semi-norms being controlled) and is thus uniformly bounded on L^2. We get

$$\widetilde{\mathrm{ad}}(b) \cdot P_t = \int_0^t P_{t-s} R_s P_s \, ds \in \mathcal{L}(L^2).$$

By induction, it is possible to write the iterated twisted commutators as sums of terms of the following type :

$$\int \cdots \int_{0 \leq s_1 \leq \ldots \leq s_N} P_{t-s_N} R_{s_N} \ldots P_{s_2 - s_1} R_{s_1} P_{s_1} \, ds_1 \ldots ds_N \in \mathcal{L}(L^2),$$

which ends the proof.

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