CHAPTER 6

More on categories and sheaves.

1. Injective objects and resolutions

Let I be an object in a category.

DEFINITION 6.1. The object I is said to be injective, if for any maps h, f such that f is a monomorphism, there exists g making the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{h} & \downarrow g \\ & & I \end{array}\qquad .$$

This is equivalent to saying that $A \to \mathrm{Mor}(A, I)$ sends monomorphisms to epimorphisms. Note that g is by no means unique ! An injective sheaf is an injective object in R-**Sheaf (X)**.

PROPOSITION 6.2. *If I is injective in an abelian category $\mathscr{C}$, the functor $A \to \mathrm{Mor}(A, I)$ from $\mathscr{C}$ to* **Ab** *is exact.*

DEFINITION 6.3. A category has enough injectives, if any object A has a monomorphism into an injective object.

EXERCICE 1. Prove that in the category **Ab** of abelian groups, the group $\mathbb{Q}/\mathbb{Z}$ is injective. Prove that **Ab** has enough injectives (prove that a sum of injectives is injective).

In a category with enough injectivs, we have the notion of **injective resolution**.

PROPOSITION 6.4 ([**Iv**], p.15). *Assume $\mathscr{C}$ has enough injectives, and let B be an object in $\mathscr{C}$. Then there is an exact sequence*

$$0 \to B \stackrel{i_B}{\to} J_0 \stackrel{d_0}{\to} J_1 \stackrel{d_1}{\to} J_2 \to \ldots.$$

where the J_k are injectives.This is called an **injective resolution** *of B. Moreover given an object A in $\mathscr{C}$ and a map $f: A \to B$ and a resolution of A (not necessarily injective), that is an exact sequence*

$$0 \to A \stackrel{i_A}{\to} L_0 \stackrel{d_0}{\to} L_1 \stackrel{d_1}{\to} L_2 \ldots.$$

and an injective resolution of B as above, then there is a morphism (i.e. a family of maps $u_k : L_k \to J_k$) such that the following diagram is commutative

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & L_0 & \xrightarrow{d_0} & L_1 & \xrightarrow{d_1} & L_2 & \xrightarrow{d_2} & \cdots . \\ & & \downarrow f & & \downarrow u_0 & & \downarrow u_1 & & \downarrow u_2 & & \\ 0 & \longrightarrow & B & \xrightarrow{i_B} & J_0 & \xrightarrow{\partial_0} & J_1 & \xrightarrow{\partial_1} & I_2 & \xrightarrow{\partial_2} & \cdots \end{array}$$

Moreover any two such maps are homotopic (i.e. $u_k - v_k = \partial_{k-1} s_k + s_{k+1}\delta_k$, *where* $s^k : I_k \to J_{k-1}$*).*

PROOF. The existence of a resolution is proved as follows: existence of J_0 is by definition of having enough injectives. Then let $M_1 = \text{Coker}(i_B)$ so that $0 \to B \overset{d_0}{\to} J_0 \overset{f_0}{\to} M_1 \to 0$ is exact. A map $0 \to M_1 \to J_1$ induces a map $0 \to B \overset{i_B}{\to} J_0 \overset{d_0}{\to} J_1$, exact at J_0. Continuing this procedure we get the injective resolution of B. Now let $f : A \to B$ and consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & L_0 \\ \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & B & \xrightarrow{i_B} & J_0 \end{array}$$

Since J_0 is injective, i_B is monomorphism and $i_A \circ f$ lifts to a map $u_0 : L_0 \to J_0$. Let us now assume inductively that the map u_k is defined, and let us define u_{k+1}. We decompose using property (2) of Definition 5.7:

$$\begin{array}{ccccc} L_{k-1} & \xrightarrow{d_{k-1}} & L_k & \xrightarrow{d_k} & L_{k+1} \\ \downarrow u_{k-1} & & \downarrow u_k & & \\ J_{k-1} & \xrightarrow{\partial_{k-1}} & J_k & \xrightarrow{\partial_k} & J_{k+1} \end{array}$$

as

$$\begin{array}{ccccccc} L_{k-1} & \xrightarrow{d_{k-1}} & L_k & \xrightarrow{d_k} & \text{Coker}(d_{k-1}) & \xrightarrow{i_k} & L_{k+1} \\ \downarrow u_{k-1} & & \downarrow u_k & & \downarrow v & & \\ J_{k-1} & \xrightarrow{\partial_{k-1}} & J_k & \xrightarrow{\partial_k} & \text{Coker}(\partial_{k-1}) & \xrightarrow{j_k} & J_{k+1} \end{array}$$

Since $(\partial_k \circ u_k) \circ d_{k-1} = 0$, there exists by definition of the cokernel a map $v_{k+1} : \text{Coker}(d_{k-1}) \to \text{Coker}(\partial_{k-1})$, making the above diagram commutative. Then since i_k is monomorphism (due to exactness at L_k) and J_{k+1} is injective, the map $j_k \circ v_{k+1}$ factors through i_k so that there exists $u_{k+1} : L_{k+1} \to J_{k+1}$ making the above diagram commutative. The construction of the homotopy is left to the reader. □

PROPOSITION 6.5. *The category* **R – Sheaf**(**X**) *has enough injectives.*

PROOF. The proposition is proved as follows.

Step 1: One proves that for each x there is an injective $\mathscr{D}(x)$ such that $\mathscr{F}_x$ injects into $D(x)$. In other words we need to show that R-mod has enough injectives. We omit this step since it is trivial for $\mathbb{C}$-sheaves (any vector space is injective).

Step 2: Construction of $\mathscr{D}$. The category **R-mod** has enough injectives, so choose for each x a map $\mathscr{F}_x \to \mathscr{D}(x)$ where $\mathscr{D}(x)$ is injective, and consider the sheaf $\mathscr{D}(U) = \prod_{x\in U}\mathscr{D}(x)$. Thus a section is the choice for each x of an element $\mathscr{D}(x)$ (without any "continuity condition"). Then for each $\mathscr{F}$ we have $\mathrm{Mor}(\mathscr{F},\mathscr{D}) = \prod_{x\in X} Hom(\mathscr{F}_x,\mathscr{D}(x))$, and clearly $\mathscr{D}$ is injective.

Step 3:Let $\mathscr{F}$ an object in R**-Sheaf(X)** and $\mathscr{D}$ be the above associated sheaf. Then the obvious map $i : \mathscr{F} \to \mathscr{D}$ induces an injection $i_x : \mathscr{F}_x \to \mathscr{D}(x)$ hence is a monomorphism. □

One should be careful. The sheaf $\mathscr{D}$ does not have $\mathscr{D}(x)$ as its stalk: the stalk of $\mathscr{D}$ is the set of germs of functions (without continuity condition) $x \mapsto \mathscr{D}(x)$ for x in a neighbourhood of x_0. Obviously, $\mathscr{D}_{x_0}$ surjects on $\mathscr{D}(x_0)$.

When R is a field, there is a unique injective sheaf with $\mathscr{D}(x) = R^q$. It is called the canonical injective R^q-sheaf. Let us now define

DEFINITION 6.6. Let $\mathscr{F}$ be a sheaf, and consider an injective resolution of $\mathscr{F}$

$$0 \to \mathscr{F} \xrightarrow{d_0} \mathscr{J}_0 \xrightarrow{d_1} \mathscr{J}_1 \xrightarrow{d_2} \mathscr{J}_2 ....$$

Then the cohomology $\mathscr{H}^*(X,\mathscr{F})$ (also denoted $R\Gamma(X,\mathscr{F})$) is the (co)homology of the sequence

$$0 \to \mathscr{J}_0(X) \xrightarrow{d_{0,X}} \mathscr{J}_2(X) \xrightarrow{d_{1,X}} \mathscr{J}_2(X) ....$$

In other words $\mathscr{H}^m(X,\mathscr{F}) = \mathrm{Ker}(\mathrm{d_{m,X}})/\mathrm{Im}(\mathrm{d_{m-1,X}})$

Check that $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$. Note that the second sequence is not an exact sequence of R-modules, because exactness of a sequence of sheafs means exactness of the sequence of R-modules obtained by taking the stalk at x (for each x). In other words, the functor from **Sheaf(X)** to R**-mod** defined by $\Gamma_x : \mathscr{F} \to \mathscr{F}_x$ is exact, but the functor $\Gamma_U : \mathscr{F} \to \mathscr{F}(U)$ is not.

This is a general construction that can be applied to any left-exact functor: take an injective resolution of an object, apply the functor to the resolution after having removed the object, and compute the cohomology. According to Proposition 6.4, this does not depend on the choice of the resolution, since two resolutions are chain homotopy equivalent, and F sends chain homotopic maps to chain homotopic maps, hence preserves chain homotopy equivalences . This is the idea of derived functors, that we are going to explain in full generality (i.e. applied to chain complexes). It is here applied to the functor Γ_X. It is a way of measuring how this left exact functor fails to be exact: if the functor is exact, then $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$ and $\mathscr{H}^m(X,\mathscr{F}) = 0$ for $m \geq 1$.

For the moment we set

DEFINITION 6.7. Let $\mathscr{C}$ be a category with enough injectives, and F be a left-exact functor. Then $R^j F(A)$ is obtained as follows: take an injective resolution of A,

$$0 \to A \xrightarrow{i_A} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \to \dots.$$

then $R^j F(A)$ is the j-th cohomology of the complex

$$0 \to F(I_0) \xrightarrow{d_0} F(I_1) \xrightarrow{d_1} F(I_2) \to \dots.$$

We say that A is F-acyclic, if $R^j F(A) = 0$ for $j \geq 1$.

Note that the left-exactness of F implies that we always have $R^0 F(A) = A$. Since according to Proposition 6.4, the $R^j F(A) = 0$ do not depend on the choice of the resolution, an injective object is acyclic: take $0 \to I \to I \to 0$ as an injective resolution, and notice that the cohomology of $0 \to I \to 0$ vanishes in degree greater than 0.

However, as we saw in the case of sheafs, injective objects do not appear naturally. So we would like to be able to use resolutions with a wider class of objects

DEFINITION 6.8. A **flabby** sheaf is a sheaf such that the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is onto for any $V \subset U$.

Notice that by composing the restriction maps, $\mathscr{F}$ is flabby if and only if $\mathscr{F}(X) \to \mathscr{F}(V)$ is onto for any $V \subset X$. This clearly implies that the restriction of a flabby sheaf is flabby.

PROPOSITION 6.9. *An injective sheaf is flabby. A flabby sheaf is Γ_X-acyclic.*

PROOF. First note that the sheaf we constructed to prove that **Sheaf(X)** has enough injectives is clearly flabby. Therefore any injective sheaf $\mathscr{I}$ injects into a flabby sheaf, $\mathscr{D}$. Moreover there is a map $p : \mathscr{D} \to \mathscr{I}$ such that $p \circ i = \mathrm{id}$, since the following diagram yields the arrow p

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{I} & \xrightarrow{i} & \mathscr{D} \\ & & & \overset{\mathrm{id}}{\searrow} & \downarrow p \\ & & & & \mathscr{I} \end{array}$$

As a result, we have diagrams

$$\begin{array}{ccc} \mathscr{D}(U) & \xrightarrow{p_U} & \mathscr{I}(U) \\ \downarrow s_{V,U} & & \downarrow r_{V,U} \\ \mathscr{D}(V) & \xrightarrow{p_V} & \mathscr{I}(V) \end{array}$$

Since $p_U \circ i_U = \mathrm{id}$, we have that p_U is onto, hence $r_{V,U}$ is onto.

We now want to prove the following: let $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ be an exact sequence, where $\mathscr{E}, \mathscr{F}$ are flabby. Then $\mathscr{G}$ is flabby.

Let us first consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ with $\mathscr{E}$ flabby. We want to prove that the map $v(X) : \mathscr{F}(X) \to \mathscr{G}(X)$ is onto. Indeed, let $s \in \Gamma(X, \mathscr{G})$, and a maximal set for inclusion, U, such that there exists a section $t \in \Gamma(U,\mathscr{F})$ such that $v(t) = s$ on U. We claim $U = X$ otherwise there exists $x \in X \setminus U$, a section t_x defined in a neighborhood V of x such that $f(t_x) = s$ on V. Then $t - t_x$ is defined in $\Gamma(U \cap V,\mathscr{F})$, but since $f(t - t_x) = 0$, we have by exactness, $t - t_x = u(z)$ for $z \in \Gamma(U \cap V,\mathscr{E})$. Since $\mathscr{E}$ is flabby, we may extend z to X, and then $t = t_x + u(z)$ on $U \cap V$. We may the find a section $\tilde{t} \in \Gamma(U \cup V,\mathscr{F})$ such that $\tilde{t} = t$ on U and $\tilde{t} = t_x + u(z)$ on V. Clearly $v(\tilde{t})_U = s_{|U}$ and $v(\tilde{t})_V = v(t_x) + vu(z) = v(t_x) = s_{|V}$, hence $v(\tilde{t}) = s$ on $U \cup V$. This contradicts the maximality of U.

As a result, we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}(X) & \xrightarrow{u(X)} & \mathscr{F}(X) & \xrightarrow{v(X)} & \mathscr{G}(X) & \longrightarrow & 0 \\ & & \downarrow \rho_{X,U} & & \downarrow \sigma_{X,U} & & \downarrow \tau_{X,U} & & \\ 0 & \longrightarrow & \mathscr{E}(U) & \xrightarrow{u(U)} & \mathscr{F}(U) & \xrightarrow{v(U)} & \mathscr{G}(U) & \longrightarrow & 0 \end{array}$$

and $\rho_{U,X}, \sigma_{U,X}$ are onto. This immediately implies that $\tau_{X,U}$ is onto. Finally, let us prove that a flabby sheaf $\mathscr{F}$ is acyclic. We consider the exact map $0 \to \mathscr{F} \to \mathscr{I}$ where $\mathscr{I}$ is injective. Using the existence of the cokernel, this yields an exact sequence $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{K} \to 0$. By the above remark, $\mathscr{K}$ is flabby. Consider then the long exact sequence associated to the short exact sequence of sheaves:

$$0 \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{I}) \to H^0(X,\mathscr{K}) \to H^1(X,\mathscr{F}) \to H^1(X,\mathscr{I}) \to H^1(X,\mathscr{K}) \to ...$$

We prove by induction on n that for any $n \geq 1$ and any flabby sheaf, $H^n(X,\mathscr{F}) = 0$. Indeed, we just proved that $H^0(X,\mathscr{I}) \to H^0(X,\mathscr{K})$ is onto, and we know that $H^1(X,\mathscr{I}) = 0$. this implies $H^1(X,\mathscr{F}) = 0$. Assume now, that for any flabby sheaf and $j \leq n$, H^j vanishes. Then the long exact sequence yields

$$.. \to H^n(X,\mathscr{K}) \to H^{n+1}(X,\mathscr{F}) \to H^{n+1}(X,\mathscr{I}) \to ...$$

Since $\mathscr{I}$ is injective, $H^{n+1}(X,\mathscr{I})$ and since $\mathscr{K}$ is flabby $H^n(X,\mathscr{K}) = 0$ hence $H^{n+1}(X,\mathscr{F})$ vanishes. □

Example: Flabby sheafs are much more natural than injective ones, and we shall see they are just as useful. The sheaf of distributions, that is $\mathscr{D}_X(U)$ is the dual of $C_0^\infty(U)$, the sheaf of differential forms with distribution coefficients, the set of singular cochain defined on U... are all flabby. A related notion is the notion of soft sheaves. A soft sheaf is a sheaf such that the map $\mathscr{F}(X) \to \mathscr{F}(K)$ is surjective for any closed set K. Of course, we define $\mathscr{F}(K) = \lim_{K \subset U} \mathscr{F}(U)$. In other words, an element defined in a neighborhood of K has an extension (maybe after reducing the neighborhood) to all of X. The sheafs of smooth functions, smooth forms, continuous functions... are all soft.

We refer to subsection 3.1 for applications of these notions.

EXERCICE 2. Prove that soft sheaves are acyclic.

2. Operations on sheaves. Sheaves in mathematics.

First of all, if $\mathscr{F}$ is sheaf over X, and U an open subset of X, we denote by $\mathscr{F}_{|U}$ the sheaf on U defined by $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ for all $V \subset U$. For clarity, we define $\Gamma(U, \bullet)$ as the functor $\mathscr{F} \to \Gamma(U, \mathscr{F}) = \mathscr{F}(U)$.

DEFINITION 6.10. Let $\mathscr{F}, \mathscr{G}$ be sheafs over X. We define $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ as the sheaf associated to the presheaf $\mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$. We define $\mathscr{F} \otimes \mathscr{G}$ to be the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U)$. The same constructions hold for sheafs of modules over a sheaf of rings $\mathscr{R}$, and we then write $\mathscr{H}om_{\mathscr{R}}(\mathscr{F}, \mathscr{G})$ and $\mathscr{F} \otimes_{\mathscr{R}} \mathscr{G}$.

REMARK 6.11.
(1) Note that $\mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U}) \neq Hom(\mathscr{F}(U), \mathscr{G}(U))$ in general, since an element f in the left hand side defines compatible $f_V \in \mathscr{H}om(\mathscr{F}(V), \mathscr{G}(V))$ for all open sets V in U, while the right-hand side does not. There is however a connection between the two definitions: $\mathrm{Mor}(\mathscr{F}, \mathscr{G}) = \Gamma(X, \mathscr{H}om(\mathscr{F}, \mathscr{G})$.
(2) Note that tensor products commute with direct limits, so $(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \otimes \mathscr{G}_x$. On the other hand Mor does not commute with direct limits, so $\mathscr{H}om(\mathscr{F}, \mathscr{G})_x$ is generally different from $\mathscr{H}om(\mathscr{F}_x, \mathscr{G}_x)$.

Let $f : X \to Y$ be a continuous map. We define a number of functors associated to f as follows.

DEFINITION 6.12. Let $f : X \to Y$ be a continuous map, $\mathscr{F} \in \mathbf{Sheaf(X)}, \mathscr{G} \in \mathbf{Sheaf(Y)}$ The sheaf $f_* \mathscr{F}$ is defined by

$$f_*(\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$$

The sheaf $f^{-1}(\mathscr{G})(U)$ is the sheaf associated to the presheaf $U \mapsto \lim_{V \supset f(U)} \mathscr{G}(V)$. We also define $\mathscr{F} \boxtimes \mathscr{G}$ as follows. If p_X, p_Y are the projections of $X \times Y$ on the respective factors, we have $\mathscr{F} \boxtimes \mathscr{G} = p_X^{-1} \mathscr{F} \otimes p_Y^{-1}(\mathscr{G})$. When $X = Y$ and d is the diagonal map, we define $d^{-1}(\mathscr{F} \boxtimes G) = \mathscr{F} \otimes \mathscr{G}$. This is the sheaf associated to the presheaf $U \to \mathscr{F}(U) \otimes \mathscr{G}(U)$.

It is also useful to have the definition of

PROPOSITION 6.13. *The functors f_*, f^{-1} are respectively left-exact and exact. Moreover , let f, g be continuous maps, then $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

PROOF. For the first statement, let us prove that f^{-1} is exact. We use the fact that $f^{-1}(\mathscr{G})_x = \mathscr{G}_{f(x)}$. Thus an exact sequence $0 \to \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \to 0$ is transformed into the sequence $0 \to f^{-1}(\mathscr{F}) \xrightarrow{u \circ f} f^{-1}(\mathscr{G}) \xrightarrow{v \circ f} f^{-1}(\mathscr{H}) \to 0$ which has germs

$$0 \to (f^{-1}(\mathscr{F}))_x \xrightarrow{u(f(x))} (f^{-1}(\mathscr{G}))_x \xrightarrow{v(f(x))} (f^{-1}(\mathscr{H}))_x \to 0$$

equal to

$$0 \to \mathscr{F}_{f(x))} \xrightarrow{u(f(x))} \mathscr{G}_{f(x)} \xrightarrow{v(f(x))} \mathscr{H}_{f(x)} \to 0$$

which is exact. Now we prove that f_* is left-exact. Indeed, consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} G$. By left-exactness of Γ_U, the sequence

$$0 \to \mathscr{E}(U) \overset{u(U)}{\to} \mathscr{F}(U) \overset{v(U)}{\to} G(U)$$

is exact, hence for any $V \subset Y$, the sequence

$$0 \to \mathscr{E}(f^{-1}(V)) \overset{v(f^{-1}(V))}{\to} \mathscr{F}(f^{-1}(V)) \overset{v(f^{-1}(V))}{\to} G(f^{-1}(V))$$

is exact, which by taking limits on $V \ni x$ implies the exactness of

$$0 \to (f_*\mathscr{E})_x \overset{(f_* u)_x}{\to} (f_*\mathscr{F})_x \overset{(f_* v)_x}{\to} (f_*\mathscr{G})_x.$$

□

PROPOSITION 6.14. *We have* $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F}) = \mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$. *We say that* f_* *is* **right-adjoint** *to* f^{-1} *or that* f^{-1} *is left adjoint to* f_*.

PROOF. We claim that an element in either space, is defined by the following data, called a f-homorphism: consider for each x a morphism $k_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$ such that for any section s of $\mathscr{G}(U)$, $k_x \circ s(f(x))$ is a (continuous) section of $\mathscr{F}(U)$. Notice that there are in general many x such that $f(x) = y$ is given, and also that a f-homomorphism is the way one defines morphisms in the category **Sheaves** of sheaves over all manifold (so that we must be able to define a morphism between a sheaf over X and a sheaf over Y). Now, we claim that an element in $\mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$ defines k_x, since $(f^{-1}(\mathscr{G}))_x = \mathscr{G}_{f(x)}$, so a map sending elements of $f^{-1}(\mathscr{G})(U)$ to elements of $\mathscr{F}(U)$ localizes to a map k_x having the above property. Conversely, given a map k_x as above, let $s \in f^{-1}(\mathscr{G})(U)$. By definition, for each point $x \in U$ there exists a section $t_{f(x)}$ defined near $f(x)$ such that $s = t_{f(x)}$ near x. Now define $s'_x = k_x t_{f(x)}$. We have that $s'_x \in F_x$, and by varying x in U, this defines a section of $\mathscr{F}(U)$. So k_x defines a morphism from $f^{-1}(\mathscr{G})$ to $\mathscr{F}$.

Now an element in $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ sends for each U, $\mathscr{G}(U)$ to $\mathscr{F}(f^{-1}(U))$, hence an element in $\mathscr{G}_{f(y)}$ to an element in some $\mathscr{F}(f^{-1}(V_{f(y)})$ which induces by restriction an element in $\mathscr{F}_y$, hence defines k_x. Vice-versa, let $s \in \mathscr{G}(V)$ then for $y \in V$ and $x \in f^{-1}(y)$, we define $s'_x = k_x s_y$. The section s'_x is defined on V_x a neghbourhood of x, and by assumption $k_x s_{f(x)}$ is continuous, so s' is continuous.

We thus identified the set of f-homorphisms both with $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ and with $\mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$, which are thus isomorphic.

EXERCICE 3. Prove that $f_*\mathscr{H}om(f^{-1}(\mathscr{G}), \mathscr{F}) = \mathscr{H}om(\mathscr{G}, f_*\mathscr{F})$.

□

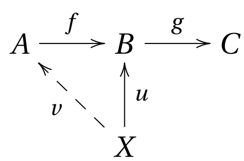
The notion of adjointness is important in view of the following.

PROPOSITION 6.15. *Any right-adjoint functor is left exact. Any left-adjoint functor is right-exact.*

PROOF. Let F be right-adjoint to G, that is $\mathrm{Mor}(A, F(B)) = \mathrm{Mor}(G(A), B)$. We wish to prove that F is left-exact. The exactness of the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is equivalent to

$$0 \to \mathrm{Mor}(X, A) \xrightarrow{f^*} \mathrm{Mor}(X, B) \xrightarrow{g^*} \mathrm{Mor}(X, A) \tag{6.1}$$

Indeed, exactness of the sequence is equivalent to the fact that $A \xrightarrow{f} B$ is the kernel of g, or else that for any X, and $u : X \to A$ such that $g \circ u = 0$, there exists a unique $v : X \to B$ such that the following diagram commutes



The existence of v implies exactness of 6.1 at $\mathrm{Mor}(X, B)$, while uniqueness yields exactness at $\mathrm{Mor}(X, A)$.

As a result, left-exactness of F is equivalent to the fact that for each X, and each exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence

$$0 \to \mathrm{Mor}(X, F(A)) \overset{F(f)^*}{\to} \mathrm{Mor}(X, F(B)) \overset{F(g)^*}{\to} \mathrm{Mor}(X, F(A))$$

is exact. But this sequence coincides with

$$0 \to \mathrm{Mor}(G(X), A) \xrightarrow{f^*} \mathrm{Mor}(G(X), B) \xrightarrow{g^*} \mathrm{Mor}(G(X), C)$$

its exactness follows from the left-exactness of $M \to \mathrm{Mor}(X, M)$.

□

Note that in the literature, f^{-1} is sometimes denoted f^*. Note also that if f is the constant map, then $f_*\mathscr{F} = \Gamma(X, \mathscr{F})$, so that $Rf_* = R\Gamma(X, \bullet)$.

COROLLARY 6.16. *The functor f_* maps injective sheafs to injective sheafs. The same holds for Γ_X.*

PROOF. Indeed, we have to check that $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is an exact functor. But this is the same as checking that $\mathscr{F} \to \mathscr{H}om(f^{-1}\mathscr{F}, \mathscr{I})$ is exact. Now $F \to f^{-1}(\mathscr{F})$ is exact, and since $\mathscr{I}$ is injective, $\mathscr{G} \to \mathscr{H}om(\mathscr{G}, \mathscr{I})$ is exact. Thus $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is the composition of two exact functors, hence is exact. The second statement is a special case of the first by taking f to be the constant map.

□

There is at least another simple functor: $f_!$ given by

DEFINITION 6.17. $f_!(\mathscr{F})(U) = \{s \in \mathscr{F}(f^{-1}(U)) \mid f : \mathrm{supp}(s) \to U \text{ is a proper map}\}$.

If f is proper, then $f_!$ and f_* coincide. Even though $f_!$ has a right-adjoint $f^!$, we shall not construct this as it requires a slightly complicated construction, extending Poincaré duality, the so-called Poincaré-Verdier duality (see [**Iv**] chapter V).

Examples:

(1) Let A be a subset of X, and k_A be the constant sheaf on A, and $j : A \to X$ be the inclusion of A in X. Then $j_*(k_A) = k_A$ and $j^{-1}(k_A) = k_X$.

Note that the above operations extend to complexes of sheaves:

DEFINITION 6.18. Let $A^\bullet, B^\bullet$ be two bounded complexes. Then we define $(A^\bullet \otimes B^\bullet)^m = \sum_j A^j \otimes B^{m-j}$ with boundary map $d_m(u_j \otimes v_{m-j}) = \partial_j u_j \otimes v_{m-j} + u_j \otimes \partial_{m-j} v_{m-j}$. and $\mathscr{H}om(A^\bullet, B^\bullet)^m = \sum_j Hom(A^j \otimes B^{m+j})$, with boundary map $d_m f = \sum_p \partial_{m+p} f^p + (-1)^{m+1} f^{p+1} \partial_p$.

Finally we define the functor Γ_Z : **Sheaf**(**X**) $\to$ **Sheaf**(**X**) defined by

DEFINITION 6.19. Let Z be a locally closed set. Let $\mathscr{F} \in$ **Sheaf**(**X**). Then the sheaf $\Gamma_Z \mathscr{F}$ is defined by $\Gamma_Z \mathscr{F}(U) = \ker(\mathscr{F}(U) \to \mathscr{F}(U \setminus Z))$

EXERCICE 4. (1) Show that the support of Γ_Z is contained in Z.
(2) Show that Γ_Z is a left exact functor from **Sheaf(X)** to **Sheaf**(**X**).
(3) Show that Γ_Z maps injectives to injectives.

PROPOSITION 6.20. *The functor Γ_Z sends flabby sheafs to flabby sheafs (an in particular injective sheafs to acyclic sheafs).*

PROOF. We must prove that $\Gamma_Z(X, \mathscr{F}) \to \Gamma_Z(U, \mathscr{F})$ is onto. Let $s \in \Gamma_Z(U, \mathscr{F})$, that is an element in $\mathscr{F}(U)$ vanishing on $U \setminus Z$. We may thus first extend s by 0 on $X \setminus Z$ to the open set $X \setminus Z) \cup U$. By flabbiness of $\mathscr{F}$ we then extend s to X. □

2.1. Sheaves and D-modules. Note that the rings we shall consider in this subsection are non-commutative, a situation we had not explicitly considered above. A D-module is a module over the ring D_X of algebraic differentials operators over an algebraic manifold X. Let O_X be the ring of holomorphic functions, Θ_X the ring of linear operators on O_X (i.e. holomorphic vector fields), and D_X the noncommutative ring generated by O_X and Θ_X, that is the sheaf of holomorphic differential operators on X. A D-module is a module over the ring D_X. More generally, given a sheaf of rings $\mathscr{R}$, we can consider $\mathscr{R}$-modules, that is for each open U, $\mathscr{F}(U)$ is an $\mathscr{R}(U)$-module and the restriction morphism is compatible with the $\mathscr{R}$-module structure. What we did for R-modules also hold for $\mathscr{R}$-modules.Let us show how D-modules appear naturally. Let P be a general differential operator, that is, locally, $Pu = (\sum_{j=1}^m P_{1,j} u_j, ..., \sum_{j=1}^m P_{q,j} u_j)$ or else $\sum_{j=1}^m P_{i,j} u_i = v_j$, and let us start with $u = 0$. The operator P yields a linear map $D_X^p \to D_X^q$ and we may consider the map

$$\begin{array}{rcl}\Phi(u) : D_X^p & \longrightarrow & O_X \\ (Q_j)_{1\le j\le p} & \longrightarrow & \sum_{j=1}^p Q_j u_j\end{array}$$

so that if $(u_1,...,u_p)$ is a solution of our equation, then $\Phi(u)$ vanishes on $D_X \cdot P_1 + ... + D_X P_q$ where

$$P_j = \begin{pmatrix} P_{1,j} \\ \vdots \\ P_{q,j}\end{pmatrix}$$

Conversely, a map $\Phi : D_X^p \longrightarrow O_X$ vanishing on $D_X \cdot P_1 + ... + D_X P_q$ yields a solution of our equation, setting $u_j = \Phi(0,..,1,0...0)$.

Then, let $\mathcal{M}$ be the D-module $D_X/(D_X \cdot P)$, the set of solutions of the equation corresponds to $\mathrm{Mor}(\mathcal{M}, O_X)$.

3. Injective and acyclic resolutions

One of the goals of this section, is to show why the injective complexes can be used to define the derived category. One of the main reasons, is that on those complexes, quasi-isomorphism coincides with chain homotopy equivalence. We also explain why acyclic resolutions are enough to compute the derived functors, and finally work out the examples of the deRham and Cech complexes, proving that they both compute the cohomology of X with coefficients in the constant sheaf.

We start with the following

PROPOSITION 6.21. *Let $f : I^\bullet \to C^\bullet$ be a quasi-isomorphism where I^p are injective. Then there exists $g : I^\bullet \to C^\bullet$ such that $g \circ f$ is homotopic to* id.

PROOF. We first construct the mapping cone of a map. Let $f^\bullet : A^\bullet \to B^\bullet$ be morphism of chain complexes, and $C(f)^\bullet = A^\bullet[1] \oplus B^\bullet$ with boundary map

$$d = \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}$$

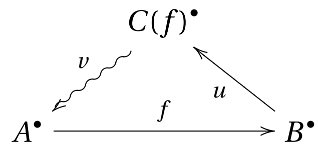
Then there is an exact sequence of chain complexes

$$0 \longrightarrow B^\bullet \xrightarrow{u=\begin{pmatrix}0\\1\end{pmatrix}} C(f)^\bullet \xrightarrow{v=\begin{pmatrix}1\\0\end{pmatrix}} A^\bullet[1] \longrightarrow 0$$

In fact there is a long (**non-exact**)sequence

$$A \xrightarrow{f} B^\bullet \xrightarrow{u} C(f)^\bullet \xrightarrow{v} A^\bullet[1] \xrightarrow{f[1]} B^\bullet[1] \xrightarrow{u[1]} \cdots$$

that we could say is "homotopy exact" if this meant something[1]. At least we can check the composition of two consecutive maps is homotopic to zero (but generally $u \circ f \neq 0$). This information is usually presented under the form of a **distinguished triangle**, and sometimes denoted as



where the wavy arrow means that indexation is shifted by 1 (i.e. the map actually goes to $A[1]$). The above exact sequence (or distinguished triangle) yields a long exact sequence in homology:

$$\longrightarrow H^n(A^\bullet, \partial_A) \xrightarrow{H^n(f_*)} H^n(B^\bullet, \partial_B) \xrightarrow{H^n(u)} H^n(C(f)^\bullet, d) \xrightarrow{\delta_f^*} H^{n+1}(A^\bullet, \partial_A) \longrightarrow \cdots$$

where the connecting map can be identified with $H^\bullet(f)$ and δ_f^* coincides with the connecting map defined in the long exact sequence of Proposition 5.14. Note that $H^n(A^\bullet[1], \partial_A) = H^{n+1}(A^\bullet, \partial_A)$. Now we see that if $H^n(f)$ is an isomorphism then $H^n(C(f)^\bullet, d) = 0$ for all n, we have an acyclic complex $(C(f)^\bullet, d)$, and a map $C(f)^\bullet \to A^\bullet[1]$. We claim that it is sufficient to prove that this map is homotopic to zero. Indeed, let s be such a homotopy. It induces a map $s^\bullet : C(f)^\bullet \to A^\bullet$ such that $-\partial_A s(a,b) + sd(a,b) = a$ or else

$$-\partial_A s(a,b) + s(-\partial_A(a), -f(a) + \partial_B(b)) = a$$

so setting $g(b) = s(0,b)$ and $t(a) = s(-a,0)$ we get (apply successively to $(0,-b)$ and $(a,0)$),

$$\partial_A g(b) - g(\partial_B b) = 0$$

so g is a chain map, and

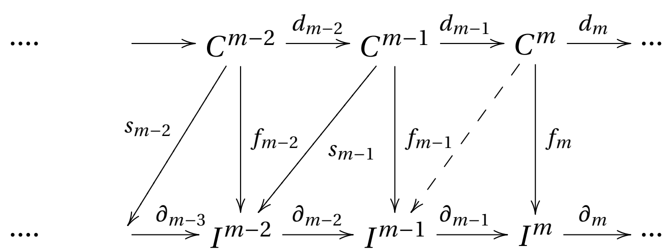
$$\partial_A t(a) + gf(a) + t\partial(a) = a$$

so gf is homotopic to Id_A

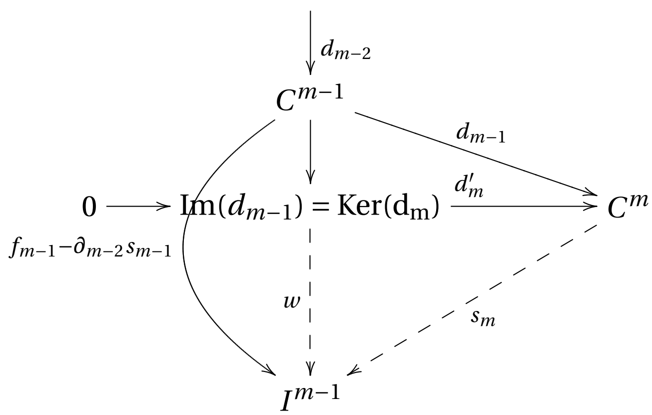
LEMMA 6.22. *Any morphism from an acyclic complex $C^\bullet$ to an injective complex $I^\bullet$ is homotopic to* 0.

Let f be the morphism. We will construct the map s such that $f = \partial s + sd$ by induction using the injectivity. Assume we have constructed the solid maps and we wish to construct the dotted one in the following (non commutative !) diagram, such that $f_{m-1} = \partial_{m-2} s_{m-1} + s_m d_{m-1}$.

[1]i.e. if the homotopy category, $\mathbf{K}(\mathscr{C})$ from Definition 7.3 was an abelian category, which is not the case.



The horizontal maps are not injective, but we may replace them by the following commutative diagram



where we first prove the existence of w and then the existence of s_m. The existence of w follows from the fact that $\mathrm{Ker}(\mathrm{d_{m-1}}) \to \mathrm{ker}(\mathrm{f_{m-1}} - \partial_{\mathrm{m-2}}\mathrm{s_{m-1}})$ since $\mathrm{Ker}(\mathrm{d_{m-1}}) = \mathrm{Im}(\mathrm{d_{m-2}})$ and we just have to check that $(f_{m-1} - \partial_{m-2}s_{m-1}) \circ d_{m-2} = 0$ which is obvious from the diagram and the induction assumption, since

$$f_{m-1} \circ d_{m-2} = \partial_{m-2} \circ f_{m-2} =$$
$$\partial_{m-2} \circ (\partial_{m-3}s_{m-2} + s_{m-1}d_{m-2}) = \partial_{m-2}s_{m-1}d_{m-2}$$

The injectivity of d'_m follows from the exactness of the sequence, and the existence of s_m follows from the injectivity of I^{m-1}. □

Notice that proposition 6.21 implies

COROLLARY 6.23. *Let I^* be an acyclic chain complex of injective elements, and F is any left-exact functor, then $F(I^\bullet)$ is also acyclic.*

PROOF OF THE COROLLARY. Indeed, since the 0 map from $I^\bullet$ to itself is a quasi-isomorphism, we get a homotopy between $\mathrm{id}_{I^\bullet}$ and 0. In other words $\mathrm{id}_{I^\bullet} = ds + sd$. As a result $F(\mathrm{id}_{I^\bullet}) = F(d)F(s) + F(s)F(d) = dF(s) + F(s)d$ and this implies that $F(\mathrm{id}_{I^\bullet}) : F(I^\bullet) \to F(I^\bullet)$ is homotopic to zero, which is equivalent to the acyclicity of $F(I^\bullet)$. □

Note that this implies that to compute the right-derived functor, we may replace the injective resolution by any F-acyclic resolution, that is resolution by objects L_m such that $H^j(L_m) = 0$ for all $j \neq 0$:

COROLLARY 6.24. *Let* $0 \to A \to L_0 \to L_1 \to \dots$ *be a resolution of* A *such that the* L_j *are* F*-acyclic, that is* $RF^m(L_j) = 0$ *for any* $m \geq 1$*. Then* $RF(A)$ *is quasi-isomorphic to the chain complex* $0 \to F(L_0) \to F(L_1) \to \dots\dots$

PROOF. Let $I^\bullet$ be an injective resolution of A. There is according to 6.4 a morphism $f : L^\bullet \to I^\bullet$ extending the identity map. Because the map f is a quasi-isomorphism (there is no homology except in degree zero, and then by assumption f_* induces the identity), according to the previous result there exists $g : I^\bullet \to L^\bullet$ such that $g \circ f$ is homotopic to the identity. But then $F(g) \circ F(f)$ is homotopic to the identity, and $F(f)$ is an isomorphism between the cohomology of $F(I^\bullet)$, that is $RF^*(A)$ and that of $F(L^\bullet)$. □

Note that the above corollary will be proved again using spectral sequences in Proposition 7.13 on page 75.

Note that if $\mathscr{I}$ is injective, $0 \to \mathscr{I} \to \mathscr{I} \to 0$ is an injective resolution, and then clearly $H^0(X, \mathscr{I}) = \Gamma(X, \mathscr{I})$ and $H^j(X, \mathscr{I}) = 0$ for $j \geq 1$. A sheaf such that $H^j(X, \mathscr{F}) = 0$ for $j \geq 1$ is said to be Γ_X**-acyclic** (or **acyclic** for short).

3.1. Complements: DeRham and Chech cohomology. We shall prove here that DeRham or Cech cohomology compute the usual cohomology.

Let $\mathbb{R}_X$ be the constant sheaf on X. Let Ω^j be the sheaf of differential forms on X, that is $\Omega^j(U)$ is the set of differential forms defined on U. This is clearly a soft sheaf, and we claim that we have a resolution

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \to 0$$

where d is the exterior differential. The fact that it is a resolution is checked by the exactness of

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0_x \xrightarrow{d} \Omega^1_x \xrightarrow{d} \Omega^2_x \xrightarrow{d} \Omega^3_x \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_x \to 0$$

which in turn follows from the Poincaré lemma, since for U contractible, we already have the exactness of

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(U) \to 0$$

and x has a fundmaental basis of contractible neighbourhoods. Since soft sheafs are acyclic, we may compute $H^*(X, \mathbb{R}_X)$ by applying Γ_X to the above resolution. That is the cohomology of

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \Omega^3(X) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(X) \to 0$$

or else the DeRham cohomology.

3.2. Cech cohomology. Let $\mathscr{F}$ be a sheaf on X.

DEFINITION 6.25. Given a covering $\mathfrak{U}$ of X by open sets U_j, an element of $C^q(\mathfrak{U},\mathscr{F})$ consists in defining for each $(q+1)$-uple $(U_{i_0},....,U_{i_q})$ an element $s(i_0,...,i_q)\in\mathscr{F}(U_{i_0}\cap ...\cap U_{i_q})$ such that $s(i_{\sigma(0)},i_{\sigma(1)},...,i_{\sigma(q)})=\varepsilon(\sigma)s(i_0,...,i_q)$.

If $s\in\check{C}^q(\mathfrak{U},\mathscr{F})$ we define $(\delta s)(i_0,i_1,...,i_{q+1})=\sum_j(-1)^j s(i_0,i_1,.,\widehat{i_j}..,i_{q+1})$. This construction defines a sheaf on X as follows: to an open set V we associate the covering of V by the $U_j\cap V$, and there is a natural map induced by restriction of the sections of $\mathscr{F}$, $\check{C}^q(\mathfrak{U},\mathscr{F})\to\check{C}^q(\mathfrak{U}\cap V,\mathscr{F})$ obtained by replacing U_j by $U_j\cup V$. Thus the Čech complex associated to a covering is a sheaf over X. We may consider the sheaf of complexes

$$0\to\mathscr{F}\xrightarrow{i}\check{C}^0(\mathfrak{U},\mathscr{F})\xrightarrow{\delta}...\xrightarrow{\delta}\check{C}^q(\mathfrak{U},\mathscr{F})\xrightarrow{\delta}\check{C}^{q+1}(\mathfrak{U},\mathscr{F})\xrightarrow{\delta}...$$

Note that this is not in general exact, hence it is not a resolution of $\mathscr{F}$.

However when the $H^j(U_{i_0}\cap...\cap U_{i_q},\mathscr{F})$ are zero for $j\geq 1$, we say we have an acyclic cover, and the cohomology of $\check{C}^q(\mathfrak{U},\mathscr{F})$ computes the cohomology of the sheaf $\mathscr{F}$.

3.3. Exercices.

(1) Let $\mathscr{A}$ be a sheaf over $\mathbb{N}$, $\mathbb{N}$ being endowed with the topology for which the open sets are $\{1,2,...,n\}$, $\mathbb{N}$ and $\emptyset$. Prove that a sheaf over $\mathbb{N}$ is equivalent to a sequence of R-modules, A_n and maps

$$...\to A_n\to A_{n-1}\to ....\to A_0$$

and that $H^0(\mathbb{N},\mathscr{A})=\lim_n A_n$. Describe $\lim^1(A_n)_{n\geq 1}\overset{def}{=}H^1(\mathbb{N},\mathscr{A})$

(2) Show that the above sheaf is flabby if and only if the maps $A_n\to A_{n-1}$ are onto, and that the sheaf is acyclic if and only if the sequence satisfies the Mittag-Leffler condition: the image of A_k in A_j is stationary as k goes to infinity.

4. Appendix: More on injective objects

Let us first show that the functor $A\to\mathrm{Mor}(A,L)$ is left exact, regardless of whether L is injective or not. Let $0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$ be an exact sequence. Since f is a monomorphism $\mathrm{Mor}(f):\mathrm{Mor}(B,L)\to\mathrm{Mor}(A,L)$ is the map $u\to u\circ f$. By definition of monomorphisms, this is injective, and we only have to prove $\mathrm{Im}(\mathrm{Mor}(g))=\mathrm{Ker}(\mathrm{Mor(f)})$. Assume $u\in\mathrm{Ker}(\mathrm{Mor(f)})$ so that $u\circ f=0$. According to proposition 5.10, $(C,g)=\mathrm{Coker}(f)$, so by definiton of the cokernel we get the factorization $u=v\circ g$.

LEMMA 6.26. *Let* $0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$ *be an exact sequence such that* A *is injective. Then there exists* $w:B\to A$ *such that* $w\circ f=\mathrm{id}_A$*. As a result there exists of* $u:C\to B$ *and* $v:B\to A$ *such that* $\mathrm{id}_B=f\circ v+u\circ g$*, and the sequence splits.*

PROOF. The existence of w follows from the definition of injectivity applied to $h = \mathrm{id}_A$. The map w is then given as the dotted map. Now since $f = f\circ w\circ f$ we get $(\mathrm{id}_A - f\circ w)\circ f = 0$, hence by definition of the Cokernel, and the fact that $C = \mathrm{Coker}(g)$, there is a map $u : C \to B$ such that $(\mathrm{id}_A - f\circ w) = u\circ g$. This proves the formula $\mathrm{id}_B = f\circ v + u\circ g$ with $v = w$. As a result, $g = g\circ \mathrm{Id}_B = g\circ f\circ v + g\circ u\circ g$, and $g\circ f = 0$, and since g is an epimorphism and $g = g\circ u\circ g$ we have $\mathrm{Id}_C = g\circ u$ and the sequence is split according to Definiton 5.9 and Exercice 3. ☐

LEMMA 6.27. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence with* A, B *injective. Then* C *is injective.*

PROOF. Indeed, the above lemma implies that the sequence splits, $B \simeq A\oplus C$, but the sum of two objects is injective if and only if they are both injectives: as injectivity is a lifting property, to lift a map to a direct sum, we must be able to lift to each factor. ☐

As a consequence any additive functor F will send a short exact sequences of injectives to a short exact sequences of injectives, since the image by F will be split, and a split sequence is exact. The same holds for a general exact sequence since it decomposes as $0 \to I_0 \to I_1 \to \mathrm{Ker}(\mathrm{d}_2) = \mathrm{Im}(\mathrm{d}_1) \to 0$. Since I_0, I_1 are injectives, so is $\mathrm{Ker}(\mathrm{d}_2) = \mathrm{Im}(\mathrm{d}_1)$. Now we use the exact sequence $0 \to \mathrm{Im}(d_1) \to I_2 \to \mathrm{Ker}(\mathrm{d}_3) = \mathrm{Im}(\mathrm{d}_2) \to 0$ to show that $\mathrm{Ker}(\mathrm{d}_3) = \mathrm{Im}(\mathrm{d}_2)$ is injective. Finally all the $\mathrm{Ker}(\mathrm{d_j})$ and $\mathrm{Im}(d_j)$ are injective. But this implies that the sequences $0 \to \mathrm{Im}(d_{m-1}) \to I_m \to \mathrm{Ker}(\mathrm{d_{m+1}}) = \mathrm{Im}(\mathrm{d_m}) \to 0$ are split, hence $0 \to F(\mathrm{Im}(d_{m-1})) \to F(I_m) \to F(\mathrm{Ker}(\mathrm{d_{m+1}})) = \mathrm{F}(\mathrm{Im}(\mathrm{d_m})) \to 0$ is split hence exact. This implies (Check !) that the sequence $0 \to F(I_0) \to F(I_1) \to F(I_2) \to F(I_3) \to$ is exact.

LEMMA 6.28 (Horseshoe lemma). *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence, and,* $I_A^\bullet$, $I_C^\bullet$ *be injective resolutions of* A *and* C. *Then there exists an injective resolution of* B, $I_B^\bullet$, *such that* $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ *is an exact sequence of complexes. Moroever, we can take* $I_B^\bullet = I_A^\bullet \oplus I_C^\bullet$.

PROOF. See [**Weib**] page 37. One can also use the Freyd-Mitchell theorem. ☐

PROPOSITION 6.29. *Let* $\mathscr{C}$ *be an abelian category with enough injectives. Let* $f : A \to B$ *be a morphism. Assume for any injective object* I, *the induced map* $f^* : \mathrm{Mor}(B, I) \to \mathrm{Mor}(A, I)$ *is an isomorphism, then* f *is an isomorphism.*

PROOF. Assume f is not a monomorphism. Then there exists a non-zero $u : K \to A$ such that $f\circ u = 0$. We first assume u is a monomorphism. Let $\pi : K \to I$ be a monomorphism into an injective I. Then there exists $v : A \to I$ such that $v\circ u = \pi$. Let $h : B \to I$ be such that $v = h\circ f$. We have $h\circ f\circ u = v\circ u = \pi$ but also $f\circ u = 0$ hence $h\circ f\circ u = 0$ which implies $\pi = 0$ a contradiction. Now we still have to prove that u may be supposed to be injective. But the map u can be factored as $t\circ s$ where $s : K \to \mathrm{Im}(u)$ and $t : \mathrm{Im}(u) \to A$ and t is mono and s is epi. Thus since $f\circ u = 0$, we have $f\circ t\circ s = 0$, but since s is epimorphisms, we have $f\circ t = 0$ with t mono. Assume now f is not an epimorphism;

Then there exists a nonzero map $v : B \to C$ such that $v \circ f = 0$. We now send C to an injective I by a monomorphism π. Then $(\pi \circ v) \circ f = 0$, and $\pi \circ v$ is nonzero, since π is a monomorphism. We thus get a non zero map $\pi \circ v \in \mathrm{Mor}(B, I)$ such that its image by f^* in $\mathrm{Mor}(A, I)$ is zero. □

As an example we consider the case of sheaves. Let $\mathscr{F}, \mathscr{G}$ be sheaves over X, and $f : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves. We consider an injective sheaf, $\mathscr{I}$, then $\mathrm{Mor}(\mathscr{F}, \mathscr{I}) = \bigcup_x \mathrm{Mor}(\mathscr{F}_x, \mathscr{I}(x))$, so that the map f^* on each component will give $f_x : \mathscr{F}_x \to \mathscr{G}_x$. If this map is an isomorphism, then f is an isomorphism.

One should be careful: the map f must be given, and the fact that $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic for all x does not imply the isomorphism of $\mathscr{F}$ and $\mathscr{G}$.

4.1. Appendix: Poincaré-Verdier Duality. Let $f : X \to Y$ be a continuous map between manifolds. We want to define the map $f^!$, and then of course $Rf^!$, adjoint of $f_!$ and $Rf_!$. This is the sheaf theoretic version of Poincaré duality.