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**Titre : Théorèmes de structure des variétés symplectiques compactes via des techniques presque complexes.**

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CHAPITRE I

# Introduction

## 1. Introduction

Une approche de la topologie symplectique qui s'est révélée extrêmement fructueuse au fil des années a pour point de départ l'observation suivante : toute variété symplectique $(X, \omega)$ peut être munie d'une *structure presque-complexe* compatible avec la structure symplectique, c'est-à-dire un endomorphisme $J$ du fibré tangent $TX$, vérifiant $J^2 = -1$, et tel que la forme bilinéaire $g(x, y) = \omega(x, Jy)$ définit une métrique riemannienne sur $X$ (voir par exemple [McS1], p. 116).

L'étude des variétés symplectiques se présente alors comme une généralisation naturelle de la géométrie kählérienne : en effet la variété $X$ est kählérienne dès lors que la structure presque-complexe $J$ est *intégrable*, c'est-à-dire permet de définir localement des coordonnées complexes sur $X$. Les variétés kählériennes fournissent un grand nombre d'exemples de variétés symplectiques, puisqu'elles englobent entre autres toutes les variétés algébriques projectives complexes. Toutefois, de nombreuses variétés symplectiques n'admettant pas de structure kählérienne ont été construites, notamment par Thurston [Th] et plus récemment par Gompf [Go].

La donnée d'une structure presque-complexe $J$ sur $X$ conduit naturellement à étudier les sous-variétés de $X$ dont l'espace tangent est en tout point $J$-invariant, c'est-à-dire un sous-espace complexe de l'espace tangent à $X$. Pour un choix générique de la structure presque-complexe compatible avec $\omega$, de telles sous-variétés n'existent qu'en dimension complexe 1 : ce sont les célèbres *courbes pseudo-holomorphes*, introduites par Gromov [Gro1] et dont la théorie a connu de constants développements (voir par exemple [McS2]).

Un autre exemple frappant de l'analogie entre variétés symplectiques compactes et variétés kählériennes compactes est donné par les invariants de Seiberg-Witten (voir par exemple [Mor]), dont l'interprétation en termes de courbes pseudo-holomorphes récemment obtenue par Taubes ([T1], [T2] et suivants) pour les variétés symplectiques présente des similarités remarquables avec l'interprétation en termes de courbes complexes donnée par Witten [W] pour les variétés kählériennes.

Néanmoins, certaines constructions de géométrie algébrique complexe semblaient jusqu'à récemment ne pas pouvoir être transposées dans un contexte symplectique : ainsi, le lieu d'annulation d'une section holomorphe générique d'un fibré très ample sur une variété projective complexe définit une sous-variété complexe, tandis que la construction analogue ne fonctionne pas en géométrie presque complexe.

L'idée introduite par Donaldson [D1] consiste à autoriser de petites variations de la structure presque-complexe : ainsi, une structure presque-complexe $J$ compatible avec $\omega$ étant fixée, il s'agit de construire non pas des sous-variétés $J$-holomorphes (qui n'existent en général pas au-delà de la dimension 1), mais plutôt des sous-variétés *approximativement $J$-holomorphes*. De telles sous-variétés sont obtenues comme lieux d'annulation de sections approximativement holomorphes de fibrés convenablement choisis : l'observation fondamentale de Donaldson est que, de même qu'un fibré holomorphe suffisamment positif sur une variété projective admet un grand nombre de sections holomorphes, un fibré en droites suffisamment positif sur une variété symplectique compacte admet de nombreuses sections approximativement holomorphes. Toutefois, contrairement au cas projectif où un argument facile de transversalité permet d'obtenir immédiatement des hypersurfaces complexes, un raisonnement assez long est nécessaire pour prouver l'existence d'une section dont les propriétés de transversalité à la section nulle garantissent que le lieu d'annulation est une sous-variété lisse et approximativement $J$-holomorphe.

Il est aisé de vérifier qu'une sous-variété approximativement $J$-holomorphe $W \subset X$ est *symplectique*, c'est-à-dire que la restriction de $\omega$ munit $W$ d'une structure symplectique. En outre, il existe une structure presque-complexe $J'$ compatible avec $\omega$, proche de $J$ (mais dépendant de $W$), telle que $W$ soit une sous-variété $J'$-holomorphe de $X$. Un intérêt majeur du résultat de Donaldson est donc de fournir le premier procédé général de construction d'hypersurfaces symplectiques (codimension réelle 2) dans une variété symplectique compacte arbitraire [D1].

Dans [A1] (voir aussi §2 et chapitre II), ce résultat a été étendu au cas de fibrés de rang supérieur : la construction de sections approximativement holomorphes vérifiant des propriétés convenables de transversalité permet alors d'obtenir des sous-variétés symplectiques (approximativement $J$-holomorphes) de codimension quelconque, dont on détermine mieux le type topologique que par simple itération du résultat de [D1]. Il a de plus été établi que, en dépit de la grande flexibilité de la construction de Donaldson, les sous-variétés que l'on est susceptible d'obtenir à partir de sections approximativement holomorphes d'un fibré suffisamment positif donné sont uniques à isotopie symplectique près [A1] (cf. §2 et chapitre II) ; en outre, ce résultat reste vrai même si l'on fait varier la structure presque-complexe. Cela signifie que les sous-variétés symplectiques construites à partir de fibrés suffisamment positifs peuvent être utilisées pour définir des invariants symplectiques de la variété considérée : ainsi des invariants topologiques définis pour des variétés de petite dimension (par exemple ceux de Seiberg-Witten en dimension 4) peuvent être utilisés pour caractériser des variétés symplectiques de dimension supérieure.

D'autres résultats classiques de géométrie projective complexe peuvent être transposés à la géométrie symplectique de façon similaire. Ainsi, Donaldson a été le premier à montrer que deux sections approximativement holomorphes convenablement choisies d'un fibré en droites suffisamment positif déterminent une structure de *pinceau de Lefschetz symplectique* sur une variété symplectique compacte [D2]. Une telle structure est l'analogue symplectique de la notion classique de pinceau de Lefschetz algébrique : la variété considérée est remplie par une famille d'hypersurfaces symplectiques indexées par $\mathbb{CP}^1$, s'intersectant le long d'une sous-variété symplectique lisse de codimension (réelle) 4 (les "points base"), toutes les hypersurfaces du pinceau étant lisses excepté un nombre fini d'entre elles dont les points singuliers sont isolés et relativement simples (des points doubles à croisement normal dans le cas de la dimension 4) ; après éclatement le long des points base, on obtient une *fibration de Lefschetz* symplectique au-dessus de $\mathbb{CP}^1$. Outre [D2], on pourra se référer au §6.2 du chapitre III où est esquissée une preuve du résultat utilisant les arguments de [A1] et [A2], ainsi qu'au texte de Sikorav [Si] ; enfin, une étude poussée des pinceaux de Lefschetz symplectiques en dimension 4 se trouve dans [BK].

Dans le même esprit, il a été établi dans [A2] (voir aussi §3 et chapitre III) que toute variété symplectique compacte $X$ de dimension 4 peut être vue comme un revêtement ramifié (singulier) de $\mathbb{CP}^2$ : le revêtement est déterminé par trois sections approximativement holomorphes soigneusement choisies d'un fibré en droites très positif sur $X$. De plus, le choix d'un fibré en droites suffisamment positif détermine canoniquement une classe d'isotopie de revêtements ramifiés (singuliers) $X \to \mathbb{CP}^2$, indépendamment de la structure presque-complexe compatible considérée [A2]. Il semble probable que de nombreux autres résultats classiques de géométrie projective admettent de la même façon des analogues symplectiques.

Le reste de ce chapitre a pour but de donner un aperçu des principaux résultats obtenus au cours de la réalisation de ce travail, et de les illustrer par divers exemples. Les énoncés décrits ci-dessous sont formulés de façon plus précise et démontrés dans les chapitres II et III ([A1] et [A2]). On pourra également se référer à [D1] pour l'argument original de Donaldson, ainsi qu'à [Si] pour une synthèse des résultats de [D1], [A1] et [D2].

## 2. Sous-variétés symplectiques : énoncés et exemples

Soit $(X, \omega)$ une variété symplectique compacte de dimension $2n$. On fera dans tout ce qui suit l'hypothèse que la classe de cohomologie $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ est entière. Cette condition d'intégralité n'est pas une restriction très forte, car dans tous les cas il existe des formes symplectiques $\omega'$ arbitrairement proches de $\omega$ et qui, après multiplication par un facteur entier, vérifient la condition requise. Une structure presque-complexe $J$ compatible avec $\omega$ et la métrique riemannienne correspondante $g$ sont également fixées.

Soit $L$ le fibré en droites complexes sur $X$ dont la classe de Chern est $c_1(L) = \frac{1}{2\pi}[\omega]$, muni d'une métrique hermitienne et d'une connexion hermitienne $\nabla^L$ dont la 2-forme de courbure est égale à $-i\omega$ (l'existence d'une telle connexion est facile à établir : la courbure d'une connexion hermitienne quelconque $\nabla$ sur $L$ diffère de $-i\omega$ par une 2-forme exacte qui peut se mettre sous la forme $i\,da$ avec $a \in \Omega^1(X, \mathbb{R})$ ; on peut alors choisir $\nabla^L = \nabla + i\,a$). L'observation fondamentale est que, pour des valeurs suffisamment grandes du paramètre entier $k$, le fibré en droites $L^k$ admet de nombreuses sections approximativement holomorphes, qui déterminent un plongement approximativement holomorphe de $X$ dans un espace projectif : il s'agit d'un analogue symplectique de la construction classique de Kodaira (voir par exemple [GH], §1.4). L'intuition dicte alors qu'un hyperplan convenablement choisi doit, par intersection avec $X$, déterminer une sous-variété symplectique approximativement $J$-holomorphe de $X$. La formulation rigoureuse de cette construction (voir [D1]) nécessite l'introduction des notions d'*holomorphie asymptotique* et de *transversalité uniforme à* 0 d'une famille de sections.

Soient $(E_k)_{k\gg 0}$ des fibrés vectoriels complexes sur $X$, tous munis d'une métrique hermitienne et d'une connexion hermitienne. La structure presque-complexe $J$ sur $X$ et la connexion sur $E_k$ déterminent des opérateurs $\partial$ et $\bar\partial$ sur $E_k$.

Les fibrés $E_k$ que l'on utilisera dans la suite sont définis à partir des fibrés en droites $L^k$ : par exemple, on s'intéressera particulièrement au cas de $E_k = E \otimes L^k$, où $E$ est un fibré vectoriel hermitien fixé muni d'une connexion hermitienne $\nabla^E$. Les structures et connexions hermitiennes dont on munit $E_k$ naturellement sont alors induites par celles de $E$ et $L$. En particulier, la connexion hermitienne induite par $\nabla^L$ sur $L^k$ a pour courbure $-ik\omega$ ; il s'ensuit que les variations des sections des fibrés $E_k$ que l'on considérera tendent naturellement à se produire à des échelles de l'ordre de $k^{-1/2}$ (à cause de l'identité liant leurs dérivées secondes à la courbure). Il est donc utile de remplacer la métrique $g$ sur $X$ par la métrique renormalisée $g_k = k.g$ : le diamètre de $X$ est alors multiplié par $k^{1/2}$, et les dérivées d'ordre $p$ des sections sont divisées par $k^{p/2}$.

Remarque : contrairement à la convention adoptée ici ainsi qu'au chapitre III et dans l'ensemble de la littérature, dans le chapitre II ci-dessous ([A1]) la métrique $g_k$ n'est pas utilisée, et toutes les estimées sont données pour la métrique $g$, ce qui introduit des facteurs $k^{1/2}$ supplémentaires dans les définitions. Par ailleurs on peut indifféremment travailler avec des estimées de type $C^r$ ([D1], [A1]) ou $C^\infty$ ([A2]).

Définition 1. *Des sections $(s_k)_{k\gg 0}$ de fibrés vectoriels complexes $E_k$ sur $X$ sont dites* asymptotiquement holomorphes *si, pour tout $p \in \mathbb{N}$, les dérivées covariantes $\nabla^p s_k$ et les quantités $k^{1/2}\nabla^p\bar\partial s_k$ sont uniformément bornées (pour la métrique $g_k$) par des constantes indépendantes de $k$, c'est-à-dire si*

$$\forall p \in \mathbb{N},\ \ |s_k|_{C^p,g_k} = O(1) \ \ et\ \ |\bar\partial s_k|_{C^p,g_k} = O(k^{-1/2}).$$

Définition 2. *Des sections $(s_k)_{k\gg 0}$ de $E_k$ sur $X$ sont dites* uniformément transverses à 0 *s'il existe une constante $\eta > 0$ telle que, pour tout $k$ et en tout point $x \in X$ tel que $|s_k(x)| < \eta$, la dérivée covariante $\nabla s_k(x) : T_xX \to (E_k)_x$ est surjective et "plus grande que $\eta$" (c'est-à-dire admet un inverse à droite de norme inférieure à $\eta^{-1}$).*

On vérifie aisément que, si des sections $(s_k)_{k\gg 0}$ de $E_k$ sont simultanément asymptotiquement holomorphes et uniformément transverses à 0, alors pour $k$ suffisamment grand le lieu d'annulation $W_k = s_k^{-1}(0)$ est une sous-variété symplectique lisse de $X$. Les sous-variétés $W_k$ sont de plus asymptotiquement $J$-holomorphes, en ce sens qu'en tout point de $W_k$ le sous-espace $J(TW_k)$ est à distance $O(k^{-1/2})$ de $TW_k$. Les résultats principaux du chapitre II peuvent alors être formulés de la façon suivante :

Théorème 1 ([A1]). *Soit $E$ un fibré vectoriel complexe quelconque sur $X$ : pour $k$ suffisamment grand, les fibrés $E \otimes L^k$ admettent des sections asymptotiquement holomorphes et uniformément transverses à 0, dont les lieux d'annulation sont des sous-variétés symplectiques lisses de $X$.*

Ce théorème (chapitre II, Corollaire 1) étend le résultat principal obtenu par Donaldson dans [D1], qui correspond au cas où $E$ est le fibré en droites trivial. Le résultat d'unicité suivant (chapitre II, Corollaire 2) est également établi :

Théorème 2 ([A1]). *Les sous-variétés symplectiques que l'on peut construire à partir de sections asymptotiquement holomorphes et uniformément transverses à 0 de $E \otimes L^k$ sont, pour toute valeur suffisamment grande de $k$, uniques à isotopie symplectique près.*

En outre, ce résultat d'unicité demeure vrai si l'on considère des sous-variétés obtenues pour différentes structures presque-complexes compatibles avec $\omega$. Pour $k$ suffisamment grand, le type topologique des sous-variétés symplectiques construites est donc un invariant symplectique de $(X, \omega)$.

Exemple. Dans le tore $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ muni de la forme symplectique standard $\omega = 4\pi(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)$, différentes topologies sont envisageables pour des sous-variétés symplectiques représentant la classe d'homologie duale de $\frac{k}{2\pi}[\omega]$ pour la dualité de Poincaré. La configuration la plus simple, qui correspond à ce que l'on obtient naturellement pour $k$ suffisamment grand à partir de sections asymptotiquement holomorphes et uniformément transverses à 0 de $L^k$, est une surface de Riemann connexe de genre $4k^2 + 1$ : une telle sous-variété peut par exemple être obtenue par désingularisation (à l'aide de sommes connexes) de la sous-variété singulière $(F_k \times T^2) \cup (T^2 \times F_k)$ où $F_k \subset T^2$ est un ensemble fini constitué de $2k$ points de $T^2$.

Toutefois, d'autres types de sous-variétés symplectiques permettent de réaliser la même classe d'homologie : par exemple des surfaces constituées de deux composantes disjointes chacune de genre $2k^2+1$. Il est aisé de vérifier sans même invoquer le Théorème 2 que de telles sous-variétés ne peuvent être obtenues par les méthodes décrites ici (Proposition 2 du chapitre II). La construction de ces sous-variétés non connexes se fonde sur l'existence d'une décomposition $\frac{1}{2\pi}\omega = \xi + \zeta$ où les 2-formes $\xi$ et $\zeta$ sont telles que $\omega \wedge \xi > 0$, $\omega \wedge \zeta > 0$ et $\xi \wedge \zeta = 0$ : les classes d'homologie duales de $[k\xi]$ et $[k\zeta]$ peuvent alors être représentées par des courbes symplectiques disjointes, chaque composante étant de genre $2k^2 + 1$ et obtenue comme ci-dessus par désingularisation de familles de 2-tores plats de $T^4$. □

Cet exemple illustre la non-trivialité du Théorème 2 : contrairement à ce qui se passe en géométrie complexe où toutes les hypersurfaces projectives lisses d'une classe d'homologie donnée sont difféomorphes, dans le cas symplectique différents types topologiques peuvent coexister dans une même classe d'homologie. Le résultat d'unicité décrit ici ainsi que plusieurs autres propriétés décrites dans [D1] et [A1] indiquent que, par de nombreux aspects, les sous-variétés approximativement holomorphes étudiées ici semblent avoir un comportement topologique plus proche de la géométrie projective complexe que de la géométrie symplectique usuelle.

La détermination du type topologique des sous-variétés construites est en général difficile, même si des invariants élémentaires tels que les nombres de Betti peuvent être calculés explicitement (Proposition 2 et §5.2 du chapitre II). En général, cette détermination complète n'est possible que dans quelques cas tels que ceux des sous-variétés de dimension 2 (ce sont des surfaces de Riemann connexes dont le genre se calcule aisément), ou parfois lorsque la variété $X$ est une variété algébrique ou encore un produit de variétés symplectiques.

Exemple. On considère le cas où $X^6 = M^4 \times \Sigma^2$ est le produit cartésien de variétés symplectiques $(M, \omega_M)$ de dimension 4 et $(\Sigma, \omega_\Sigma)$ de dimension 2 vérifiant toutes deux la condition d'intégralité requise. Par le Théorème 1, on peut construire pour $k$ suffisamment grand des courbes symplectiques $\Sigma_k \subset M$ de classe fondamentale $\frac{k}{2\pi}[\omega_M]$ (le genre de ces courbes connexes se calcule par la formule d'adjonction), ainsi que des parties finies $F_k$ constituées de $f_k = \frac{k}{2\pi} \int_\Sigma \omega_\Sigma$ points de $\Sigma$.

La variété $X$ étant munie de la structure symplectique produit $\omega = \pi_1^* \omega_M + \pi_2^* \omega_\Sigma$, on peut alors établir (à l'aide du Théorème 2 du chapitre II) que les fibrés $L^k$ sur $X$ admettent des sections asymptotiquement holomorphes et uniformément transverses à 0 dont les lieux d'annulation $W_k$ sont arbitrairement proches des sous-variétés singulières $V_k = (\Sigma_k \times \Sigma) \cup (M \times F_k)$. La topologie naturelle des sous-variétés symplectiques de $X$ de classe fondamentale duale de $\frac{k}{2\pi}[\omega]$ décrites par le Théorème 1 correspond donc à une désingularisation (au voisinage de $\Sigma_k \times F_k$) de la sous-variété $V_k$. On peut montrer que la construction qui permet d'obtenir $W_k$ à partir de $\Sigma_k \times \Sigma$ et de $M \times F_k$ est une opération de *somme connexe symplectique avec éclatements* le long de $\Sigma_k \times F_k$ : si on note $M_k$ la variété obtenue à partir de $M$ par éclatement de points de $\Sigma_k$ jusqu'à rendre trivial le fibré normal de $\Sigma_k$, la sous-variété $W_k$ s'obtient en recollant à $\Sigma_k \times \Sigma$ un exemplaire de la variété éclatée $M_k$ le long de chacune des $f_k$ composantes de $\Sigma_k \times F_k$. On se référera à [Go] pour une description plus précise du procédé de somme connexe symplectique. □

La construction de sous-variétés symplectiques de dimension 4 (canoniques à isotopie près d'après le Théorème 2) est particulièrement intéressante, car les nombreux invariants de dimension 4 définis pour ces sous-variétés fournissent autant d'invariants symplectiques de la variété ambiante. Dans le cas des invariants de Seiberg-Witten, cette approche est toutefois décevante, car les invariants des sous-variétés construites pour $k$ grand semblent contenir très peu d'information : dans le cas des variétés projectives, les sous-variétés obtenues sont des surfaces algébriques de type général, dont les invariants de Seiberg-Witten sont peu intéressants [FM] (ils ne décrivent que la classe de Chern $c_1(TX)$ et la présence d'éventuelles sphères exceptionnelles), et dans le cas décrit ci-dessus du produit $X^6 = M^4 \times \Sigma^2$, le calcul partiel à l'aide de formules pour les sommes connexes telles que celle de [MST] n'est pas plus fructueux. Toutefois, il est probable que des invariants plus fins tels que ceux qui décrivent la topologie des structures de pinceaux de Lefschetz symplectiques en dimension 4 ([D2], voir aussi [Si] et [BK]), appliqués aux sous-variétés de dimension 4 données par le Théorème 1, permettent d'obtenir des informations plus précises sur la topologie de la variété ambiante.

## 3. Revêtements ramifiés de $\mathbb{CP}^2$

Les théorèmes d'existence de sections asymptotiquement holomorphes et uniformément transverses à 0 décrits ci-dessus constituent également un premier pas vers l'obtention de structures plus élaborées : ainsi, il est possible pour $k$ suffisamment grand de construire une section approximativement holomorphe de $\mathbb{C}^2 \otimes L^k$ (c'est-à-dire un couple de sections de $L^k$) dont les propriétés de transversalité entrainent l'existence d'une structure de *pinceau de Lefschetz symplectique* sur $X$ ([D2], voir également le §6.2 du chapitre III). Dans cette partie nous nous intéresserons plus particulièrement à la construction de *trois* sections approximativement holomorphes de $L^k$ (c'est-à-dire une section de $\mathbb{C}^3 \otimes L^k$) lorsque $X$ est de dimension 4, ce qui permet d'aboutir au résultat que toute variété symplectique compacte de dimension 4 est un revêtement ramifié (singulier) de $\mathbb{CP}^2$ [A2]. Les énoncés et définitions ci-dessous sont formulés plus précisément et démontrés dans le chapitre III.

Il existe un lien naturel entre sections de $\mathbb{C}^3 \otimes L^k$ et applications à valeurs dans $\mathbb{CP}^2$ : la donnée d'une section $s = (s_0, s_1, s_2)$ de $\mathbb{C}^3 \otimes L^k$ qui ne s'annule pas sur $X$ permet de définir une application $f(x) = [s_0(x) : s_1(x) : s_2(x)]$ (en coordonnées homogènes) de $X$ dans $\mathbb{CP}^2$. Lorsque $k$ est assez grand, il est possible de construire des sections approximativement holomorphes de $\mathbb{C}^3 \otimes L^k$ qui ne s'annulent pas et dont les propriétés de généricité et de compatibilité avec la structure presque-complexe suffisent à assurer que l'application projective correspondante est un *revêtement ramifié singulier* approximativement holomorphe (la formulation précise des propriétés requises étant relativement compliquée, on se réfèrera aux Définitions 5, 6 et 7 du chapitre III).

Le terme de revêtement *ramifié* fait référence au fait que l'on autorise des feuillets du revêtement à se rejoindre le long d'une sous-variété $R$ appelée *lieu de ramification* : l'exemple le plus simple est l'application $(x, y) \mapsto (x^2, y)$ de $\mathbb{C}^2$ dans $\mathbb{C}^2$, dont le lieu de ramification est $\mathbb{C} \times 0$. En outre, le revêtement décrit ici est *singulier* de par la présence de points isolés où le lieu de ramification $R$ devient "vertical", c'est-à-dire que la restriction de $f$ à $R$ cesse d'être une immersion et l'image $f(R)$ présente alors un *cusp* : un exemple type est l'application $(x, y) \mapsto (x^3 - xy, y)$ de $\mathbb{C}^2$ dans $\mathbb{C}^2$ au voisinage de l'origine. Une description plus précise de la notion de revêtement ramifié singulier est donnée au §1 du chapitre III. En outre, dans le cas que l'on considère ici l'application de revêtement est approximativement holomorphe, ce qui implique en particulier que le lieu de ramification $R$ est une sous-variété symplectique approximativement holomorphe de $X$ et que son image par $f$ est une sous-variété symplectique singulière de $\mathbb{CP}^2$.

Le résultat principal de [A2] peut se formuler de la façon suivante (cf. Théorèmes 1 et 4 du chapitre III) :

Théorème 3. *Pour tout $k$ suffisamment grand, il est possible de construire des sections de $\mathbb{C}^3 \otimes L^k$ qui donnent à $X$ une structure de revêtement ramifié singulier approximativement holomorphe au-dessus de $\mathbb{CP}^2$. En outre, pour chaque valeur suffisamment grande de $k$ la topologie d'un tel revêtement est canoniquement déterminée à isotopie près.*

De même que les structures de pinceaux de Lefschetz symplectiques des variétés symplectiques de dimension 4 permettent, par l'étude de la monodromie de la fibration de Lefschetz correspondante au-dessus de $\mathbb{CP}^1$, de définir des invariants symplectiques très fins ([D2], [BK]), il est possible d'exploiter le Théorème 3 pour construire de nouveaux invariants symplectiques.

La topologie d'un revêtement ramifié $f : X \to \mathbb{CP}^2$ est en grande partie déterminée par celle de l'image du lieu de ramification $D = f(R) \subset \mathbb{CP}^2$. Dans notre cas, $D$ est une courbe symplectique singulière dans $\mathbb{CP}^2$, dont les seules singularités sont génériquement des cusps et des points doubles (*a priori*, il n'est pas certain que l'on puisse exclure les points doubles à auto-intersection négative comme c'est le cas en géométrie projective). L'étude des invariants qui caractérisent la topologie d'une telle sous-variété de $\mathbb{CP}^2$ est actuellement l'objet d'un travail en collaboration avec L. Katzarkov, en faisant appel à des techniques introduites et développées en géométrie complexe par Moishezon dans les années 80 ([Moi1], [Moi2], ...) et qui permettent de se ramener à l'étude d'une factorisation dans un groupe de tresses. Les perspectives offertes par de tels invariants pour la résolution de divers problèmes ouverts importants en topologie symplectique de la dimension 4 semblent d'ores et déjà prometteuses, même si un travail important reste à fournir avant que la topologie des revêtements ramifiés singuliers de $\mathbb{CP}^2$ soit entièrement comprise.

CHAPITRE II

# Asymptotically Holomorphic Families of Symplectic Submanifolds

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Abstract. We construct a wide range of symplectic submanifolds in a compact symplectic manifold as zero sets of asymptotically holomorphic sections of vector bundles obtained by tensoring an arbitrary vector bundle by large powers of the complex line bundle whose first Chern class is the symplectic form (assuming a suitable integrality condition). We also show that, asymptotically, all sequences of submanifolds constructed from a given vector bundle are isotopic. Furthermore, we prove a result analogous to the Lefschetz hyperplane theorem for the constructed submanifolds.

## 1. Introduction

In a recent paper [D1], Donaldson has exhibited an elementary construction of symplectic submanifolds of codimension 2 in a compact symplectic manifold, where the submanifolds are seen as zero sets of asymptotically holomorphic sections of well-chosen line bundles. In this paper, we extend this construction to higher rank bundles as well as to one-parameter families, and obtain as a consequence an important isotopy result.

In all the following, $(X, \omega)$ will be a compact symplectic manifold of dimension $2n$, such that the cohomology class $\frac{1}{2\pi}[\omega]$ is integral. A compatible almost-complex structure $J$ and the corresponding riemannian metric $g$ are fixed. Let $L$ be the complex line bundle on $X$ whose first Chern class is $c_1(L) = \frac{1}{2\pi}[\omega]$. Fix a Hermitian structure on $L$, and let $\nabla^L$ be a Hermitian connection on $L$ whose curvature 2-form is equal to $-i\omega$ (it is clear that such a connection always exists).

We will consider families of sections of bundles of the form $E \otimes L^k$ on $X$, defined for all large values of an integer parameter $k$, where $E$ is any Hermitian vector bundle over $X$. The connection $\nabla^L$ induces a connection of curvature $-ik\omega$ on $L^k$, and together with any given Hermitian connection $\nabla^E$ on $E$ this yields a Hermitian connection on $E \otimes L^k$ for any $k$. We are interested in sections which satisfy the following two properties :

Definition 1. *A sequence of sections $s_k$ of $E \otimes L^k$ (for large $k$) is said to be asymptotically holomorphic with respect to the given connections and almost-complex structure if the following bounds hold :*

$$|s_k| = O(1), \qquad |\nabla s_k| = O(k^{1/2}), \qquad |\bar\partial s_k| = O(1),$$
$$|\nabla\nabla s_k| = O(k), \qquad |\nabla\bar\partial s_k| = O(k^{1/2}).$$

Since $X$ is compact, up to a change by a constant factor in the estimates, the notion of asymptotic holomorphicity does not actually depend on the chosen Hermitian structures and on the chosen connection $\nabla^E$. On the contrary, the connection $\nabla^L$ is essentially determined by the symplectic form $\omega$, and the positivity property of its curvature is the fundamental ingredient that makes the construction possible.

DEFINITION 2. *A section $s$ of a vector bundle $E \otimes L^k$ is said to be $\eta$-transverse to 0 if whenever $|s(x)| < \eta$, the covariant derivative $\nabla s(x) : T_xX \to (E \otimes L^k)_x$ is surjective and admits a right inverse whose norm is smaller than $\eta^{-1}.k^{-1/2}$. A family of sections is transverse to 0 if there exists an $\eta > 0$ such that $\eta$-transversality to 0 holds for all large values of $k$.*

In the case of line bundles, $\eta$-transversality to 0 simply means that the covariant derivative of the section is larger than $\eta k^{1/2}$ wherever the section is smaller than $\eta$. Also note that transversality to 0 is an *open* property : if $s$ is $\eta$-transverse to 0, then any section $\sigma$ such that $|s - \sigma| < \epsilon$ and $|\nabla s - \nabla \sigma| < k^{1/2}\epsilon$ is automatically $(\eta - \epsilon)$-transverse to 0. The following holds clearly, independently of the choice of the connections on the vector bundles :

PROPOSITION 1. *Let $s_k$ be sections of the vector bundles $E \otimes L^k$ which are simultaneously asymptotically holomorphic and transverse to 0. Then for all large enough $k$, the zero sets $W_k$ of $s_k$ are embedded symplectic submanifolds in $X$. Furthermore, the submanifolds $W_k$ are asymptotically $J$-holomorphic, i.e. $J(TW_k)$ is within $O(k^{-1/2})$ of $TW_k$.*

The result obtained by Donaldson [D1] can be expressed as follows :

THEOREM 1. *For all large $k$ there exist sections of the line bundles $L^k$ which are transverse to 0 and asymptotically holomorphic (with respect to connections with curvature $-ik\omega$ on $L^k$).*

Our main result is the following (the extension to almost-complex structures that depend on the parameter $t$ was suggested by the referee) :

THEOREM 2. *Let $E$ be a complex vector bundle of rank $r$ over $X$, and let a parameter space $T$ be either $\{0\}$ or $[0, 1]$. Let $(J_t)_{t\in T}$ be a family of almost-complex structures on $X$ compatible with $\omega$. Fix a constant $\epsilon > 0$, and let $(s_{t,k})_{t\in T, k\geq K}$ be a sequence of families of asymptotically $J_t$-holomorphic sections of $E \otimes L^k$ defined for all large $k$, such that the sections $s_{t,k}$ and their derivatives depend continuously on $t$.*

*Then there exist constants $\tilde{K} \geq K$ and $\eta > 0$ (depending only on $\epsilon$, the geometry of $X$ and the bounds on the derivatives of $s_{t,k}$), and a sequence $(\sigma_{t,k})_{t\in T, k\geq \tilde{K}}$ of families of asymptotically $J_t$-holomorphic sections of $E \otimes L^k$ defined for all $k \geq \tilde{K}$, such that*

*(a) the sections $\sigma_{t,k}$ and their derivatives depend continuously on $t$,*
*(b) for all $t \in T$, $|\sigma_{t,k} - s_{t,k}| < \epsilon$ and $|\nabla\sigma_{t,k} - \nabla s_{t,k}| < k^{1/2}\epsilon$,*
*(c) for all $t \in T$, $\sigma_{t,k}$ is $\eta$-transverse to 0.*

Note that, since we allow the almost-complex structure on $X$ to depend on $t$, great care must be taken as to the choice of the metric on $X$ used for the estimates on derivatives. The most reasonable choice, and the one which will be made in the proof, is to always use the same metric, independently of $t$ (so, there is no relation between $g$, $\omega$ and $J_t$). However, it is clear from the statement of the theorem that, since the spaces $X$ and $T$ are compact, any change in the choice of metric can be absorbed by simply changing the constants $\tilde{K}$ and $\eta$, and so the result holds in all generality.

Theorem 2 has many consequences. Among them, we mention the following extension of Donaldson's result to higher rank bundles :

Corollary 1. *For any complex vector bundle $E$ over $X$ and for all large $k$, there exist asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, and thus whose zero sets are embedded symplectic submanifolds in $X$. Furthermore given a sequence of asymptotically holomorphic sections of $E \otimes L^k$ and a constant $\epsilon > 0$, we can require that the transverse sections lie within $\epsilon$ in $C^0$ sense (and $k^{1/2}\epsilon$ in $C^1$ sense) of the given sections.*

Therefore, homology classes that one can realize by this construction include all classes whose Poincaré dual is of the form $(\frac{k}{2\pi}[\omega])^r + c_1.(\frac{k}{2\pi}[\omega])^{r-1} + \ldots + c_r$, with $c_1, \ldots, c_r$ the Chern classes of any complex vector bundle and $k$ any sufficiently large integer.

An important result that one can obtain on the sequences of submanifolds constructed using Corollary 1 is the following isotopy result derived from the case where $T = [0,1]$ in Theorem 2 and which had been conjectured by Donaldson in the case of line bundles :

Corollary 2. *Let $E$ be any complex vector bundle over $X$, and let $s_{0,k}$ and $s_{1,k}$ be two sequences of sections of $E \otimes L^k$. Assume that these sections are asymptotically holomorphic with respect to almost-complex structures $J_0$ and $J_1$ respectively, and that they are $\epsilon$-transverse to 0. Then for all large $k$ the zero sets of $s_{0,k}$ and $s_{1,k}$ are isotopic through asymptotically holomorphic symplectic submanifolds. Moreover, this isotopy can be realized through symplectomorphisms of $X$.*

This result follows from Theorem 2 by defining sections $s_{t,k}$ and almost-complex structures $J_t$ that interpolate between $(s_{0,k}, J_0)$ and $(s_{1,k}, J_1)$ in the following way : for $t \in [0, \frac{1}{3}]$, let $s_{t,k} = (1 - 3t)s_{0,k}$ and $J_t = J_0$ ; for $t \in [\frac{1}{3}, \frac{2}{3}]$, let $s_{t,k} = 0$ and take $J_t$ to be a path of compatible almost-complex structures from $J_0$ to $J_1$ (this is possible since the space of compatible almost-complex structures is connected) ; and for $t \in [\frac{2}{3}, 1]$, let $s_{t,k} = (3t - 2)s_{1,k}$ and $J_t = J_1$. One can then apply Theorem 2 and obtain for all large $k$ and for all $t \in [0,1]$ sections $\sigma_{t,k}$ that differ from $s_{t,k}$ by at most $\epsilon/2$ and are $\eta$-transverse to 0 for some $\eta$. Since transversality to 0 is an open property, the submanifolds cut out by $\sigma_{0,k}$ and $\sigma_{1,k}$ are clearly isotopic to those cut out by $s_{0,k}$ and $s_{1,k}$. Moreover, the family $\sigma_{t,k}$ gives an isotopy between the zero sets of $\sigma_{0,k}$ and $\sigma_{1,k}$. So the constructed submanifolds are isotopic. The proof that this isotopy can be realized through symplectomorphisms of $X$ will be given in Section 4.

As a first step in the characterization of the topology of the constructed submanifolds, we also prove the following statement, extending the result obtained by Donaldson in the case of the line bundles $L^k$ :

Proposition 2. *Let $E$ be a vector bundle of rank $r$ over $X$, and let $W_k$ be a sequence of symplectic submanifolds of $X$ constructed as the zero sets of asymptotically holomorphic sections $s_k$ of $E \otimes L^k$ which are transverse to 0, for all large $k$. Then when $k$ is sufficiently large, the inclusion $i : W_k \to X$ induces an isomorphism on homotopy groups $\pi_p$ for $p < n - r$, and a surjection on $\pi_{n-r}$. The same property also holds for homology groups.*

Section 2 contains the statement and proof of the local result on which the whole construction relies. Section 3 deals with the proof of a semi-global statement, using a globalization process to obtain results on large subsets of $X$ from the local picture. The proofs of Theorem 2 and Corollary 2 are then completed in Section 4. Section 5 contains miscellaneous results on the topology and geometry of the obtained submanifolds, including Proposition 2.

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## 2. The local result

The proof of Theorem 2 relies on a local transversality result for approximatively holomorphic functions, which we state and prove immediately.

Proposition 3. *There exists an integer $p$ depending only on the dimension $n$, with the following property : let $\delta$ be a constant with $0 < \delta < \frac{1}{2}$, and let $\sigma = \delta.\log(\delta^{-1})^{-p}$. Let $(f_t)_{t \in T}$ be a family of complex-valued functions over the ball $B^+$ of radius $\frac{11}{10}$ in $\mathbb{C}^n$, depending continuously on the parameter $t \in T$ and satisfying for all $t$ the following bounds over $B^+$ :*

$$|f_t| \leq 1, \qquad |\bar{\partial} f_t| \leq \sigma, \qquad |\nabla \bar{\partial} f_t| \leq \sigma.$$

*Then there exists a family of complex numbers $w_t \in \mathbb{C}$, depending continuously on $t$, such that for all $t \in T$, $|w_t| \leq \delta$, and $f_t - w_t$ has a first derivative larger than $\sigma$ at any point of the interior ball $B$ of radius 1 where its norm is smaller than $\sigma$.*

Proposition 3 extends a similar result proved in detail in [D1], which corresponds to the case where $T = \{0\}$. The proof of Proposition 3 is based on the same ideas as Donaldson's proof, which is in turn based on considerations from real algebraic geometry following the method of Yomdin [Y][Gro2], with the only difference that we must get everything to depend continuously on $t$. Note that this statement is false for more general parameter spaces $T$ than $\{0\}$ and $[0,1]$, since for example when $T$ is the unit disc in $\mathbb{C}$ and $f_t(z) = t$, one looks for a continuous map $t \mapsto w_t$ of the disc to itself without a fixed point, in contradiction with Brouwer's theorem.

The idea is to deal with polynomial functions $g_t$ approximating $f_t$, for which a general result on the complexity of real semi-algebraic sets gives constraints on the near-critical levels. This part of the proof is similar to that given in [D1], so we skip the details. To obtain polynomial functions, we approximate $f_t$ first by a continuous family of holomorphic functions $\tilde{f}_t$ differing from $f_t$ by at most a fixed multiple of $\sigma$ in $C^1$ sense, using that $\bar{\partial} f_t$ is small. The polynomials $g_t$ are then obtained by truncating the Taylor series expansion of $\tilde{f}_t$ to a given degree. It can be shown that by this method one can obtain polynomial functions $g_t$ of degree $d$ less than a constant times $\log(\sigma^{-1})$, such that $g_t$ differs from $f_t$ by at most $c.\sigma$ in $C^1$ sense, where $c$ is a fixed constant (see [D1]). This approximation process does not hold on the whole ball where $f_t$ is defined, which is why we needed $f_t$ to be defined on $B^+$ to get a result over the slightly smaller ball $B$ (see Lemmas 27 and 28 of [D1]).

For a given complex-valued function $h$ over $B$, call $Y_{h,\epsilon}$ the set of all points in $B$ where the derivative of $h$ has norm less than $\epsilon$, and call $Z_{h,\epsilon}$ the $\epsilon$-tubular neighborhood of $h(Y_{h,\epsilon})$. What we wish to construct is a path $w_t$ avoiding by at least $\sigma$ all near-critical levels of $f_t$, i.e. consisting of values that lie outside of $Z_{f_t,\sigma}$. Since $g_t$ is within $c.\sigma$ of $f_t$, it is clear that $Z_{f_t,\sigma}$ is contained in $Z_t = Z_{g_t,(c+1)\sigma}$. However a general result on the complexity of real semi-algebraic sets yields constraints on the set $Y_{g_t,(c+1)\sigma}$. The precise statement which one applies to the real polynomial $|dg_t|^2$ is the following (Proposition 25 of [D1]) :

LEMMA 1. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function of degree $d$, and let $S(\theta) \subset \mathbb{R}^m$ be the subset $S(\theta) = \{x \in \mathbb{R}^m : |x| \leq 1, F(x) \leq 1 + \theta\}$. Then for arbitrarily small $\theta > 0$ there exist fixed constants $C$ and $\nu$ depending only on the dimension $m$ such that $S(0)$ may be decomposed into pieces $S(0) = S_1 \cup S_2 \cdots \cup S_A$, where $A \leq Cd^\nu$, in such a way that any pair of points in the same piece $S_r$ can be joined by a path in $S(\theta)$ of length less than $Cd^\nu$.*

So, as described in [D1], given any fixed $t$, the set $Y_{g_t,(c+1)\sigma}$ of near-critical points of the polynomial function $g_t$ of degree $d$ can be subdivided into at most $P(d)$ subsets, where $P$ is a fixed polynomial, in such a way that two points lying in the same subset can be joined by a path of length at most $P(d)$ inside $Y_{g_t,2(c+1)\sigma}$. It follows that the image by $g_t$ of $Y_{g_t,(c+1)\sigma}$ is contained in the union of $P(d)$ discs of radius at most $2(c+1)\sigma P(d)$, so that the set $Z_t$ of values which we wish to avoid is contained in the union $Z_t^+$ of $P(d)$ discs of radius $\sigma Q(d)$, where $Q = 3(c+1)P$ is a fixed polynomial and $d = O(\log \sigma^{-1})$.

If one assumes $\delta$ to be larger than $\sigma Q(d) P(d)^{1/2}$, it follows immediately from this constraint on $Z_t$ that $Z_t$ cannot fill the disc $D$ of all complex numbers of norm at most $\delta$ : this immediately proves the case $T = \{0\}$. However, when $T = [0, 1]$, we also need $w_t$ to depend continuously on $t$. For this purpose, we show that if $\delta$ is large enough, $D - Z_t^+$, when decomposed into connected components, splits into several small components and only *one* large component.

Indeed, given a component $C$ of $D - Z_t^+$, the simplest situation is that it does not meet the boundary of $D$. Then its boundary is a curve consisting of pieces of the boundaries of the balls making up $Z_t^+$, so its length is at most $2\pi P(d)Q(d)\sigma$, and it follows that $C$ has diameter less than $\pi P(d)Q(d)\sigma$. Considering two components $C_1$ and $C_2$ which meet the boundary of $D$ at points $z_1$ and $z_2$, we can consider an arc $\gamma$ joining the boundary of $D$ to itself that separates $C_1$ from $C_2$ and is contained in the boundary of $Z_t^+$. Assuming that $\delta$ is larger than e.g. $100P(d)Q(d)\sigma$, since the length of $\gamma$ is at most $2\pi P(d)Q(d)\sigma$, it must stay close to either $z_1$ or $z_2$ in order to separate them : $\gamma$ must remain within a distance of at most $10P(d)Q(d)\sigma$ from one of them. It follows that there exists $i \in \{1, 2\}$ such that $C_i$ is contained in the ball of radius $10P(d)Q(d)\sigma$ centered at $z_i$. So all components of $D - Z_t^+$ except at most one are contained in balls of radius $R(d)\sigma$, for some fixed polynomial $R$. Furthermore, the number of components of $D - Z_t^+$ is bounded by a value directly related to the number of balls making up $Z_t^+$, so that, increasing $R$ if necessary, the number of components of $D - Z_t^+$ is also bounded by $R(d)$.

Assuming that $\delta$ is much larger than $R(d)^{3/2}\sigma$, the area $\pi\delta^2$ of $D$ is much larger than $\pi R(d)^3\sigma^2$, so that the small components of $D - Z_t^+$ cannot fill it, and there must be a single large component. Getting back to $D - Z_t$, which was the set in which we had to choose $w_t$, it contains $D - Z_t^+$ and differs from it by at most $Q(d)\sigma$, so that, letting $U(t)$ be the component of $D - Z_t$ containing the large component of $D - Z_t^+$, it is the only large component of $D - Z_t$. The component $U(t)$ is characterized by the property that it is the only component of diameter more than $2R(d)\sigma$ in $D - Z_t$.

So the existence of a single large component $U(t)$ in $D - Z_t$ is proved upon the assumption that $\delta$ is large enough, namely larger than $\sigma.\Phi(d)$ where $\Phi$ is a given fixed polynomial that can be expressed in terms of $P$, $Q$ and $R$ (so $\Phi$ depends only on the dimension $n$). Since $d$ is bounded by a constant times $\log \sigma^{-1}$, it is not hard to see that there exists an integer $p$ such that, for all $0 < \delta < \frac{1}{2}$, the relation $\sigma = \delta. \log(\delta^{-1})^{-p}$ implies that $\delta > \sigma.\Phi(d)$. This is the value of $p$ which we choose in the statement of the proposition, thus ensuring that the above statements always hold.

Since $\bigcup_t \{t\} \times Z_t$ is a closed subset of $T \times D$, the open set $U(t)$ depends semi-continuously on $t$ : let $U^-(t, \epsilon)$ be the set of all points of $U(t)$ at distance more than $\epsilon$ from $Z_t \cup \partial D$. We claim that, given any $t$ and any small $\epsilon > 0$, for all $\tau$ close enough to $t$, $U(\tau)$ contains $U^-(t, \epsilon)$. To see this, we first show for all $\tau$ close to $t$, $U^-(t, \epsilon) \cap Z_\tau = \emptyset$. Assuming that this is not the case, one can get a sequence of points of $Z_\tau$ for $\tau \to t$ that belong to $U^-(t, \epsilon)$. From this sequence one can extract a convergent subsequence, whose limit belongs to $\bar{U}^-(t, \epsilon)$ and thus lies outside of $Z_t$, in contradiction with the fact that $\bigcup_t \{t\} \times Z_t$ is closed. So $U^-(t, \epsilon) \subset D - Z_\tau$ for all $\tau$ close enough to $t$. Making $\epsilon$ smaller if necessary, one may assume that $U^-(t, \epsilon)$ is connected, so that for $\tau$ close to $t$, $U^-(t, \epsilon)$ is necessarily contained in the large component of $D - Z_\tau$, namely $U(\tau)$.

It follows that $U = \bigcup_t \{t\} \times U(t)$ is an open connected subset of $T \times D$, and is thus path-connected. So we get a path $s \mapsto (t(s), w(s))$ joining $(0, w(0))$ to $(1, w(1))$ inside $U$, for any given $w(0)$ and $w(1)$ in $U(0)$ and $U(1)$. We then only have to make sure that $s \mapsto t(s)$ is strictly increasing in order to define $w_{t(s)} = w(s)$.

Getting the $t$ component to increase strictly is in fact quite easy. Indeed, we first get it to be weakly increasing, by considering values $s_1 < s_2$ of the parameter such that $t(s_1) = t(s_2) = t$ and simply replacing the portion of the path between $s_1$ and $s_2$ by a path joining $w(s_1)$ to $w(s_2)$ in the connected set $U(t)$. Then, we slightly shift the path, using the fact that $U$ is open, to get the $t$ component to increase slightly over the parts where it was constant. Thus we can define $w_{t(s)} = w(s)$ and end the proof of Proposition 3.

## 3. The globalization process

**3.1. Statement of the result.** We will now prove a semi-global result using Proposition 3. The globalization process we describe here is based on that used by Donaldson in [D1], but a significantly higher amount of work is required because we have to deal with bundles of rank larger than one. The important fact we use is that transversality to 0 is, as expected, a *local* and *open* property.

THEOREM 3. *Let $U$ be any open subset of $X$, and let $E$ be a complex vector bundle of rank $r \geq 0$ over $U$. Let $(J_t)_{t\in T}$ be a family of almost-complex structures on $X$ compatible with $\omega$. Fix a constant $\epsilon > 0$. Let $W_{t,k}$ be a family of symplectic submanifolds in $U$, obtained as the zero sets of asymptotically $J_t$-holomorphic sections $w_{t,k}$ of the vector bundles $E \otimes L^k$ which are $\eta$-transverse to 0 over $U$ for some $\eta > 0$ and depend continuously on $t \in T$ (if the rank is $r = 0$, then we simply define $W_{t,k} = U$). Finally, let $(\sigma_{t,k})$ be a family of asymptotically $J_t$-holomorphic sections of $L^k$ which depend continuously on $t$. Define $U_k^-$ to be the set of all points of $U$ at distance more than $4k^{-1/3}$ from the boundary of $U$.*

*Then for some $\tilde{\eta} > 0$ and for all large $k$, there exist asymptotically $J_t$-holomorphic sections $\tilde{\sigma}_{t,k}$ of $L^k$ over $U$, depending continuously on $t$, and such that*

*(a) for all $t \in T$, $\tilde{\sigma}_{t,k}$ is equal to $\sigma_{t,k}$ near the boundary of $U$,*

*(b) $|\tilde{\sigma}_{t,k} - \sigma_{t,k}| < \epsilon$ and $|\nabla\tilde{\sigma}_{t,k} - \nabla\sigma_{t,k}| < k^{1/2}\,\epsilon$ for all $t$,*

*(c) the sections $(w_{t,k} + \tilde{\sigma}_{t,k})$ of $(E \oplus \mathbb{C}) \otimes L^k$ are $\tilde{\eta}$-transverse to 0 over $U_k^-$ for all $t$.*

Basically, this result states that the construction of Theorem 2 can be carried out, in the line bundle case, in such a way that the resulting sections are transverse to a given family of symplectic submanifolds.

As remarked in the introduction, the choice of the metric in the statement of the theorem is not obvious. We choose to use always the same metric $g$ on $X$, rather than trying to work directly with the metrics $g_t$ induced by $\omega$ and $J_t$.

**3.2. Local coordinates and sections.** The proof of Theorem 3 is based on the existence of highly localized asymptotically holomorphic sections of $L^k$ near every point $x \in X$. First, we notice that near any point $x \in X$, we can define local *complex* Darboux coordinates $(z_i)$, that is to say a symplectomorphism from a neighborhood of $x$ in $(X, \omega)$ to a neighborhood of 0 in $\mathbb{C}^n$ with the standard symplectic form. Furthermore it is well-known that, by composing the coordinate map with a ($\mathbb{R}$-linear) symplectic transformation of $\mathbb{C}^n$, one can ensure that its differential at $x$ induces a *complex linear* map from $(T_xX, J_t)$ to $\mathbb{C}^n$ with its standard complex structure.

Since the almost-complex structure $J_t$ is not integrable, the coordinate map cannot be made pseudo-holomorphic on a whole neighborhood of $x$. However, since the manifold $X$ and the parameter space $T$ are compact, the Nijenhuis tensor, which is the obstruction to the integrability of the complex structure $J_t$ on $X$, is bounded by a fixed constant, and so are its derivatives. It follows that for a suitable choice of the Darboux coordinates, the coordinate map can be made nearly pseudo-holomorphic around $x$, in the sense that the antiholomorphic part of its differential vanishes at $x$ and grows no faster than a constant times the distance to $x$. Furthermore, it is easy to check that the coordinate map can be chosen to depend continuously on the parameter $t$. So, we have the following lemma :

LEMMA 2. *Near any point $x \in X$, there exist for all $t \in T$ complex Darboux coordinates depending continuously on $t$, such that the inverse $\psi_t : (\mathbb{C}^n, 0) \to (X, x)$ of the coordinate map is nearly pseudo-holomorphic with respect to the almost-complex structure $J_t$ on $X$ and the canonical complex structure on $\mathbb{C}^n$. Namely, the map $\psi_t$, which trivially satisfies $|\nabla\psi_t| = O(1)$ and $|\nabla\nabla\psi_t| = O(1)$ on a ball of fixed radius around 0, fails to be pseudo-holomorphic by an amount that vanishes at 0 and thus grows no faster than the distance to the origin, i.e. $|\bar\partial\psi_t(z)| = O(|z|)$, and $|\nabla\bar\partial\psi_t| = O(1)$.*

Fix a certain value of the parameter $t \in T$, and consider the Hermitian connections with curvature $-ik\omega$ that we have put on $L^k$ in the introduction. Near any point $x \in X$, using the local complex Darboux coordinates $(z_i)$ we have just constructed, a suitable choice of a local trivialization of $L^k$ leads to the following connection 1-form :

$$A_k = \frac{k}{4}\sum_{j=1}^{n}(z_j d\overline{z}_j - \overline{z}_j dz_j)$$

(it can be readily checked that $dA_k = -ik\omega$).

On the standard $\mathbb{C}^n$ with connection $A_k$, the function $s(z) = \exp(-k|z|^2/4)$ satisfies the equation $\bar\partial_{A_k}s = 0$ and the bound $|\nabla_{A_k}s| = O(k^{1/2})$. Multiplying this section by a cut-off function at distance $k^{-1/3}$ from the origin whose derivative is small enough, we get a section $\tilde{s}$ with small compact support. Since the coordinate map near $x$ has small antiholomorphic part where $\tilde{s}$ is large, the local sections $\tilde{s} \circ \psi_t^{-1}$ of $L^k$ defined near $x$ by pullback of $\tilde{s}$ through the coordinate map can be easily checked to be asymptotically holomorphic with respect to $J_t$ and $A_k$. Thus, for all large $k$ and for any point $x \in X$, extending $\tilde{s} \circ \psi_t^{-1}$ by 0 away from $x$, we obtain asymptotically holomorphic sections $s_{t,k,x}$ of $L^k$.

Since $T$ is compact, the metrics $g_t$ induced on $X$ by $\omega$ and $J_t$ differ from the chosen reference metric $g$ by a bounded factor. Therefore, it is clear from the way we constructed the sections $s_{t,k,x}$ that the following statement holds :

LEMMA 3. *There exist constants $\lambda > 0$ and $c_s > 0$ such that, given any $x \in X$, for all $t \in T$ and large $k$, there exist sections $s_{t,k,x}$ of $L^k$ over $X$ with the following properties : the sections $s_{t,k,x}$ are asymptotically $J_t$-holomorphic ; they depend continuously on $t$ ; the bound $|s_{t,k,x}| \geq c_s$ holds over the ball of radius $10\,k^{-1/2}$ around $x$ ; and finally, $|s_{t,k,x}| \leq \exp(-\lambda k\,\mathrm{dist}_g(x,.)^2)$ everywhere on $X$.*

**3.3. General setup and strategy of proof.** In a first step, we wish to obtain sections $\tilde\sigma_{t,k}$ of $L^k$ over $U$ satisfying all the requirements of Theorem 3, except that we replace *(c)* by the weaker condition that the restriction of $\tilde\sigma_{t,k}$ to $W_{t,k}$ must be $\hat\eta$-transverse to 0 over $W_{t,k} \cap U_k^-$ for some $\hat\eta > 0$, where $U_k^-$ is the set of all points of $U$ at distance more than $2k^{-1/3}$ from the boundary of $U$. It will be shown later that the transversality to 0 of the restriction to $W_{t,k} \cap U_k^-$ of $\tilde\sigma_{t,k}$, together with the bounds on the second derivatives, implies the transversality to 0 of $(w_{t,k} + \tilde{\sigma}_{t,k})$ over $U_k^=$.

To start with, we notice that there exists a constant $c > 0$ such that $W_{t,k}$ is trivial at small scale, namely in the ball of radius $10\,c\,k^{-1/2}$ around any point. Indeed, if $r = 0$ we just take $c = 1$, and otherwise we use the fact that $w_{t,k}$ is $\eta$-transverse to 0, which implies that at any $x \in W_{t,k}$, $|\nabla w_{t,k}(x)| > \eta\, k^{1/2}$. Since $|\nabla\nabla w_{t,k}| < C_2\, k$ for some constant $C_2$, defining $c = \frac{1}{100}\eta\, C_2^{-1}$, the derivative $\nabla w_{t,k}$ varies by a factor of at most $\frac{1}{10}$ in the ball $B$ of radius $10\,c\,k^{-1/2}$ around $x$. It follows that $B \cap W_{t,k}$ is diffeomorphic to a ball.

In all the following, we work with a given fixed value of $k$, while keeping in mind that all constants appearing in the estimates have to be independent of $k$.

For fixed $k$, we consider a finite set of points $x_i$ of $U_k^- \subset U$ such that the balls of radius $c\,k^{-1/2}$ centered around $x_i$ cover $U_k^-$. A suitable choice of the points ensures that their number is $O(k^n)$. For fixed $D > 0$, this set can be subdivided into $N$ subsets $S_j$ such that the distance between two points in the same subset is at least $D\,k^{-1/2}$. Furthermore, $N = O(D^{2n})$ can be chosen independent of $k$. The precise value of $D$ (and consequently of $N$) will be determined later in the proof.

The idea is to start with the sections $\sigma_{t,k}$ of $L^k$ and proceed in steps. Let $\mathcal{N}_j$ be the union of all balls of radius $c\,k^{-1/2}$ around the points of $S_i$ for all $i < j$. During the $j$-th step, we start from asymptotically $J_t$-holomorphic sections $\sigma_{t,k,j}$ which satisfy conditions *(a)* and *(b)*, and such that the restriction of $\sigma_{t,k,j}$ to $W_{t,k}$ is $\eta_j$-transverse to 0 over $W_{t,k} \cap \mathcal{N}_j$, for some constant $\eta_j$ independent of $k$. For the first step, this requirement is void, but we choose $\eta_0 = \frac{\epsilon}{2}$ in order to obtain a total perturbation smaller than $\epsilon$ at the end of the process. We wish to construct $\sigma_{t,k,j+1}$ from $\sigma_{t,k,j}$ by subtracting small multiples $c_{t,k,x}\, s_{t,k,x}$ of the sections $s_{t,k,x}$ for $x \in S_j$, in such a way that the restrictions of the resulting sections are $\eta_{j+1}$-transverse to 0, for some small $\eta_{j+1}$, over the intersection of $W_{t,k}$ with all balls of radius $c\,k^{-1/2}$ around points in $S_j$. Furthermore, if the coefficients of the linear combination are chosen much smaller than $\eta_j$, transversality to 0 still holds over $W_{t,k} \cap \mathcal{N}_j$. Also, since the coefficients $c_{t,k,x}$ are bounded, the resulting sections, which are sums of asymptotically holomorphic sections, remain asymptotically holomorphic. So we need to find, for all $x \in S_j$, small coefficients $c_{t,k,x}$ so that $\sigma_{t,k,j} - c_{t,k,x}\, s_{t,k,x}$ has the desired properties near $x$.

**3.4. Obtaining transversality near a point of $S_j$.** In what follows, $x$ is a given point in $S_j$, and $B_x$ is the ball of radius $c\,k^{-1/2}$ around $x$. Let $\Omega$ be the closure of the open subset of $T$ containing all $t$ such that $B_x \cap W_{t,k}$ is not empty (when $r = 0$, one gets $\Omega = T$). When $\Omega$ is empty, it is sufficient to define $c_{t,k,x} = 0$ for all $t$. Otherwise, $\Omega = \{0\}$ when $T = \{0\}$, and when $T = [0,1]$ clearly $\Omega$ is a union of disjoint closed intervals. In any case, we choose a component $I$ of $\Omega$, i.e. either a closed interval or a point.

We can then define for all $t \in I$ a point $x_t$ belonging to $\overline{B_x} \cap W_{t,k}$, in such a way that $x_t$ depends continuously on $t$, since $W_{t,k}$ depends continuously on $t$ and always intersects $B_x$ in a nice way (when $r = 0$ one can simply choose $x_t = x$). Let $\hat{B}_t$ be the ball in $W_{t,k}$ of radius $3\,c\,k^{-1/2}$ (for the metric induced by $g$) centered at $x_t$.

Because of the bounds on the second derivatives of $w_{t,k}$, we know that $\hat{B}_t$ contains $B_x \cap W_{t,k}$ for all $t \in I$. We now want to define a nearly holomorphic diffeomorphism from a neighborhood of 0 in $\mathbb{C}^{n-r}$ to $\hat{B}_t$.

Let $\hat{B}$ be the ball of radius $4ck^{-1/2}$ around 0 in $\mathbb{C}^{n-r}$, and let $\hat{B}^-$ be the smaller ball of radius $3ck^{-1/2}$ around 0. We claim the following :

LEMMA 4. *For all $t \in I$, there exist diffeomorphisms $\theta_t$ from $\hat{B}$ to a neighborhood of $x_t$ in $W_{t,k}$, depending continuously on $t$, such that $\theta_t(0) = x_t$ and $\theta_t(\hat{B}^-) \supset \hat{B}_t$, and satisfying the following estimates over $\hat{B}$ :*

$$|\bar{\partial}\theta_t| = O(k^{-1/2}), \quad |\nabla\theta_t| = O(1), \quad |\nabla\bar{\partial}\theta_t| = O(1), \quad |\nabla\nabla\theta_t| = O(k^{1/2}).$$

PROOF. Recall that, by Lemma 2, there exist local complex Darboux coordinates on $X$ near $x$ depending continuously on $t$ with the property that the inverse map $\psi_t : (\mathbb{C}^n, 0) \to (X, x)$ satisfies the following bounds at all points at distance $O(k^{-1/2})$ from $x$ :

$$|\bar{\partial}\psi_t| = O(k^{-1/2}), \quad |\nabla\psi_t| = O(1), \quad |\nabla\bar{\partial}\psi_t| = O(1), \quad |\nabla\nabla\psi_t| = O(1).$$

Let $\mathcal{T}_t$ be the kernel of the complex linear map $\partial w_{t,k}(x_t)$ in $T_{x_t}X$ : it is within $O(k^{-1/2})$ of the tangent space to $W_{t,k}$ at $x_t$, but $\mathcal{T}_t$ is preserved by $J_t$. Composing $\psi_t$ with a translation and a rotation in $\mathbb{C}^n$, one gets maps $\tilde{\psi}_t$ satisfying the same requirements as $\psi_t$, but with $\tilde{\psi}_t(0) = x_t$ and such that the differential of $\tilde{\psi}_t$ at 0 maps the span of the $n-r$ first coordinates to $\mathcal{T}_t$.

Furthermore, $X$ and $T$ are compact, so the metrics $g_t$ induced by $\omega$ and $J_t$ differ from the reference metric $g$ by at most a fixed constant. It follows that, composing $\tilde{\psi}_t$ with a fixed dilation of $\mathbb{C}^n$ if necessary, one may also require that the image by $\tilde{\psi}_t$ of the ball of radius $3ck^{-1/2}$ around 0 contains the ball of radius $4ck^{-1/2}$ around $x$ for the reference metric $g$. The only price to pay is that $\tilde{\psi}_t$ is no longer a local symplectomorphism ; all other properties still hold.

Since by definition of $c$ the submanifolds $W_{t,k}$ are trivial over the considered balls, it follows from the implicit function theorem that $W_{t,k}$ can be parametrized around $x_t$ in the chosen coordinates as the set of points of the form $\tilde{\psi}_t(z, \tau_t(z))$ for $z \in \mathbb{C}^{n-r}$, where $\tau_t : \mathbb{C}^{n-r} \to \mathbb{C}^r$ satisfies $\tau_t(0) = 0$ and $\nabla\tau_t(0) = O(k^{-1/2})$. The derivatives of $\tau_t$ can be easily computed, since it is characterized by the equation

$$w_{t,k}(\tilde{\psi}_t(z, \tau_t(z))) = 0.$$

Notice that it follows from the transversality to 0 of $w_{t,k}$ that $|\nabla w_{t,k} \circ d\tilde{\psi}_t(v))|$ is larger than a constant times $k^{1/2}|v|$ for all $v \in 0 \times \mathbb{C}^r$. Combining this estimate with the bounds on the derivatives of $w_{t,k}$ given by asymptotic holomorphicity and the above bounds on those of $\tilde{\psi}_t$, one gets the following estimates for $\tau_t$ over the ball $\hat{B}$ :

$$|\bar{\partial}\tau_t| = O(k^{-1/2}), \quad |\nabla\tau_t| = O(1), \quad |\nabla\bar{\partial}\tau_t| = O(1), \quad |\nabla\nabla\tau_t| = O(k^{1/2}).$$

It is then clear that $\theta_t(z) = \tilde{\psi}_t(z, \tau_t(z))$ satisfies all the required properties. □

Now that a local identification between $W_{t,k}$ and $\mathbb{C}^{n-r}$ is available, we define the restricted sections $\hat{s}_{t,k,x}(z) = s_{t,k,x}(\theta_t(z))$ and $\hat{\sigma}_{t,k,j}(z) = \sigma_{t,k,j}(\theta_t(z))$. Since

$s_{t,k,x}$ and $\sigma_{t,k,j}$ are both asymptotically holomorphic, the estimates on $\theta_t$ imply that $\hat{s}_{t,k,x}$ and $\hat{\sigma}_{t,k,j}$, as sections of the pull-back of $L^k$ over the ball $\hat{B}$, are also asymptotically holomorphic. Furthermore, they clearly depend continuously on $t \in I$, and $\hat{s}_{t,k,x}$ remains larger than a fixed constant $c_s > 0$ over $\hat{B}$. We can then define the complex-valued functions $f_{t,k,x} = \hat{\sigma}_{t,k,j}/\hat{s}_{t,k,x}$ over $\hat{B}$, which are clearly asymptotically holomorphic too.

After dilation of $\hat{B}$ by a factor of $3ck^{1/2}$, all hypotheses of Proposition 3 are satisfied with $\delta$ as small as desired, provided that $k$ is large enough. Indeed, the asymptotic holomorphicity of $f_{t,k,x}$ implies that, for large $k$, the antiholomorphic part of the function over the dilated ball is smaller than $\sigma = \delta(\log \delta^{-1})^{-p}$. So the local result implies that there exist complex numbers $c_{t,k,x}$ of norm less than $\delta$ and depending continuously on $t \in I$, such that the functions $f_{t,k,x} - c_{t,k,x}$ are $\sigma$-transverse to 0 over the ball $\hat{B}^-$ of radius $3ck^{-1/2}$ around 0 in $\mathbb{C}^{n-r}$. We now notice that the sections $\hat{g}_{t,k,x} = \hat{\sigma}_{t,k,j} - c_{t,k,x}\,\hat{s}_{t,k,x}$, which clearly depend continuously on $t$ and are asymptotically holomorphic, are $\sigma'$-transverse to 0 over $\hat{B}^-$, for some $\sigma'$ differing from $\sigma$ by at most a constant factor. Indeed,

$$\nabla \hat{g}_{t,k,x} = \nabla(\hat{s}_{t,k,x}(f_{t,k,x} - c_{t,k,x})) = \hat{s}_{t,k,x} \nabla f_{t,k,x} - (f_{t,k,x} - c_{t,k,x}) \nabla \hat{s}_{t,k,x}.$$

Wherever $\hat{g}_{t,k,x}$ is very small, so is $f_{t,k,x} - c_{t,k,x}$, and $\nabla f_{t,k,x}$ is thus large. Since $\hat{s}_{t,k,x}$ remains larger than some $c_s > 0$ and $\nabla \hat{s}_{t,k,x}$ is bounded by a constant times $k^{1/2}$, it follows that $\nabla \hat{g}_{t,k,x}$ is large wherever $\hat{g}_{t,k,x}$ is very small. Putting the right constants in the right places, one easily checks that $\hat{g}_{t,k,x}$ is $\sigma'$-transverse to 0 with $\sigma/\sigma'$ bounded by a fixed constant.

We now notice that the restrictions to $W_{t,k}$ of the sections $g_{t,k,x} = \sigma_{t,k,j} - c_{t,k,x}\, s_{t,k,x}$ of $L^k$ over $U$, which clearly are asymptotically $J_t$-holomorphic and depend continuously and $t$, are also $\sigma''$-transverse to 0 over $\hat{B}_t$ for some $\sigma''$ differing from $\sigma'$ by at most a constant factor. Indeed, $\hat{B}_t$ is contained in the set of all points of the form $\theta_t(z)$ for $z \in \hat{B}^-$, and

$$g_{t,k,x}(\theta_t(z)) = \hat{\sigma}_{t,k,j}(z) - c_{t,k,x}\, \hat{s}_{t,k,x}(z) = \hat{g}_{t,k,x}(z),$$

so wherever $g_{t,k,x}$ is smaller than $\sigma'$, the derivative of $\hat{g}_{t,k,x}$ is larger than $\sigma'.k^{1/2}$, and since $\nabla\theta_t$ is bounded by a fixed constant, $\nabla g_{t,k,x}$ is large too.

Next we extend the definition of $c_{t,k,x}$ to all $t \in T$, in the case of $T = [0, 1]$, since we have defined it only over the components of $\Omega$. However, when $t \notin \Omega$, $W_{t,k}$ does not meet the ball $B_x$, so that there is no transversality requirement. Thus the only constraints are that $c_{t,k,x}$ must depend continuously on $t$ and remain smaller than $\delta$ for all $t$. These conditions are easy to satisfy, so we have proved the following :

LEMMA 5. *For all large $k$ there exist complex numbers $c_{t,k,x}$ smaller than $\delta$ and depending continuously on $t \in T$ such that the restriction to $W_{t,k}$ of $\sigma_{t,k,j} - c_{t,k,x}\, s_{t,k,x}$ is $\sigma''$-transverse to 0 over $W_{t,k} \cap B_x$. Furthermore, for some constant $p'$ depending only on the dimension, $\sigma''$ is at least $\delta(\log \delta^{-1})^{-p'}$.*

**3.5. Constructing $\sigma_{t,k,j+1}$ from $\sigma_{t,k,j}$.** We can now define the sections $\sigma_{t,k,j+1}$ of $L^k$ over $U$ by

$$\sigma_{t,k,j+1} = \sigma_{t,k,j} - \sum_{x \in S_j} c_{t,k,x}\, s_{t,k,x}.$$

Clearly the sections $\sigma_{t,k,j+1}$ are asymptotically holomorphic and depend continuously on $t \in T$. Furthermore, any two points in $S_j$ are distant of at least $D\,k^{-1/2}$ with $D > 0$, so the total size of the perturbation is bounded by a fixed multiple of $\delta$. So, choosing $\delta$ smaller than $\eta_j$ over a constant factor (recall that $\eta_j$ is the transversality estimate of the previous step of the iterative process), we can ensure that $|\sigma_{t,k,j+1} - \sigma_{t,k,j}| < \frac{\eta_j}{2}$ and $|\nabla\sigma_{t,k,j+1} - \nabla\sigma_{t,k,j}| < \frac{\eta_j}{2} k^{1/2}$. As a direct consequence, the restriction to $W_{t,k}$ of $\sigma_{t,k,j+1}$ is $\frac{\eta_j}{2}$-transverse to 0 wherever the restriction of $\sigma_{t,k,j}$ is $\eta_j$-transverse to 0, including over $W_{t,k} \cap \mathcal{N}_j$ (recall that $\mathcal{N}_j = \bigcup_{i<j} \bigcup_{x\in S_i} B_x$).

Letting $\eta_{j+1} = \frac{1}{2}\sigma''$, it is known that for all $x \in S_j$ the restriction to $W_{t,k}$ of $\sigma_{t,k,j} - c_{t,k,x}\, s_{t,k,x}$ is $2\eta_{j+1}$-transverse to 0 over $B_x \cap W_{t,k}$. So, in order to prove that the restriction to $W_{t,k}$ of $\sigma_{t,k,j+1}$ is $\eta_{j+1}$-transverse to 0 over $W_{t,k} \cap \mathcal{N}_{j+1}$, it is sufficient to check that given $x \in S_j$, over $B_x$, the sum of the perturbations corresponding to all points $y \in S_j$ distinct from $x$ is smaller than $\eta_{j+1}$, and the sum of their derivatives is smaller than $\eta_{j+1} k^{1/2}$. In other words, since several contributions were added at the same time (one at each point of $S_j$), we have to make sure that they cannot interfere.

This is where the parameter $D$ (minimum distance between two points in $S_j$) is important : indeed, over $B_x$, by Lemma 3, each of the contributions of the other points in $S_j$ is at most of the order of $\delta \exp(-\lambda D^2)$, and the sum of these terms is $O(\eta_j \exp(-\lambda D^2))$. Similarly, the derivative of that sum is $O(\eta_j \exp(-\lambda D^2)\, k^{1/2})$. So the requirement that the sum of the contributions of all points of $S_j$ distinct from $x$ be smaller than $\eta_{j+1}$ corresponds to an inequality of the form $K_0 \exp(-\lambda D^2) < \eta_{j+1}/\eta_j$, where $K_0$ is a fixed constant depending only on the geometry of $X$. Recalling that $\eta_{j+1}$ is no smaller than $\eta_j \log(\eta_j^{-1})^{-P}$ for some fixed integer $P$, the required inequality is

$$\exp(\lambda D^2) > K_0 \log(\eta_j^{-1})^P.$$

This inequality, which *does not depend on k*, must be satisfied by every $\eta_j$, for each of the $N$ steps of the process.

To check that the condition on $D$ can be enforced at all steps, we must recall that the number of steps in the process is $N = O(D^{2n})$, and study the sequence $(\eta_j)$ given by a fixed $\eta_0 > 0$ and the inductive definition described above. It can be shown (see Lemma 24 of [D1]) that the sequence $(\eta_j)$ satisfies for all $j$ a bound of the type $\log(\eta_j^{-1}) = O(j \log(j))$. It follows that $\log(\eta_N^{-1})^P = O(D^{2nP} \log(D^{2n})^P)$, which is clearly subexponential : a choice of sufficiently large $D$ thus ensures that the required inequality holds at all steps. So the inductive process described above is valid, and leads to sections $\tilde\sigma_{t,k} = \sigma_{t,k,N}$ which are asymptotically $J_t$-holomorphic, depend continuously on $t$, and whose restrictions to $W_{t,k}$ are $\hat\eta$-transverse to 0 over $U_k^-$ for $\hat\eta = \eta_N$. Furthermore, $\tilde\sigma_{t,k}$ is equal to $\sigma_{t,k}$ near the boundary of $U$ because we only added a linear combination of sections $s_{t,k,x}$ for $x \in U_k^-$, and $s_{t,k,x}$ vanishes by construction outside of the ball of radius $k^{-1/3}$ around $x$. Moreover, $\tilde\sigma_{t,k}$ differs from $\sigma_{t,k}$ by at most $\sum_j \eta_j$, which is less than $2\eta_0 = \epsilon$. So to complete the proof of Theorem 3 we only have to show that the transversality result on $\tilde{\sigma}_{t,k|W_{t,k}}$ implies the transversality to 0 of $(w_{t,k} + \tilde{\sigma}_{t,k})$ over $U_k^=$.

**3.6. Transversality to 0 over $U_k^=$.** At a point $x \in W_{t,k} \cap U_k^-$ where $|\tilde{\sigma}_{t,k}| < \hat{\eta}$, we know that $\nabla w_{t,k}$ is surjective and vanishes in all directions tangential to $W_{t,k}$, while $\nabla \tilde{\sigma}_{t,k}$ has a tangential component larger than $\hat{\eta}\, k^{1/2}$. It follows that $\nabla(w_{t,k} + \tilde{\sigma}_{t,k})$ is surjective. We now construct a right inverse $R : (E_x \oplus \mathbb{C}) \otimes L_x^k \to T_xX$ whose norm is $O(k^{-1/2})$.

Considering a unit length element $u$ of $L_x^k$, there exists a vector $\hat{u} \in T_xW_{t,k}$ of norm at most $(\hat{\eta}\, k^{1/2})^{-1}$ such that $\nabla \tilde{\sigma}_{t,k}(\hat{u}) = u$. Clearly $\nabla w_{t,k}(\hat{u}) = 0$ because $\hat{u} \in T_xW_{t,k}$, so we define $R(u) = \hat{u}$. Now consider an orthonormal frame $(v_i)$ in $E_x \otimes L_x^k$. It follows from the $\eta$-transversality to 0 of $w_{t,k}$ that $\nabla_x w_{t,k}$ has a right inverse of norm smaller than $(\eta\, k^{1/2})^{-1}$, so we obtain vectors $\hat{v}_i$ in $T_xX$ such that $\nabla w_{t,k}(\hat{v}_i) = v_i$ and $|\hat{v}_i| < (\eta\, k^{1/2})^{-1}$. There exist coefficients $\lambda_i$ such that $\nabla \tilde{\sigma}_{t,k}(\hat{v}_i) = \lambda_i\, u$, with $|\lambda_i| < C\, k^{1/2}\, |\hat{v}_i| < C\, \eta^{-1}$, for some constant $C$ such that $|\nabla \tilde{\sigma}_{t,k}| < C\, k^{1/2}$ everywhere. So we define $R(v_i) = \hat{v}_i - \lambda_i \hat{u}$, which completes the determination of $R$.

The norm of $R$ is, by construction, smaller than $K.k^{-1/2}$ for some $K$ depending only on the constants above ($C$, $\eta$ and $\hat{\eta}$). We thus know that $\nabla(w_{t,k} + \tilde{\sigma}_{t,k})$ has a right inverse smaller than $K\, k^{-1/2}$ at any point of $W_{t,k} \cap U_k^-$ where $|\tilde{\sigma}_{t,k}| < \hat{\eta}$. Furthermore we know, from the definition of asymptotic holomorphicity, that $|\nabla\nabla(w_{t,k} + \tilde{\sigma}_{t,k})| < K'\, k$ for some constant $K'$.

Consider a point $x$ of $U_k^=$ where $|w_{t,k}|$ and $|\tilde{\sigma}_{t,k}|$ are both smaller than some $\alpha$ which is simultaneously smaller than $\frac{\hat{\eta}}{2}$, $\frac{\eta\hat{\eta}}{2C}$ and $\frac{\eta}{2KK'}$. From the $\eta$-transversality to 0 of $w_{t,k}$, we know that $\nabla w_{t,k}$ is surjective at $x$ and has a right inverse smaller than $(\eta\, k^{1/2})^{-1}$. Since the connection $\nabla$ is unitary, applying the right inverse to $w_{t,k}$ itself, we can follow the downward gradient flow of $|w_{t,k}|$, and we are certain to reach a point $y$ of $W_{t,k}$ at a distance $d$ from the starting point $x$ no larger than $\alpha\, (\eta\, k^{1/2})^{-1}$, which is simultaneously smaller than $\frac{1}{2KK'}\, k^{-1/2}$ and $\frac{\hat{\eta}}{2C}\, k^{-1/2}$. Furthermore if $k$ is large enough, $d < 2k^{-1/3}$ so that $y \in U_k^-$.

Since $|\nabla \tilde{\sigma}_{t,k}| < C.k^{1/2}$ everywhere, $|\tilde{\sigma}_{t,k}(y)| - |\tilde{\sigma}_{t,k}(x)| < C\, k^{1/2}\, d < \frac{\hat{\eta}}{2}$, so that $|\tilde{\sigma}_{t,k}(y)| < \hat{\eta}$, and the previous results apply at $y$. Also, since the second derivatives are bounded by $K'\, k$ everywhere, $\nabla_x(w_{t,k} + \tilde{\sigma}_{t,k})$ differs from $\nabla_y(w_{t,k} + \tilde{\sigma}_{t,k})$ by at most $K' k\, d$, which is smaller than $\frac{1}{2K}\, k^{1/2}$, so that it is still surjective and admits a right inverse of norm $O(k^{-1/2})$. From this we infer immediately that $(w_{t,k} + \tilde{\sigma}_{t,k})$ is transverse to 0 over all of $U_k^=$, and the proof of Theorem 3 is complete.

## 4. The main result

**4.1. Proof of Theorem 2.** Theorem 2 follows from Theorem 3 by a simple induction argument. Indeed, to obtain asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0 over $X$ for any vector bundle $E$, we start from the fact that $E$ is locally trivial, so that there exists a finite covering of $X$ by $N$ open subsets $U_j$ such that $E$ is a trivial bundle on a small neighborhood of each $U_j$.

We start initially from the sections $s_{t,k,0} = s_{t,k}$ of $E \otimes L^k$, and proceed iteratively, assuming at the beginning of the $j$-th step that we have constructed, for all large $k$, asymptotically holomorphic sections $s_{t,k,j}$ of $E \otimes L^k$ which are $\eta_j$-transverse to 0 on $\bigcup_{i<j} U_i$ for some $\eta_j > 0$ and differ from $s_{t,k}$ by at most $j\epsilon/N$.

Over a small neighborhood of $U_j$, we trivialize $E \simeq \underline{\mathbb{C}}^r$ and decompose the sections $s_{t,k,j}$ into their $r$ components for this trivialization. Recall that, in order to define the connections on $E \otimes L^k$ for which asymptotic holomorphicity and transversality to 0 are expected, we have used a Hermitian connection $\nabla^E$ on $E$. Because $X$ is compact the connection 1-form of $\nabla^E$ in the chosen trivializations can be safely assumed to be bounded by a fixed constant. It follows that, up to a change in the constants, asymptotic holomorphicity and transversality to 0 over $U_j$ with respect to the connections on $E \otimes L^k$ induced by $\nabla^E$ and $\nabla^L$ are equivalent to asymptotic holomorphicity and transversality to 0 with respect to the connections induced by $\nabla^L$ and the trivial connection on $E$ in the chosen trivialization. So we actually do not have to worry about $\nabla^E$.

Now, let $\alpha$ be a constant smaller than both $\epsilon/rN$ and $\eta_j/2r$. First, using Theorem 3, we perturb the first component of $s_{t,k,j}$ over a neighborhood of $U_j$ by at most $\alpha$ to make it transverse to 0 over a slightly smaller neighborhood. Next, using again Theorem 3, we perturb the second component by at most $\alpha$ so that the sum of the two first components is transverse to 0, and so on, perturbing the $i$-th component by at most $\alpha$ to make the sum of the $i$ first components transverse to 0. The result of this process is a family of asymptotically $J_t$-holomorphic sections $s_{t,k,j+1}$ of $E \otimes L^k$ which are transverse to 0 over $U_j$. Furthermore, since the total perturbation is smaller than $r\alpha \leq \eta_j/2$, transversality to 0 still holds over $U_i$ for $i < j$, so that the hypotheses of the next step are satisfied. The construction thus leads to sections $\sigma_{t,k} = s_{t,k,N}$ which are transverse to 0 over all of $X$. Since at each of the $N$ steps the total perturbation is less than $\epsilon/N$, the sections $\sigma_{t,k}$ differ from $s_{t,k}$ by less than $\epsilon$, and Theorem 2 is proved.

**4.2. Symplectic isotopies.** We now give the remaining part of the proof of Corollary 2, namely the following statement :

PROPOSITION 4. *Let $(W_t)_{t\in[0,1]}$ be a family of symplectic submanifolds in $X$. Then there exist symplectomorphisms $\Phi_t : X \to X$ depending continuously on $t$, such that $\Phi_0 = \mathrm{Id}$ and $\Phi_t(W_0) = W_t$.*

The following strategy of proof, based on Moser's ideas, was suggested to me by M. Gromov. The reader unfamiliar with these techniques may use [McS1] (pp. 91-101) as a reference.

It follows immediately from Moser's stability theorem that there exists a continuous family of symplectomorphisms $\phi_t : (W_0, \omega_{|W_0}) \to (W_t, \omega_{|W_t})$. Since the symplectic normal bundles to $W_t$ are all isomorphic, Weinstein's symplectic neighborhood theorem allows one to extend these maps to symplectomorphisms $\psi_t : U_0 \to U_t$ such that $\psi_t(W_0) = W_t$, where $U_t$ is a small tubular neighborhood of $W_t$ for all $t$.

Let $\rho_t$ be any family of *diffeomorphisms* of $X$ extending $\psi_t$. Let $\omega_t = \rho_t^*\omega$ and $\Omega_t = -d\omega_t/dt$. We want to find vector fields $\xi_t$ on $X$ such that the 1-forms $\alpha_t = \iota_{\xi_t}\omega_t$ satisfy $d\alpha_t = \Omega_t$ and such that $\xi_t$ is tangent to $W_0$ at any point of $W_0$.

If this is possible, then define diffeomorphisms $\Psi_t$ as the flow of the vector fields $\xi_t$, and notice that

$$\frac{d}{dt}(\Psi_t^*\rho_t^*\omega) = \Psi_t^*\left(\frac{d}{dt}(\rho_t^*\omega) + L_{\xi_t}(\rho_t^*\omega)\right) = \Psi_t^*(-\Omega_t + d\iota_{\xi_t}\omega_t) = 0.$$

So the diffeomorphisms $\rho_t \circ \Psi_t$ are actually symplectomorphisms of $X$. Furthermore $\Psi_t$ preserves $W_0$ by construction, so $\rho_t \circ \Psi_t$ maps $W_0$ to $W_t$, thus giving the desired result.

So we are left with the problem of finding $\xi_t$, or equivalently $\alpha_t$, such that $d\alpha_t = \Omega_t$ and $\xi_{t|W_0}$ is tangent to $W_0$. Note that, since $\rho_t$ extends the symplectomorphisms $\psi_t$, one has $\omega_t = \omega$ and $\Omega_t = 0$ over $U_0$. It follows that the condition on $\xi_{t|W_0}$ is equivalent to the requirement that at any point $x \in W_0$, the $\omega$-symplectic orthogonal $N_xW_0$ to $T_xW_0$ lies in the kernel of the 1-form $\alpha_t$.

Since the closed 2-forms $\omega_t$ are all cohomologous, one has $[\Omega_t] = 0$ in $H^2(X, \mathbb{R})$, so there exist 1-forms $\beta_t$ on $X$ such that $d\beta_t = \Omega_t$. Remark that, although $\Omega_t = 0$ over $U_0$, one cannot ensure that $\beta_{t|U_0} = 0$ unless the class $[\Omega_t]$ also vanishes in the relative cohomology group $H^2(X, U_0; \mathbb{R})$. So we need to work a little more to find the proper 1-forms $\alpha_t$.

Over $U_0$ one has $d\beta_t = \Omega_t = 0$, so $\beta_t$ defines a class in $H^1(U_0, \mathbb{R})$. By further restriction, the forms $\beta_{t|W_0}$ are also closed 1-forms on $W_0$. Let $\pi$ be a projection map $U_0 \to W_0$ such that at any point $x \in W_0$ the tangent space to $\pi^{-1}(x)$ is the symplectic normal space $N_xW_0$, and let $\gamma_t = \pi^*(\beta_{t|W_0})$. First we notice that, by construction, the 1-form $\gamma_t$ is closed over $U_0$, and at any point $x \in W_0$ the space $N_xW_0$ lies in the kernel of $\gamma_t$. Furthermore the composition of $\pi^*$ and the restriction map induces the identity map over $H^1(U_0, \mathbb{R})$, so $[\gamma_t] = [\beta_{t|U_0}]$ in $H^1(U_0, \mathbb{R})$. Therefore there exist functions $f_t$ over $U_0$ such that $\gamma_t = \beta_t + df_t$ at any point of $U_0$.

Let $g_t$ be any smooth functions over $X$ extending $f_t$, and let $\alpha_t = \beta_t + dg_t$. The 1-forms $\alpha_t$ satisfy $d\alpha_t = d\beta_t = \Omega_t$, and since $\alpha_{t|U_0} = \gamma_t$ the space $N_xW_0$ also lies in the kernel of $\alpha_t$ at any $x \in W_0$. So Proposition 4 is proved.

## 5. Properties of the constructed submanifolds

**5.1. Proof of Proposition 2.** This proof is based on that of a similar result obtained by Donaldson [D1] for the submanifolds obtained from Theorem 1 ($r = 1$). The result comes from a Morse theory argument, as described in [D1]. Indeed, consider the real valued function $f = \log |s|^2$ over $X - W$ (where $W = s^{-1}(0)$). We only have to show that, if $k$ is large enough, all its critical points are of index at least $n - r + 1$. For this purpose, let $x$ be a critical point of $f$, and let us compute the derivative $\bar\partial\partial f$ at $x$.

First we notice that $x$ is also a critical point of $|s|^2$, so that $s$ itself is not in the image of $\nabla_x s$. Recalling that $s$ is $\eta$-transverse to 0 for some $\eta > 0$, it follows that $\nabla_x s$ is not surjective and thus $|s(x)| \geq \eta$.

Recalling that the scalar product is linear in the first variable and antilinear in the second variable, we compute the derivative

$$\partial \log |s|^2 = \frac{1}{|s|^2}(\langle \partial s, s\rangle + \langle s, \bar\partial s\rangle),$$

which equals zero at $x$. A first consequence is that, at $x$, $|\langle\partial s, s\rangle| = |\langle\bar\partial s, s\rangle| < C|s|$, where $C$ is a constant bounding $\bar\partial s$ independently of $k$.

A second derivation, omitting the quantities that vanish at a critical point, yields that, at $x$,

$$\bar\partial\partial\log|s|^2 = \frac{1}{|s|^2}(\langle\bar\partial\partial s, s\rangle - \langle\partial s, \partial s\rangle + \langle\bar\partial s, \bar\partial s\rangle + \langle s, \partial\bar\partial s\rangle).$$

Recall that $\bar\partial\partial + \partial\bar\partial$ is equal to the part of type (1,1) of the curvature of the bundle $E\otimes L^k$. This is equal to $-ik\omega\otimes \mathrm{Id} + R$, where $R$ is the part of type (1,1) of the curvature of $E$, so that at $x$,

$$\bar\partial\partial\log|s|^2 = -ik\omega + \frac{1}{|s|^2}(\langle R.s, s\rangle - \langle\partial\bar\partial s, s\rangle + \langle s, \partial\bar\partial s\rangle - \langle\partial s, \partial s\rangle + \langle\bar\partial s, \bar\partial s\rangle).$$

To go further, we have to restrict our choice of vectors to a subspace of the tangent space $T_xX$ at $x$. Call $\Theta$ the space of all vectors $v$ in $T_xX$ such that $\partial s(v)$ belongs to the complex line generated by $s$ in $(E\otimes L^k)_x$. The subspace $\Theta$ of $T_xX$ is clearly stable by the almost-complex structure, and its complex dimension is at least $n-r+1$. For any vector $v\in\Theta$, $|\langle\partial s(v), s\rangle| = |\partial s(v)|\,|s|$ is smaller than $|v|\,|\langle\partial s, s\rangle| < C|v|\,|s|$ where $C$ is the same constant as above, so that $\partial s$ is $O(1)$ over $\Theta$.

Since $\bar\partial s = O(1)$ and $\partial\bar\partial s = O(k^{1/2})$ because of asymptotic holomorphicity, it is now known that the restriction to $\Theta$ of $\bar\partial\partial\log|s|^2$ is equal to $-ik\,\omega + O(k^{1/2})$. It follows that, for all large $k$, given any unit length vector $u\in\Theta$, the quantity $-2i\,\bar\partial\partial f(u, Ju)$, which equals $H_f(u) + H_f(Ju)$ where $H_f$ is the Hessian of $f$ at $x$, is negative. If the index of the critical point at $x$ were less than $n-r+1$, there would exist a subspace $P\subset T_xX$ of real dimension at least $n+r$ over which $H_f$ is non-negative, and the subspace $P\cap JP$ of real dimension at least $2r$ would necessarily intersect non-trivially $\Theta$ whose real dimension is at least $2n-2r+2$, contradicting the previous remark. The index of the critical point $x$ of $f$ is thus at least $n-r+1$.

A standard Morse theory argument then implies that the inclusion $W\to X$ induces an isomorphism on all homotopy (and homology) groups up to $\pi_{n-r-1}$ (resp. $H_{n-r-1}$), and a surjection on $\pi_{n-r}$ (resp. $H_{n-r}$), which completes the proof of Proposition 2.

**5.2. Homology and Chern numbers of the submanifolds.** Proposition 2 allows one to compute the middle-dimensional Betti number $b_{n-r} = \dim H_{n-r}(W_k, \mathbb{R})$ of the constructed submanifolds. Indeed the tangent bundle $TW_k$ and the normal bundle $NW_k$ (isomorphic to the restriction to $W_k$ of $E\otimes L^k$) are both symplectic vector bundles over $W_k$. So it is well-known (see e.g. [McS1], p. 67) that they admit underlying structures of complex vector bundles, uniquely determined up to homotopy (in our case there exist $J$-stable subspaces in $TX$ very close to $TW_k$ and $NW_k$, so after a small deformation one can think of these complex structures as induced by $J$). Furthermore one has $TW_k\oplus NW_k \simeq TX_{|W_k}$. It follows that, calling $i$ the inclusion map $W_k\to X$, the Chern classes of the bundle $TW_k$ can be computed from the relation

$$i^*c(TX) = i^*c(E\otimes L^k).c(TW_k).$$

Since $c_{n-r}(TW_k).[W_k]$ is equal to the Euler-Poincaré characteristic of $W_k$, and since the spaces $H_i(W_k, \mathbb{R})$ have the same dimension as $H_i(X, \mathbb{R})$ for $i < n - r$, the dimension of $H_{n-r}(W_k, \mathbb{R})$ follows immediately.

For further computations, we need an estimate on this dimension :

PROPOSITION 5. *For any sequence of symplectic submanifolds $W_k \subset X$ of real codimension $2r$ obtained as the zero sets of asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, the Chern classes of $W_k$ are given by*

$$c_l(TW_k) = (-1)^l \tbinom{r+l-1}{l} (k\hat{\omega})^l + O(k^{l-1}),$$

*where $\hat{\omega}$ denotes the class of $\frac{1}{2\pi}\omega$ in the cohomology of $W_k$.*

This can be proved by induction on $l$, starting from $c_0(TW_k) = 1$, since the above equality implies that

$$c_l(TW_k) = i^* c_l(TX) - \sum_{j=0}^{l-1} i^* c_{l-j}(E \otimes L^k).c_j(TW_k).$$

It can be checked that $i^* c_{l-j}(E \otimes L^k) = \tbinom{r}{l-j} (k\hat{\omega})^{l-j} + O(k^{l-j-1})$, so that the result follows from a combinatorial calculation showing that $\sum_{j=0}^{l} (-1)^j \tbinom{r}{l-j} \tbinom{r+j-1}{j}$ is equal to 0. $\square$

Since $[W_k]$ is Poincaré dual in $X$ to $c_r(E \otimes L^k)$, Proposition 5 yields that $\chi(W_k) = c_{n-r}(TW_k).[W_k] = (-1)^{n-r} \tbinom{n-1}{n-r} (k\hat{\omega})^{n-r}.(k\hat{\omega})^r.[X] + O(k^{n-1})$. Finally, Proposition 2 implies that $\chi(W_k) = (-1)^{n-r} \dim H_{n-r}(W_k, \mathbb{R}) + O(1)$, so that

$$\dim H_{n-r}(W_k, \mathbb{R}) = \tbinom{n-1}{n-r} \left(\tfrac{1}{2\pi}[\omega]\right)^n k^n + O(k^{n-1}).$$

**5.3. Geometry of the submanifolds.** Aside from the above topological information on the submanifolds, one can also try to characterize the *geometry* of $W_k$ inside $X$. We prove the following result, expressing the fact that the middle-dimensional homology of $W_k$ has many generators that are very “localized” around any given point of $X$ :

PROPOSITION 6. *There exists a constant $C > 0$ depending only on the geometry of the manifold $X$ with the following property : let $B$ be any ball of small enough radius $\rho > 0$ in $X$. For any sequence of symplectic submanifolds $W_k \subset X$ of real codimension $2r$ obtained as the zero sets of asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, let $N_k(B)$ be the number of independent generators of $H_{n-r}(W_k, \mathbb{R})$ which can be realized by cycles that are entirely included in $W_k \cap B$. Then, if $k$ is large enough, one has*

$$N_k(B) > C\, \rho^{2n} \dim H_{n-r}(W_k, \mathbb{R}).$$

As a consequence, we can state that when $k$ becomes large the submanifolds $W_k$ tend to “fill out” all of $X$, since they must intersect non-trivially with any given ball.

The proof of Proposition 6 relies on the study of what happens when we perform a symplectic blow-up on the manifold $X$ inside the ball $B$. Recall that the blown-up manifold $\tilde{X}$ is endowed with a symplectic form $\tilde{\omega}$ which is equal to $\omega$ outside of $B$, and can be described inside $B$ using the following model on $\mathbb{C}^n$ around $0$ : define on $\mathbb{C}^n \times (\mathbb{C}^n - \{0\})$ the 2-form

$$\phi = i\partial\bar\partial\big((\beta \circ p_1)(\log \|\cdot\|^2 \circ p_2)\big),$$

where $p_1$ is the projection map to $\mathbb{C}^n$, $\beta$ is a cut-off function around the blow-up point, and $p_2$ is the projection on the factor $\mathbb{C}^n - \{0\}$. The 2-form $\phi$ projects to $\mathbb{C}^n \times \mathbb{CP}^{n-1}$, and after restriction to the graph of the blown-up manifold (i.e. the set of all $(x, y)$ such that $x$ belongs to the complex line in $\mathbb{C}^n$ defined by $y$) one obtains a closed 2-form whose restriction to the exceptional divisor is positive. Calling $\theta$ the 2-form on $\tilde X$ supported in $B$ defined by this procedure, it can be checked that, if $\epsilon > 0$ is small enough and $\pi$ is the projection map $\tilde X \to X$, the 2-form $\tilde\omega = \pi^*\omega + \epsilon\theta$ is symplectic on $\tilde X$.

If we call $e \in H^2(\tilde X, \mathbb{Z})$ the Poincaré dual of the exceptional divisor, since its normal bundle is the inverse of the standard bundle over $\mathbb{CP}^{n-1}$, we have $(-e)^{n-1}.e.[X] = 1$, so that $e^n.[X] = (-1)^{n-1}$. Furthermore, the cohomology class of $\tilde\omega$ is given by $\frac{1}{2\pi}[\tilde\omega] = \frac{1}{2\pi}\pi^*[\omega] - \epsilon\, e$. Now we consider the sections $s_k$ of $E \otimes L^k$ over $X$ which define $W_k$, and assuming $\epsilon^{-1}$ to be an integer we write $k = K + \tilde k$ with $0 \le \tilde k < \epsilon^{-1}$ and $\epsilon K \in \mathbb{N}$. Notice that $\tilde\omega = \pi^*\omega$ outside $B$ and that we can safely choose a metric on $\tilde X$ with the same property. Considering that the line bundle $\tilde L^K$ on $\tilde X$ whose first Chern class is $\frac{K}{2\pi}[\tilde\omega]$ is isomorphic to $\pi^* L^K$ over $\tilde X - B$, the sections $\pi^* s_k$ of $\pi^*(E \otimes L^k) = \pi^*(E \otimes L^{\tilde k}) \otimes \pi^* L^K$ obtained by pull-back of $s_k$ satisfy all desired conditions outside $B$, namely asymptotic holomorphicity and transversality to 0. If we multiply $\pi^* s_k$ by a cut-off function equal to 1 over $\tilde X - B$ and vanishing over the support of $\theta$, we now obtain asymptotically holomorphic sections of $\pi^*(E \otimes L^{\tilde k}) \otimes \tilde L^K$ over $\tilde X$ which are transverse to 0 over $\tilde X - B$. So, if $K$ is large enough, we can use the construction described in Theorems 2 and 3 to perturb these sections *over $B$ only* to make them transverse to 0 over all of $\tilde X$. Since there are only finitely many values of $\tilde k$, the bounds on $K$ required for each $\tilde k$ translate as a single bound on $k$. Considering the zero sets of the resulting sections, we thus obtain symplectic submanifolds $\tilde W_k \subset \tilde X$ to which we can again apply Propositions 2 and 5. The interesting remark is that, using the above estimate for $\dim H_{n-r}(\tilde W_k, \mathbb{R})$, since $(\frac{1}{2\pi}[\tilde\omega])^n = (\frac{1}{2\pi}[\omega])^n - \epsilon^n$ (symplectic blow-ups decrease the symplectic volume), we get for all large $k$

$$\dim H_{n-r}(\tilde W_k, \mathbb{R}) = \dim H_{n-r}(W_k, \mathbb{R}) - \epsilon^n \tbinom{n-1}{n-r} k^n + O(k^{n-1}).$$

This means that we have decreased the dimension of $H_{n-r}(W_k, \mathbb{R})$ by changing the picture only inside the ball $B$. To continue we need an estimate on the dependence of $\epsilon$ on the radius $\rho$ of the ball. The main constraint on $\epsilon$ is that $\epsilon\theta$ should be much smaller than $\pi^*\omega$ so that the perturbation does not affect the positivity of $\pi^*\omega$. The norm of $\theta$ is directly related to that of the second derivative $\partial\bar\partial\beta$ of the cut-off function $\beta$. Since the only constraint on $\beta$ is that it should be 0 outside $B$ and 1 near the blow-up point, an appropriate choice of $\beta$ leads to a bound of the type $|\partial\bar\partial\beta| = O(\rho^{-2})$. It follows that $\epsilon$ can be chosen equal at least to a constant times $\rho^2$. So we obtain that, for a suitable value of $C$ and for all large enough $k$,

$$\dim H_{n-r}(\tilde{W}_k, \mathbb{R}) < (1 - 2C\rho^{2n}) \dim H_{n-r}(W_k, \mathbb{R}).$$

Proposition 6 now follows immediately from the following general lemma by decomposing $W_k$ into $(W_k - B) \cup (W_k \cap B)$ and perturbing slightly $\rho$ if necessary so that the boundary of $B$ is transverse to $W_k$ :

Lemma 6. *Let $W$ be a $2d$-dimensional compact manifold which decomposes into two pieces $W = A \cup B$ glued along their common boundary $S$, which is a smooth codimension 1 submanifold in $W$. Assume that there exists a manifold $\tilde{W}$ which is identical to $W$ outside of $B$, and such that $\dim H_d(\tilde{W}, \mathbb{R}) \le \dim H_d(W, \mathbb{R}) - N$. Then there exists an $\frac{N}{2}$-dimensional subspace in $H_d(W, \mathbb{R})$ consisting of classes which can be represented by cycles contained in $B$.*

To prove this lemma, let $H = H_d(W, \mathbb{R})$ and consider its subspaces $F$ consisting of all classes which can be represented by a cycle contained in $A$ and $G$ consisting of all classes representable in $B$. We have to show that $\dim G \ge \frac{N}{2}$. Let $G^\perp$ be the subspace of $H$ orthogonal to $G$ with respect to the intersection pairing, namely the set of classes which intersect trivially with all classes in $G$. We claim that $G^\perp \subset F + G$.

Indeed, let $\alpha$ be a cycle realizing a class in $G^\perp$. Subdividing $\alpha$ along its intersection with the common boundary $S$ of $A$ and $B$, we have $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1$ and $\alpha_2$ are chains respectively in $A$ and $B$, such that $\partial\alpha_1 = -\partial\alpha_2 = \beta$ is a $(d-1)$-cycle contained in $S$. However $\beta$ intersects trivially with any $d$-cycle in $S$ since $\alpha$ intersects trivially with all cycles that have a representative in $B$. So the homology class represented by $\beta$ in $H_{d-1}(S, \mathbb{R})$ is trivial, and we have $\beta = \partial\gamma$ for some $d$-chain $\gamma$ in $S$. Writing $\alpha = (\alpha_1 - \gamma) + (\alpha_2 + \gamma)$ and shifting slightly the two copies of $\gamma$ on either side of $S$, we get that $[\alpha] \in F + G$.

It follows that, if $F_G$ is a supplementary of $F \cap G$ in $F$, $\dim F_G + \dim G = \dim(F + G)$ is larger than $\dim G^\perp \ge \dim H - \dim G$, so that $\dim G \ge \frac{1}{2}(\dim H - \dim F_G)$. Thus it only remains to show that $\dim F_G \le \dim H_d(\tilde{W}, \mathbb{R})$ to complete the proof of the lemma. To do this, we remark that the morphism $h : H_d(W; \mathbb{R}) \to H_d(W, B; \mathbb{R})$ in the relative homology sequence is injective on $F_G$, since its kernel is precisely $G$. However, if we define $\tilde{F}$ and $\tilde{G}$ inside $H_d(\tilde{W}, \mathbb{R})$ similarly to $F$ and $G$, the subspace $\tilde{F}_{\tilde{G}}$ similarly injects into $H_d(\tilde{W}, \tilde{B}; \mathbb{R})$. Furthermore, the images of the two injections are both equal to the image of the morphism $H_d(A; \mathbb{R}) \to H_d(A, S; \mathbb{R})$ under the identification $H_d(\tilde{W}, \tilde{B}; \mathbb{R}) \simeq H_d(A, S; \mathbb{R}) \simeq H_d(W, B; \mathbb{R})$, so that $\dim H_d(\tilde{W}, \mathbb{R}) \ge \dim \tilde{F}_{\tilde{G}} = \dim F_G$ and the proof is complete.

## 6. Conclusion

This paper has extended the field of applicability of the construction outlined by Donaldson [D1] to more general vector bundles. It is in fact probable that similar methods can be used in other situations involving sequences of vector bundles whose curvatures become very positive.

The statement that, in spite of the high flexibility of the construction, the submanifolds obtained as zero sets of asymptotically holomorphic sections of $E \otimes L^k$

which are transverse to 0 are all isotopic for a given large enough $k$, has important consequences. Indeed, as suggested by Donaldson, it may allow the definition of relatively easily computable invariants of higher-dimensional symplectic manifolds from the topology of their submanifolds, for example from the Seiberg-Witten invariants of 4-dimensional submanifolds [T1][W]. Furthermore, it facilitates the characterization of the topology of the constructed submanifolds in many cases, thus leading the way to many examples of symplectic manifolds, some of them possibly new.

CHAPITRE III

# Symplectic 4-manifolds as branched coverings of $\mathbb{CP}^2$

Abstract. We show that every compact symplectic 4-manifold $X$ can be topologically realized as a covering of $\mathbb{CP}^2$ branched along a smooth symplectic curve in $X$ which projects as an immersed curve with cusps in $\mathbb{CP}^2$. Furthermore, the covering map can be chosen to be approximately pseudo-holomorphic with respect to any given almost-complex structure on $X$.

## 1. Introduction

It has recently been shown by Donaldson [D2] that the existence of approximately holomorphic sections of very positive line bundles over compact symplectic manifolds allows the construction not only of symplectic submanifolds ([D1], see also [A1],[Pa]) but also of symplectic Lefschetz pencil structures. The aim of this paper is to show how similar techniques can be applied in the case of 4-manifolds to obtain maps to $\mathbb{CP}^2$, thus proving that every compact symplectic 4-manifold is topologically a (singular) branched covering of $\mathbb{CP}^2$.

Let $(X,\omega)$ be a compact symplectic 4-manifold such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X,\mathbb{R})$ is integral. This integrality condition does not restrict the diffeomorphism type of $X$ in any way, since starting from an arbitrary symplectic structure one can always perturb it so that $\frac{1}{2\pi}[\omega]$ becomes rational, and then multiply $\omega$ by a constant factor to obtain integrality. A compatible almost-complex structure $J$ on $X$ and the corresponding Riemannian metric $g$ are also fixed.

Let $L$ be the complex line bundle on $X$ whose first Chern class is $c_1(L) = \frac{1}{2\pi}[\omega]$. Fix a Hermitian structure on $L$, and let $\nabla^L$ be a Hermitian connection on $L$ whose curvature 2-form is equal to $-i\omega$ (it is clear that such a connection always exists). The key observation is that, for large values of an integer parameter $k$, the line bundles $L^k$ admit many approximately holomorphic sections, thus making it possible to obtain sections which have nice transversality properties.

For example, one such section can be used to define an approximately holomorphic symplectic submanifold in $X$ [D1]. Similarly, constructing two sections satisfying certain transversality requirements yields a Lefschetz pencil structure [D2]. In our case, the aim is to construct, for large enough $k$, *three* sections $s_k^0$, $s_k^1$ and $s_k^2$ of $L^k$ satisfying certain transversality properties, in such a way that the three sections do not vanish simultaneously and that the map from $X$ to $\mathbb{CP}^2$ defined by $x \mapsto [s_k^0(x) : s_k^1(x) : s_k^2(x)]$ is a branched covering.

Let us now describe more precisely the notion of approximately holomorphic singular branched covering. Fix a constant $\epsilon > 0$, and let $U$ be a neighborhood of a point $x$ in an almost-complex 4-manifold. We say that a local complex coordinate map $\phi : U \to \mathbb{C}^2$ is *$\epsilon$-approximately holomorphic* if, at every point, $|\phi_* J - \mathbb{J}_0| \leq \epsilon$, where $\mathbb{J}_0$ is the canonical complex structure on $\mathbb{C}^2$. Another equivalent way to state the same property is the bound $|\bar\partial\phi(u)| \leq \epsilon|\nabla\phi(u)|$ for every tangent vector $u$.

DEFINITION 1. *A map $f : X \to \mathbb{CP}^2$ is* locally $\epsilon$-holomorphically modelled *at $x$ on a map $g : \mathbb{C}^2 \to \mathbb{C}^2$ if there exist neighborhoods $U$ of $x$ in $X$ and $V$ of $f(x)$ in $\mathbb{CP}^2$, and $\epsilon$-approximately holomorphic $C^1$ coordinate maps $\phi : U \to \mathbb{C}^2$ and $\psi : V \to \mathbb{C}^2$ such that $f = \psi^{-1} \circ g \circ \phi$ over $U$.*

DEFINITION 2. *A map $f : X \to \mathbb{CP}^2$ is an* $\epsilon$-holomorphic singular covering *branched along a submanifold $R \subset X$ if its differential is surjective everywhere except at the points of $R$, where* $\mathrm{rank}(df) = 2$*, and if at any point $x \in X$ it is locally $\epsilon$-holomorphically modelled on one of the three following maps :*
(i) *local diffeomorphism : $(z_1, z_2) \mapsto (z_1, z_2)$ ;*
(ii) *branched covering : $(z_1, z_2) \mapsto (z_1^2, z_2)$ ;*
(iii) *cusp covering : $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$.*

In particular it is clear that the cusp model occurs only in a neighborhood of a finite set of points $\mathcal{C} \subset R$, and that the branched covering model occurs only in a neighborhood of $R$ (away from $\mathcal{C}$), while $f$ is a local diffeomorphism everywhere outside of a neighborhood of $R$. Moreover, the set of branch points $R$ and its projection $f(R)$ can be described as follows in the local models : for the branched covering model, $R = \{(z_1, z_2),\ z_1 = 0\}$ and $f(R) = \{(x, y),\ x = 0\}$ ; for the cusp covering model, $R = \{(z_1, z_2),\ 3z_1^2 - z_2 = 0\}$ and $f(R) = \{(x, y),\ 27x^2 - 4y^3 = 0\}$.

It follows that, if $\epsilon < 1$, $R$ is a *smooth 2-dimensional submanifold in $X$, approximately $J$-holomorphic, and therefore symplectic*, and that $f(R)$ is an *immersed symplectic curve in $\mathbb{CP}^2$ except for a finite number of cusps.*

We now state our main result :

THEOREM 1. *For any $\epsilon > 0$ there exists an $\epsilon$-holomorphic singular covering map $f : X \to \mathbb{CP}^2$.*

The techniques involved in the proof of this result are similar to those introduced by Donaldson in [D1] : the first ingredient is a local transversality result stating roughly that, given approximately holomorphic sections of certain bundles, it is possible to ensure that they satisfy certain transversality estimates over a small ball in $X$ by adding to them small and localized perturbations. The other ingredient is a globalization principle, which, if the small perturbations providing local transversality are sufficiently well localized, ensures that a transversality estimate can be obtained over all of $X$ by combining the local perturbations.

We now define more precisely the notions of approximately holomorphic sections and of transversality with estimates. We will be considering sequences of sections of complex vector bundles $E_k$ over $X$, for all large values of the integer $k$, where each of the bundles $E_k$ carries naturally a Hermitian metric and a Hermitian connection. These connections together with the almost complex structure $J$ on $X$ yield $\partial$ and $\bar\partial$ operators on $E_k$. Moreover, we choose to rescale the metric on $X$, and use $g_k = k.g$ : for example, the diameter of $X$ is multiplied by $k^{1/2}$, and all derivatives of order

$p$ are divided by $k^{p/2}$. The reason for this rescaling is that the vector bundles $E_k$ we will consider are derived from $L^k$, on which the natural Hermitian connection induced by $\nabla^L$ has curvature $-ik\omega$.

DEFINITION 3. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of complex vector bundles $E_k$ over $X$. The sections $s_k$ are said to be* asymptotically holomorphic *if there exist constants $(C_p)_{p\in\mathbb{N}}$ such that, for all $k$ and at every point of $X$, $|s_k| \le C_0$, $|\nabla^p s_k| \le C_p$ and $|\nabla^{p-1}\bar\partial s_k| \le C_p k^{-1/2}$ for all $p \ge 1$, where the norms of the derivatives are evaluated with respect to the metrics $g_k = k\,g$.*

DEFINITION 4. *Let $s_k$ be a section of a complex vector bundle $E_k$, and let $\eta > 0$ be a constant. The section $s_k$ is said to be $\eta$-*transverse to 0 *if, at any point $x \in X$ where $|s_k(x)| < \eta$, the covariant derivative $\nabla s_k(x) : T_xX \to (E_k)_x$ is surjective and has a right inverse of norm less than $\eta^{-1}$ w.r.t. the metric $g_k$.*

We will often say that a sequence $(s_k)_{k\gg 0}$ of sections of $E_k$ is *transverse to* 0 (without precising the constant) if there exists a constant $\eta > 0$ independent of $k$ such that $\eta$-transversality to 0 holds for all large $k$.

In this definition of transversality, two cases are of specific interest. First, when $E_k$ is a line bundle, and if one assumes the sections to be asymptotically holomorphic, transversality to 0 can be equivalently expressed by the property

$$\forall x \in X,\ |s_k(x)| < \eta \Rightarrow |\nabla s_k(x)|_{g_k} > \eta.$$

Next, when $E_k$ has rank greater than 2 (or more generally than the complex dimension of $X$), the property actually means that $|s_k(x)| \ge \eta$ for all $x \in X$.

An important point to keep in mind is that transversality to 0 is an *open* property : if $s$ is $\eta$-transverse to 0, then any section $\sigma$ such that $|s - \sigma|_{C^1} < \epsilon$ is $(\eta - \epsilon)$-transverse to 0.

The interest of such a notion of transversality with estimates is made clear by the following observation :

LEMMA 1. *Let $\gamma_k$ be asymptotically holomorphic sections of vector bundles $E_k$ over $X$, and assume that the sections $\gamma_k$ are transverse to* 0*. Then, for large enough $k$, the zero set of $\gamma_k$ is a smooth symplectic submanifold in $X$.*

This lemma follows from the observation that, where $\gamma_k$ vanishes, $|\bar\partial\gamma_k| = O(k^{-1/2})$ by the asymptotic holomorphicity property while $\partial\gamma_k$ is bounded from below by the transversality property, thus ensuring that for large enough $k$ the zero set is smooth and symplectic, and even asymptotically $J$-holomorphic. We can now write our second result, which is a one-parameter version of Theorem 1 :

THEOREM 2. *Let $(J_t)_{t\in[0,1]}$ be a family of almost-complex structures on $X$ compatible with $\omega$. Fix a constant $\epsilon > 0$, and let $(s_{t,k})_{t\in[0,1],k\gg 0}$ be asymptotically $J_t$-holomorphic sections of $\mathbb{C}^3 \otimes L^k$, such that the sections $s_{t,k}$ and their derivatives depend continuously on $t$.*

*Then, for all large enough values of $k$, there exist asymptotically $J_t$-holomorphic sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, nowhere vanishing, depending continuously on $t$, and such that, for all $t \in [0,1]$, $|\sigma_{t,k} - s_{t,k}|_{C^3,g_k} \le \epsilon$ and the map $X \to \mathbb{CP}^2$ defined by $\sigma_{t,k}$ is an approximately holomorphic singular covering with respect to $J_t$.*

Note that, although we allow the almost-complex structure on $X$ to depend on $t$, we always use the same metric $g_k = k\,g$ independently of $t$. Therefore, there is no special relation between $g_k$ and $J_t$. However, since the parameter space $[0,1]$ is compact, we know that the metric defined by $\omega$ and $J_t$ differs from $g$ by at most a constant factor, and therefore up to a change in the constants this has no real influence on the transversality and holomorphicity properties.

We now describe more precisely the properties of the approximately holomorphic singular coverings constructed in Theorems 1 and 2, in order to state a uniqueness result for such coverings.

Definition 5. *Let $s_k$ be nowhere vanishing asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Define the corresponding projective maps $f_k = \mathbb{P}s_k$ from $X$ to $\mathbb{CP}^2$ by $f_k(x) = [s_k^0(x) : s_k^1(x) : s_k^2(x)]$. Define the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$, which is a section of the line bundle $\Lambda^{2,0}T^*X \otimes f_k^*\Lambda^{2,0}T\mathbb{CP}^2 = K_X \otimes L^{3k}$. Finally, define $R(s_k)$ to be the set of points of $X$ where $\mathrm{Jac}(f_k)$ vanishes, i.e. where $\partial f_k$ is not surjective.*

*Given a constant $\gamma > 0$, we say that $s_k$ satisfies the transversality property $\mathcal{P}_3(\gamma)$ if $|s_k| \geq \gamma$ and $|\partial f_k|_{g_k} \geq \gamma$ at every point of $X$, and if $\mathrm{Jac}(f_k)$ is $\gamma$-transverse to $0$.*

If $s_k$ satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$ and if $k$ is large enough, then it follows from Lemma 1 that $R(s_k)$ is a smooth symplectic submanifold in $X$. By analogy with the expected properties of the set of branch points, it is therefore natural to require such a property for the sections which define our covering maps.

Furthermore, recall that one expects the projection to $\mathbb{CP}^2$ of the set of branch points to be an immersed curve except at only finitely many non-degenerate cusps. Forget temporarily the antiholomorphic derivative $\bar\partial f_k$, and consider only the holomorphic part. Then the cusps correspond to the points of $R(s_k)$ where the kernel of $\partial f_k$ and the tangent space to $R(s_k)$ coincide (in other words, the points where the tangent space to $R(s_k)$ becomes "vertical"). Since $R(s_k)$ is the set of points where $\mathrm{Jac}(f_k) = 0$, the cusp points are those where the quantity $\partial f_k \wedge \partial \mathrm{Jac}(f_k)$ vanishes.

Note that, along $R(s_k)$, $\partial f_k$ has complex rank 1 and so is actually a nowhere vanishing $(1,0)$-form with values in the rank 1 subbundle $\mathrm{Im}\,\partial f_k \subset f_k^*T\mathbb{CP}^2$. In a neighborhood of $R(s_k)$, this is no longer true, but one can project $\partial f_k$ onto a rank 1 subbundle in $f_k^*T\mathbb{CP}^2$, thus obtaining a nonvanishing $(1,0)$-form $\pi(\partial f_k)$ with values in a line bundle. Cusp points are then characterized in $R(s_k)$ by the vanishing of $\pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$, which is a section of a line bundle. Therefore, it is natural to require that the restriction to $R(s_k)$ of this last quantity be transverse to 0, since it implies that the cusp points are isolated and in some sense non-degenerate.

It is worth noting that, up to a change of constants in the estimates, this transversality property is actually independent of the choice of the subbundle of $f_k^*T\mathbb{CP}^2$ on which one projects $\partial f_k$, as long as $\pi(\partial f_k)$ remains bounded from below.

For convenience, we introduce the following notations :

Definition 6. *Let $s_k$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ and $f_k = \mathbb{P}s_k$. Assume that $s_k$ satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$. Consider the rank one subbundle $(\mathrm{Im}\,\partial f_k)_{|R(s_k)}$ of $f_k^*T\mathbb{CP}^2$ over $R(s_k)$, and define $\mathcal{L}(s_k)$ to be its extension over a neighborhood of $R(s_k)$ as a subbundle of $f_k^*T\mathbb{CP}^2$, constructed by* *radial parallel transport along directions normal to* $R(s_k)$*. Finally, define, over the same neighborhood of* $R(s_k)$*,* $\mathcal{T}(s_k) = \pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$*, where* $\pi : f_k^* T\mathbb{CP}^2 \to \mathcal{L}(s_k)$ *is the orthogonal projection.*

*We say that asymptotically holomorphic sections* $s_k$ *of* $\mathbb{C}^3 \otimes L^k$ *are* $\gamma$-generic *if they satisfy* $\mathcal{P}_3(\gamma)$ *and if the restriction to* $R(s_k)$ *of* $\mathcal{T}(s_k)$ *is* $\gamma$*-transverse to* 0 *over* $R(s_k)$*. We then define the set of cusp points* $\mathcal{C}(s_k)$ *as the set of points of* $R(s_k)$ *where* $\mathcal{T}(s_k) = 0$*.*

In a holomorphic setting, such a genericity property would be sufficient to ensure that the map $f_k = \mathbb{P}s_k$ is a singular branched covering. However, in our case, extra difficulties arise because we only have approximately holomorphic sections. This means that at a point of $R(s_k)$, although $\partial f_k$ has rank 1, we have no control over the rank of $\bar\partial f_k$, and the local picture may be very different from what one expects. Therefore, we need to control the antiholomorphic part of the derivative along the set of branch points by adding the following requirement :

Definition 7. *Let* $s_k$ *be* $\gamma$*-generic asymptotically* $J$*-holomorphic sections of* $\mathbb{C}^3 \otimes L^k$*. We say that* $s_k$ *is* $\bar\partial$-tame *if there exist constants* $(C_p)_{p\in\mathbb{N}}$ *and* $c > 0$*, depending only on the geometry of* $X$ *and the bounds on* $s_k$ *and its derivatives, and an* $\omega$*-compatible almost complex structure* $\tilde J_k$*, such that the following properties hold :*

(1) $\forall p \in \mathbb{N},\ |\nabla^p(\tilde J_k - J)|_{g_k} \le C_p k^{-1/2}$ *;*

(2) *the almost-complex structure* $\tilde J_k$ *is integrable over the set of points whose* $g_k$*-distance to* $\mathcal{C}_{\tilde J_k}(s_k)$ *is less than c (the subscript indicates that one uses* $\partial_{\tilde J_k}$ *rather than* $\partial_J$ *to define* $\mathcal{C}(s_k)$*) ;*

(3) *the map* $f_k = \mathbb{P}s_k$ *is* $\tilde J_k$*-holomorphic at every point of* $X$ *whose* $g_k$*-distance to* $\mathcal{C}_{\tilde J_k}(s_k)$ *is less than c ;*

(4) *at every point of* $R_{\tilde J_k}(s_k)$*, the antiholomorphic derivative* $\bar\partial_{\tilde J_k}(\mathbb{P}s_k)$ *vanishes over the kernel of* $\partial_{\tilde J_k}(\mathbb{P}s_k)$*.*

Note that since $\tilde J_k$ is within $O(k^{-1/2})$ of $J$, the notions of asymptotic $J$-holomorphicity and asymptotic $\tilde J_k$-holomorphicity actually coincide, because the $\partial$ and $\bar\partial$ operators differ only by $O(k^{-1/2})$. Furthermore, if $k$ is large enough, then $\gamma$-genericity for $J$ implies $\gamma'$-genericity for $\tilde J_k$ as well for some $\gamma'$ slightly smaller than $\gamma$ ; and, because of the transversality properties, the sets $R_{\tilde J_k}(s_k)$ and $\mathcal{C}_{\tilde J_k}(s_k)$ lie within $O(k^{-1/2})$ of $R_J(s_k)$ and $\mathcal{C}_J(s_k)$.

In the case of families of sections depending continuously on a parameter $t \in [0,1]$, it is natural to also require that the almost complex structures $\tilde J_{t,k}$ close to $J_t$ for every $t$ depend continuously on $t$. We claim the following :

Theorem 3. *Let* $s_k$ *be asymptotically* $J$*-holomorphic sections of* $\mathbb{C}^3 \otimes L^k$*. Assume that the sections* $s_k$ *are* $\gamma$*-generic and* $\bar\partial$*-tame. Then, for all large enough values of* $k$*, the maps* $f_k = \mathbb{P}s_k$ *are* $\epsilon_k$*-holomorphic singular branched coverings, for some constants* $\epsilon_k = O(k^{-1/2})$*.*

Therefore, in order to prove Theorems 1 and 2 it is sufficient to construct $\gamma$-generic and $\bar\partial$-tame sections (resp. one-parameter families of sections) of $\mathbb{C}^3 \otimes L^k$.

Even better, we have the following uniqueness result for these particular singular branched coverings :

THEOREM 4. *Let $s_{0,k}$ and $s_{1,k}$ be sections of $\mathbb{C}^3 \otimes L^k$, asymptotically holomorphic with respect to $\omega$-compatible almost-complex structures $J_0$ and $J_1$ respectively. Assume that $s_{0,k}$ and $s_{1,k}$ are $\gamma$-generic and $\bar\partial$-tame. Then there exist almost-complex structures $(J_t)_{t\in[0,1]}$ interpolating between $J_0$ and $J_1$, and a constant $\eta > 0$, with the following property : for all large enough $k$, there exist sections $(s_{t,k})_{t\in[0,1],k\gg 0}$ of $\mathbb{C}^3 \otimes L^k$ interpolating between $s_{0,k}$ and $s_{1,k}$, depending continuously on $t$, which are, for all $t \in [0,1]$, asymptotically $J_t$-holomorphic, $\eta$-generic and $\bar\partial$-tame with respect to $J_t$.*

*In particular, for large $k$ the two approximately holomorphic singular branched coverings $\mathbb{P}s_{0,k}$ and $\mathbb{P}s_{1,k}$ are isotopic among approximately holomorphic singular branched coverings.*

Therefore, there exists for all large $k$ a canonical isotopy class of singular branched coverings $X \to \mathbb{CP}^2$, which could potentially be used to define symplectic invariants of $X$.

The remainder of this article is organized as follows : Section 2 describes the process of perturbing asymptotically holomorphic sections of bundles of rank greater than 2 to make sure that they remain away from zero. Section 3 deals with further perturbation in order to obtain $\gamma$-genericity. Section 4 describes a way of achieving $\bar\partial$-tameness, and therefore completes the proofs of Theorems 1, 2 and 4. Finally, Theorem 3 is proved in Section 5, and Section 6 deals with various related remarks.

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## 2. Nowhere vanishing sections

**2.1. Non-vanishing of $s_k$.** Our strategy to prove Theorem 1 is to start with given asymptotically holomorphic sections $s_k$ (for example $s_k = 0$) and perturb them in order to obtain the required properties ; the proof of Theorem 2 then relies on the same arguments, with the added difficulty that all statements must apply to 1-parameter families of sections.

The first step is to ensure that the three components $s_k^0$, $s_k^1$ and $s_k^2$ do not vanish simultaneously, and more precisely that, for some constant $\eta > 0$ independent of $k$, the sections $s_k$ are $\eta$-transverse to 0, i.e. $|s_k| \geq \eta$ over all of $X$. Therefore, the first ingredient in the proof of Theorems 1 and 2 is the following result :

PROPOSITION 1. *Let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, and fix a constant $\epsilon > 0$. Then there exists a constant $\eta > 0$ such that, for all large enough values of $k$, there exist asymptotically holomorphic sections $\sigma_k$ of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \leq \epsilon$ and that $|\sigma_k| \geq \eta$ at every point of $X$. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 1 is a direct consequence of the main theorem in [A1], where it is proved that, given any complex vector bundle $E$, asymptotically holomorphic sections of $E\otimes L^k$ (or 1-parameter families of such sections) can be made transverse to 0 by small perturbations : Proposition 1 follows simply by considering the case where $E$ is the trivial bundle of rank 3. However, for the sake of completeness and in order to introduce tools which will also be used in later parts of the proof, we give here a shorter argument dealing with the specific case at hand.

There are three ingredients in the proof of Proposition 1. The first one is the existence of many localized asymptotically holomorphic sections of the line bundle $L^k$ for sufficiently large $k$.

DEFINITION 8. *A section $s$ of a vector bundle $E_k$ has* Gaussian decay *in $C^r$ norm away from a point $x \in X$ if there exists a polynomial $P$ and a constant $\lambda > 0$ such that for all $y \in X$, $|s(y)|$, $|\nabla s(y)|_{g_k}, \dots, |\nabla^r s(y)|_{g_k}$ are all bounded by $P(d(x,y))\exp(-\lambda\, d(x,y)^2)$, where $d(.,.)$ is the distance induced by $g_k$.*

*The decay properties of a family of sections are said to be* uniform *if there exist $P$ and $\lambda$ such that the above bounds hold for all sections of the family, independently of $k$ and of the point $x$ at which decay occurs for a given section.*

LEMMA 2 ([D1],[A1]). *Given any point $x \in X$, for all large enough $k$, there exist asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x}$ of $L^k$ over $X$ satisfying the following bounds : $|s^{\mathrm{ref}}_{k,x}| \geq c_s$ at every point of the ball of $g_k$-radius 1 centered at $x$, for some universal constant $c_s > 0$ ; and the sections $s^{\mathrm{ref}}_{k,x}$ have uniform Gaussian decay away from $x$ in $C^3$ norm.*

*Moreover, given a one-parameter family of $\omega$-compatible almost-complex structures $(J_t)_{t\in[0,1]}$, there exist one-parameter families of sections $s^{\mathrm{ref}}_{t,k,x}$ which are asymptotically $J_t$-holomorphic for all $t$, depend continuously on $t$ and satisfy the same bounds.*

The first part of this statement is Proposition 11 of [D1], while the extension to one-parameter families is carried out in Lemma 3 of [A1]. Note that here we require decay with respect to the $C^3$ norm instead of $C^0$, but the bounds on all derivatives follow immediately from the construction of these sections : indeed, they are modelled on $f(z) = \exp(-|z|^2/4)$ in a local approximately holomorphic Darboux coordinate chart for $k\omega$ at $x$ and in a suitable local trivialization of $L^k$ where the connection 1-form is $\frac{1}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$. Therefore, it is sufficient to notice that the model function has Gaussian decay and that all derivatives of the coordinate map are uniformly bounded because of the compactness of $X$.

More precisely, the result of existence of local approximately holomorphic Darboux coordinate charts needed for Lemma 2 (and throughout the proofs of the main theorems as well) is the following (see also [D1]) :

LEMMA 3. *Near any point $x \in X$, for any integer $k$, there exist local complex Darboux coordinates $(z^1_k, z^2_k) : (X,x) \to (\mathbb{C}^2, 0)$ for the symplectic structure $k\omega$ (i.e. such that the pullback of the standard symplectic structure of $\mathbb{C}^2$ is $k\omega$) such that, denoting by $\psi_k : (\mathbb{C}^2,0) \to (X,x)$ the inverse of the coordinate map, the following bounds hold uniformly in $x$ and $k$ : $|z^1_k(y)| + |z^2_k(y)| = O(dist_{g_k}(x,y))$ on a ball of fixed radius around $x$ ; $|\nabla^r \psi_k|_{g_k} = O(1)$ for all $r \geq 1$ on a ball of fixed radius around $0$ ; and, with respect to the almost-complex structure $J$ on $X$ and the canonical* *complex structure* $\mathbb{J}_0$ *on* $\mathbb{C}^2$, $|\bar\partial\psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$ *and* $|\nabla^r\bar\partial\psi|_{g_k} = O(k^{-1/2})$ *for all* $r \geq 1$ *on a ball of fixed radius around* $0$.

*Moreover, given a continuous 1-parameter family of* $\omega$*-compatible almost-complex structures* $(J_t)_{t\in[0,1]}$ *and a continuous family of points* $(x_t)_{t\in[0,1]}$*, one can find for all* $t$ *coordinate maps near* $x_t$ *satisfying the same estimates and depending continuously on* $t$.

PROOF. By Darboux's theorem, there exists a local symplectomorphism $\phi$ from a neighborhood of $0$ in $\mathbb{C}^2$ with its standard symplectic structure to a neighborhood of $x$ in $(X, \omega)$. It is well-known that the space of symplectic $\mathbb{R}$-linear endomorphisms of $\mathbb{C}^2$ which intertwine the complex structures $\mathbb{J}_0$ and $\phi^*J(x)$ is non-empty (and actually isomorphic to $\mathrm{U}(2)$). So, choosing such a linear map $\Psi$ and defining $\psi = \phi\circ\Psi$, one gets a local symplectomorphism such that $\bar\partial\psi(0) = 0$. Moreover, because of the compactness of $X$, it is possible to carry out the construction in such a way that, with respect to the metric $g$, all derivatives of $\psi$ are bounded over a neighborhood of $x$ by uniform constants which do not depend on $x$. Therefore, over a neighborhood of $x$ one can assume that $|\nabla(\psi^{-1})|_g = O(1)$, as well as $|\nabla^r\psi|_g = O(1)$ and $|\nabla^r\bar\partial\psi|_g = O(1)$ $\forall r \geq 1$.

Define $\psi_k(z) = \psi(k^{-1/2}z)$, and switch to the metric $g_k$ : then $\bar\partial\psi_k(0) = 0$, and at every point near $x$, $|\nabla(\psi_k^{-1})|_{g_k} = |\nabla(\psi^{-1})|_g = O(1)$. Moreover, $|\nabla^r\psi_k|_{g_k} = O(k^{(1-r)/2}) = O(1)$ and $|\nabla^r\bar\partial\psi_k|_{g_k} = O(k^{-r/2}) = O(k^{-1/2})$ for all $r \geq 1$. Finally, since $|\nabla\bar\partial\psi_k|_{g_k} = O(k^{-1/2})$ and $\bar\partial\psi_k(0) = 0$ we have $|\bar\partial\psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$, so that all expected estimates hold. Because of the compactness of $X$, the estimates are uniform in $x$, and because the maps $\psi_k$ for different values of $k$ differ only by a rescaling, the estimates are also uniform in $k$.

In the case of a one-parameter family of almost-complex structures, there is only one thing to check in order to carry out the same construction for every value of $t \in [0,1]$ while ensuring continuity in $t$ : given a one-parameter family of local Darboux maps $\phi_t$ near $x_t$ (the existence of such maps depending continuously on $t$ is trivial), one must check the existence of a continuous one-parameter family of $\mathbb{R}$-linear symplectic endomorphisms $\Psi_t$ of $\mathbb{C}^2$ intertwining the complex structures $\mathbb{J}_0$ and $\phi_t^*J_t(x_t)$ on $\mathbb{C}^2$. To prove this, remark that for every $t$ the set of these endomorphisms of $\mathbb{C}^2$ can be identified with the group $\mathrm{U}(2)$. Therefore, what we are looking for is a continuous section $(\Psi_t)_{t\in[0,1]}$ of a principal $\mathrm{U}(2)$-bundle over $[0,1]$. Since $[0,1]$ is contractible, this bundle is necessarily trivial and therefore has a continuous section. This proves the existence of the required maps $\Psi_t$, so one can define $\psi_t = \phi_t\circ\Psi_t$, and set $\psi_{t,k}(z) = \psi_t(k^{-1/2}z)$ as above. The expected bounds follow naturally ; the estimates are uniform in $t$ because of the compactness of $[0,1]$. □

The second tool we need for Proposition 1 is the following local transversality result, which involves ideas similar to those in [D1] and in §2 of [A1] but applies to maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ with $m > n$ rather than $m = 1$ :

PROPOSITION 2. *Let* $f$ *be a function defined over the ball* $B^+$ *of radius* $\frac{11}{10}$ *in* $\mathbb{C}^n$ *with values in* $\mathbb{C}^m$*, with* $m > n$*. Let* $\delta$ *be a constant with* $0 < \delta < \frac{1}{2}$*, and let*

$\eta = \delta \log(\delta^{-1})^{-p}$ *where* $p$ *is a suitable fixed integer depending only on the dimension* $n$. *Assume that* $f$ *satisfies the following bounds over* $B^+$ :

$$|f| \leq 1, \qquad |\bar\partial f| \leq \eta, \qquad |\nabla \bar\partial f| \leq \eta.$$

*Then, there exists* $w \in \mathbb{C}^m$, *with* $|w| \leq \delta$, *such that* $|f - w| \geq \eta$ *over the interior ball* $B$ *of radius* 1.

*Moreover, if one considers a one-parameter family of functions* $(f_t)_{t\in[0,1]}$ *satisfying the same bounds, then one can find for all* $t$ *elements* $w_t \in \mathbb{C}^m$ *depending continuously on* $t$ *such that* $|w_t| \leq \delta$ *and* $|f_t - w_t| \geq \eta$ *over* $B$.

This statement is proved in Section 2.3. The last, and most crucial, ingredient of the proof of Proposition 1 is a globalization principle due to Donaldson [D1] which we state here in a general form.

Definition 9. *A family of properties* $\mathcal{P}(\epsilon, x)_{x\in X, \epsilon>0}$ *of sections of bundles over* $X$ *is* local and $C^r$-open *if, given a section* $s$ *satisfying* $\mathcal{P}(\epsilon, x)$, *any section* $\sigma$ *such that* $|\sigma(x) - s(x)|$, $|\nabla\sigma(x) - \nabla s(x)|, \dots, |\nabla^r\sigma(x) - \nabla^r s(x)|$ *are smaller than* $\eta$ *satisfies* $\mathcal{P}(\epsilon - C\eta, x)$, *where* $C$ *is a constant (independent of* $x$ *and* $\epsilon$*).*

For example, the property $|s(x)| \geq \epsilon$ is local and $C^0$-open ; $\epsilon$-transversality to 0 of $s$ at $x$ is local and $C^1$-open.

Proposition 3 ([D1]). *Let* $\mathcal{P}(\epsilon, x)_{x\in X,\epsilon>0}$ *be a local and* $C^r$*-open family of properties of sections of vector bundles* $E_k$ *over* $X$. *Assume that there exist constants* $c$, $c'$ *and* $p$ *such that, given any* $x \in X$, *any small enough* $\delta > 0$, *and asymptotically holomorphic sections* $s_k$ *of* $E_k$, *there exist, for all large enough* $k$, *asymptotically holomorphic sections* $\tau_{k,x}$ *of* $E_k$ *with the following properties :* (a) $|\tau_{k,x}|_{C^r,g_k} < \delta$, (b) *the sections* $\frac{1}{\delta}\tau_{k,x}$ *have uniform Gaussian decay away from* $x$ *in* $C^r$*-norm, and* (c) *the sections* $s_k + \tau_{k,x}$ *satisfy the property* $\mathcal{P}(\eta, y)$ *for all* $y \in B_{g_k}(x, c)$, *with* $\eta = c'\delta\log(\delta^{-1})^{-p}$.

*Then, given any* $\alpha > 0$ *and asymptotically holomorphic sections* $s_k$ *of* $E_k$, *there exist, for all large enough* $k$, *asymptotically holomorphic sections* $\sigma_k$ *of* $E_k$ *such that* $|s_k - \sigma_k|_{C^r,g_k} < \alpha$ *and the sections* $\sigma_k$ *satisfy* $\mathcal{P}(\epsilon, x)$ $\forall x \in X$ *for some* $\epsilon > 0$ *independent of* $k$.

*Moreover the same result holds for one-parameter families of sections, provided the existence of sections* $\tau_{t,k,x}$ *satisfying properties* (a), (b), (c) *and depending continuously on* $t \in [0, 1]$.

This result is a general formulation of the argument contained in Section 3 of [D1] (see also [A1], §3.3 and 3.5). For the sake of completeness, let us recall just a brief outline of the construction. To achieve property $\mathcal{P}$ over all of $X$, the idea is to proceed iteratively : in step $j$, one starts from sections $s_k^{(j)}$ satisfying $\mathcal{P}(\delta_j, x)$ for all $x$ in a certain (possibly empty) subset $U_k^{(j)} \subset X$, and perturbs them by less than $\frac{1}{2C}\delta_j$ (where $C$ is the same constant as in Definition 9) to get sections $s_k^{(j+1)}$ satisfying $\mathcal{P}(\delta_{j+1}, x)$ over certain balls of $g_k$-radius $c$, with $\delta_{j+1} = c'(\frac{\delta_j}{2C})\log((\frac{\delta_j}{2C})^{-1})^{-p}$. Because the property $\mathcal{P}$ is open, $s_k^{(j+1)}$ also satisfies $\mathcal{P}(\delta_{j+1}, x)$ over $U_k^{(j)}$, therefore allowing one to obtain $\mathcal{P}$ everywhere after a certain number $N$ of steps.

The catch is that, since the value of $\delta_j$ decreases after each step and we want $\mathcal{P}(\epsilon, x)$ with $\epsilon$ independent of $k$, the number of steps needs to be bounded independently of $k$. However, the size of $X$ for the metric $g_k$ increases as $k$ increases, and the number of balls of radius $c$ needed to cover $X$ therefore also increases. The key observation due to Donaldson [D1] is that, because of the Gaussian decay of the perturbations, if one chooses a sufficiently large constant $D$, one can in a single step carry out perturbations centered at as many points as one wants, provided that any two of these points are distant of at least $D$ with respect to $g_k$ : the idea is that each of the perturbations becomes sufficiently small in the vicinity of the other perturbations in order to have no influence on property $\mathcal{P}$ there (up to a slight decrease of $\delta_{j+1}$). Therefore the construction is possible with a bounded number of steps $N$ and yields property $\mathcal{P}(\epsilon, x)$ for all $x \in X$ and for all large enough $k$, with $\epsilon = \delta_N$ independent of $k$. $\square$

We now show how to derive Proposition 1 from Lemma 2 and Propositions 2 and 3, following the ideas contained in [D1]. Proposition 1 follows directly from Proposition 3 by considering the property $\mathcal{P}$ defined as follows : say that a section $s_k$ of $\mathbb{C}^3 \otimes L^k$ satisfies $\mathcal{P}(\epsilon, x)$ if $|s_k(x)| \geq \epsilon$. This property is local and open in $C^0$-sense, and therefore also in $C^3$-sense. So it is sufficient to check that the assumptions of Proposition 3 hold for $\mathcal{P}$.

Let $x \in X$, $0 < \delta < \frac{1}{2}$, and consider asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$ (or 1-parameter families of sections $s_{t,k}$). Recall that Lemma 2 provides asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x}$ of $L^k$ which have Gaussian decay away from $x$ and remain larger than a constant $c_s$ over $B_{g_k}(x, 1)$. Therefore, dividing $s_k$ by $s^{\mathrm{ref}}_{k,x}$ yields asymptotically holomorphic functions $u_k$ on $B_{g_k}(x, 1)$ with values in $\mathbb{C}^3$. Next, one uses a local approximately holomorphic coordinate chart as given by Lemma 3 to obtain, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions $v_k$ defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}^3$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$.

Let $C_0$ be a constant bounding $|s^{\mathrm{ref}}_{k,x}|_{C^3, g_k}$, and let $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Then, provided that $k$ is large enough, Proposition 2 yields constants $w_k \in \mathbb{C}^3$, with $|w_k| \leq \frac{\delta}{C_0}$, such that $|v_k - w_k| \geq \alpha$ over the unit ball in $\mathbb{C}^2$. Equivalently, one has $|u_k - w_k| \geq \alpha$ over $B_{g_k}(x, c)$ for some constant $c$. Multiplying by $s^{\mathrm{ref}}_{k,x}$ again, one gets that $|s_k - w_k\, s^{\mathrm{ref}}_{k,x}| \geq c_s\alpha$ over $B_{g_k}(x, c)$.

The assumptions of Proposition 3 are therefore satisfied if one chooses $\eta = c_s\alpha$ (larger than $c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$) and $\tau_{k,x} = -w_k\, s^{\mathrm{ref}}_{k,x}$. Moreover, the same argument applies to one-parameter families of sections $s_{t,k}$ (one similarly constructs perturbations $\tau_{t,k,x} = -w_{t,k}\, s^{\mathrm{ref}}_{t,k,x}$). So Proposition 3 applies, which ends the proof of Proposition 1.

**2.2. Non-vanishing of $\partial f_k$.** We have constructed asymptotically holomorphic sections $s_k = (s^0_k, s^1_k, s^2_k)$ of $\mathbb{C}^3 \otimes L^k$ for all large enough $k$ which remain away from zero. Therefore, the maps $f_k = \mathbb{P}s_k$ from $X$ to $\mathbb{CP}^2$ are well defined, and they are asymptotically holomorphic, because the lower bound on $|s_k|$ implies that the derivatives of $f_k$ are $O(1)$ and that $\bar\partial f_k$ and its derivatives are $O(k^{-1/2})$ (taking the metric $g_k$ on $X$ and the standard metric on $\mathbb{CP}^2$). Our next step is to ensure, by further perturbation of the sections $s_k$, that $\partial f_k$ vanishes nowhere and remains far from zero :

PROPOSITION 4. *Let $\delta$ and $\gamma$ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k \gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ such that $|s_k| \geq \gamma$ at every point of $X$ and for all $k$. Then there exists a constant $\eta > 0$ such that, for all large enough values of $k$, there exist asymptotically holomorphic sections $\sigma_k$ of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \delta$ and that the maps $f_k = \mathbb{P}\sigma_k$ satisfy the bound $|\partial f_k|_{g_k} \geq \eta$ at every point of $X$. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 4 is proved in the same manner as Proposition 1 and uses the same three ingredients, namely Lemma 2 and Propositions 2 and 3. Proposition 4 follows directly from Proposition 3 by considering the following property : say that a section $s$ of $\mathbb{C}^3 \otimes L^k$ of norm everywhere larger than $\frac{\gamma}{2}$ satisfies $\mathcal{P}(\eta, x)$ if the map $f = \mathbb{P}s$ satisfies $|\partial f(x)|_{g_k} \geq \eta$. This property is local and open in $C^1$-sense, and therefore also in $C^3$-sense, because the lower bound on $|s|$ makes $f$ depend nicely on $s$ (by the way, note that the bound $|s| \geq \frac{\gamma}{2}$ is always satisfied in our setting since one considers only sections differing from $s_k$ by less than $\frac{\gamma}{4}$). So one only needs to check that the assumptions of Proposition 3 hold for this property $\mathcal{P}$.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider nonvanishing asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over $X$), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x) = s_k^2(x) = 0$ and therefore $|s_k^0(x)| \geq \frac{\gamma}{2}$. Because one has a uniform bound on $|\nabla s_k|$, there exists a constant $r > 0$ (independent of $k$) such that $|s_k^0| \geq \frac{\gamma}{3}$ over $B_{g_k}(x,r)$. Therefore, over this ball one can define a map to $\mathbb{C}^2$ by

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

It is quite easy to see that, at any point $y \in B_{g_k}(x,r)$, the ratio between $|\partial h_k(y)|$ and $|\partial f_k(y)|$ is bounded by a uniform constant. Therefore, what one actually needs to prove is that, for large enough $k$, a perturbation of $s_k$ with Gaussian decay and smaller than $\delta$ can make $|\partial h_k|$ larger than $\eta = c'\delta\,(\log \delta^{-1})^{-p}$ over a ball $B_{g_k}(x,c)$, for some constants $c$, $c'$ and $p$.

Recall that Lemma 2 provides asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of $L^k$ which have Gaussian decay away from $x$ and remain larger than a constant $c_s$ over $B_{g_k}(x,1)$. Moreover, consider a local approximately holomorphic coordinate chart (as given by Lemma 3) on a neighborhood of $x$, and call $z_k^1$ and $z_k^2$ the two complex coordinate functions. Define the two 1-forms

$$\mu_k^1 = \partial\Big(\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \quad \text{and} \quad \mu_k^2 = \partial\Big(\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big),$$

and notice that at $x$ they are both of norm larger than a fixed constant (which can be expressed as a function of $c_s$ and the uniform $C^0$ bound on $s_k$), and mutually orthogonal. Moreover $\mu_k^1$, $\mu_k^2$ and their derivatives are uniformly bounded because of the bounds on $s^{\mathrm{ref}}_{k,x}$, on $s^0_k$ and on the coordinate map ; these bounds are independent of $k$. Finally, $\mu^1_k$ and $\mu^2_k$ are asymptotically holomorphic because all the ingredients in their definition are asymptotically holomorphic and $|s^0_k|$ is bounded from below.

If follows that, for some constant $r'$, one can express $\partial h_k$ on the ball $B_{g_k}(x,r')$ as $(\partial h^1_k, \partial h^2_k) = (u^{11}_k\mu^1_k + u^{12}_k\mu^2_k, u^{21}_k\mu^1_k + u^{22}_k\mu^2_k)$, thus defining a function $u_k$ on $B_{g_k}(x,r')$ with values in $\mathbb{C}^4$. The properties of $\mu^i_k$ described above imply that the ratio between $|\partial h_k|$ and $|u_k|$ is bounded between two constants which do not depend on $k$ (because of the bounds on $\mu^1_k$ and $\mu^2_k$, and of their orthogonality at $x$), and that the map $u_k$ is asymptotically holomorphic (because of the asymptotic holomorphicity of $\mu^i_k$).

Next, one uses the local approximately holomorphic coordinate chart to obtain from $u_k$, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions $v_k$ defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}^4$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$. Let $C_0$ be a constant larger than $|z^i_k s^{\mathrm{ref}}_{k,x}|_{C^3,g_k}$, and let $\alpha = \frac{\delta}{4C_0}.\log((\frac{\delta}{4C_0})^{-1})^{-p}$. Then, by Proposition 2, for all large enough $k$ there exist constants $w_k = (w^{11}_k, w^{12}_k, w^{21}_k, w^{22}_k) \in \mathbb{C}^4$, with $|w_k| \leq \frac{\delta}{4C_0}$, such that $|v_k - w_k| \geq \alpha$ over the unit ball in $\mathbb{C}^2$.

Equivalently, one has $|u_k - w_k| \geq \alpha$ over $B_{g_k}(x,c)$ for some constant $c$. Multiplying by $\mu^i_k$, one therefore gets that, over $B_{g_k}(x,c)$,

$$\left|\left(\partial\Big(h^1_k - w^{11}_k\frac{z^1_k s^{\mathrm{ref}}_{k,x}}{s^0_k} - w^{12}_k\frac{z^2_k s^{\mathrm{ref}}_{k,x}}{s^0_k}\Big), \partial\Big(h^2_k - w^{21}_k\frac{z^1_k s^{\mathrm{ref}}_{k,x}}{s^0_k} - w^{22}_k\frac{z^2_k s^{\mathrm{ref}}_{k,x}}{s^0_k}\Big)\right)\right| \geq \frac{\alpha}{C}$$

where $C$ is a fixed constant determined by the bounds on $\mu^i_k$. In other terms, letting

$$(\tau^0_{k,x}, \tau^1_{k,x}, \tau^2_{k,x}) = (0, -(w^{11}_k z^1_k + w^{12}_k z^2_k)s^{\mathrm{ref}}_{k,x}, -(w^{21}_k z^1_k + w^{22}_k z^2_k)s^{\mathrm{ref}}_{k,x}),$$

and defining $\tilde h_k$ similarly to $h_k$ starting with $s_k + \tau_{k,x}$ instead of $s_k$, the above formula can be rewritten as $|\partial\tilde h_k| \geq \frac{\alpha}{C}$. Therefore, one has managed to make $|\partial\tilde h_k|$ larger than $\eta = \frac{\alpha}{C}$ over $B_{g_k}(x,c)$ by adding to $s_k$ the perturbation $\tau_{k,x}$. Moreover, $|\tau_{k,x}| \leq \sum |w^{ij}_k|.|z^i_k s^{\mathrm{ref}}_{k,x}| \leq \delta$, and the sections $z^i_k s^{\mathrm{ref}}_{k,x}$ have uniform Gaussian decay away from $x$.

As remarked above, setting $\tilde f_k = \mathbb{P}(s_k + \tau_{k,x})$, the bound $|\partial\tilde h_k| \geq \eta$ implies that $|\partial\tilde f_k|$ is larger than some $\eta'$ differing from $\eta$ by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since one has $\eta' \geq c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$. Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well (considering one-parameter families of coordinate charts, reference sections $s^{\mathrm{ref}}_{t,k,x}$, and constants $w_{t,k}$). So Proposition 3 applies. This ends the proof of Proposition 4.

**2.3. Proof of Proposition 2.** The proof of Proposition 2 goes along the same lines as that of the local transversality result introduced in [D1] and extended to one-parameter families in [A1] (see Proposition 6 below). To start with, notice that it is sufficient to prove the result in the case where $m = n+1$. Indeed, given a map $f = (f^1, \dots, f^m) : B^+ \to \mathbb{C}^m$ with $m > n+1$ satisfying the hypotheses of Proposition 2, one can define $f' = (f^1, \dots, f^{n+1}) : B^+ \to \mathbb{C}^{n+1}$, and notice that $f'$ also satisfies the required bounds. Therefore, if it is possible to find $w' = (w^1, \dots, w^{n+1}) \in \mathbb{C}^{n+1}$ of norm at most $\delta$ such that $|f' - w'| \geq \eta$ over the unit ball

$B$, then setting $w = (w^1, \dots, w^{n+1}, 0, \dots, 0) \in \mathbb{C}^m$ one gets $|w| = |w'| \le \delta$ and $|f - w| \ge |f' - w'| \ge \eta$ at all points of $B$, which is the desired result. The same argument applies to one-parameter families $(f_t)_{t\in[0,1]}$.

So we are now reduced to the case $m = n + 1$. Let us start with the case of a single map $f$, before moving on to the case of one-parameter families. The first step in the proof is to replace $f$ by a complex polynomial $g$ approximating $f$. For this, one approximates each of the $n+1$ components $f^i$ by a polynomial $g^i$, in such a way that $g$ differs from $f$ by at most a fixed multiple of $\eta$ over the unit ball $B$ and that the degree $d$ of $g$ is less than a constant times $\log(\eta^{-1})$. The process is the same as the one described in [D1] for asymptotically holomorphic maps to $\mathbb{C}$, so we skip the details. To obtain polynomial functions, one first constructs holomorphic functions $\tilde{f}^i$ differing from $f^i$ by at most a fixed multiple of $\eta$, using the given bounds on $\bar\partial f^i$. The polynomials $g^i$ are then obtained by truncating the Taylor series expansion of $\tilde{f}^i$ to a given degree. It can be shown that by this method one can obtain polynomial functions $g^i$ of degree less than a constant times $\log(\eta^{-1})$ and differing from $\tilde{f}^i$ by at most a constant times $\eta$ (see Lemmas 27 and 28 of [D1]). The approximation process does not hold on the whole ball where $f$ is defined ; this is why one needs $f$ to be defined on $B^+$ to get a result over the slightly smaller ball $B$.

Therefore, we now have a polynomial map $g$ of degree $d = O(\log(\eta^{-1}))$ such that $|f - g| \le c\,\eta$ for some constant $c$. In particular, if one finds $w \in \mathbb{C}^{n+1}$ with $|w| \le \delta$ such that $|g - w| \ge (c+1)\eta$ over the ball $B$, then it follows immediately that $|f - w| \ge \eta$ everywhere, which is the desired result. The key observation for finding such a $w$ is that the image $g(B) \subset \mathbb{C}^{n+1}$ is contained in an algebraic hypersurface $H$ in $\mathbb{C}^{n+1}$ of degree at most $D = (n+1)d^n$. Indeed, if such were not the case, then for every nonzero polynomial $P$ of degree at most $D$ in $n+1$ variables, $P(g^1, \dots, g^{n+1})$ would be a non identically zero polynomial function of degree at most $dD$ in $n$ variables ; since the space of polynomials of degree at most $D$ in $n+1$ variables is of dimension $\binom{D+n+1}{n+1}$ while the space of polynomials of degree at most $dD$ in $n$ variables is of dimension $\binom{dD+n}{n}$, the injectivity of the map $P \mapsto P(g^1, \dots, g^{n+1})$ from the first space to the second would imply that $\binom{D+n+1}{n+1} \le \binom{dD+n}{n}$. However since $D = (n+1)d^n$ one has

$$\frac{\binom{D+n+1}{n+1}}{\binom{dD+n}{n}} = \frac{(n+1)d^n + (n+1)}{n+1} \cdot \frac{D+n}{dD+n} \cdots \frac{D+1}{dD+1} \ge (d^n+1) \cdot \left(\frac{1}{d}\right)^n > 1,$$

which gives a contradiction. So $g(B) \subset H$ for a certain hypersurface $H \subset \mathbb{C}^{n+1}$ of degree at most $D = (n+1)d^n$. Therefore the following classical result of algebraic geometry (see e.g. [Gri], pp. 11–15) can be used to provide control on the size of $H$ inside any ball in $\mathbb{C}^{n+1}$ :

LEMMA 4. *Let $H \subset \mathbb{C}^{n+1}$ be a complex algebraic hypersurface of degree $D$. Then, given any $r > 0$ and any $x \in \mathbb{C}^{n+1}$, the $2n$-dimensional volume of $H \cap B(x, r)$ is at most $DV_0\, r^{2n}$, where $V_0$ is the volume of the unit ball of dimension $2n$. Moreover, if $x \in H$, then one also has $vol_{2n}(H \cap B(x, r)) \ge V_0\, r^{2n}$.*

In particular, we are interested in the intersection of $H$ with the ball $\hat{B}$ of radius $\delta$ centered at the origin. Lemma 4 implies that the volume of this intersection is bounded by $(n+1)V_0\, d^n\delta^{2n}$. Cover $\hat{B}$ by a finite number of balls $B(x_i,\eta)$, in such a way that no point is contained in more than a fixed constant number (depending only on $n$) of the balls $B(x_i,2\eta)$. Then, for every $i$ such that $B(x_i,\eta)\cap H$ is non-empty, $B(x_i,2\eta)$ contains a ball of radius $\eta$ centered at a point of $H$, so by Lemma 4 the volume of $B(x_i,2\eta)\cap H$ is at least $V_0\,\eta^{2n}$. Summing the volumes of these intersections and comparing with the total volume of $H\cap\hat{B}$, one gets that the number of balls $B(x_i,\eta)$ which meet $H$ is bounded by $N=Cd^n\delta^{2n}\eta^{-2n}$, where $C$ is a constant depending only on $n$. Therefore, $H\cap\hat{B}$ is contained in the union of $N$ balls of radius $\eta$.

Since our goal is to find $w\in\hat{B}$ at distance more than $(c+1)\eta$ of $g(B)\subset H$, the set $Z$ of values which we want to avoid is contained in a set $Z^+$ which is the union of $N=Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The volume of $Z^+$ is bounded by $C'd^n\delta^{2n}\eta^2$ for some constant $C'$ depending only on $n$. Therefore, there exists a constant $C''$ such that, if one assumes $\delta$ to be larger than $C''d^{n/2}\eta$, the volume of $\hat{B}$ is strictly larger than that of $Z^+$, and so $\hat{B}-Z^+$ is not empty. Calling $w$ any element of $\hat{B}-Z^+$, one has $|w|\le\delta$, and $|g-w|\ge(c+1)\eta$ at every point of $B$, and therefore $|f-w|\ge\eta$ at every point of $B$, which is the desired result.

Since $d$ is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer $p$ such that, for all $0<\delta<\frac{1}{2}$, the relation $\eta=\delta\log(\delta^{-1})^{-p}$ implies that $\delta>C''d^{n/2}\eta$. This is the value of $p$ which we choose in the statement of the proposition, thus ensuring that $\hat{B}-Z^+$ is not empty and therefore that there exists $w$ with $|w|\le\delta$ such that $|f-w|\ge\eta$ at every point of $B$.

We now consider the case of a one-parameter family of functions $(f_t)_{t\in[0,1]}$. The first part of the above argument also applies to this case, so there exist polynomial maps $g_t$ of degree $d=O(\log(\eta^{-1}))$, depending continuously on $t$, such that $|f_t-g_t|\le c\,\eta$ for some constant $c$ and for all $t$. In particular, if one finds $w_t\in\mathbb{C}^{n+1}$ with $|w_t|\le\delta$ and depending continuously on $t$ such that $|g_t-w_t|\ge(c+1)\eta$ over the ball $B$, then it follows immediately that $|f_t-w_t|\ge\eta$ everywhere, which is the desired result.

As before, $g_t(B)$ is contained in a hypersurface of degree at most $(n+1)d^n$ in $\mathbb{C}^{n+1}$, and the same argument as above implies that the set $Z_t$ of values which we want to avoid for $w_t$ (i.e. all the points of $\hat{B}$ at distance less than $(c+1)\eta$ from $g_t(B)$) is contained in a set $Z_t^+$ which is the union of $N=Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The rest of the proof is now a higher-dimensional analogue of the argument used in [A1] : the crucial point is to show that, if $\delta$ is large enough, $\hat{B}-Z_t^+$ splits into several small connected components and only *one* large component, because the boundary $Y_t=\partial Z_t^+$ is much smaller than a $(2n+1)$-ball of radius $\delta$ and therefore cannot split $\hat{B}$ into components of comparable sizes.

Each component of $\hat{B}-Z_t^+$ is delimited by a subset of the sphere $\partial\hat{B}$ and by a union of components of $Y_t$. Each component $Y_{t,i}$ of $Y_t$ is a real hypersurface in $\hat{B}$ (with corners at the points where the boundaries of the various balls of $Z_t^+$ intersect) whose boundary is contained in $\partial\hat{B}$, and therefore splits $\hat{B}$ into two components $C_i'$ and $C_i''$. So each component of $\hat{B}-Z_t^+$ is an intersection of components $C_i'$ or

$C''_i$ where $i$ ranges over a certain subset of the set of components of $Y_t$. Let us now state the following isoperimetric inequality :

LEMMA 5. *Let $Y$ be a connected (singular) submanifold of real codimension 1 in the unit ball of dimension $2n+2$, with (possibly empty) boundary contained in the boundary of the ball. Let $A$ be the $(2n+1)$-dimensional area of $Y$. Then the volume $V$ of the smallest of the two components delimited by $Y$ in the ball satisfies the bound $V \leq K\, A^{(2n+2)/(2n+1)}$, where $K$ is a fixed constant depending only on the dimension.*

PROOF. The stereographic projection maps the unit ball quasi-isometrically onto a half-sphere. Therefore, up to a change in the constant, it is sufficient to prove the result on the half-sphere. By doubling $Y$ along its intersection with the boundary of the half-sphere, which doubles both the volume delimited by $Y$ and its area, one reduces to the case of a closed connected (singular) real hypersurface in the sphere $S^{2n+2}$ (if $Y$ does not meet the boundary, then it is not necessary to consider the double). Next, one notices that the singular hypersurfaces we consider can be smoothed in such a way that the area of $Y$ and the volume it delimits are changed by less than any fixed constant ; therefore, Lemma 5 follows from the classical spherical isoperimetric inequality (see e.g. [Sch]). □

It follows that, letting $A_i$ be the $(2n+1)$-dimensional area of $Y_{t,i}$, the smallest of the two components delimited by $Y_{t,i}$, e.g. $C'_i$, has volume $V_i \leq K\, A_i^{(2n+2)/(2n+1)}$. Therefore, the volume of the set $\bigcup_i C'_i$ is bounded by $K \sum_i A_i^{(2n+2)/(2n+1)}$, which is less than $K\,(\sum_i A_i)^{(2n+2)/(2n+1)}$. However, $\sum_i A_i$ is the total area of the boundary $Y_t$ of $Z_t^+$, so it is less than the total area of the boundaries of the balls composing $Z_t^+$, which is at most a fixed constant times $Cd^n\delta^{2n}\eta^{-2n}((c+2)\eta)^{2n+1}$, i.e. at most a fixed constant times $d^n\delta^{2n}\eta$. Therefore, one has

$$\mathrm{vol}(\bigcup_i C'_i) \leq K' \left(d^n \frac{\eta}{\delta}\right)^{\frac{2n+2}{2n+1}} \delta^{2n+2}$$

for some constant $K'$ depending only on $n$. So there exists a constant $K''$ depending only on $n$ such that, if $\delta > K'' d^n \eta$, then $\mathrm{vol}(\bigcup_i C'_i) \leq \frac{1}{10}\mathrm{vol}(\hat{B})$, and therefore $\mathrm{vol}(\bigcap_i C''_i) \geq \frac{8}{10}\mathrm{vol}(\hat{B})$.

Since $d$ is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer $p$ such that, for all $0 < \delta < \frac{1}{2}$, the relation $\eta = \delta \log(\delta^{-1})^{-p}$ implies that $\delta > K'' d^n \eta$. This is the value of $p$ which we choose in the statement of the proposition, thus ensuring that the above volume bounds on $\bigcup_i C'_i$ and $\bigcap_i C''_i$ hold.

Now, recall that every component of $\hat{B} - Z_t^+$ is an intersection of sets $C'_i$ and $C''_i$ for certain values of $i$. Therefore, every component of $\hat{B} - Z_t^+$ either is contained in $\bigcup_i C'_i$ or contains $\bigcap_i C''_i$. However, because $\bigcup_i C'_i$ is much smaller than the ball $\hat{B}$, one cannot have $\hat{B} - Z_t^+ \subset \bigcup_i C'_i$. Therefore, there exists a component in $\hat{B} - Z_t^+$ containing $\bigcup_i C''_i$. Since its volume is at least $\frac{8}{10}\mathrm{vol}(\hat{B})$, this large component is necessarily unique.

Let $U(t)$ be the connected component of $\hat{B}-Z_t$ which contains the large component of $\hat{B}-Z_t^+$ : it is the only large component of $\hat{B}-Z_t$. We now follow the same argument as in [A1]. Since $g_t(B)$ depends continuously on $t$, so does its $(c+1)\eta$-neighborhood $Z_t$, and the set $\bigcup_t \{t\}\times Z_t$ is therefore a closed subset of $[0,1]\times\hat{B}$. Let $U^-(t,\epsilon)$ be the set of all points of $U(t)$ at distance more than $\epsilon$ from $Z_t\cup\partial\hat{B}$. Then, given any $t$ and any small $\epsilon>0$, for all $\tau$ close to $t$, $U(\tau)$ contains $U^-(t,\epsilon)$. To see this, we first notice that, for all $\tau$ close to $t$, $U^-(t,\epsilon)\cap Z_\tau=\emptyset$. Indeed, if such were not the case, one could take a sequence of points of $Z_\tau\cap U^-(t,\epsilon)$ for $\tau\to t$, and extract a convergent subsequence whose limit belongs to $\overline{U}^-(t,\epsilon)$ and therefore lies outside of $Z_t$, in contradiction with the fact that $\bigcup_t\{t\}\times Z_t$ is closed. So $U^-(t,\epsilon)\subset\hat{B}-Z_\tau$ for all $\tau$ close enough to $t$. Making $\epsilon$ smaller if necessary, one may assume that $U^-(t,\epsilon)$ is connected, so that for all $\tau$ close to $t$, $U^-(t,\epsilon)$ is necessarily contained in the large component of $\hat{B}-Z_\tau$, namely $U(\tau)$.

It follows that $U=\bigcup_t\{t\}\times U(t)$ is an open connected subset of $[0,1]\times\hat{B}$, and is therefore path-connected. So we get a path $s\mapsto(t(s),w(s))$ joining $(0,w(0))$ to $(1,w(1))$ inside $U$, for any given $w(0)$ and $w(1)$ in $U(0)$ and $U(1)$. We then only have to make sure that $s\mapsto t(s)$ is strictly increasing in order to define $w_{t(s)}=w(s)$.

Getting the $t$ component to increase strictly is not hard. Indeed, one first gets it to be weakly increasing, by considering values $s_1<s_2$ of the parameter such that $t(s_1)=t(s_2)=t$ and replacing the portion of the path between $s_1$ and $s_2$ by a path joining $w(s_1)$ to $w(s_2)$ in the connected set $U(t)$. Then, we slightly shift the path, using the fact that $U$ is open, to get the $t$ component to increase slightly over the parts where it was constant. Thus we can define $w_{t(s)}=w(s)$ and end the proof of Proposition 2.

## 3. Transversality of derivatives

**3.1. Transversality to 0 of** $\mathrm{Jac}(f_k)$**.** At this point in the proofs of Theorems 1 and 2, we have constructed for all large $k$ asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3\otimes L^k$ (or families of sections), bounded away from 0, and such that the holomorphic derivative of the map $f_k=\mathbb{P}s_k$ is bounded away from 0. The next property we wish to ensure by perturbing the sections $s_k$ is the transversality to 0 of the $(2,0)$-Jacobian $\mathrm{Jac}(f_k)=\det(\partial f_k)$. The main result of this section is :

PROPOSITION 5. *Let $\delta$ and $\gamma$ be two constants such that $0<\delta<\frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3\otimes L^k$ such that $|s_k|\geq\gamma$ and $|\partial(\mathbb{P}s_k)|_{g_k}\geq\gamma$ at every point of $X$. Then there exists a constant $\eta>0$ such that, for all large enough values of $k$, there exist asymptotically holomorphic sections $\sigma_k$ of $\mathbb{C}^3\otimes L^k$ such that $|\sigma_k-s_k|_{C^3,g_k}\leq\delta$ and $\mathrm{Jac}(\mathbb{P}\sigma_k)$ is $\eta$-transverse to $0$. Moreover, the same statement holds for families of sections indexed by a parameter $t\in[0,1]$.*

The proof of Proposition 5 uses once more the same techniques and globalization argument as Propositions 1 and 4. The local transversality result one uses in conjunction with Proposition 3 is now the following statement for complex valued functions :

PROPOSITION 6 ([D1],[A1]). *Let $f$ be a function defined over the ball $B^+$ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}$. Let $\delta$ be a constant such that $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-p}$ where $p$ is a suitable fixed integer depending only on the dimension $n$. Assume that $f$ satisfies the following bounds over $B^+$ :*

$$|f| \leq 1, \qquad |\bar{\partial} f| \leq \eta, \qquad |\nabla \bar{\partial} f| \leq \eta.$$

*Then there exists $w \in \mathbb{C}$, with $|w| \leq \delta$, such that $f - w$ is $\eta$-transverse to $0$ over the interior ball $B$ of radius $1$, i.e. $f - w$ has derivative larger than $\eta$ at any point of $B$ where $|f - w| < \eta$.*

*Moreover, the same statement remains true for a one-parameter family of functions $(f_t)_{t\in[0,1]}$ satisfying the same bounds, i.e. for all $t$ one can find elements $w_t \in \mathbb{C}$ depending continuously on $t$ such that $|w_t| \leq \delta$ and $f_t - w_t$ is $\eta$-transverse to $0$ over $B$.*

The first part of this statement is exactly Theorem 20 of [D1], and the version for one-parameter families is Proposition 3 of [A1]. □

Proposition 5 is proved by applying Proposition 3 to the following property : say that a section $s$ of $\mathbb{C}^3 \otimes L^k$ everywhere larger than $\frac{\gamma}{2}$ and such that $|\partial \mathbb{P}s| \geq \frac{\gamma}{2}$ everywhere satisfies $\mathcal{P}(\eta, x)$ if $\mathrm{Jac}(\mathbb{P}s)$ is $\eta$-transverse to $0$ at $x$, i.e. either $|\mathrm{Jac}(\mathbb{P}s)(x)| \geq \eta$ or $|\nabla \mathrm{Jac}(\mathbb{P}s)(x)| > \eta$. This property is local and $C^2$-open, and therefore also $C^3$-open, because the lower bound on $s$ makes $\mathrm{Jac}(\mathbb{P}s)$ depend nicely on $s$. Note that, since one considers only sections differing from $s_k$ by less than $\delta$ in $C^3$ norm, decreasing $\delta$ if necessary, one can safely assume that the two hypotheses $|s| \geq \frac{\gamma}{2}$ and $|\partial(\mathbb{P}s)| \geq \frac{\gamma}{2}$ are satisfied everywhere by all the sections appearing in the construction of $\sigma_k$. So one only needs to check that the assumptions of Proposition 3 hold for the property $\mathcal{P}$ defined above.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$, such that $|s_k| \geq \frac{\gamma}{2}$ and $|\partial f_k| \geq \frac{\gamma}{2}$ everywhere. The setup is similar to that of Section 2.2. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over $X$), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x) = s_k^2(x) = 0$ and therefore $|s_k^0(x)| \geq \frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r > 0$ (independent of $k$) such that $|s_k^0| \geq \frac{\gamma}{3}$, $|s_k^1| < \frac{\gamma}{3}$ and $|s_k^2| < \frac{\gamma}{3}$ over the ball $B_{g_k}(x, r)$. Therefore, over this ball one can define the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Note that $f_k$ is the composition of $h_k$ with the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y \in B_{g_k}(x, r)$, the bound $|\partial f_k(y)| \geq \frac{\gamma}{2}$ implies that $|\partial h_k(y)| \geq \gamma'$ for some constant $\gamma' > 0$. Moreover, the $(2, 0)$-Jacobians $\mathrm{Jac}(f_k) = \det(\partial f_k)$ and $\mathrm{Jac}(h_k) = \det(\partial h_k)$ are related to each other : $\mathrm{Jac}(f_k)(y) = \phi(y)\,\mathrm{Jac}(h_k)(y)$, where $\phi(y)$ is the Jacobian of $\iota$ at $h_k(y)$. In particular, $|\phi|$ is bounded between two universal constants over $B_{g_k}(x, r)$, and $\nabla\phi$ is also bounded.

Since $\nabla \mathrm{Jac}(h_k) = \phi^{-1}\nabla \mathrm{Jac}(f_k) - \phi^{-2}\mathrm{Jac}(f_k)\nabla\phi$, it follows from the bounds on $\phi$ that, if $\mathrm{Jac}(f_k)$ fails to be $\alpha$-transverse to 0 at $y$ for some $\alpha$, i.e. if $|\mathrm{Jac}(f_k)(y)| < \alpha$ and $|\nabla \mathrm{Jac}(f_k)(y)| \le \alpha$, then $|\mathrm{Jac}(h_k)(y)| < C\alpha$ and $|\nabla \mathrm{Jac}(h_k)(y)| \le C\alpha$ for some constant $C$ independent of $k$ and $\alpha$. This means that, if $\mathrm{Jac}(h_k)$ is $C\alpha$-transverse to 0 at $y$, then $\mathrm{Jac}(f_k)$ is $\alpha$-transverse to 0 at $y$. Therefore, what one actually needs to prove is that, for large enough $k$, a perturbation of $s_k$ with Gaussian decay and smaller than $\delta$ allows one to obtain the $\eta$-transversality to 0 of $\mathrm{Jac}(h_k)$ over a ball $B_{g_k}(x, c)$, with $\eta = c'\delta\,(\log\delta^{-1})^{-p}$, for some constants $c$, $c'$ and $p$ ; the $\frac{\eta}{C}$-transversality to 0 of $\mathrm{Jac}(f_k)$ then follows by the above remark.

Since $|\partial h_k(x)| \ge \gamma'$, one can assume, after composing with a rotation in $\mathbb{C}^2$ (constant over $X$) acting on the two components $(s_k^1, s_k^2)$ or equivalently on $(h_k^1, h_k^2)$, that $|\partial h_k^2(x)| \ge \frac{\gamma'}{2}$. As in Section 2.2, consider the asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of $L^k$ with Gaussian decay away from $x$ given by Lemma 2, and the complex coordinate functions $z_k^1$ and $z_k^2$ of a local approximately holomorphic Darboux coordinate chart on a neighborhood of $x$. Recall that the two asymptotically holomorphic 1-forms

$$\mu_k^1 = \partial\Big(\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \quad \text{and} \quad \mu_k^2 = \partial\Big(\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big)$$

are, at $x$, both of norm larger than a fixed constant and mutually orthogonal, and that $\mu_k^1$, $\mu_k^2$ and their derivatives are uniformly bounded independently of $k$.

Because $\mu_k^1(x)$ and $\mu_k^2(x)$ define an orthogonal frame in $\Lambda^{1,0}T_x^*X$, there exist complex numbers $a_k$ and $b_k$ such that $\partial h_k^2(x) = a_k\mu_k^1(x) + b_k\mu_k^2(x)$. Let $\lambda_{k,x} = (\bar b_k z_k^1 - \bar a_k z_k^2)s_{k,x}^{\mathrm{ref}}$. The properties of $\lambda_{k,x}$ of importance to us are the following : the sections $\lambda_{k,x}$ are asymptotically holomorphic because the coordinates $z_k^i$ are asymptotically holomorphic ; they are uniformly bounded in $C^3$ norm by a constant $C_0$, because of the bounds on $s_{k,x}^{\mathrm{ref}}$, on the coordinate chart and on $\partial h_k^2(x)$ ; they have uniform Gaussian decay away from $x$ ; and, letting

$$\Theta_{k,x} = \partial\Big(\frac{\lambda_{k,x}}{s_k^0}\Big) \wedge \partial h_k^2,$$

one has $|\Theta_{k,x}(x)| = |(\bar b_k\mu_k^1(x) - \bar a_k\mu_k^2(x)) \wedge (a_k\mu_k^1(x) + b_k\mu_k^2(x))| \ge \gamma''$ for some constant $\gamma'' > 0$, because of the lower bounds on $|\mu_k^i(x)|$ and $|\partial h_k^2(x)|$.

Because $\nabla\Theta_{k,x}$ is uniformly bounded and $|\Theta_{k,x}(x)| \ge \gamma''$, there exists a constant $r' > 0$ independent of $k$ such that $|\Theta_{k,x}|$ remains larger than $\frac{\gamma''}{2}$ over the ball $B_{g_k}(x, r')$. Define on $B_{g_k}(x, r')$ the function $u_k = \Theta_{k,x}^{-1}\mathrm{Jac}(h_k)$ with values in $\mathbb{C}$ : because $\Theta_{k,x}$ is bounded from above and below and has bounded derivative, the transversality to 0 of $u_k$ is equivalent to that of $\mathrm{Jac}(h_k)$. Moreover, for any $w_k \in \mathbb{C}$, adding $w_k\lambda_{k,x}$ to $s_k^1$ is equivalent to adding $w_k\Theta_{k,x}$ to $\mathrm{Jac}(h_k) = \partial h_k^1 \wedge \partial h_k^2$, i.e. adding $w_k$ to $u_k$. Therefore, to prove Proposition 5 we only need to find $w_k \in \mathbb{C}$ with $|w_k| \le \frac{\delta}{C_0}$ such that the functions $u_k - w_k$ are transverse to 0.

Using the local approximately holomorphic coordinate chart, one can obtain from $u_k$, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions $v_k$ defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}$, and satisfying the estimates $|v_k| = O(1)$,

$|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$. One can then apply Proposition 6, provided that $k$ is large enough, to obtain constants $w_k \in \mathbb{C}$, with $|w_k| \le \frac{\delta}{C_0}$, such that $v_k - w_k$ is $\alpha$-transverse to 0 over the unit ball in $\mathbb{C}^2$, where $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Therefore, $u_k - w_k$ is $\frac{\alpha}{C'}$-transverse to 0 over $B_{g_k}(x,c)$ for some constants $c$ and $C'$. Multiplying by $\Theta_{k,x}$, one finally gets that, over $B_{g_k}(x,c)$, $\mathrm{Jac}(h_k) - w_k\Theta_{k,x}$ is $\eta$-transverse to 0, where $\eta = \frac{\alpha}{C''}$ for some constant $C''$.

In other terms, let $(\tau^0_{k,x}, \tau^1_{k,x}, \tau^2_{k,x}) = (0, -w_k\lambda_{k,x}, 0)$, and define $\tilde h_k$ similarly to $h_k$ starting with $s_k + \tau_{k,x}$ instead of $s_k$ : then the above discussion shows that $\mathrm{Jac}(\tilde h_k)$ is $\eta$-transverse to 0 over $B_{g_k}(x,c)$. Moreover, $|\tau_{k,x}|_{C^3} = |w_k|\,|\lambda_{k,x}|_{C^3} \le \delta$, and the sections $\tau_{k,x}$ have uniform Gaussian decay away from $x$. As remarked above, the $\eta$-transversality to 0 of $\mathrm{Jac}(\tilde h_k)$ implies that $\mathrm{Jac}(\mathbb{P}(s_k + \tau_{k,x}))$ is $\eta'$-transverse to 0 for some $\eta'$ differing from $\eta$ by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since $\eta' \ge c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$.

Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well. The only nontrivial point to check, in order to apply the above construction for each $t \in [0,1]$ in such a way that everything depends continuously on $t$, is the existence of a continuous family of rotations of $\mathbb{C}^2$ acting on $(h^1_k, h^2_k)$ allowing one to assume that $|\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$ for all $t$. For this, observe that, for every $t$, such rotations in $\mathrm{SU}(2)$ are in one-to-one correspondence with pairs $(\alpha,\beta) \in \mathbb{C}^2$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha\,\partial h^1_{t,k}(x) + \beta\,\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$. The set $\Gamma_t$ of such pairs $(\alpha,\beta)$ is non-empty because $|\partial h_{t,k}(x)| \ge \gamma'$ ; let us now prove that it is connected.

First, notice that $\Gamma_t$ is invariant under the diagonal $\mathrm{S}^1$ action on $\mathbb{C}^2$. Therefore, it is sufficient to prove that the set of $(\alpha:\beta) \in \mathbb{CP}^1$ such that

$$\phi(\alpha:\beta) := \frac{|\alpha\,\partial h^1_{t,k}(x) + \beta\,\partial h^2_{t,k}(x)|^2}{|\alpha|^2 + |\beta|^2} > \frac{(\gamma')^2}{4}$$

is connected. For this, consider a critical point of $\phi$ over $\mathbb{CP}^1$. Composing with a rotation in $\mathbb{CP}^1$, one may assume that this critical point is $(1:0)$. Then it follows from the property $\frac{\partial}{\partial\beta}\phi(1:\beta)_{|\beta=0} = 0$ that $\partial h^1_{t,k}(x)$ and $\partial h^2_{t,k}(x)$ must necessarily be orthogonal to each other. Therefore, one has

$$\phi(1:\beta) = \frac{|\partial h^1_{t,k}(x)|^2 + |\beta|^2|\partial h^2_{t,k}(x)|^2}{1 + |\beta|^2},$$

and it follows that either $\phi$ is constant over $\mathbb{CP}^1$ (if $|\partial h^1_{t,k}(x)| = |\partial h^2_{t,k}(x)|$), or the critical point is nondegenerate of index 0 (if $|\partial h^1_{t,k}(x)| < |\partial h^2_{t,k}(x)|$), or it is nondegenerate of index 2 (if $|\partial h^1_{t,k}(x)| > |\partial h^2_{t,k}(x)|$). As a consequence, since $\phi$ has no critical point of index 1, all nonempty sets of the form $\{(\alpha:\beta) \in \mathbb{CP}^1,\ \phi(\alpha,\beta) > \text{constant}\}$ are connected.

Lifting back from $\mathbb{CP}^1$ to the unit sphere in $\mathbb{C}^2$, it follows that $\Gamma_t$ is connected. Therefore, for each $t$ the open set $\Gamma_t \subset \mathrm{SU}(2)$ of admissible rotations of $\mathbb{C}^2$ is connected. Since $h_{t,k}$ depends continuously on $t$, the sets $\Gamma_t$ also depend continuously on $t$ (with respect to nearly every conceivable topology), and therefore $\bigcup_t \{t\} \times \Gamma_t$ is connected. The same argument as in the end of §2.3 then implies the existence of a continuous section of $\bigcup_t \{t\} \times \Gamma_t$ over $[0,1]$, i.e. the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ which allows one to ensure that $|\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$ for all $t$. Therefore, the argument described in this section also applies to the case of one-parameter families, and the assumptions of Proposition 3 are satisfied by the property $\mathcal{P}$ even in the case of one-parameter families of sections. Proposition 5 follows immediately.

**3.2. Nondegeneracy of cusps.** At this point in the proof, we have obtained sections satisfying the transversality property $\mathcal{P}_3(\gamma)$. The only missing property in order to obtain $\eta$-genericity for some $\eta > 0$ is the transversality to 0 of the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$. The main result of this section is therefore the following :

PROPOSITION 7. *Let $\delta$ and $\gamma$ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\gamma)$ for all $k$. Then there exists a constant $\eta > 0$ such that, for all large enough values of $k$, there exist asymptotically holomorphic sections $\sigma_k$ of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \leq \delta$ and that the restrictions to $R(\sigma_k)$ of the sections $\mathcal{T}(\sigma_k)$ are $\eta$-transverse to 0 over $R(\sigma_k)$. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Note that, decreasing $\delta$ if necessary in the statement of Proposition 7, it is safe to assume that all sections lying within $\delta$ of $s_k$ in $C^3$ norm, and in particular the sections $\sigma_k$, satisfy $\mathcal{P}_3(\frac{\gamma}{2})$.

There are several ways of obtaining transversality to 0 of certain sections restricted to asymptotically holomorphic symplectic submanifolds : for example, one such technique is described in the main argument of [A1]. However in our case, the perturbations we will add to $s_k$ in order to get the transversality to 0 of $\mathcal{T}(s_k)$ have the side effect of moving the submanifolds $R(s_k)$ along which the transversality conditions have to hold, which makes things slightly more complicated. Therefore, we choose to use the equivalence between two different transversality properties :

LEMMA 6. *Let $\sigma_k$ and $\sigma'_k$ be asymptotically holomorphic sections of vector bundles $E_k$ and $E'_k$ respectively over $X$. Assume that $\sigma'_k$ is $\gamma$-transverse to 0 over $X$ for some $\gamma > 0$, and let $\Sigma'_k$ be its (smooth) zero set. Fix a constant $r > 0$ and a point $x \in X$. Then :*

(1) *There exists a constant $c > 0$, depending only on $r$, $\gamma$ and the bounds on the sections, such that, if the restriction of $\sigma_k$ to $\Sigma'_k \cap B_{g_k}(x,r)$ is $\eta$-transverse to 0 for some $\eta < \gamma$, then $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at $x$ as a section of $E_k \oplus E'_k$.*

(2) *If $\sigma_k \oplus \sigma'_k$ is $\eta$-transverse to 0 at $x$ and $x$ belongs to $\Sigma'_k$, then the restriction of $\sigma_k$ to $\Sigma'_k$ is $\eta$-transverse to 0 at $x$.*

PROOF. We start with (1), whose proof follows the ideas of §3.6 of [A1] with improved estimates. Let $C_1$ be a constant bounding $|\nabla \sigma_k|$ everywhere, and let $C_2$ be a constant bounding $|\nabla\nabla\sigma_k|$ and $|\nabla\nabla\sigma'_k|$ everywhere. Fix two constants $0 < c < c' < \frac{1}{2}$, such that the following inequalities hold : $c < r$, $c < \frac{1}{2}\gamma\, C_1^{-1}$, $c' < (2+\gamma^{-1}C_1)^{-1}$, and $(2C_2\gamma^{-1}+1)c < c'$. Clearly, these constants depend only on $r$, $\gamma$, $C_1$ and $C_2$.

Assume that $|\sigma_k(x)|$ and $|\sigma'_k(x)|$ are both smaller than $c\,\eta$. Because of the $\gamma$-transversality to 0 of $\sigma'_k$ and because $|\sigma'_k(x)| < c\,\eta < \gamma$, the covariant derivative of $\sigma'_k$ is surjective at $x$, and admits a right inverse $(E'_k)_x \to T_xX$ of norm less than $\gamma^{-1}$. Since the connection is unitary, applying this right inverse to $\sigma'_k$ itself one can follow the downward gradient flow of $|\sigma'_k|$, and since one remains in the region where $|\sigma'_k| < \gamma$ this gradient flow converges to a point $y$ where $\sigma'_k$ vanishes, at a distance $d$ from the starting point $x$ no larger than $\gamma^{-1}c\,\eta$. In particular, $d < c < r$, so $y \in B_{g_k}(x,r) \cap \Sigma'_k$, and the restriction of $\sigma_k$ to $\Sigma'_k$ is $\eta$-transverse to 0 at $y$.

Since $c < \frac{1}{2}\gamma\,C_1^{-1}$, the norm of $\sigma_k(y)$ differs from that of $\sigma_k(x)$ by at most $C_1 d < \frac{\eta}{2}$, and so $|\sigma_k(y)| < \eta$. Since $y \in B_{g_k}(x,r) \cap \Sigma'_k$, we therefore know that $\nabla\sigma'_k$ is surjective at $y$ and vanishes in all directions tangential to $\Sigma'_k$, while $\nabla\sigma_k$ restricted to $T_y\Sigma'_k$ is surjective and larger than $\eta$. It follows that $\nabla(\sigma_k \oplus \sigma'_k)$ is surjective at $y$. Let $\rho : (E_k)_y \to T_y\Sigma'_k$ and $\rho' : (E'_k)_y \to T_yX$ be the right inverses of $\nabla_y\sigma_{k|\Sigma'_k}$ and $\nabla_y\sigma'_k$ given by the transversality properties of $\sigma_{k|\Sigma'_k}$ and $\sigma'_k$. We now construct a right inverse $\hat\rho : (E_k \oplus E'_k)_y \to T_yX$ of $\nabla_y(\sigma_k \oplus \sigma'_k)$ with bounded norm.

Considering any element $u \in (E_k)_y$, the vector $\hat u = \rho(u) \in T_y\Sigma'_k$ has norm at most $\eta^{-1}|u|$ and satisfies $\nabla\sigma_k(\hat u) = u$. Clearly $\nabla\sigma'_k(\hat u) = 0$ because $\hat u$ is tangent to $\Sigma'_k$, so we define $\hat\rho(u) = \hat u$. Now consider an element $v$ of $(E'_k)_y$, and let $\hat v = \rho'(v)$ : we have $|\hat v| \le \gamma^{-1}|v|$ and $\nabla\sigma'_k(\hat v) = v$. Let $\hat w = \rho(\nabla\sigma_k(\hat v))$ : then $\nabla\sigma_k(\hat w) = \nabla\sigma_k(\hat v)$ and $\nabla\sigma'_k(\hat w) = 0$, while $|\hat w| \le \eta^{-1}C_1|\hat v| \le \eta^{-1}\gamma^{-1}C_1|v|$. Therefore $\nabla(\sigma_k \oplus \sigma'_k)(\hat v - \hat w) = v$, and we define $\hat\rho(v) = \hat v - \hat w$.

Therefore $\nabla(\sigma_k \oplus \sigma'_k)$ admits at $y$ a right inverse $\hat\rho$ of norm bounded by $\eta^{-1} + \gamma^{-1} + \eta^{-1}\gamma^{-1}C_1 \le (2 + \gamma^{-1}C_1)\eta^{-1} < (c'\eta)^{-1}$. Finally, note that $\nabla_x(\sigma_k \oplus \sigma'_k)$ differs from $\nabla_y(\sigma_k \oplus \sigma'_k)$ by at most $2C_2 d < 2C_2\gamma^{-1}c\,\eta < (c'-c)\eta$. Therefore, $\nabla_x(\sigma_k \oplus \sigma'_k)$ is also surjective, and is larger than $(c'\eta) - ((c'-c)\eta) = c\,\eta$. In other terms, we have shown that $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at $x$, which is what we sought to prove.

The proof of (2) is much easier : we know that $x \in \Sigma'_k$, i.e. $\sigma'_k(x) = 0$, and let us assume that $|\sigma_k(x)| < \eta$. Then $|\sigma_k(x) \oplus \sigma'_k(x)| = |\sigma_k(x)| < \eta$, and the $\eta$-transversality to 0 of $\sigma_k \oplus \sigma'_k$ at $x$ implies that $\nabla_x(\sigma_k \oplus \sigma'_k)$ has a right inverse $\hat\rho$ of norm less than $\eta^{-1}$. Choose any $u \in (E_k)_x$, and let $\rho(u) = \hat\rho(u \oplus 0)$. One has $\nabla\sigma'_k(\rho(u)) = 0$, therefore $\rho(u)$ lies in $T_x\Sigma'_k$, and $\nabla\sigma_k(\rho(u)) = u$ by construction. So $(\nabla\sigma_k)_{|T_x\Sigma'_k}$ is surjective and admits $\rho$ as a right inverse. Moreover, $|\rho(u)| = |\hat\rho(u \oplus 0)| \le \eta^{-1}|u|$, so the norm of $\rho$ is less than $\eta^{-1}$, which shows that $\sigma_{k|\Sigma'_k}$ is $\eta$-transverse to 0 at $x$. ☐

It follows from assertion (2) of Lemma 6 that, in order to obtain the transversality to 0 of $\mathcal{T}(\sigma_k)_{|R(\sigma_k)}$, it is sufficient to make $\mathcal{T}(\sigma_k) \oplus \mathrm{Jac}(\mathbb{P}\sigma_k)$ transverse to 0 over a neighborhood of $R(\sigma_k)$. Therefore, we can use once more the globalization principle of Proposition 3 to prove Proposition 7. Indeed, consider a section $s$ of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$, a point $x \in X$ and a constant $\eta > 0$, and say that $s$ satisfies the property $\mathcal{P}(\eta, x)$ if either $x$ is at distance more than $\eta$ of $R(s)$, or $x$ lies close to $R(s)$ and $\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s)$ is $\eta$-transverse to 0 at $x$ (i.e. one of the two quantities $|(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ and $|\nabla(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ is larger than $\eta$). Since $\mathrm{Jac}(\mathbb{P}s) \oplus \mathcal{T}(s)$ is, under the assumption $\mathcal{P}_3(\frac{\gamma}{2})$, a smooth function of $s$ and its first two derivatives, and since $R(s)$ depends nicely on $s$, it is easy to show that the property $\mathcal{P}$ is local and $C^3$-open. So one only needs to check that $\mathcal{P}$ satisfies the assumptions of Proposition 3. Our next remark is :

LEMMA 7. *There exists a constant $r'_0 > 0$ (independent of $k$) with the following property : choose $x \in X$ and $r' < r'_0$, and let $s_k$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$. Assume that $\overline{B}_{g_k}(x, r')$ intersects $R(s_k)$. Then there exists an approximately holomorphic map $\theta_{k,x}$ from the disc $D^+$ of radius $\frac{11}{10}$ in $\mathbb{C}$ to $R(s_k)$ such that :* (i) *the image by $\theta_{k,x}$ of the unit disc $D$ contains $B_{g_k}(x, r') \cap R(s_k)$ ;* (ii) *$|\nabla\theta_{k,x}|_{C^1,g_k} = O(1)$ and $|\bar\partial\theta_{k,x}|_{C^1,g_k} = O(k^{-1/2})$ ;* (iii) *$\theta_{k,x}(D^+)$ is contained in a ball of radius $O(r')$ centered at $x$.*

*Moreover the same statement holds for one-parameter families of sections : given sections $(s_{t,k})_{t\in[0,1]}$ depending continuously on $t$, satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and such that $B_{g_k}(x, r')$ intersects $R(s_{t,k})$ for all $t$, there exist approximately $J_t$-holomorphic maps $\theta_{t,k,x}$ depending continuously on $t$ and with the same properties as above.*

PROOF. We work directly with the case of one-parameter families (the result for isolated sections follows trivially) and let $j_{t,k} = \mathrm{Jac}(\mathbb{P}s_{t,k})$. First note that $R(s_{t,k})$ is the zero set of $j_{t,k}$, which is $\frac{\gamma}{2}$-transverse to 0 and has uniformly bounded second derivative. So, given any point $y \in R(s_{t,k})$, $|\nabla j_{t,k}(y)| > \frac{\gamma}{2}$, and therefore there exists $c > 0$, depending only on $\gamma$ and the bound on $\nabla\nabla j_{t,k}$, such that $\nabla j_{t,k}$ varies by a factor of at most $\frac{1}{10}$ in the ball of radius $c$ centered at $y$. It follows that $\overline{B}_{g_k}(y, c) \cap R(s_{t,k})$ is diffeomorphic to a ball (in other words, $R(s_{t,k})$ is "trivial at small scale").

Assume first that $3r' < c$. For all $t$, choose a point $y_{t,k}$ (not necessarily depending continuously on $t$) in $\overline{B}_{g_k}(x, r') \cap R(s_{t,k}) \neq \emptyset$. The intersection $B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is diffeomorphic to a ball and therefore connected, and contains $\overline{B}_{g_k}(x, r') \cap R(s_{t,k})$ which is nonempty and depends continuously on $t$. Therefore, the set $\bigcup_t \{t\} \times B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is connected, which implies the existence of points $x_{t,k} \in B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k}) \subset B_{g_k}(x, 4r') \cap R(s_{t,k})$ which depend continuously on $t$.

Consider local approximately $J_t$-holomorphic coordinate charts over a neighborhood of $x_{t,k}$, depending continuously on $t$, as given by Lemma 3, and call $\psi_{t,k} : (\mathbb{C}^2, 0) \to (X, x_{t,k})$ the inverse of the coordinate map. Because of asymptotic holomorphicity, the tangent space to $R(s_{t,k})$ at $x_{t,k}$ lies within $O(k^{-1/2})$ of the complex subspace $\tilde T_{x_{t,k}} R(s_{t,k}) = \mathrm{Ker}\ \partial j_{t,k}(x_{t,k})$ of $T_{x_{t,k}}X$. Composing $\psi_{t,k}$ with a rotation in $\mathbb{C}^2$, one can get maps $\psi'_{t,k}$ satisfying the same bounds as $\psi_{t,k}$ and such that the differential of $\psi'_{t,k}$ at 0 maps $\mathbb{C} \times \{0\}$ to $\tilde T_{x_{t,k}} R(s_{t,k})$.

The estimates of Lemma 3 imply that there exists a constant $\lambda = O(r')$ such that $\psi'_{t,k}(B_{\mathbb{C}^2}(0, \lambda)) \supset B_{g_k}(x, r')$. Define $\tilde\psi_{t,k}(z) = \psi'_{t,k}(\lambda z)$ : if $r'$ is sufficiently small, this map is well-defined over the ball $B_{\mathbb{C}^2}(0, 2)$. Over $B_{\mathbb{C}^2}(0, 2)$ the estimates of Lemma 3 imply that $|\bar\partial\tilde\psi_{t,k}|_{C^1,g_k} = O(\lambda k^{-1/2})$ and $|\nabla\tilde\psi_{t,k}|_{C^1,g_k} = O(\lambda)$. Moreover, because $\lambda = O(r')$ the image by $\tilde\psi_{t,k}$ of $B_{\mathbb{C}^2}(0, 2)$ is contained in a ball of radius $O(r')$ around $x$.

Assuming $r'$ to be sufficiently small, one can also require that the image of $B_{\mathbb{C}^2}(0, 2)$ by $\tilde\psi_{t,k}$ has diameter less than $c$. The submanifolds $R(s_{t,k})$ are then trivial over the considered balls, so it follows from the implicit function theorem that

$R(s_{t,k}) \cap \tilde{\psi}_{t,k}(D^+ \times D^+)$ can be parametrized in the chosen coordinates as the set of points of the form $\tilde{\psi}_{t,k}(z, \tau_{t,k}(z))$ for $z \in D^+$, where $\tau_{t,k} : D^+ \to D^+$ satisfies $\tau_{t,k}(0) = 0$ and $\nabla\tau_{t,k}(0) = O(k^{-1/2})$.

The derivatives of $\tau_{t,k}$ can be easily computed, since they are characterized by the equation $j_{t,k}(\tilde{\psi}_{t,k}(z, \tau_{t,k}(z))) = 0$. Notice that, if $r'$ is small enough, it follows from the transversality to 0 of $j_{t,k}$ that $|\nabla j_{t,k} \circ d\tilde{\psi}_{t,k}(v)|$ is larger than a constant times $\lambda|v|$ for all $v \in \{0\}\times\mathbb{C}$ and at any point of $D^+ \times D^+$. Combining this estimate with the bounds on the derivatives of $j_{t,k}$ given by asymptotic holomorphicity and the above bounds on the derivatives of $\tilde{\psi}_{t,k}$, one gets that $|\nabla\tau_{t,k}|_{C^1} = O(1)$ and $|\bar{\partial}\tau_{t,k}|_{C^1} = O(k^{-1/2})$ over $D^+$.

One then defines $\theta_{t,k}(z) = \tilde{\psi}_{t,k}(z, \tau_{t,k}(z))$ over $D^+$, which satisfies all the required properties : the image $\theta_{t,k}(D^+)$ is contained in $R(s_{t,k})$ and in a ball of radius $O(r')$ centered at $x$ ; $\theta_{t,k}(D)$ contains the intersection of $R(s_{t,k})$ with $\tilde{\psi}_{t,k}(D \times D^+) \supset \psi'_{t,k}(B_{\mathbb{C}^2}(0,\lambda)) \supset B_{g_k}(x, r')$ ; and the required bounds on derivatives follow directly from those on derivatives of $\tau_{t,k}$ and $\tilde{\psi}_{t,k}$. Therefore, Lemma 7 is proved under the assumption that $r'$ is small enough. We set $r'_0$ in the statement of the lemma to be the bound on $r'$ which ensures that all the assumptions we have made on $r'$ are satisfied. □

We now prove that the assumptions of Proposition 3 hold for property $\mathcal{P}$ in the case of single sections $s_k$ (the case of one-parameter families is discussed later). Let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and the corresponding maps $f_k = \mathbb{P}s_k$. We have to show that, for large enough $k$, a perturbation of $s_k$ with Gaussian decay and smaller than $\delta$ in $C^3$ norm can make property $\mathcal{P}$ hold over a ball centered at $x$. Because of assertion (1) of Lemma 6, it is actually sufficient to show that there exist constants $c$, $c'$ and $p$ independent of $k$ and $\delta$ such that, if $x$ lies within distance $c$ of $R(s_k)$, then $s_k$ can be perturbed to make the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ $\eta$-transverse to 0 over the intersection of $R(s_k)$ with a ball $B_{g_k}(x, c)$, where $\eta = c'\delta\,(\log\delta^{-1})^{-p}$. Such a result is then sufficient to imply the transversality to 0 of $\mathcal{T}(s_k) \oplus \mathrm{Jac}(f_k)$ over the ball $B_{g_k}(x, \frac{c}{2})$, with a transversality constant decreased by a bounded factor.

As in previous sections, composing with a rotation in $\mathbb{C}^3$ (constant over $X$), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s^1_k(x) = s^2_k(x) = 0$ and therefore $|s^0_k(x)| \geq \frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r > 0$ (independent of $k$) such that $|s^0_k| \geq \frac{\gamma}{3}$, $|s^1_k| < \frac{\gamma}{3}$ and $|s^2_k| < \frac{\gamma}{3}$ over the ball $B_{g_k}(x, r)$. Therefore, over this ball one can define the map

$$h_k(y) = (h^1_k(y), h^2_k(y)) = \Big(\frac{s^1_k(y)}{s^0_k(y)}, \frac{s^2_k(y)}{s^0_k(y)}\Big).$$

The map $f_k$ is the composition of $h_k$ with the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y \in B_{g_k}(x, r)$, the bound $|\partial f_k(y)| \geq \frac{\gamma}{2}$ implies that $|\partial h_k(y)| \geq \gamma'$ for some constant $\gamma' > 0$. Moreover, one has $\mathrm{Jac}(f_k) = \phi\,\mathrm{Jac}(h_k)$, where $\phi(y)$ is the Jacobian of $\iota$ at $h_k(y)$. In particular, $\mathrm{Jac}(h_k)$ vanishes at exactly the same points of $B_{g_k}(x, r)$ as $\mathrm{Jac}(f_k)$. Since $|\phi|$ is bounded between two universal constants over $B_{g_k}(x, r)$

and $\nabla\phi$ is bounded too, it follows from the $\frac{\gamma}{2}$-transversality to 0 of $\mathrm{Jac}(f_k)$ that, decreasing $\gamma'$ if necessary, $\mathrm{Jac}(h_k)$ is $\gamma'$-transverse to 0 over $B_{g_k}(x,r)$.

Since $|\partial h_k(x)| \ge \gamma'$, after composing with a rotation in $\mathbb{C}^2$ (constant over $X$) acting on the two components $(s_k^1, s_k^2)$ one can assume that $|\partial h_k^2(x)| \ge \frac{\gamma'}{2}$. Since $\nabla\nabla h_k$ is uniformly bounded, decreasing $r$ if necessary one can ensure that $|\partial h_k^2|$ remains larger than $\frac{\gamma'}{4}$ at every point of $B_{g_k}(x,r)$.

Let us now show that, over $\hat{R}_x(s_k) = B_{g_k}(x,r) \cap R(s_k)$, the transversality to 0 of $\mathcal{T}(s_k)$ follows from that of $\hat{\mathcal{T}}(s_k) = \partial h_k^2 \wedge \partial\mathrm{Jac}(h_k)$.

It follows from the identity $\mathrm{Jac}(f_k) = \phi\,\mathrm{Jac}(h_k)$ and the vanishing of $\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$ that $\partial\mathrm{Jac}(f_k) = \phi\,\partial\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$. Moreover the two $(1,0)$-forms $\partial f_k$ and $\partial h_k$ have complex rank one at any point of $\hat{R}_x(s_k)$ and are related by $\partial f_k = d\iota(\partial h_k)$, so they have the same kernel (in some sense they are "colinear"). Because $|\partial h_k^2|$ is bounded from below over $B_{g_k}(x,r)$, the ratio between $|\partial h_k|$ and $|\partial h_k^2|$ is bounded. Because the line bundle $\mathcal{L}(s_k)$ on which one projects $\partial f_k$ coincides with $\mathrm{Im}\,\partial f_k$ over $R(s_k)$, we have $|\pi(\partial f_k)| = |\partial f_k|$ over $R(s_k)$. Since $\iota$ is a quasi-isometry over the unit ball, it follows that the ratio between $|\pi(\partial f_k)|$ and $|\partial h_k^2|$ is bounded from above and below over $\hat{R}_x(s_k)$. Moreover, the two 1-forms $\pi(\partial f_k)$ and $\partial h_k^2$ have same kernel, so one can write $\pi(\partial f_k) = \psi\,\partial h_k^2$ over $\hat{R}_x(s_k)$, with $\psi$ bounded from above and below. Because of the uniform bounds on derivatives of $s_k$ and therefore $f_k$ and $h_k$, it is easy to check that the derivatives of $\psi$ are bounded.

So $\mathcal{T}(s_k) = \phi\psi\,\hat{\mathcal{T}}(s_k)$ over $\hat{R}_x(s_k)$. Therefore, assume that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is $\eta$-transverse to 0 at a given point $y \in \hat{R}_x(s_k)$, and let $C > 1$ be a constant such that $\frac{1}{C} < |\phi\psi| < C$ and $|\nabla(\phi\psi)| < C$ over $\hat{R}_x(s_k)$. If $|\mathcal{T}(s_k)(y)| < \frac{\eta}{2C^3}$, then $|\hat{\mathcal{T}}(s_k)(y)| < \frac{\eta}{2C^2} < \eta$, and therefore $|\partial(\hat{\mathcal{T}}(s_k))(y)| > \eta$, so at $y$ one has $|\partial(\mathcal{T}(s_k))| \ge |\phi\psi\,\partial(\hat{\mathcal{T}}(s_k))| - |\hat{\mathcal{T}}(s_k)\partial(\phi\psi)| > \frac{1}{C}\eta - \frac{\eta}{2C^2}C = \frac{\eta}{2C} > \frac{\eta}{2C^3}$. In other terms, the restriction to $R(s_k)$ of $\mathcal{T}(s_k)$ is $\frac{\eta}{2C^3}$-transverse to 0 at $y$.

Therefore, we only need to show that there exists a constant $c > 0$ such that, if $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, then by perturbing $s_k$ it is possible to ensure that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is transverse to 0 over $B_{g_k}(x,c) \cap R(s_k)$.

By Lemma 7, given any sufficiently small constant $c > 0$ and assuming that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, there exists an approximately holomorphic map $\theta_k$ from $D^+$ to $R(s_k)$ such that $\theta_k(D)$ contains $B_{g_k}(x,c) \cap R(s_k)$ and satisfying bounds $|\nabla\theta_k|_{C^1,g_k} = O(1)$ and $|\bar\partial\theta_k|_{C^1,g_k} = O(k^{-1/2})$. We call $\bar{c} = O(c)$ the size of the ball such that $\theta_k(D^+) \subset B_{g_k}(x,\bar{c})$, and assume that $c$ is small enough to have $\bar{c} < r$.

From now on, we assume that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$.

Let $s_{k,x}^{\mathrm{ref}}$ be the asymptotically holomorphic sections of $L^k$ with Gaussian decay away from $x$ given by Lemma 2, and let $z_k^1$ and $z_k^2$ be the complex coordinate functions of a local approximately holomorphic Darboux coordinate chart on a neighborhood of $x$. There exist two complex numbers $a$ and $b$ such that $\partial h_k^2(x) = a\,\partial z_k^1(x) + b\,\partial z_k^2(x)$. Composing the coordinate chart $(z_k^1, z_k^2)$ with the rotation

$$\frac{1}{|a|^2 + |b^2|}\begin{pmatrix}\bar{b} & -\bar{a} \\ a & b\end{pmatrix},$$

we can actually write $\partial h_k^2(x) = \lambda\, \partial z_k^2(x)$, with $|\lambda|$ bounded from below independently of $k$ and $x$. We now define $Q_{k,x} = \big(0, (z_k^1)^2 s_{k,x}^{\mathrm{ref}}, 0\big)$ and study the behavior of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ for small $w \in \mathbb{C}$.

First we look at how adding $wQ_{k,x}$ to $s_k$ affects the submanifold $R(s_k)$ : for small enough $w$, $R(s_k + wQ_{k,x})$ is a small deformation of $R(s_k)$ and can therefore be seen as a section of $TX_{|R(s_k)}$. Because the derivative of $\mathrm{Jac}(h_k)$ is uniformly bounded and $B_{g_k}(x,c) \cap R(s_k)$ is not empty, if $c$ is small enough then $|\mathrm{Jac}(h_k)|$ remains less than $\gamma'$ over $B_{g_k}(x,\bar{c})$. Recall that $\mathrm{Jac}(h_k)$ is $\gamma'$-transverse to 0 over $B_{g_k}(x,r)$ : therefore, at every point $y \in B_{g_k}(x,\bar{c})$, $\nabla \mathrm{Jac}(h_k)$ admits a right inverse $\rho : \Lambda^{2,0} T_y^* X \to T_y X$ of norm less than $\frac{1}{\gamma'}$. Adding $wQ_{k,x}$ to $s_k$ increases $\mathrm{Jac}(h_k)$ by $w\Delta_{k,x}$, where

$$\Delta_{k,x} = \partial\Big(\frac{(z_k^1)^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \wedge \partial h_k^2.$$

Therefore, $R(s_k + wQ_{k,x})$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta_{k,x}) + O(|w\Delta_{k,x}|^2)$. It follows that the value of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ at a point of $R(s_k + wQ_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta_{k,x}(w) = w\, \partial h_k^2 \wedge \partial \Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) + O(w^2).$$

Our aim is therefore to show that, if $c$ is small enough, for a suitable value of $w$ the quantity $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$.

Notice that the quantities $\hat{\mathcal{T}}(s_k)$ and $\mathrm{Jac}(h_k)$ are asymptotically holomorphic, so that $\nabla(\hat{\mathcal{T}}(s_k))$ and $\rho$ are approximately complex linear. Therefore,

$$\nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) = w\nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}) + O(k^{-1/2}).$$

It follows that $\Theta_{k,x}(w) = w\Theta_{k,x}^0 + O(w^2) + O(k^{-1/2})$, where

$$\Theta_{k,x}^0 = \partial h_k^2 \wedge \partial \Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}).$$

We start by computing the value of $\Theta_{k,x}^0$ at $x$, using the fact that $\partial h_k^2(x) = \lambda\, \partial z_k^2(x)$ while $z_k^1(x) = 0$ and therefore $\Delta_{k,x}(x) = 0$. Because of the identity $\Delta_{k,x} = \frac{s_{k,x}^{\mathrm{ref}}}{s_k^0}\, 2 z_k^1 \partial z_k^1 \wedge \partial h_k^2 + O(|z_k^1|^2)$, an easy calculation yields that

$$\partial \Delta_{k,x} = 2\frac{s_{k,x}^{\mathrm{ref}}}{s_k^0}(\partial z_k^1 \wedge \partial h_k^2)\, \partial z_k^1 + O(|z_k^1|)$$

and therefore

$$\Theta_{k,x}^0(x) = -2\lambda^2\, \frac{s_{k,x}^{\mathrm{ref}}(x)}{s_k^0(x)} \big(\partial z_k^1(x) \wedge \partial z_k^2(x)\big)^2.$$

The important point is that there exists a constant $\gamma'' > 0$ independent of $k$ and $x$ such that $|\Theta_{k,x}^0(x)| \geq \gamma''$.

Since the derivatives of $\Theta_{k,x}^0$ are uniformly bounded, $|\Theta_{k,x}^0|$ remains larger than $\frac{\gamma''}{2}$ at every point of $B_{g_k}(x,\bar{c})$ if $c$ is small enough. It follows that, over $R(s_k) \cap B_{g_k}(x,c)$, the transversality to 0 of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is equivalent to that of the function

$(\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w))/\Theta^0_{k,x}$. The value of $c$ we finally choose to use in Lemma 7 for the construction of $\theta_k$ is one small enough to ensure that all the above statements hold (but still independent of $k$, $x$ and $\delta$). Now define, over the disc $D^+ \subset \mathbb{C}$, the function

$$v_k(z) = \frac{\hat{\mathcal{T}}(s_k)(\theta_k(z))}{\Theta^0_{k,x}(\theta_k(z))}$$

with values in $\mathbb{C}$. Because $\Theta^0_{k,x}$ is bounded from below over $B_{g_k}(x,\bar{c})$ and because of the bounds on the derivatives of $\theta_k$ given by Lemma 7, the functions $v_k : D^+ \to \mathbb{C}$ satisfy the hypotheses of Proposition 6 for all large enough $k$. Therefore, if $C_0$ is a constant larger than $|Q_{k,x}|_{C^3,g_k}$, and if $k$ is large enough, there exists $w_k \in \mathbb{C}$, with $|w_k| \le \frac{\delta}{C_0}$, such that $v_k + w_k$ is $\alpha$-transverse to 0 over the unit disc $D$ in $\mathbb{C}$, where $\alpha = \frac{\delta}{C_0} \log((\frac{\delta}{C_0})^{-1})^{-p}$.

Multiplying again by $\Theta^0_{k,x}$ and recalling that $\theta_k$ maps diffeomorphically $D$ to a subset of $R(s_k)$ containing $R(s_k) \cap B_{g_k}(x,c)$, we get that the restriction to $R(s_k)$ of $\hat{\mathcal{T}}(s_k) + w_k\Theta^0_{k,x}$ is $\alpha'$-transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$ for some $\alpha'$ differing from $\alpha$ by at most a constant factor. Recall that $\Theta_{k,x}(w_k) = w_k\Theta^0_{k,x} + O(|w_k|^2) + O(k^{-1/2})$, and note that $|w_k|^2$ is at most of the order of $\delta^2$, while $\alpha'$ is of the order of $\delta \log(\delta^{-1})^{-p}$ : so, if $\delta$ is small enough, one can assume that $|w_k|^2$ is much smaller than $\alpha'$. If $k$ is large enough, $k^{-1/2}$ is also much smaller than $\alpha'$, so that $\hat{\mathcal{T}}(s_k)+\Theta_{k,x}(w_k)$ differs from $\hat{\mathcal{T}}(s_k) + w_k\Theta^0_{k,x}$ by less than $\frac{\alpha'}{2}$, and is therefore $\frac{\alpha'}{2}$-transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$.

Next, recall that $R(s_k + w_kQ_{k,x})$ is obtained by shifting $R(s_k)$ by an amount $-\rho(w_k\Delta_{k,x})+O(|w_k\Delta_{k,x}|^2) = O(|w_k|)$ (because $|\Delta_{k,x}|$ is uniformly bounded, or more generally because the perturbation of $s_k$ is $O(|w_k|)$ in $C^3$ norm). So, if $\delta$ is small enough, one can safely assume that the distance by which one shifts the points of $R(s_k)$ is less than $\frac{c}{2}$. Therefore, given any point in $R(s_k + w_kQ_{k,x}) \cap B_{g_k}(x,\frac{c}{2})$, the corresponding point in $R(s_k)$ belongs to $B_{g_k}(x,c)$.

We have seen above that the value of $\hat{\mathcal{T}}(s_k+w_kQ_{k,x})$ at a point of $R(s_k+w_kQ_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by $\Theta_{k,x}(w_k)$ ; therefore it follows from the transversality properties of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w_k)$ that the restriction to $R(s_k + w_kQ_{k,x})$ of $\hat{\mathcal{T}}(s_k + w_kQ_{k,x})$ is $\alpha''$-transverse to 0 over $R(s_k + w_kQ_{k,x}) \cap B_{g_k}(x,\frac{c}{2})$ for some $\alpha'' > 0$ differing from $\alpha'$ by at most a constant factor.

By the remarks above, this transversality property implies transversality to 0 of the restriction of $\mathcal{T}(s_k + w_kQ_{k,x})$ over $R(s_k + w_kQ_{k,x}) \cap B_{g_k}(x,\frac{c}{2})$ ; therefore, by Lemma 6, $\mathcal{T}(s_k + w_kQ_{k,x}) \oplus \mathrm{Jac}(\mathbb{P}(s_k + w_kQ_{k,x}))$ is $\eta$-transverse to 0 over $B_{g_k}(x,\frac{c}{4})$, with a transversality constant $\eta$ differing from $\alpha''$ by at most a constant factor. So, if $\delta$ is small enough and $k$ large enough, in the case where $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, we have constructed $w_k$ such that $s_k + w_kQ_{k,x}$ satisfies the required property $\mathcal{P}(\eta,y)$ at every point $y \in B_{g_k}(x,\frac{c}{4})$. By construction, $|w_kQ_{k,x}|_{C^3,g_k} \le \delta$, the asymptotically holomorphic sections $Q_{k,x}$ have uniform Gaussian decay away from $x$, and $\eta$ is larger than $c'\delta\log(\delta^{-1})^{-p}$ for some constant $c' > 0$, so all required properties hold in this case.

Moreover, in the case where $B_{g_k}(x,c)$ does not intersect $R(s_k)$, the section $s_k$ already satisfies the property $\mathcal{P}(\frac{3}{4}c,y)$ at every point $y$ of $B_{g_k}(x,\frac{c}{4})$ and no perturbation is necessary. Therefore, the property $\mathcal{P}$ under consideration satisfies the hypotheses of Proposition 3 whether $B_{g_k}(x,c)$ intersects $R(s_k)$ or not. This ends the proof of Proposition 7 for isolated sections $s_k$.

In the case of one-parameter families of sections, the argument still works similarly : we are now given sections $s_{t,k}$ depending continuously on a parameter $t \in [0,1]$, and try to perform the same construction as above for each value of $t$, in such a way that everything depends continuously on $t$. As previously, we have to show that one can perturb $s_{t,k}$ in order to ensure that, for all $t$ such that $x$ lies in a neighborhood of $R(s_{t,k})$, $\mathcal{T}(s_{t,k})_{|R(s_{t,k})}$ is transverse to 0 over the intersection of $R(s_{t,k})$ with a ball centered at $x$.

As before, a continuous family of rotations of $\mathbb{C}^3$ can be used to ensure that $s^1_{t,k}(x)$ and $s^2_{t,k}(x)$ vanish for all $t$, allowing one to define $h_{t,k}$ for all $t$. Moreover the argument at the end of Section 3.1 proves the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ acting on the two components $(s^1_{t,k},s^2_{t,k})$ allowing one to assume that $|\partial h^2_{t,k}(x)| \geq \frac{\gamma'}{2}$ for all $t$. Therefore, as in the case of isolated sections, the problem is reduced to that of perturbing $s_{t,k}$ when $x$ lies in a neighborhood of $R(s_{t,k})$ in order to obtain the transversality to 0 of $\hat{\mathcal{T}}(s_{t,k})_{|R(s_{t,k})}$ over the intersection of $R(s_{t,k})$ with a ball centered at $x$.

Because Lemma 7 and Proposition 6 also apply in the case of 1-parameter families of sections, the argument used above to obtain the expected transversality result for isolated sections also works here for all $t$ such that $x$ lies in the neighborhood of $R(s_{t,k})$. However, the ball $B_{g_k}(x,c)$ intersects $R(s_{t,k})$ only for certain values of $t \in [0,1]$, which makes it necessary to work more carefully.

Define $\Omega_k \subset [0,1]$ as the set of all $t$ for which $B_{g_k}(x,c) \cap R(s_{t,k}) \neq \emptyset$. For all large enough $k$ and for all $t \in \Omega_k$, Lemma 7 allows one to define maps $\theta_{t,k} : D^+ \to R(s_{t,k})$ depending continuously on $t$ and with the same properties as in the case of isolated sections. Using local coordinates $z^i_{t,k}$ depending continuously on $t$ given by Lemma 3 and sections $s^{\mathrm{ref}}_{t,k,x}$ given by Lemma 2, the quantities $Q_{t,k,x}$, $\Delta_{t,k,x}$, $\Theta_{t,k,x}(w)$, $\Theta^0_{t,k,x}$ and $v_{t,k}$ can be defined for all $t \in \Omega_k$ by the same formulae as above and depend continuously on $t$.

Proposition 6 then gives, for all large $k$ and for all $t \in \Omega_k$, complex numbers $w_{t,k}$ of norm at most $\frac{\delta}{C_0}$ and depending continuously on $t$, such that the functions $v_{t,k}+w_{t,k}$ are transverse to 0 over $D$. As in the case of isolated sections, this implies that $s_{t,k}+w_{t,k}Q_{t,k,x}$ satisfies the required transversality property over $B_{g_k}(x,\frac{c}{4})$.

Our problem is to define asymptotically holomorphic sections $\tau_{t,k,x}$ of $\mathbb{C}^3 \otimes L^k$ *for all values of* $t \in [0,1]$, of $C^3$-norm less than $\delta$ and with Gaussian decay away from $x$, in such a way that the sections $s_{t,k}+\tau_{t,k,x}$ depend continuously on $t \in [0,1]$ and satisfy the property $\mathcal{P}$ over $B_{g_k}(x,\frac{c}{4})$ for all $t$. For this, let $\beta : \mathbb{R}_+ \to [0,1]$ be a continuous cut-off function equal to 1 over $[0,\frac{3c}{4}]$ and to 0 over $[c,+\infty)$. Define, for all $t \in \Omega_k$,

$$\tau_{t,k,x} = \beta\big(dist_{g_k}(x,R(s_{t,k}))\big)w_{t,k}Q_{t,k,x},$$

and $\tau_{t,k,x} = 0$ for all $t \notin \Omega_k$. It is clear that, for all $t \in [0,1]$, the sections $\tau_{t,k,x}$ are asymptotically holomorphic, have Gaussian decay away from $x$, depend continuously on $t$ and are smaller than $\delta$ in $C^3$ norm. Moreover, for all $t$ such that $dist_{g_k}(x, R(s_{t,k})) \leq \frac{3c}{4}$, one has $\tau_{t,k,x} = w_{t,k} Q_{t,k,x}$, so the sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over $B_{g_k}(x, \frac{c}{4})$ for all such values of $t$.

For the remaining values of $t$, namely those such that $x$ is at distance more than $\frac{3c}{4}$ from $R(s_{t,k})$, the argument is the following : since the perturbation $\tau_{t,k,x}$ is smaller than $\delta$, every point of $R(s_{t,k} + \tau_{t,k,x})$ lies within distance $O(\delta)$ of $R(s_{t,k})$. Therefore, decreasing the maximum allowable value of $\delta$ in Proposition 3 if necessary, one can safely assume that this distance is less than $\frac{c}{4}$. It follows that $x$ is at distance more than $\frac{c}{2}$ of $R(s_{t,k} + \tau_{t,k,x})$, and so that the property $\mathcal{P}(\frac{c}{4}, y)$ holds at every point $y \in B_{g_k}(x, \frac{c}{4})$.

Therefore, for all large enough $k$ and for all $t \in [0,1]$, the perturbed sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over the ball $B_{g_k}(x, \frac{c}{4})$. It follows that the assumptions of Proposition 3 also hold for $\mathcal{P}$ in the case of one-parameter families, and so Proposition 7 is proved.

## 4. Dealing with the antiholomorphic part

**4.1. Holomorphicity in the neighborhood of cusp points.** At this point in the proof, we have constructed asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying all the required transversality properties. We now need to show that, by further perturbation, one can obtain $\bar{\partial}$-tameness. We first handle the case of cusp points :

PROPOSITION 8. *Let $(s_k)_{k \gg 0}$ be $\gamma$-generic asymptotically $J$-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Then there exist constants $(C_p)_{p \in \mathbb{N}}$ and $c > 0$ such that, for all large $k$, there exist $\omega$-compatible almost-complex structures $\tilde{J}_k$ on $X$ and asymptotically $J$-holomorphic sections $\sigma_k$ of $\mathbb{C}^3 \otimes L^k$ with the following properties : at any point whose $g_k$-distance to $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ is less than $c$, the almost-complex structure $\tilde{J}_k$ is integrable and the map $\mathbb{P}\sigma_k$ is $\tilde{J}_k$-holomorphic ; and for all $p \in \mathbb{N}$, $|\tilde{J}_k - J|_{C^p, g_k} \leq C_p k^{-1/2}$ and $|\sigma_k - s_k|_{C^p, g_k} \leq C_p k^{-1/2}$.*

*Furthermore, the result also applies to 1-parameter families of $\gamma$-generic asymptotically $J_t$-holomorphic sections $(s_{t,k})_{t \in [0,1], k \gg 0}$ : for all large $k$ there exist almost-complex structures $\tilde{J}_{t,k}$ and asymptotically $J_t$-holomorphic sections $\sigma_{t,k}$ depending continuously on $t$ and such that the above properties hold for all values of $t$. Moreover, if $s_{0,k}$ and $s_{1,k}$ already satisfy the required properties, and if one assumes that, for some $\epsilon > 0$, $J_t$ and $s_{t,k}$ are respectively equal to $J_0$ and $s_{0,k}$ for all $t \in [0, \epsilon]$ and to $J_1$ and $s_{1,k}$ for all $t \in [1-\epsilon, 1]$, then it is possible to ensure that $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.*

The proof of this result relies on the following analysis lemma, which states that any approximately holomorphic complex-valued function defined over the ball $B^+$ of radius $\frac{11}{10}$ in $\mathbb{C}^2$ can be approximated over the interior ball $B$ of unit radius by a holomorphic function :

LEMMA 8. *There exist an operator $P : C^\infty(B^+, \mathbb{C}) \to C^\infty(B, \mathbb{C})$ and constants $(K_p)_{p\in\mathbb{N}}$ such that, given any function $f \in C^\infty(B^+, \mathbb{C})$, the function $\tilde f = P(f)$ is holomorphic over the unit ball $B$ and satisfies $|f - \tilde f|_{C^p(B)} \le K_p\, |\bar\partial f|_{C^p(B^+)}$ for every $p \in \mathbb{N}$.*

PROOF. (see also [D1]). This is a standard fact which can be proved e.g. using the Hörmander theory of weighted $L^2$ spaces. Using a suitable weighted $L^2$ norm on $B^+$ which compares uniformly with the standard norm on the interior ball $B'$ of radius $1 + \frac{1}{20}$ ($B \subset B' \subset B^+$), one obtains a bounded solution to the Cauchy-Riemann equation : for any $\bar\partial$-closed $(0,1)$-form $\rho$ on $B^+$ there exists a function $T(\rho)$ such that $\bar\partial T(\rho) = \rho$ and $|T(\rho)|_{L^2(B')} \le C|\rho|_{L^2(B^+)}$ for some constant $C$.

Take $\rho = \bar\partial f$ and let $h = T(\rho)$ : since $\bar\partial h = \rho = \bar\partial f$, the function $\tilde f = f - h$ is holomorphic (in other words, we set $P = \mathrm{Id} - T\bar\partial$). Moreover the $L^2$ norm of $h$ and the $C^p$ norm of $\bar\partial h = \bar\partial f$ over $B'$ are bounded by multiples of $|\bar\partial f|_{C^p(B^+)}$ ; therefore, by standard elliptic theory, the same is true for the $C^p$ norm of $h$ over the interior ball $B$, which gives the desired result. □

We first prove Proposition 8 in the case of isolated sections $s_k$, where the argument is fairly easy. Because $s_k$ is $\gamma$-generic, the set of points of $R(s_k)$ where $\mathcal{T}(s_k)$ vanishes, i.e. $\mathcal{C}_J(s_k)$, is finite. Moreover $\nabla\mathcal{T}(s_k)_{|R(s_k)}$ is larger than $\gamma$ at all cusp points and $\nabla\nabla\mathcal{T}(s_k)$ is uniformly bounded, so there exists a constant $r > 0$ such that the $g_k$-distance between any two points of $\mathcal{C}_J(s_k)$ is larger than $4r$.

Let $x$ be a point of $\mathcal{C}_J(s_k)$, and consider a local approximately $J$-holomorphic Darboux map $\psi_k : (\mathbb{C}^2, 0) \to (X, x)$ as given by Lemma 3. Because of the bounds on $\bar\partial\psi_k$, the $\omega$-compatible almost-complex structure $J'_k$ on the ball $B_{g_k}(x, 2r)$ defined by pulling back the standard complex structure of $\mathbb{C}^2$ satisfies bounds of the type $|J'_k - J|_{C^p,g_k} = O(k^{-1/2})$ over $B_{g_k}(x, 2r)$ for all $p \in \mathbb{N}$.

Recall that the set of $\omega$-skew-symmetric endomorphisms of square $-1$ of the tangent bundle $TX$ (i.e. $\omega$-compatible almost-complex structures) is a subbundle of $\mathrm{End}(TX)$ whose fibers are contractible. Therefore, there exists a one-parameter family $(J^\tau_k)_{\tau\in[0,1]}$ of $\omega$-compatible almost-complex structures over $B_{g_k}(x, 2r)$ depending smoothly on $\tau$ and such that $J^0_k = J$ and $J^1_k = J'_k$. Also, let $\tau_x : B_{g_k}(x, 2r) \to [0, 1]$ be a smooth cut-off function with bounded derivatives such that $\tau_x = 1$ over $B_{g_k}(x, r)$ and $\tau_x = 0$ outside of $B_{g_k}(x, \frac{3}{2}r)$.

Then, define $\tilde J_k$ to be the almost-complex structure which equals $J$ outside of the $2r$-neighborhood of $\mathcal{C}_J(s_k)$, and which at any point $y$ of a ball $B_{g_k}(x, 2r)$ centered at $x \in \mathcal{C}_J(s_k)$ coincides with $J^{\tau_x(y)}_k$ : it is quite easy to check that $\tilde J_k$ is integrable over the $r$-neighborhood of $\mathcal{C}_J(s_k)$ where it coincides with $J'_k$, and satisfies bounds of the type $|\tilde J_k - J|_{C^p,g_k} = O(k^{-1/2})$ $\forall p \in \mathbb{N}$.

Let us now return to a neighborhood of $x \in \mathcal{C}_J(s_k)$, where we need to perturb $s_k$ to make the corresponding projective map locally $\tilde J_k$-holomorphic. First notice that, by composing with a rotation of $\mathbb{C}^3$ (constant over $X$), one can safely assume that $s^1_k(x) = s^2_k(x) = 0$. Therefore, $|s^0_k(x)| \ge \gamma$, and decreasing $r$ if necessary one can assume that $|s^0_k|$ remains larger than $\frac{\gamma}{2}$ at every point of $B_{g_k}(x, r)$. The $\tilde J_k$-holomorphicity of $\mathbb{P}s_k$ over a neighborhood of $x$ is then equivalent to that of the map $h_k$ with values in $\mathbb{C}^2$ defined by

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Because of the properties of the map $\psi_k$ given by Lemma 3, there exist constants $\lambda > 0$ and $r' > 0$, independent of $k$, such that $\psi_k(B_{\mathbb{C}^2}(0, \frac{11}{10}\lambda))$ is contained in $B_{g_k}(x,r)$ while $\psi_k(B_{\mathbb{C}^2}(0, \frac{1}{2}\lambda))$ contains $B_{g_k}(x,r')$. We now define the two complex-valued functions $f_k^1(z) = h_k^1(\psi_k(\lambda z))$ and $f_k^2(z) = h_k^2(\psi_k(\lambda z))$ over the ball $B^+ \subset \mathbb{C}^2$. By definition of $\tilde{J}_k$, the map $\psi_k$ intertwines the almost-complex structure $\tilde{J}_k$ over $B_{g_k}(x,r)$ and the standard complex structure of $\mathbb{C}^2$, so our goal is to make the functions $f_k^1$ and $f_k^2$ holomorphic in the usual sense over a ball in $\mathbb{C}^2$.

This is where we use Lemma 8. Remark that, because of the estimates on $\bar\partial_J \psi_k$ given by Lemma 3 and those on $\bar\partial_J h_k$ coming from asymptotic holomorphicity, we have $|\bar\partial f_k^i|_{C^p(B^+)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$. Therefore, by Lemma 8 there exist two holomorphic functions $\tilde{f}_k^1$ and $\tilde{f}_k^2$, defined over the unit ball $B \subset \mathbb{C}^2$, such that $|f_k^i - \tilde{f}_k^i|_{C^p(B)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$.

Let $\beta : [0,1] \to [0,1]$ be a smooth cut-off function such that $\beta = 1$ over $[0, \frac{1}{2}]$ and $\beta = 0$ over $[\frac{3}{4}, 1]$, and define, for all $z \in B$ and $i \in \{1,2\}$, $\hat{f}_k^i(z) = \beta(|z|)\tilde{f}_k^i(z) + (1-\beta(|z|))f_k^i(z)$. By construction, the functions $\hat{f}_k^i$ are holomorphic over the ball of radius $\frac{1}{2}$ and differ from $f_k^i$ by $O(k^{-1/2})$.

Going back through the coordinate map, let $\hat{h}_k^i$ be the functions on the neighborhood $U_x = \psi_k(B_{\mathbb{C}^2}(0,\lambda))$ of $x$ which satisfy $\hat{h}_k^i(\psi_k(\lambda z)) = \hat{f}_k^i(z)$ for every $z \in B$. Define $\hat{s}_k^0 = s_k^0$, $\hat{s}_k^1 = \hat{h}_k^1 s_k^0$ and $\hat{s}_k^2 = \hat{h}_k^2 s_k^0$ over $U_x$, and let $\sigma_k$ be the global section of $\mathbb{C}^3 \otimes L^k$ which $\forall x \in \mathcal{C}_J(s_k)$ equals $\hat{s}_k$ over $U_x$ and which coincides with $s_k$ away from $\mathcal{C}_J(s_k)$.

Because $\hat{f}_k^i = f_k^i$ near the boundary of $B$, $\hat{s}_k$ coincides with $s_k$ near the boundary of $U_x$, and $\sigma_k$ is therefore a smooth section of $\mathbb{C}^3 \otimes L^k$. For every $p \in \mathbb{N}$, it follows from the bound $|\hat{f}_k^i - f_k^i|_{C^p(B)} = O(k^{-1/2})$ that $|\sigma_k - s_k|_{C^p, g_k} = O(k^{-1/2})$. Moreover, the functions $\hat{f}_k^i$ are holomorphic over $B_{\mathbb{C}^2}(0, \frac{1}{2})$ where they coincide with $\tilde{f}_k^i$, so the functions $\hat{h}_k^i$ are $\tilde{J}_k$-holomorphic over $\psi_k(B_{\mathbb{C}^2}(0, \frac{1}{2}\lambda)) \supset B_{g_k}(x,r')$, and it follows that $\mathbb{P}\sigma_k$ is $\tilde{J}_k$-holomorphic over $B_{g_k}(x,r')$.

Therefore, the almost-complex structures $\tilde{J}_k$ and the sections $\sigma_k$ satisfy all the required properties, except that the integrability of $\tilde{J}_k$ and the holomorphicity of $\mathbb{P}\sigma_k$ are proved to hold on the $r'$-neighborhood of $\mathcal{C}_J(s_k)$ rather than on a neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$.

However, the $C^p$ bounds $|\tilde{J}_k - J_k| = O(k^{-1/2})$ and $|\sigma_k - s_k| = O(k^{-1/2})$ imply that $|\mathrm{Jac}_{\tilde{J}_k}(\mathbb{P}\sigma_k) - \mathrm{Jac}_J(\mathbb{P}s_k)| = O(k^{-1/2})$ and $|\mathcal{T}_{\tilde{J}_k}(\sigma_k) - \mathcal{T}_J(s_k)| = O(k^{-1/2})$. Therefore it follows from the transversality properties of $s_k$ that the points of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ lie within $g_k$-distance $O(k^{-1/2})$ of $\mathcal{C}_J(s_k)$. In particular, if $k$ is large enough, the $\frac{r'}{2}$-neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ is contained in the $r'$-neighborhood of $\mathcal{C}_J(s_k)$, which ends the proof of Proposition 8 in the case of isolated sections.

In the case of one-parameter families of sections, the argument is similar. One first notices that, because of $\gamma$-genericity, there exists $r > 0$ such that, for every $t \in [0,1]$, the set $\mathcal{C}_{J_t}(s_{t,k})$ consists of finitely many points, any two of which are mutually distant of at least $4r$. Therefore, the points of $\mathcal{C}_{J_t}(s_{t,k})$ depend continuously on $t$, and their number remains constant.

Consider a continuous family $(x_t)_{t\in[0,1]}$ of points of $\mathcal{C}_{J_t}(s_{t,k})$ : Lemma 3 provides approximately $J_t$-holomorphic Darboux maps $\psi_{t,k}$ depending continuously on $t$ on a neighborhood of $x_t$. By pulling back the standard complex structure of $\mathbb{C}^2$, one obtains integrable almost-complex structures $J'_{t,k}$ over $B_{g_k}(x_t, 2r)$, depending continuously on $t$ and differing from $J_t$ by $O(k^{-1/2})$. As previously, because the set of $\omega$-compatible almost-complex structures is contractible, one can define a continuous family of almost-complex structures $\tilde{J}_{t,k}$ on $X$ by gluing together $J_t$ with the almost-complex structures $J'_{t,k}$ defined over $B_{g_k}(x_t, 2r)$, using a cut-off function at distance $r$ from $\mathcal{C}_{J_t}(s_{t,k})$. By construction, the almost-complex structures $\tilde{J}_{t,k}$ are integrable over the $r$-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$, and $|\tilde{J}_{t,k} - J_t|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, we perturb $s_{t,k}$ near $x_t \in \mathcal{C}_{J_t}(s_{t,k})$ in order to make the corresponding projective map locally $\tilde{J}_{t,k}$-holomorphic. As before, composing with a rotation of $\mathbb{C}^3$ (constant over $X$ and depending continuously on $t$) and decreasing $r$ if necessary, we can assume that $s^1_{t,k}(x_t) = s^2_{t,k}(x_t) = 0$ and therefore that $|s^0_{t,k}|$ remains larger than $\frac{\gamma}{2}$ over $B_{g_k}(x_t, r)$. The $\tilde{J}_{t,k}$-holomorphicity of $\mathbb{P}s_{t,k}$ over $B_{g_k}(x_t, r)$ is then equivalent to that of the map $h_{t,k}$ with values in $\mathbb{C}^2$ defined as above.

As previously, there exist constants $\lambda$ and $r'$ such that $\psi_{t,k}(B_{\mathbb{C}^2}(0, \frac{11}{10}\lambda))$ is contained in $B_{g_k}(x_t, r)$ and $\psi_{t,k}(B_{\mathbb{C}^2}(0, \frac{1}{2}\lambda)) \supset B_{g_k}(x_t, r')$ ; once again, our goal is to make the functions $f^i_{t,k} : B^+ \to \mathbb{C}$ defined by $f^i_{t,k}(z) = h^i_{t,k}(\psi_{t,k}(\lambda z))$ holomorphic in the usual sense.

Because of the estimates on $\bar\partial_{J_t}\psi_{t,k}$ and $\bar\partial_{J_t}h_{t,k}$, we have $|\bar\partial f^i_{t,k}|_{C^p(B^+)} = O(k^{-1/2})$ $\forall p \in \mathbb{N}$, so Lemma 8 provides holomorphic functions $\tilde{f}^i_{t,k}$ over $B$ which differ from $f^i_{t,k}$ by $O(k^{-1/2})$. By the same cut-off procedure as above, we can thus define functions $\hat{f}^i_{t,k}$ which are holomorphic over $B_{\mathbb{C}^2}(0, \frac{1}{2})$ and coincide with $f^i_{t,k}$ near the boundary of $B$. Going back through the coordinate maps, we define as previously functions $\hat{h}^i_{t,k}$ and sections $\hat{s}_{t,k}$ over the neighborhood $U_{t,x_t} = \psi_{t,k}(B_{\mathbb{C}^2}(0, \lambda))$ of $x_t$. Since $\hat{s}_{t,k}$ coincides with $s_{t,k}$ near the boundary of $U_{t,x_t}$, we can obtain smooth sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ by gluing $s_{t,k}$ together with the various sections $\hat{s}_{t,k}$ defined near the points of $\mathcal{C}_{J_t}(s_{t,k})$.

As previously, the maps $\mathbb{P}\sigma_{t,k}$ are $\tilde{J}_{t,k}$-holomorphic over the $r'$-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$ and satisfy $|\sigma_{t,k} - s_{t,k}|_{C^p,g_k} = O(k^{-1/2})$ ; therefore the desired result follows from the observation that, for large enough $k$, $\mathcal{C}_{\tilde{J}_{t,k}}(\sigma_{t,k})$ lies within distance $\frac{r'}{2}$ of $\mathcal{C}_{J_t}(s_{t,k})$.

We now consider the special case where $s_{0,k}$ already satisfies the required conditions, i.e. there exists an almost-complex structure $\bar{J}_{0,k}$ within $O(k^{-1/2})$ of $J_0$, integrable near $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$, and such that $\mathbb{P}s_{0,k}$ is $\bar{J}_{0,k}$-holomorphic near $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$. Although this is actually not necessary for the result to hold, we also assume, as in the statement of Proposition 8, that $s_{t,k} = s_{0,k}$ and $J_t = J_0$ for every $t \le \epsilon$, for some $\epsilon > 0$. We want to prove that one can take $\sigma_{0,k} = s_{0,k}$ in the above construction.

We first show that one can assume that $\tilde{J}_{0,k}$ coincides with $\bar{J}_{0,k}$ over a small neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$. For this, remark that $\mathcal{C}_{J_0}(s_{0,k})$ lies within $O(k^{-1/2})$ of $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$, so there exists a constant $\delta$ such that, for large enough $k$, $\bar{J}_{0,k}$ is integrable and $\mathbb{P}s_{0,k}$ is $\bar{J}_{0,k}$-holomorphic over the $\delta$-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

Fix points $(x_t)_{t\in[0,1]}$ in $\mathcal{C}_{J_t}(s_{t,k})$, and consider, for all $t \ge \epsilon$, the approximately $J_t$-holomorphic Darboux coordinates $(z^1_{t,k}, z^2_{t,k})$ on a neighborhood of $x_t$ and the inverse map $\psi_{t,k}$ given by Lemma 3 and which are used to define the almost-complex structures $J'_{t,k}$ and $\tilde{J}_{t,k}$ near $x_t$. We want to show that one can extend the family $\psi_{t,k}$ to all $t \in [0,1]$ in such a way that the map $\psi_{0,k}$ is $\bar{J}_{0,k}$-holomorphic. The hypothesis that $J_t$ and $s_{t,k}$ are the same for all $t \in [0,\epsilon]$ makes things easier to handle because $J_\epsilon = J_0$ and $x_\epsilon = x_0$.

Since $\bar{J}_{0,k}$ is integrable over $B_{g_k}(x_0,\delta)$ and $\omega$-compatible, there exist local complex Darboux coordinates $Z_k = (Z^1_k, Z^2_k)$ at $x_0$ which are $\bar{J}_{0,k}$-holomorphic. It follows from the approximate $J_0$-holomorphicity of the coordinates $z_{\epsilon,k} = (z^1_{\epsilon,k}, z^2_{\epsilon,k})$ and from the bound $|J_0 - \bar{J}_{0,k}| = O(k^{-1/2})$ that, composing with a linear endomorphism of $\mathbb{C}^2$ if necessary, one can assume that the differentials at $x_0$ of the two coordinate maps, namely $\nabla_{x_0} z_{\epsilon,k}$ and $\nabla_{x_0} Z_k$, lie within $O(k^{-1/2})$ of each other. For all $t \in [0,\epsilon]$, $\check{z}_{t,k} = \frac{t}{\epsilon} z_{\epsilon,k} + (1 - \frac{t}{\epsilon}) Z_k$ defines local coordinates on a neighborhood of $x_0$ ; however, for $t \in (0,\epsilon)$ this map fails to be symplectic by an amount which is $O(k^{-1/2})$. So we apply Moser's argument to $\check{z}_{t,k}$ in order to get local Darboux coordinates $z_{t,k}$ over a neighborhood of $x_0$ which interpolate between $Z_k$ and $z_{\epsilon,k}$ and which differ from $\check{z}_{t,k}$ by $O(k^{-1/2})$. It is easy to check that, if $k$ is large enough, then the coordinates $z_{t,k}$ are well-defined over the ball $B_{g_k}(x_t, 2r)$. Since $\bar\partial_{J_0} Z_k$ and $\bar\partial_{J_0} z_{\epsilon,k}$ are $O(k^{-1/2})$, and because $z_{t,k}$ differs from $\check{z}_{t,k}$ by $O(k^{-1/2})$, the coordinates defined by $z_{t,k}$ are approximately $J_0$-holomorphic (in the sense of Lemma 3) for all $t \in [0,\epsilon]$.

Defining $\psi_{t,k}$ as the inverse of the map $z_{t,k}$ for every $t \in [0,\epsilon]$, it follows immediately that the maps $\psi_{t,k}$, which depend continuously on $t$, are approximately $J_t$-holomorphic over a neighborhood of 0 for every $t \in [0,1]$, and that $\psi_{0,k}$ is $\bar{J}_{0,k}$-holomorphic.

We can then define $J'_{t,k}$ as previously on $B_{g_k}(x_t, 2r)$, and notice that $J'_{0,k}$ coincides with $\bar{J}_{0,k}$. Therefore, the corresponding almost-complex structures $\tilde{J}_{t,k}$ over $X$, in addition to all the properties described previously, also satisfy the equality $\tilde{J}_{0,k} = \bar{J}_{0,k}$ over the $r$-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

It follows that, constructing the sections $\sigma_{t,k}$ from $s_{t,k}$ as previously, we have $\sigma_{0,k} = s_{0,k}$. Indeed, since $\mathbb{P}s_{0,k}$ is already $\tilde{J}_{0,k}$-holomorphic over the $r$-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$, we get that, in the above construction, $h^1_{0,k}$ and $h^2_{0,k}$ are $\tilde{J}_{0,k}$-holomorphic, and so $f^1_{0,k}$ and $f^2_{0,k}$ are holomorphic. Therefore, by definition of the operator $P$ of Lemma 8, we have $\tilde{f}^1_{0,k} = f^1_{0,k}$ and $\tilde{f}^2_{0,k} = f^2_{0,k}$, which clearly implies that $\sigma_{0,k} = s_{0,k}$.

The same argument applies near $t = 1$ to show that, if $s_{1,k}$ already satisfies the expected properties and if $J_t$ and $s_{t,k}$ are the same for all $t \in [1-\epsilon, 1]$, then one can take $\sigma_{1,k} = s_{1,k}$. This ends the proof of Proposition 8.

**4.2. Holomorphicity at generic branch points.** Our last step in order to obtain $\bar\partial$-tame sections is to ensure, by further perturbation, the vanishing of $\bar\partial_{\tilde J_k}(\mathbb{P}s_k)$ over the kernel of $\partial_{\tilde J_k}(\mathbb{P}s_k)$ at every branch point.

PROPOSITION 9. *Let $(s_k)_{k\gg 0}$ be $\gamma$-generic asymptotically $J$-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Assume that there exist $\omega$-compatible almost-complex structures $\tilde J_k$ such that $|\tilde J_k - J|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and such that, for some constant $c > 0$, $f_k = \mathbb{P}s_k$ is $\tilde J_k$-holomorphic over the $c$-neighborhood of $\mathcal{C}_{\tilde J_k}(s_k)$. Then, for all large $k$, there exist sections $\sigma_k$ such that the following properties hold : $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ ; $\sigma_k$ coincides with $s_k$ over the $\frac{c}{2}$-neighborhood of $\mathcal{C}_{\tilde J_k}(\sigma_k) = \mathcal{C}_{\tilde J_k}(s_k)$ ; and, at every point of $R_{\tilde J_k}(\sigma_k)$, $\bar\partial_{\tilde J_k}(\mathbb{P}\sigma_k)$ vanishes over the kernel of $\partial_{\tilde J_k}(\mathbb{P}\sigma_k)$.*

*Moreover, the same result holds for one-parameter families of asymptotically $J_t$-holomorphic sections $(s_{t,k})_{t\in[0,1],k\gg 0}$ satisfying the above properties. Furthermore, if $s_{0,k}$ and $s_{1,k}$ already satisfy the properties required of $\sigma_{0,k}$ and $\sigma_{1,k}$, then one can take $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.*

The role of the almost-complex structure $J$ in the statement of this result may seem ambiguous, as the sections $s_k$ are also asymptotically holomorphic and generic with respect to the almost-complex structures $\tilde J_k$. The point is that, by requiring that all the almost-complex structures $\tilde J_k$ lie within $O(k^{-1/2})$ of a fixed almost-complex structure, one ensures the existence of uniform bounds on the geometry of $\tilde J_k$ independently of $k$.

We now prove Proposition 9 in the case of isolated sections. In all the following, we use the almost complex structure $\tilde J_k$ implicitly. Consider a point $x \in R(s_k)$ at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, and let $K_x$ be the one-dimensional complex subspace $\mathrm{Ker}\ \partial f_k(x)$ of $T_xX$. Because $x \notin \mathcal{C}(s_k)$, we have $T_xX = T_xR(s_k) \oplus K_x$. Therefore, there exists a unique 1-form $\theta_x \in T^*_xX \otimes T_{f_k(x)}\mathbb{CP}^2$ such that the restriction of $\theta_x$ to $T_xR(s_k)$ is zero and the restriction of $\theta_x$ to $K_x$ is equal to $\bar\partial f_k(x)_{|K_x}$.

Because the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ is transverse to 0 and because $x$ is at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, the quantity $|\mathcal{T}(s_k)(x)|$ is bounded from below by a uniform constant, and therefore the angle between $T_xR(s_k)$ and $K_x$ is also bounded from below. So there exists a constant $C$ independent of $k$ and $x$ such that $|\theta_x| \le Ck^{-1/2}$. Moreover, because $\bar\partial f_k$ vanishes over the $c$-neighborhood of $\mathcal{C}(s_k)$, the 1-form $\theta_x$ vanishes at all points $x$ close to $\mathcal{C}(s_k)$ ; therefore we can extend $\theta$ into a section of $T^*X \otimes f_k^*T\mathbb{CP}^2$ over $R(s_k)$ which vanishes over the $c$-neighborhood of $\mathcal{C}(s_k)$, and which satisfies bounds of the type $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, use the exponential map of the metric $g$ to identify a tubular neighborhood of $R(s_k)$ with a neighborhood of the zero section in the normal bundle $NR(s_k)$. Given $\delta > 0$ sufficiently small, we define a section $\chi$ of $f_k^*T\mathbb{CP}^2$ over the $\delta$-tubular neighborhood of $R(s_k)$ by the following identity : given any point $x \in R(s_k)$ and any vector $\xi \in N_xR(s_k)$ of norm less than $\delta$,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,\theta_x(\xi),$$

where the fibers of $f_k^*T\mathbb{CP}^2$ at $x$ and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport, and $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function equal to 1 over

$[0, \frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta, \delta]$. Since $\chi$ vanishes near the boundary of the chosen tubular neighborhood, we can extend it into a smooth section over all of $X$ which vanishes at distance more than $\delta$ from $R(s_k)$.

Decreasing $\delta$ if necessary, we can assume that $\delta < \frac{c}{2}$ : it then follows from the vanishing of $\theta$ over the $c$-neighborhood of $\mathcal{C}(s_k)$ that $\chi$ vanishes over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$. Moreover, because $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and because the cut-off function $\beta$ is smooth, $\chi$ also satisfies bounds $|\chi|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Fix a point $x \in R(s_k)$ : $\chi$ is identically zero over $R(s_k)$ by construction, so $\nabla\chi(x)$ vanishes over $T_xR(s_k)$ ; and, because $\beta \equiv 1$ near the origin and by definition of the exponential map, $\nabla\chi(x)_{|N_xR(s_k)} = \theta_{x|N_xR(s_k)}$. Since $T_xR(s_k)$ and $N_xR(s_k)$ generate $T_xX$, we conclude that $\nabla\chi(x) = \theta_x$. In particular, restricting to $K_x$, we get that $\nabla\chi(x)_{|K_x} = \theta_{x|K_x} = \bar\partial f_k(x)_{|K_x}$. Equivalently, since $K_x$ is a complex subspace of $T_xX$, we have $\bar\partial\chi(x)_{|K_x} = \bar\partial f_k(x)_{|K_x}$ and $\partial\chi(x)_{|K_x} = 0 = \partial f_k(x)_{|K_x}$.

Recall that, for all $x \in X$, the tangent space to $\mathbb{CP}^2$ at $f_k(x) = \mathbb{P}s_k(x)$ canonically identifies with the space of complex linear maps from $\mathbb{C}s_k(x)$ to $(\mathbb{C}s_k(x))^\perp \subset \mathbb{C}^3 \otimes L^k_x$. This allows us to define $\sigma_k(x) = s_k(x) - \chi(x).s_k(x)$.

It follows from the properties of $\chi$ described above that $\sigma_k$ coincides with $s_k$ over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$ and that $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$. Because of the transversality properties of $s_k$, we get that the points of $\mathcal{C}(\sigma_k)$ lie within distance $O(k^{-1/2})$ of $\mathcal{C}(s_k)$, and therefore if $k$ is large enough that $\mathcal{C}(\sigma_k) = \mathcal{C}(s_k)$.

Let $\tilde f_k = \mathbb{P}\sigma_k$, and consider a point $x \in R(s_k)$ : since $\chi(x) = 0$ and therefore $\tilde f_k(x) = f_k(x)$, it is easy to check that $\nabla\tilde f_k(x) = \nabla f_k(x) - \nabla\chi(x)$ in $T^*_xX \otimes T_{f_k(x)}\mathbb{CP}^2$. Therefore, setting $K_x = \mathrm{Ker}\ \partial f_k(x)$ as above, we get that $\partial\tilde f_k(x) = \partial f_k(x) - \partial\chi(x)$ and $\bar\partial\tilde f_k(x) = \bar\partial f_k(x) - \bar\partial\chi(x)$ both vanish over $K_x$. A first consequence is that $\partial\tilde f_k(x)$ also has rank one, i.e. $x \in R(\sigma_k)$ : therefore $R(s_k) \subset R(\sigma_k)$. However, because $\sigma_k$ differs from $s_k$ by $O(k^{-1/2})$, it follows from the transversality properties of $s_k$ that, for large enough $k$, $R(\sigma_k)$ is contained in a small neighborhood of $R(s_k)$, and so $R(\sigma_k) = R(s_k)$.

Moreover, recall that at every point $x$ of $R(\sigma_k) = R(s_k)$ one has $\bar\partial\tilde f_k(x)_{|K_x} = \partial\tilde f_k(x)_{|K_x} = 0$. Therefore $\bar\partial\tilde f_k(x)$ vanishes over the kernel of $\partial\tilde f_k(x)$, and so the sections $\sigma_k$ satisfy all the required properties.

To handle the case of one-parameter families, remark that the above construction consists of explicit formulae, so it is easy to check that $\theta$, $\chi$ and $\sigma_k$ depend continuously on $s_k$ and $\tilde J_k$. Therefore, starting from one-parameter families $s_{t,k}$ and $\tilde J_{t,k}$, the above construction yields for all $t \in [0,1]$ sections $\sigma_{t,k}$ which satisfy the required properties and depend continuously on $t$.

Moreover, if $s_{0,k}$ already satisfies the required properties, i.e. $\bar\partial f_{0,k}(x)_{|K_x}$ vanishes at any point $x \in R(s_{0,k})$, then the above definitions give $\theta \equiv 0$, and therefore $\chi \equiv 0$ and $\sigma_{0,k} = s_{0,k}$ ; similarly for $t = 1$, which ends the proof of Proposition 9.

**4.3. Proof of the main theorems.** Assuming that Theorem 3 holds, Theorems 1 and 2 follow directly from the results we have proved so far : combining Propositions 1, 4, 5 and 7, one gets, for all large $k$, asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which are $\gamma$-generic for some constant $\gamma > 0$ ; Propositions 8 and 9 imply that these sections can be made $\bar\partial$-tame by perturbing them by $O(k^{-1/2})$ (which preserves the genericity properties if $k$ is large enough) ; and Theorem 3 implies that the corresponding projective maps are then approximately holomorphic singular branched coverings.

Let us now prove Theorem 4. We are given two sequences $s_{0,k}$ and $s_{1,k}$ of sections of $\mathbb{C}^3 \otimes L^k$ which are asymptotically holomorphic, $\gamma$-generic and $\bar\partial$-tame with respect to almost-complex structures $J_0$ and $J_1$, and want to show the existence of a one-parameter family of almost-complex structures $J_t$ interpolating between $J_0$ and $J_1$ and of generic and $\bar\partial$-tame asymptotically $J_t$-holomorphic sections interpolating between $s_{0,k}$ and $s_{1,k}$.

One starts by defining sections $s_{t,k}$ and compatible almost-complex structures $J_t$ interpolating between $(s_{0,k}, J_0)$ and $(s_{1,k}, J_1)$ in the following way : for $t \in [0, \frac{2}{7}]$, let $s_{t,k} = s_{0,k}$ and $J_t = J_0$ ; for $t \in [\frac{2}{7}, \frac{3}{7}]$, let $s_{t,k} = (3-7t)s_{0,k}$ and $J_t = J_0$ ; for $t \in [\frac{3}{7}, \frac{4}{7}]$, let $s_{t,k} = 0$ and take $J_t$ to be a path of $\omega$-compatible almost-complex structures from $J_0$ to $J_1$ (recall that the space of compatible almost-complex structures is connected) ; for $t \in [\frac{4}{7}, \frac{5}{7}]$, let $s_{t,k} = (7t-4)s_{1,k}$ and $J_t = J_1$ ; and for $t \in [\frac{5}{7}, 1]$, let $s_{t,k} = s_{1,k}$ and $J_t = J_1$. Clearly, $J_t$ and $s_{t,k}$ depend continuously on $t$, and the sections $s_{t,k}$ are asymptotically $J_t$-holomorphic for all $t \in [0,1]$.

Since $\gamma$-genericity is a local and $C^3$-open property, there exists $\alpha > 0$ such that any section differing from $s_{0,k}$ by less than $\alpha$ in $C^3$ norm is $\frac{\gamma}{2}$-generic, and similarly for $s_{1,k}$. Applying Propositions 1, 4, 5 and 7, we get for all large $k$ asymptotically $J_t$-holomorphic sections $\sigma_{t,k}$ which are $\eta$-generic for some $\eta > 0$, and such that $|\sigma_{t,k} - s_{t,k}|_{C^3, g_k} < \alpha$ for all $t \in [0,1]$.

We now set $s'_{t,k} = s_{0,k}$ for $t \in [0, \frac{1}{7}]$ ; $s'_{t,k} = (2-7t)s_{0,k} + (7t-1)\sigma_{\frac{2}{7},k}$ for $t \in [\frac{1}{7}, \frac{2}{7}]$ ; $s'_{t,k} = \sigma_{t,k}$ for $t \in [\frac{2}{7}, \frac{5}{7}]$ ; $s'_{t,k} = (7t-5)s_{1,k} + (6-7t)\sigma_{\frac{5}{7},k}$ for $t \in [\frac{5}{7}, \frac{6}{7}]$ ; and $s'_{t,k} = s_{1,k}$ for $t \in [\frac{6}{7}, 1]$. By construction, the sections $s'_{t,k}$ are asymptotically $J_t$-holomorphic for all $t \in [0,1]$ and depend continuously on $t$. Moreover, they are $\frac{\gamma}{2}$-generic for $t \in [0, \frac{2}{7}]$ because $s'_{t,k}$ then lies within $\alpha$ in $C^3$ norm of $s_{0,k}$, and similarly for $t \in [\frac{5}{7}, 1]$ because $s'_{t,k}$ then lies within $\alpha$ in $C^3$ norm of $s_{1,k}$. They are also $\eta$-generic for $t \in [\frac{2}{7}, \frac{5}{7}]$ because $s'_{t,k}$ is then equal to $\sigma_{t,k}$. Therefore the sections $s'_{t,k}$ are $\eta'$-generic for all $t \in [0,1]$, where $\eta' = \min(\eta, \frac{\gamma}{2})$.

Next, we apply Proposition 8 to the sections $s'_{t,k}$ : since $s'_{0,k} = s_{0,k}$ and $s'_{1,k} = s_{1,k}$ are already $\bar\partial$-tame, and since the families $s'_{t,k}$ and $J_t$ are constant over $[0, \frac{1}{7}]$ and $[\frac{6}{7}, 1]$, one can require of the sections $s''_{t,k}$ given by Proposition 8 that $s''_{0,k} = s'_{0,k} = s_{0,k}$ and $s''_{1,k} = s'_{1,k} = s_{1,k}$. Finally, we apply Proposition 9 to the sections $s''_{t,k}$ to obtain sections $\sigma''_{t,k}$ which simultaneously have genericity and $\bar\partial$-tameness properties. Since $s''_{0,k}$ and $s''_{1,k}$ are already $\bar\partial$-tame, one can require that $\sigma''_{0,k} = s''_{0,k} = s_{0,k}$ and $\sigma''_{1,k} = s''_{1,k} = s_{1,k}$. The sections $\sigma''_{t,k}$ interpolating between $s_{0,k}$ and $s_{1,k}$ therefore satisfy all the required properties, which ends the proof of Theorem 4.

## 5. Generic tame maps and branched coverings

**5.1. Structure near cusp points.** In order to prove Theorem 3, we need to check that, given any generic and $\bar\partial$-tame asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$, the corresponding maps $f_k = \mathbb{P}s_k : X \to \mathbb{CP}^2$ are, at any point of $X$, locally approximately holomorphically modelled on one of the three model maps of Definition 2. We start with the case of the neighborhood of a cusp point.

Let $x_0 \in X$ be a cusp point of $f_k$, i.e. an element of $\mathcal{C}_{\tilde J_k}(s_k)$, where $\tilde J_k$ is the almost-complex structure involved in the definition of $\bar\partial$-tameness. By definition, $\tilde J_k$ differs from $J$ by $O(k^{-1/2})$ and is integrable over a neighborhood of $x_0$, and $f_k$ is $\tilde J_k$-holomorphic over a neighborhood of $x_0$. Therefore, choose $\tilde J_k$-holomorphic local complex coordinates on $X$ near $x_0$, and local complex coordinates on $\mathbb{CP}^2$ near $f_k(x_0)$ : the map $h$ corresponding to $f_k$ in these coordinate charts is, locally, *holomorphic*. Because the coordinate map on $X$ is within $O(k^{-1/2})$ of being $J$-holomorphic, we can restrict ourselves to the study of the holomorphic map $h = (h_1, h_2)$ defined over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, which satisfies transversality properties following from the genericity of $s_k$. Our aim will be to show that, composing $h$ with holomorphic local diffeomorphisms of the source space $\mathbb{C}^2$ or of the target space $\mathbb{C}^2$, we can get $h$ to be of the form $(z_1, z_2) \mapsto (z_1^3 - z_1z_2, z_2)$ over a neighborhood of 0.

First, because $|\partial f_k|$ is bounded from below and $x_0$ is a cusp point, the derivative $\partial h(0)$ does not vanish and has rank one. Therefore, composing with a rotation of the target space $\mathbb{C}^2$ if necessary, we can assume that its image is directed along the second coordinate, i.e. $\mathrm{Im}\,(\partial h(0)) = \{0\} \times \mathbb{C}$.

Calling $Z_1$ and $Z_2$ the two coordinates on the target space $\mathbb{C}^2$, it follows immediately that the function $z_2 = h^*Z_2$ over the source space has a non-vanishing differential at 0, and can therefore be considered as a local coordinate function on the source space. Choose $z_1$ to be any linear function whose differential at the origin is linearly independent with $dz_2(0)$, so that $(z_1, z_2)$ define holomorphic local coordinates on a neighborhood of 0 in $\mathbb{C}^2$. In these coordinates, $h$ is of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$ where $h_1$ is a holomorphic function such that $h_1(0) = 0$ and $\partial h_1(0) = 0$.

Next, notice that, because $\mathrm{Jac}(f_k)$ vanishes transversely at $x_0$, the quantity $\mathrm{Jac}(h) = \det(\partial h) = \partial h_1/\partial z_1$ vanishes transversely at the origin, i.e.

$$\left(\frac{\partial^2 h_1}{\partial z_1^2}(0), \frac{\partial^2 h_1}{\partial z_1 \partial z_2}(0)\right) \neq (0,0).$$

Moreover, an argument similar to that of Section 3.2 shows that locally, because we have arranged for $|\partial h_2|$ to be bounded from below, the ratio between the quantities $\mathcal{T}(s_k)$ and $\hat{\mathcal{T}} = \partial h_2 \wedge \partial \mathrm{Jac}(h)$ is bounded from above and below. In particular, the fact that $x_0 \in \mathcal{C}_{\tilde J_k}(s_k)$ implies that the restriction of $\hat{\mathcal{T}}$ to the set of branch points vanishes transversely at the origin.

In our case, $\hat{\mathcal{T}} = dz_2 \wedge \partial(\frac{\partial h_1}{\partial z_1}) = -(\partial^2 h_1/\partial z_1^2)\, dz_1 \wedge dz_2$. Therefore, the vanishing of $\hat{\mathcal{T}}(0)$ implies that $\partial^2 h_1/\partial z_1^2\,(0) = 0$. It follows that $\partial^2 h_1/\partial z_1 \partial z_2\,(0)$ must be non-zero ; rescaling the coordinate $z_1$ by a constant factor if necessary, this derivative can be assumed to be equal to $-1$. Therefore, the map $h$ can be written as

$$\begin{aligned} h(z_1, z_2) &= (-z_1 z_2 + \lambda z_2^2 + O(|z|^3), z_2) \\ &= (-z_1 z_2 + \lambda z_2^2 + \alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2 + \delta z_2^3 + O(|z|^4), z_2) \end{aligned}$$

where $\lambda$, $\alpha$, $\beta$, $\gamma$ and $\delta$ are complex coefficients.

We now consider the following coordinate changes : on the target space $\mathbb{C}^2$, define $\psi(Z_1, Z_2) = (Z_1 - \lambda Z_2^2 - \delta Z_2^3, Z_2)$, and on the source space $\mathbb{C}^2$, define $\phi(z_1, z_2) = (z_1 + \beta z_1^2 + \gamma z_1 z_2, z_2)$. Clearly, these two maps are local diffeomorphisms near the origin. Therefore, one can replace $h$ by $\psi \circ h \circ \phi$, which has the effect of killing most terms of the above expansion : this allows us to consider that $h$ is of the form

$$h(z_1, z_2) = (-z_1 z_2 + \alpha z_1^3 + O(|z|^4), z_2).$$

Next, recall that the set of branch points is, in our local setting, the set of points where $\mathrm{Jac}(h) = \partial h_1/\partial z_1 = -z_2 + 3\alpha z_1^2 + O(|z|^3)$ vanishes. Therefore, the tangent direction to the set of branch points at the origin is the $z_1$ axis, and the transverse vanishing of $\hat{\mathcal{T}}$ at the origin implies that $\frac{\partial}{\partial z_1}\hat{\mathcal{T}}(0) \neq 0$. Using the above formula for $\hat{\mathcal{T}}$, we conclude that $\partial^3 h_1/\partial z_1^3 \neq 0$, i.e. $\alpha \neq 0$.

Rescaling the two coordinates $z_1$ and $Z_1$ by a constant factor, we can assume that $\alpha$ is equal to 1. Therefore, we have used all the transversality properties of $h$ to show that, on a neighborhood of $x_0$, it is of the form

$$h(z_1, z_2) = (-z_1 z_2 + z_1^3 + O(|z|^4), z_2).$$

The uniform bounds and transversality estimates on $s_k$ can be used to show that all the rescalings and transformations we have used are "nice", i.e. they have bounded derivatives and their inverses have bounded derivatives.

Our next task is to show that further coordinate changes can kill the higher order terms still present in the expression of $h$. For this, we first prove the following lemma :

LEMMA 9. *Let $\mathcal{D}$ be the space of holomorphic local diffeomorphisms of $\mathbb{C}^2$ near the origin, and let $\mathcal{H}$ be the space of holomorphic maps from a neighborhood of $0$ in $\mathbb{C}^2$ to a neighborhood of $0$ in $\mathbb{C}^2$. Let $h_0 \in \mathcal{H}$ be the map $(x, y) \mapsto (x^3 - xy, y)$. Then the differential at the point $(\mathrm{Id}, \mathrm{Id})$ of the map $\mathcal{F} : \mathcal{D} \times \mathcal{D} \to \mathcal{H}$ defined by $\mathcal{F}(\Phi, \Psi) = \Psi \circ h_0 \circ \Phi$ is surjective.*

PROOF. Let $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ be two tangent vectors to $\mathcal{D}$ at Id (i.e. holomorphic functions over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$). The differential of $\mathcal{F}$ at $(\mathrm{Id}, \mathrm{Id})$ is given by

$$\begin{aligned} D\mathcal{F}_{(\mathrm{Id},\mathrm{Id})}(\phi, \psi)(x, y) &= \frac{d}{dt}_{|t=0}\Big[(\mathrm{Id} + t\psi) \circ h_0 \circ (\mathrm{Id} + t\phi)(x, y)\Big] \\ &= \Big(\psi_1(x^3 - xy, y) + (3x^2 - y)\,\phi_1(x, y) - x\,\phi_2(x, y), \psi_2(x^3 - xy, y) + \phi_2(x, y)\Big). \end{aligned}$$

Proving the surjectivity of $D\mathcal{F}$ at $(\mathrm{Id}, \mathrm{Id})$ is equivalent to checking that, given any tangent vector $(\epsilon_1, \epsilon_2) \in T_{h_0}\mathcal{H}$ (i.e. a holomorphic function over a neighborhood of

0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$), there exist $\phi$ and $\psi$ such that $D\mathcal{F}_{(\mathrm{Id},\mathrm{Id})}(\phi,\psi)(x,y) = (\epsilon_1(x,y),\epsilon_2(x,y))$. Projecting this equality on the second factor, one gets

$$\psi_2(x^3-xy,y)+\phi_2(x,y)=\epsilon_2(x,y),$$

which implies that $\phi_2(x,y)=\epsilon_2(x,y)-\psi_2(x^3-xy,y)$. Replacing $\phi_2$ by its expression in the first component, and setting $\epsilon(x,y)=\epsilon_1(x,y)+x\,\epsilon_2(x,y)$, the equation which we need to solve finally rewrites as

$$\psi_1(x^3-xy,y)+x\,\psi_2(x^3-xy,y)+(3x^2-y)\,\phi_1(x,y)=\epsilon(x,y),$$

where the parameter $\epsilon$ can be any holomorphic function, and $\psi_1$, $\psi_2$ and $\phi_1$ are the unknown quantities.

Solving this equation is *a priori* difficult, so in order to get an idea of the general solution it is best to first work in the ring of formal power series in the two variables $x$ and $y$. Since the equation is linear, it is sufficient to find a solution when $\epsilon$ is a monomial of the form $\epsilon(x,y)=x^py^q$ with $(p,q)\in\mathbb{N}^2$.

First note that, for $\epsilon(x,y)=y^q$ (i.e. when $p=0$), a trivial solution is given by $\psi_1(x^3-xy,y)=y^q$, $\psi_2=0$ and $\phi_1=0$. Next, remark that, if there exists a solution for a given $\epsilon(x,y)$, then there also exists a solution for $x\,\epsilon(x,y)$ : indeed, if $\psi_1(x^3-xy,y)+x\,\psi_2(x^3-xy,y)+(3x^2-y)\,\phi_1(x,y)=\epsilon(x,y)$, then setting $\tilde\psi_1=\frac{1}{3}y\,\psi_2$, $\tilde\psi_2=\psi_1$ and $\tilde\phi_1(x,y)=x\,\phi_1(x,y)+\frac{1}{3}\psi_2(x^3-xy,y)$ one gets

$$\tilde\psi_1(x^3-xy,y)+x\,\tilde\psi_2(x^3-xy,y)+(3x^2-y)\,\tilde\phi_1(x,y)=x\,\epsilon(x,y).$$

Therefore, by induction on $p$, the equation has a solution for all monomials $x^py^q$, and by linearity there exists a formal solution for all power series $\epsilon(x,y)$. A short calculation gives the following explicit solution of the equation for $\epsilon(x,y)=x^py^q$ : if $p=2k$ is even,

$$\psi_1(x^3-xy,y)=3^{-k}y^{k+q},\quad \psi_2=0,\quad \phi_1(x,y)=\sum_{j=0}^{k-1}3^{-(j+1)}y^{j+q}x^{2k-2-2j},$$

and if $p=2k+1$ is odd,

$$\psi_1=0,\quad \psi_2(x^3-xy,y)=3^{-k}y^{k+q},\quad \phi_1(x,y)=\sum_{j=0}^{k-1}3^{-(j+1)}y^{j+q}x^{2k-1-2j}.$$

In particular, $\psi_1$ and $\psi_2$ actually only depend on the second variable $y$.

The above formulae make it possible to compute a general solution for any holomorphic $\epsilon$, given by the following expressions, where $\gamma_+$ and $\gamma_-$ are by definition the two square roots of $\frac{1}{3}y$ (exchanging $\gamma_+$ and $\gamma_-$ clearly does not affect the result) :

$$\begin{aligned}\psi_1(x^3-xy,y)&=\tfrac{1}{2}\big(\epsilon(\gamma_+,y)+\epsilon(\gamma_-,y)\big),\\ \psi_2(x^3-xy,y)&=\tfrac{1}{2\gamma_+}\big(\epsilon(\gamma_+,y)-\epsilon(\gamma_-,y)\big),\\ \phi_1(x,y)&=\tfrac{1}{6\gamma_+}\left[\frac{\epsilon(x,y)-\epsilon(\gamma_+,y)}{x-\gamma_+}-\frac{\epsilon(x,y)-\epsilon(\gamma_-,y)}{x-\gamma_-}\right].\end{aligned}$$

Note that these functions are actually smooth, although they depend on $\gamma_\pm$ which are not smooth functions of $y$, because the odd powers of $\gamma_\pm$ cancel each other in the expressions. Similarly, one easily checks that, when $y\to 0$ or $x\to\gamma_\pm$, the vanishing of a term in the formula for $\phi_1$ always makes up for the singularity of the denominator, so that $\phi_1$ is actually well-defined everywhere. Another way to see these smoothness properties is to observe that, because these formulae are simply a rewriting of the formal solution computed previously for power series, the functions they define admit power series expansions at the origin. Lemma 9 is therefore proved. □

Lemma 9 implies the desired result. Indeed, endow the space of holomorphic maps from a neighborhood $D$ of 0 in $\mathbb{C}^2$ to $\mathbb{C}^2$ with a structure of Hilbert space given by a suitable Sobolev norm, e.g. the $L^2_4$ norm which is stronger than the $C^1$ norm : then, since the differential at $(\mathrm{Id},\mathrm{Id})$ of $\mathcal{F}$ is a surjective continuous linear map, the submersion theorem for Hilbert spaces implies the existence of a constant $\alpha > 0$ with the property that, given any holomorphic function $\epsilon$ such that $|\epsilon|_{L^2_4(D)} < \alpha$, there exist holomorphic local diffeomorphisms $\Phi$ and $\Psi$ of $\mathbb{C}^2$ near 0, $L^2_4$-close to the identity, such that $\Psi \circ h_0 \circ \Phi = h_0 + \epsilon$.

Recall that we are trying to remove the higher order terms from $h(z_1, z_2) = (z_1^3 - z_1 z_2 + \theta(z_1, z_2), z_2)$, where $\theta(z_1, z_2) = O(|z|^4)$. There is no reason for the $L^2_4$ norm of $\theta$ to be smaller than $\alpha$ over the fixed domain $D$. However the required bound can be achieved by rescaling all the coordinates : let $\lambda$ be a small positive constant, and consider the diffeomorphisms $\Phi_\lambda : (z_1, z_2) \mapsto (\lambda z_1, \lambda^2 z_2)$ of the source space and $\Psi_\lambda : (Z_1, Z_2) \mapsto (\lambda^{-3} Z_1, \lambda^{-2} Z_2)$ of the target space. Then we have $\Psi_\lambda \circ h_0 \circ \Phi_\lambda = h_0$, and $\Psi_\lambda \circ h \circ \Phi_\lambda(z_1, z_2) = (z_1^3 - z_1 z_2 + \tilde{\theta}_\lambda(z_1, z_2), z_2)$ where $\tilde{\theta}_\lambda(z_1, z_2) = \lambda^{-3}\theta(\lambda z_1, \lambda^2 z_2)$.

Let $R$ be a constant such that $D \subset B(0, R)$, and let $\delta > 0$ be a constant such that $\delta^2(1 + R^2 + R^4 + R^6 + R^8)\,\mathrm{vol}(D) < \alpha^2$. It follows from the bound $|\nabla^4 \tilde{\theta}_\lambda(z_1, z_2)| \le \lambda\, |\nabla^4 \theta(\lambda z_1, \lambda^2 z_2)|$ that, if $\lambda$ is small enough, the fourth derivative of $\tilde{\theta}_\lambda$ remains smaller than $\delta$ over $D$. Since $\tilde{\theta}_\lambda$ and its first three derivatives vanish at the origin, by integrating the bound $|\nabla^4 \tilde{\theta}_\lambda| < \delta$ one gets that $|\tilde{\theta}_\lambda|_{L^2_4(D)} < \alpha$. Therefore, if $\lambda$ is small enough there exist local diffeomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ such that $\tilde{\Psi} \circ h_0 \circ \tilde{\Phi} = \Psi_\lambda \circ h \circ \Phi_\lambda$ over the domain $D$. Equivalently, setting $\Psi = \Psi_\lambda^{-1} \circ \tilde{\Psi} \circ \Psi_\lambda$ and $\Phi = \Phi_\lambda \circ \tilde{\Phi} \circ \Phi_\lambda^{-1}$, we have $\Psi \circ h_0 \circ \Phi = h$ over a small neighborhood of 0 in $\mathbb{C}^2$, which is what we wanted to prove.

Moreover, because of the uniform transversality estimates and bounds on the derivatives of $s_k$, the derivatives of $h$ are uniformly bounded. Therefore one can choose the constant $\lambda$ to be independent of $k$ and of the given point $x_0 \in \mathcal{C}_{\tilde{J}_k}(s_k)$ : it follows that the neighborhood of $x_0$ over which the map $f_k$ has been shown to be $O(k^{-1/2})$-approximately holomorphically modelled on the map $h_0$ can be assumed to contain a ball of fixed radius (depending on the bounds and transversality estimates, but independent of $x_0$ and $k$).

**5.2. Structure near generic branch points.** We now consider a branch point $x_0 \in R_{\tilde{J}_k}(s_k)$, which we assume to be at distance more than a fixed constant $\delta$ from the set of cusp points $\mathcal{C}_{\tilde{J}_k}(s_k)$. We want to show that, over a neighborhood of $x_0$, $f_k = \mathbb{P}s_k$ is approximately holomorphically modelled on the map $(z_1, z_2) \mapsto (z_1^2, z_2)$.

From now on, we implicitly use the almost-complex structure $\tilde{J}_k$ and write $R$ for the intersection of $R_{\tilde{J}_k}(s_k)$ with the ball $B_{g_k}(x_0, \frac{\delta}{2})$. First note that, since $R$ remains at distance more than $\frac{\delta}{2}$ from the cusp points, the tangent space to $R$ remains everywhere away from the kernel of $\partial f_k$. Therefore, the restriction of $f_k$ to $R$ is a local diffeomorphism over a neighborhood of $x_0$, and so $f_k(R)$ is locally a smooth approximately holomorphic submanifold in $\mathbb{CP}^2$. It follows that there exist approximately holomorphic coordinates $(Z_1, Z_2)$ on a neighborhood of $f_k(x_0)$ in $\mathbb{CP}^2$ such that $f_k(R)$ is locally defined by the equation $Z_1 = 0$.

Define the approximately holomorphic function $z_2 = f_k^* Z_2$ over a neighborhood of $x_0$, and notice that its differential $dz_2 = dZ_2 \circ df_k$ does not vanish, because by construction $Z_2$ is a coordinate on $f_k(R)$. Therefore, $z_2$ can be considered as a local complex coordinate function on a neighborhood of $x_0$. In particular, the level sets of $z_2$ are smooth and intersect $R$ transversely at a single point.

Take $z_1$ to be an approximately holomorphic function on a neighborhood of $x_0$ which vanishes at $x_0$ and whose differential at $x_0$ is linearly independent with that of $z_2$ (e.g. take the two differentials to be mutually orthogonal), so that $(z_1, z_2)$ define approximately holomorphic coordinates on a neighborhood of $x_0$. From now on we use the local coordinates $(z_1, z_2)$ on $X$ and $(Z_1, Z_2)$ on $\mathbb{CP}^2$.

Because $dz_{2|TR}$ remains away from 0, $R$ has locally an equation of the form $z_1 = \rho(z_2)$ for some approximately holomorphic function $\rho$ (satisfying $\rho(0) = 0$ since $x_0 \in R$). Therefore, shifting the coordinates on $X$ in order to replace $z_1$ by $z_1 - \rho(z_2)$, one can assume that $z_1 = 0$ is a local equation of $R$. In the chosen local coordinates, $f_k$ is therefore modelled on an approximately holomorphic map $h$ from a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$, with the following properties.

First, because $R = \{z_1 = 0\}$ is mapped to $f_k(R) = \{Z_1 = 0\}$, we have $h_1(0, z_2) = 0$ for all $z_2$. Next, recall that the differential of $f_k$ has real rank 2 at any point of $R$ (because $\partial f_k$ has complex rank 1 and $\bar{\partial} f_k$ vanishes over the kernel of $\partial f_k$), so its image is exactly the tangent space to $f_k(R)$. It follows that $\nabla h_1 = 0$ at every point $(0, z_2) \in R$.

Finally, because the chosen coordinates are approximately holomorphic the quantity $\mathrm{Jac}(f_k)$ is within $O(k^{-1/2})$ of $\det(\partial h) = (\partial h_1/\partial z_1)\, \partial z_1 \wedge \partial z_2$. Therefore, the transversality to 0 of $\mathrm{Jac}(f_k)$ implies that $(\partial^2 h_1/\partial z_1^2, \partial^2 h_1/\partial z_1 \partial z_2)$ has a norm which remains larger than a fixed constant along $R$. However $\partial^2 h_1/\partial z_1 \partial z_2$ vanishes at any point of $R$ because $\partial h_1/\partial z_1\, (0, z_2) = 0$ for all $z_2$. Therefore the quantity $\partial^2 h_1/\partial z_1^2$ remains bounded away from 0 on $R$.

The above properties imply that $h$ can be written as

$$h(z_1, z_2) = \big(\alpha(z_2) z_1^2 + \beta(z_2) z_1 \bar{z}_1 + \gamma(z_2) \bar{z}_1^2 + \epsilon(z_1, z_2), z_2\big)\,,$$

where $\alpha$ is approximately holomorphic and bounded away from 0, while $\beta$ and $\gamma$ are $O(k^{-1/2})$ (because of asymptotic holomorphicity), and $\epsilon(z_1, z_2) = O(|z_1|^3)$ is approximately holomorphic. Moreover, composing with the coordinate change $(Z_1, Z_2) \mapsto (\alpha(Z_2)^{-1} Z_1, Z_2)$ (which is approximately holomorphic and has bounded derivatives because $\alpha$ is bounded away from 0), one reduces to the case where $\alpha$ is identically equal to 1.

We now want to reduce further the problem by removing the $\beta$ and $\gamma$ terms in the above expression : for this, we first remark that, given any small enough complex numbers $\beta$ and $\gamma$, there exists a complex number $\lambda$, of norm less than $|\beta| + |\gamma|$ and depending smoothly on $\beta$ and $\gamma$, such that

$$\lambda = -\gamma\bar{\lambda} + \frac{\beta}{2}(1 + |\lambda|^2).$$

Indeed, if $|\beta| + |\gamma| < \frac{1}{2}$ the right hand side of this equation is a contracting map of the unit disc to itself, so the existence of a solution $\lambda$ in the unit disc follows immediately from the fixed point theorem. Furthermore, using the bound $|\lambda| < 1$ in the right hand side, one gets that $|\lambda| < |\beta| + |\gamma|$. Finally, the smooth dependence of $\lambda$ upon $\beta$ and $\gamma$ follows from the implicit function theorem.

Assuming again that $|\beta| + |\gamma| < \frac{1}{2}$ and defining $\lambda$ as above, let

$$A = \frac{1 - \bar{\lambda}^2\gamma}{1 - |\lambda|^4} \quad \text{and} \quad B = \frac{\gamma - \lambda^2}{1 - |\lambda|^4}.$$

The complex numbers $A$ and $B$ are also smooth functions of $\beta$ and $\gamma$, and it is clear that $|A - 1| = O(|\beta| + |\gamma|)$ and $|B| = O(|\beta| + |\gamma|)$. Moreover, one easily checks that, in the ring of polynomials in $z$ and $\bar{z}$,

$$A(z + \lambda\bar{z})^2 + B(\bar{z} + \bar{\lambda}z)^2 = z^2 + 2\frac{\lambda + \gamma\bar{\lambda}}{1 + |\lambda|^2}z\bar{z} + \gamma\bar{z}^2 = z^2 + \beta z\bar{z} + \gamma\bar{z}^2.$$

Therefore, if one assumes $k$ to be large enough, recalling that the quantities $\beta(z_2)$ and $\gamma(z_2)$ which appear in the above expression of $h$ are bounded by $O(k^{-1/2})$, there exist $\lambda(z_2)$, $A(z_2)$ and $B(z_2)$, depending smoothly on $z_2$, such that $|A(z_2) - 1| = O(k^{-1/2})$, $|B(z_2)| = O(k^{-1/2})$, $|\lambda(z_2)| = O(k^{-1/2})$ and

$$A(z_2)(z_1 + \lambda(z_2)\bar{z}_1)^2 + B(z_2)(\overline{z_1 + \lambda(z_2)\bar{z}_1})^2 = z_1^2 + \beta(z_2)z_1\bar{z}_1 + \gamma(z_2)\bar{z}_1^2.$$

So, let $h_0$ be the map $(z_1, z_2) \mapsto (z_1^2, z_2)$, and let $\Phi$ and $\Psi$ be the two approximately holomorphic local diffeomorphisms of $\mathbb{C}^2$ defined by $\Phi(z_1, z_2) = (z_1 + \lambda(z_2)\bar{z}_1, z_2)$ and $\Psi(Z_1, Z_2) = (A(Z_2)Z_1 + B(Z_2)\bar{Z}_1, Z_2)$ : then

$$h(z_1, z_2) = \Psi \circ h_0 \circ \Phi(z_1, z_2) + (\epsilon(z_1, z_2), 0).$$

It follows immediately that $\Psi^{-1} \circ h \circ \Phi^{-1}(z_1, z_2) = (z_1^2 + O(|z_1|^3), z_2)$. Therefore, this new coordinate change allows us to consider only the case where $h$ is of the form $(z_1, z_2) \mapsto (z_1^2 + \tilde{\epsilon}(z_1, z_2), z_2)$, where $\tilde{\epsilon}(z_1, z_2) = O(|z_1|^3)$.

Because $\tilde{\epsilon}(z_1, z_2) = O(|z_1|^3)$, the bound $|\tilde{\epsilon}(z_1, z_2)| < \frac{1}{2}|z_1|^2$ holds over a neighborhood of the origin whose size can be bounded from below independently of $k$ and $x_0$ by using the uniform estimates on all derivatives. Over this neighborhood, define

$$\phi(z_1, z_2) = z_1\sqrt{1 + \frac{\tilde{\epsilon}(z_1, z_2)}{z_1^2}}$$

for $z_1 \neq 0$, where the square root is determined without ambiguity by the condition that $\sqrt{1} = 1$. Setting $\phi(0, z_2) = 0$, it follows from the bound $|\phi(z_1, z_2) - z_1| = O(|z_1|^2)$ that the function $\phi$ is $C^1$. In general $\phi$ is not $C^2$, because $\tilde{\epsilon}$ may contain terms involving $\bar{z}_1^2 z_1$ or $\bar{z}_1^3$.

Because $\phi(z_1, z_2) = z_1 + O(|z_1|^2)$, the map $\Theta : (z_1, z_2) \mapsto (\phi(z_1, z_2), z_2)$ is a $C^1$ local diffeomorphism of $\mathbb{C}^2$ over a neighborhood of the origin. As previously, the uniform bounds on all derivatives imply that the size of this neighborhood can be bounded from below independently of $k$ and $x_0$. Moreover, it follows from the asymptotic holomorphicity of $s_k$ that $\tilde{\epsilon}$ has antiholomorphic derivatives bounded by $O(k^{-1/2})$, and so $|\bar{\partial}\phi| = O(k^{-1/2})$. Therefore $\Theta$ is $O(k^{-1/2})$-approximately holomorphic, and we have

$$h_0 \circ \Theta(z_1, z_2) = h(z_1, z_2),$$

which finally gives the desired result.

**5.3. Proof of Theorem 3.** Theorem 3 follows readily from the above arguments : indeed, consider $\gamma$-generic and $\bar{\partial}$-tame asymptotically holomorphic sections $s_k$ of $\mathbb{C}^3 \otimes L^k$, and let $\tilde{J}_k$ be the almost-complex structures involved in the definition of $\bar{\partial}$-tameness. We need to show that, at any point $x \in X$, the maps $f_k = \mathbb{P}s_k$ are approximately holomorphically modelled on one of the three maps of Definition 2.

First consider the case where $x$ lies close to a point $y \in \mathcal{C}_{\tilde{J}_k}(s_k)$. The argument of Section 5.1 implies the existence of a constant $\delta > 0$ independent of $k$ and $y$ such that, over the ball $B_{g_k}(y, 2\delta)$, the map $f_k$ is $\tilde{J}_k$-holomorphically modelled on the cusp covering map $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$. If $x$ lies within distance $\delta$ of $y$, $B_{g_k}(y, 2\delta)$ is a neighborhood of $x$ ; therefore the expected result follows at every point within distance $\delta$ of $\mathcal{C}_{\tilde{J}_k}(s_k)$ from the observation that, because $|\tilde{J}_k - J| = O(k^{-1/2})$, the relevant coordinate chart on $X$ is $O(k^{-1/2})$-approximately $J$-holomorphic.

Next, consider the case where $x$ lies close to a point $y$ of $R_{\tilde{J}_k}(s_k)$ which is itself at distance more than $\delta$ from $\mathcal{C}_{\tilde{J}_k}(s_k)$. The argument of Section 5.2 then implies the existence of a constant $\delta' > 0$ independent of $k$ and $y$ such that, over the ball $B_{g_k}(y, 2\delta')$, the map $f_k$ is, in $O(k^{-1/2})$-approximately holomorphic $C^1$ coordinate charts, locally modelled on the branched covering map $(z_1, z_2) \mapsto (z_1^2, z_2)$. Therefore, if one assumes the distance between $x$ and $y$ to be less than $\delta'$, the given ball is a neighborhood of $x$, and the expected result follows.

So we are left only with the case where $x$ is at distance more than $\delta'$ from $R_{\tilde{J}_k}(s_k)$. Assuming $k$ to be large enough, it then follows from the bound $|\tilde{J}_k - J| = O(k^{-1/2})$ that $x$ is at distance more than $\frac{1}{2}\delta'$ from $R_J(s_k)$. Therefore, the $\gamma$-transversality to 0 of $\mathrm{Jac}(f_k)$ implies that $|\mathrm{Jac}(f_k)(x)|$ is larger than $\alpha = \min(\frac{1}{2}\delta'\gamma, \gamma)$ (otherwise, the downward gradient flow of $|\mathrm{Jac}(f_k)|$ would reach a point of $R_J(s_k)$ at distance less than $\frac{1}{2}\delta'$ from $x$).

Recalling that $|\bar{\partial} f_k| = O(k^{-1/2})$, one gets that $f_k$ is a $O(k^{-1/2})$-approximately holomorphic local diffeomorphism over a neighborhood of $x$. Therefore, choose holomorphic complex coordinates on $\mathbb{CP}^2$ near $f_k(x)$ and pull them back by $f_k$ to obtain $O(k^{-1/2})$-approximately holomorphic local coordinates over a neighborhood of $x$ : in these coordinates, the map $f_k$ becomes the identity map, which ends the proof of Theorem 3.

## 6. Further remarks

**6.1. Branched coverings of $\mathbb{CP}^2$.** A natural question to ask about the results obtained in this paper is whether the property of being a (singular) branched covering of $\mathbb{CP}^2$, i.e. the existence of a map to $\mathbb{CP}^2$ which is locally modelled at every point on one of the three maps of Definition 2, strongly restricts the topology of a general compact 4-manifold. Since the notion of approximately holomorphic coordinate chart on $X$ no longer has a meaning in this case, we relax Definition 2 by only requiring the existence of a local identification of the covering map with one of the model maps in a smooth local coordinate chart on $X$. However we keep requiring that the corresponding local coordinate chart on $\mathbb{CP}^2$ be approximately holomorphic, so that the branch locus in $\mathbb{CP}^2$ remains an immersed symplectic curve with cusps. Call such a map a *topological singular branched covering of* $\mathbb{CP}^2$. Then the following holds :

PROPOSITION 10. *Let $X$ be a compact 4-manifold and consider a topological singular covering $f : X \to \mathbb{CP}^2$ branched along a submanifold $R \subset X$. Then $X$ carries a symplectic structure arbitrarily close to $f^*\omega_0$, where $\omega_0$ is the standard symplectic structure of* $\mathbb{CP}^2$.

PROOF. The closed 2-form $f^*\omega_0$ on $X$ defines a symplectic structure on $X - R$ which degenerates along $R$. Therefore, one needs to perturb it by adding a small multiple of a closed 2-form with support in a neighborhood of $R$ in order to make it nondegenerate. This perturbation can be constructed as follows.

Call $C$ the set of cusp points, i.e. the points of $R$ where the tangent space to $R$ lies in the kernel of the differential of $f$, or equivalently the points around which $f$ is modelled on the map $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$. Consider a point $x \in C$, and work in local coordinates such that $f$ identifies with the model map. In these coordinates, a local equation of $R$ is $z_2 = 3z_1^2$, and the kernel $K$ of the differential of $f$ coincides at every point of $R$ with the subspace $\mathbb{C} \times \{0\}$ of the tangent space ; this complex identification determines a natural orientation of $K$. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0, 2\rho_x) \times B_{\mathbb{C}}(0, 2\rho_x^2)$ is contained in the local coordinate patch, and choose cut-off functions $\chi_1$ and $\chi_2$ over $\mathbb{C}$ in such a way that $\chi_1$ equals 1 over $B_{\mathbb{C}}(0, \rho_x)$ and vanishes outside of $B_{\mathbb{C}}(0, 2\rho_x)$, and that $\chi_2$ equals 1 over $B_{\mathbb{C}}(0, \rho_x^2)$ and vanishes outside of $B_{\mathbb{C}}(0, 2\rho_x^2)$. Then, let $\psi_x$ be the 2-form which equals $d(\chi_1(z_1)\, \chi_2(z_2)\, x_1\, dy_1)$ over the local coordinate patch, where $x_1$ and $y_1$ are the real and imaginary parts of $z_1$, and which vanishes over the remainder of $X$ : the 2-form $\psi_x$ coincides with $dx_1 \wedge dy_1$ over a neighborhood of $x$. More importantly, it follows from the choice of the cut-off functions that the restriction of $\psi_x$ to $K = \mathbb{C} \times \{0\}$ is non-negative at every point of $R$, and positive non-degenerate at every point of $R$ which lies sufficiently close to $x$.

Similarly, consider a point $x \in R$ away from $C$ and local coordinates such that $f$ identifies with the model map $(z_1, z_2) \mapsto (z_1^2, z_2)$. In these coordinates, $R$ identifies with $\{0\} \times \mathbb{C}$, and the kernel $K$ of the differential of $f$ coincides at every point of $R$ with the subspace $\mathbb{C} \times \{0\}$ of the tangent space. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0, 2\rho_x) \times B_{\mathbb{C}}(0, 2\rho_x)$ is contained in the local coordinate patch, and choose a cut-off function $\chi$ over $\mathbb{C}$ which equals 1 over $B_{\mathbb{C}}(0, \rho_x)$ and 0 outside of $B_{\mathbb{C}}(0, 2\rho_x)$. Then, let $\psi_x$ be the 2-form which equals $d(\chi(z_1)\, \chi(z_2)\, x_1\, dy_1)$ over the local coordinate patch, where $x_1$ and $y_1$ are the real and imaginary parts of $z_1$, and which vanishes over the remainder of $X$ : as previously, the restriction of $\psi_x$ to

$K = \mathbb{C} \times \{0\}$ is non-negative at every point of $R$, and positive non-degenerate at every point of $R$ which lies sufficiently close to $x$.

Choose a finite collection of points $x_i$ of $R$ (including all the cusp points) in such a way that the neighborhoods of $x_i$ over which the 2-forms $\psi_{x_i}$ restrict positively to $K$ cover all of $R$, and define $\alpha$ as the sum of all the 2-forms $\psi_{x_i}$. Then it follows from the above definitions that the 2-form $\alpha$ is exact, and that at any point of $R$ its restriction to the kernel of the differential of $f$ is positive and non-degenerate. Therefore, the 4-form $f^*\omega_0 \wedge \alpha$ is a positive volume form at every point of $R$.

Now choose any metric on a neighborhood of $R$, and let $d_R$ be the distance function to $R$. It follows from the compactness of $X$ and $R$ and from the general properties of the map $f$ that, using the orientation induced by $f$ and the chosen metric to implicitly identify 4-forms with functions, there exist positive constants $K$, $C$, $C'$ and $M$ such that the following bounds hold over a neighborhood of $R$ : $f^*\omega_0 \wedge f^*\omega_0 \geq K d_R$, $f^*\omega_0 \wedge \alpha \geq C - C' d_R$, and $|\alpha \wedge \alpha| \leq M$. Therefore, for all $\epsilon > 0$ one gets over a neighborhood of $R$ the bound

$$(f^*\omega_0 + \epsilon\,\alpha) \wedge (f^*\omega_0 + \epsilon\,\alpha) \geq (2\epsilon\,C - \epsilon^2 M) + (K - 2\epsilon\,C')d_R.$$

If $\epsilon$ is chosen sufficiently small, the coefficients $2\epsilon\,C - \epsilon^2 M$ and $K - 2\epsilon\,C'$ are both positive, which implies that the closed 2-form $f^*\omega_0 + \epsilon\,\alpha$ is everywhere nondegenerate, and therefore symplectic. □

Another interesting point is the compatibility of our approximately holomorphic singular branched coverings with respect to the symplectic structures $\omega$ on $X$ and $\omega_0$ in $\mathbb{CP}^2$ (as opposed to the compatibility with the almost-complex structures, which has been a major preoccupation throughout the previous sections).

It is easy to check that given a covering map $f : X \to \mathbb{CP}^2$ defined by a section of $\mathbb{C}^3 \otimes L^k$, the number of preimages of a generic point is equal to $\frac{1}{4\pi^2}k^2(\omega^2.[X])$, while the homology class of the preimage of a generic line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is Poincaré dual to $\frac{1}{2\pi}k[\omega]$. If we normalize the standard symplectic structure $\omega_0$ on $\mathbb{CP}^2$ in such a way that the symplectic area of a line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is equal to $2\pi$, it follows that the cohomology class of $f^*\omega_0$ is $[f^*\omega_0] = k[\omega]$.

As we have said above, the pull-back $f^*\omega_0$ of the standard symplectic form of $\mathbb{CP}^2$ by the covering map degenerates along the set of branch points, so there is no chance of $(X, f^*\omega_0)$ being symplectic and symplectomorphic to $(X, k\omega)$. However, one can prove the following result which is nearly as good :

PROPOSITION 11. *The 2-forms $\tilde{\omega}_t = t f^*\omega_0 + (1-t)k\omega$ on $X$ are symplectic for all $t \in [0, 1)$. Moreover, for $t \in [0, 1)$ the manifolds $(X, \tilde{\omega}_t)$ are all symplectomorphic to $(X, k\omega)$.*

This means that $f^*\omega_0$ is, in some sense, a degenerate limit of the symplectic structure defined by $k\omega$ : therefore the covering map $f$ behaves quite reasonably with respect to the symplectic structures.

PROOF. The 2-forms $\tilde{\omega}_t$ are all closed and lie in the same cohomology class. We have to show that they are non-degenerate for $t < 1$. For this, let $x$ be any point of $X$ and let $v$ be a nonzero tangent vector at $x$. It is sufficient to prove that there exists a vector $w \in T_xX$ such that $\omega(v,w) > 0$ and $f^*\omega_0(v,w) \geq 0$ : then $\tilde{\omega}_t(v,w) > 0$ for all $t < 1$, which implies the non-degeneracy of $\tilde{\omega}_t$.

Recall that, by definition, there exist local approximately holomorphic coordinate maps $\phi$ over a neighborhood of $x$ and $\psi$ over a neighborhood of $f(x)$ such that locally $f = \psi^{-1} \circ g \circ \phi$ where $g$ is a holomorphic map from a subset of $\mathbb{C}^2$ to $\mathbb{C}^2$. Define $w = \phi_*^{-1}\mathbb{J}_0\phi_*v$, where $\mathbb{J}_0$ is the standard complex structure on $\mathbb{C}^2$ : then we have $w = (\phi^*\mathbb{J}_0)v$ and, because $g$ is holomorphic, $f_*w = (\psi^*\mathbb{J}_0)f_*v$.

Because the coordinate maps are $O(k^{-1/2})$-approximately holomorphic, we have $|w - Jv| \leq Ck^{-1/2}|v|$ and $|f_*w - J_0f_*v| \leq Ck^{-1/2}|f_*v|$, where $C$ is a constant and $J_0$ is the standard complex structure on $\mathbb{CP}^2$. It follows that $\omega(v,w) \geq |v|^2 - Ck^{-1/2}|v|^2 > 0$, and that $\omega_0(f_*v, f_*w) \geq |f_*v|^2 - Ck^{-1/2}|f_*v|^2 \geq 0$. Therefore, $\tilde{\omega}_t(v,w) > 0$ for all $t \in [0,1)$ ; since the existence of such a $w$ holds for every nonzero vector $v$, this proves that the closed 2-forms $\tilde{\omega}_t$ are non-degenerate, and therefore symplectic.

Moreover, these symplectic forms all lie in the cohomology class $[k\omega]$, so it follows from Moser's stability theorem that the symplectic structures defined on $X$ by $\tilde{\omega}_t$ for $t \in [0,1)$ are all symplectomorphic. $\square$

**6.2. Symplectic Lefschetz pencils.** The techniques used in this paper can also be applied to the construction of sections of $\mathbb{C}^2 \otimes L^k$ (i.e. pairs of sections of $L^k$) satisfying appropriate transversality properties : this is the existence result for Lefschetz pencil structures (and uniqueness up to isotopy for a given value of $k$) obtained by Donaldson [D2].

For the sake of completeness, we give here an overview of a proof of Donaldson's theorem using the techniques described in the above sections. Let $(X,\omega)$ be a compact symplectic manifold (of arbitrary dimension $2n$) such that $\frac{1}{2\pi}[\omega]$ is integral, and as before consider a compatible almost-complex structure $J$, the corresponding metric $g$, and the line bundle $L$ whose first Chern class is $\frac{1}{2\pi}[\omega]$, endowed with a Hermitian connection of curvature $-i\,\omega$. The required properties of the sections we wish to construct are determined by the following statement :

Proposition 12. *Let $s_k = (s_k^0, s_k^1)$ be asymptotically holomorphic sections of $\mathbb{C}^2 \otimes L^k$ over $X$ for all large $k$, which we assume to be $\eta$-transverse to $0$ for some $\eta > 0$. Let $F_k = s_k^{-1}(0)$ (it is a real codimension $4$ symplectic submanifold of $X$), and define the map $f_k = \mathbb{P}s_k = (s_k^0 : s_k^1)$ from $X - F_k$ to $\mathbb{CP}^1$. Assume furthermore that $\partial f_k$ is $\eta$-transverse to $0$, and that $\bar\partial f_k$ vanishes at every point where $\partial f_k = 0$. Then, for all large $k$, the section $s_k$ and the map $f_k$ define a structure of symplectic Lefschetz pencil on $X$.*

Indeed, $F_k$ corresponds to the set of base points of the pencil, while the hypersurfaces $(\Sigma_{k,u})_{u\in\mathbb{CP}^1}$ forming the pencil are defined to be $\Sigma_{k,u} = f_k^{-1}(u) \cup F_k$, i.e. $\Sigma_{k,u}$ is the set of all points where $(s_k^0, s_k^1)$ belongs to the complex line in $\mathbb{C}^2$ determined by $u$. The transversality to $0$ of $s_k$ gives the expected pencil structure near the base points, and the asymptotic holomorphicity implies that, near any point of $X - F_k$ where $\partial f_k$ is not too small, the hypersurfaces $\Sigma_{k,u}$ are smooth and symplectic (and even approximately $J$-holomorphic).

Moreover, the transversality to 0 of $\partial f_k$ implies that $\partial f_k$ becomes small only in the neighborhood of finitely many points where it vanishes, and that at these points the holomorphic Hessian $\partial\partial f_k$ is large enough and nondegenerate. Because $\bar\partial f_k$ also vanishes at these points, an argument similar to that of §5.2 shows that, near its critical points, $f_k$ behaves like a complex Morse function, i.e. it is locally approximately holomorphically modelled on the map $(z_1, \dots, z_n) \mapsto \sum z_i^2$ from $\mathbb{C}^n$ to $\mathbb{C}$. The approximate holomorphicity of $f_k$ and its structure at the critical points can be easily shown to imply that the hypersurfaces $\Sigma_{k,u}$ are all symplectic, and that only finitely many of them have isolated singular points, which correspond to the critical points of $f_k$ and whose structure is therefore completely determined.

Therefore, the construction of a Lefschetz pencil structure on $X$ can be carried out in three steps. The first step is to obtain for all large $k$ sections $s_k$ of $\mathbb{C}^2 \otimes L^k$ which are asymptotically holomorphic and transverse to 0 : for example, the existence of such sections follows immediately from the main result of [A1]. As a consequence, the required properties are satisfied on a neighborhood of $F_k = s_k^{-1}(0)$.

The second step is to perturb $s_k$, away from $F_k$, in order to obtain the transversality to 0 of $\partial f_k$. For this purpose, one uses an argument similar to that of §2.2, but where Proposition 2 has to be replaced by a similar result for approximately holomorphic functions defined over a ball of $\mathbb{C}^n$ with values in $\mathbb{C}^n$ which has been announced by Donaldson (see [D2]). Over a neighborhood of any given point $x \in X - F_k$, composing with a rotation of $\mathbb{C}^2$ in order to ensure the nonvanishing of $s_k^0$ over a ball centered at $x$ and defining $h_k = (s_k^0)^{-1}s_k^1$, one remarks that the transversality to 0 of $\partial f_k$ is locally equivalent to that of $\partial h_k$. Choosing local approximately holomorphic coordinates $z_k^i$, it is possible to write $\partial h_k$ as a linear combination $\sum_{i=1}^n u_k^i \mu_k^i$ of the 1-forms $\mu_k^i = \partial(z_k^i.(s_k^0)^{-1}s_{k,x}^{\mathrm{ref}})$. The existence of $w_k \in \mathbb{C}^n$ of norm less than a given $\delta$ ensuring the transversality to 0 of $u_k - w_k$ over a neighborhood of $x$ is then given by the suitable local transversality result, and it follows easily that the section $(s_k^0, s_k^1 - \sum w_k^i z_k^i s_{k,x}^{\mathrm{ref}})$ satisfies the required transversality property over a ball around $x$. The global result over the complement in $X$ of a small neighborhood of $F_k$ then follows by applying Proposition 3.

An alternate strategy allows one to proceed without proving the local transversality result for functions with values in $\mathbb{C}^n$, if one assumes $s_k^0$ and $s_k^1$ to be linear combinations of sections with uniform Gaussian decay (this is not too restrictive since the iterative process described in [A1] uses precisely the sections $s_{k,x}^{\mathrm{ref}}$ as building blocks). In that case, it is possible to locally trivialize the cotangent bundle $T^*X$, and therefore work component by component to get the desired transversality result ; in a manner similar to the argument of [A1], one uses Lemma 6 to reduce the problem to the transversality of sections of line bundles over submanifolds of $X$, and Proposition 6 as local transversality result. The assumption on $s_k$ is used to prove the existence of asymptotically holomorphic sections which approximate $s_k$ very well over a neighborhood of a given point $x \in X$ and have Gaussian decay away from $x$ : this makes it possible to find perturbations with Gaussian decay which at the same time behave nicely with respect to the trivialization of $T^*X$. This way of obtaining the transversality to 0 of $\partial f_k$ is very technical, so we don't describe the details.

The last step in the proof of Donaldson's theorem is to ensure that $\bar\partial f_k$ vanishes at the points where $\partial f_k$ vanishes, by perturbing $s_k$ by $O(k^{-1/2})$ over a neighborhood of these points. The argument is a much simpler version of §4.2 : on a neighborhood of a point $x$ where $\partial f_k$ vanishes, one defines a section $\chi$ of $f_k^* T\mathbb{CP}^1$ by $\chi(\exp_x(\xi)) = \beta(|\xi|)\,\bar\partial f_{k(x)}(\xi)$, where $\beta$ is a cut-off function, and one uses $\chi$ as a perturbation of $s_k$ in order to cancel the antiholomorphic derivative at $x$.

**6.3. Symplectic ampleness.** We have seen that similar techniques apply in various situations involving very positive bundles over a compact symplectic manifold, such as constructing symplectic submanifolds ([D1],[A1]), Lefschetz pencils [D2], or covering maps to $\mathbb{CP}^2$. In all these cases, the result is the exact approximately holomorphic analogue of a classical result of complex projective geometry. Therefore, it is natural to wonder if there exists a symplectic analogue of the notion of ampleness : for example, the line bundle $L$ endowed with a connection of curvature $-i\,\omega$, when raised to a sufficiently large power, admits many approximately holomorphic sections, and so it turns out that some of these sections behave like generic sections of a very ample bundle over a complex projective manifold.

Let $(X,\omega)$ be a compact $2n$-dimensional symplectic manifold endowed with a compatible almost-complex structure, and fix an integer $r$ : it seems likely that any sufficiently positive line bundle over $X$ admits $r+1$ approximately holomorphic sections whose behavior is similar to that of generic sections of a very ample line bundle over a complex projective manifold of dimension $n$. For example, the zero set of a suitable section is a smooth approximately holomorphic submanifold of $X$ ; two well-chosen sections define a Lefschetz pencil ; for $r=n$, one expects that $n+1$ well-chosen sections determine an approximately holomorphic singular covering $X \to \mathbb{CP}^n$ (this is what we just proved for $n=2$) ; for $r=2n$, it should be possible to construct an approximately holomorphic immersion $X \to \mathbb{CP}^{2n}$, and for $r>2n$ a projective embedding. Moreover, in all known cases, the space of "good" sections is connected when the line bundle is sufficiently positive, so that the structures thus defined are in some sense canonical up to isotopy.

However, the constructions tend to become more and more technical when one gets to the more sophisticated cases, and the development of a general theory of symplectic ampleness seems to be a necessary step before the relations between the approximately holomorphic geometry of compact symplectic manifolds and the ordinary complex projective geometry can be fully understood.

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