Brieskorn lattices for families of Laurent polynomials (after Givental and Iritani)

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March 18, 2010

Abstract

In these notes, we give an overview on the construction of the so called B-model quantum-$\mathcal{D}$-module of a smooth toric weak Fano manifold, as studied by Givental and described recently by Iritani ([Iri09]).

1 The mirror of a toric weak Fano manifold

We first explain how to construct from toric data a mirror fibration. Let $N = \oplus_{k=1}^{n} \mathbb{Z} n_k$ be a free abelian group of rank n and $\Sigma \subset N \otimes \mathbb{R}$ be a fan. This means that $\Sigma = \{\sigma\}$, where σ is a strongly convex (i.e., $\sigma \cap (-\sigma) = \{0\}$) polyhedral cone (i.e., $\sigma = \sum \mathbb{R}_{\geq 0} b_i$ for some b_i's in N). Being a fan means that for any $\sigma \in \Sigma$, any face of σ is again a cone in Σ, and that for any two cones $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both τ and σ. The fan Σ defines a toric variety X_Σ. Recall that X_Σ is covered by affine charts $X_\sigma :=$ Spec $\mathbb{C}[M \cap \sigma^\vee]$, here $M := Hom_{\mathbb{Z}}(N, \mathbb{Z})$ and $\sigma^\vee := \{m \in M \otimes \mathbb{R} \,|\, m(n) \geq 0 \quad \forall n \in N \otimes \mathbb{R}\}$, and that X_Σ is obtained from these affine pieces by gluing X_σ and X_τ along $X_{\sigma\cap\tau}$.
We will suppose in the remainder of these notes that the fan Σ is *smooth* and *complete*, which means by definition that any cone $\sigma \in \Sigma$ can be generated by elements b_i which can be completed to a $\mathbb{Z}$-basis of N and that the *support* $\mathrm{Supp}(\Sigma) = \bigcup_{\sigma\in\Sigma} \sigma$ is all of $N \otimes \mathbb{R}$. It is well-known that this translates into X_Σ being smooth and complete. Notice that the smoothness condition can be weakened by requiring Σ to be only *simplicial*, which means that the generators of each cone are linearly independent over $\mathbb{R}$. In this case X_Σ can have quotient singularities, i.e., it is the underlying topological space of a an orbifold.
We have an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow N \longrightarrow 0 \tag{1}$$

where $\Sigma(1)$ are the one-dimensional cones of Σ, called rays, the last map sends a generator e_i of $\mathbb{Z}^{\Sigma(1)}$ to a primitive integral generator $b_i \in N$ of a ray, and where the lattice $\mathbb{L}$ is the free submodule of $\mathbb{Z}^{\Sigma(1)}$ of relations between the elements $b_i \in N$. Dualizing yields the sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathbb{L}^\vee \longrightarrow 0.$$

It is well known (see, e.g., [Ful93, p. 106]) that for a smooth toric manifold X_Σ, we have $H^2(X_\Sigma, \mathbb{Z}) \cong \mathbb{L}^\vee$. Inside $\mathbb{L}^\vee \otimes \mathbb{R}$ we have the cone $K(X_\Sigma)$ of *Kähler classes*, which can be defined by saying that $a \in K(X_\Sigma)$ iff $a(\beta) \geq 0$ for all effective 1-cycles in $H_2(X_\Sigma, \mathbb{R})$ (The latter set of cycles also forms a cone, called the Mori cone). We write $K^0(X_\Sigma)$ for the interior of $K(X)$, i.e., for all elements $a \in \mathbb{L}^\vee$ with $a(\beta) > 0$. Write $D_i \in \mathbb{L}^\vee$ for the components of the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$, then the canonical divisor K_{X_Σ} is $\sum_{i=1}^m D_i \in \mathbb{L}^\vee$. X_Σ is called a Fano variety iff $-K_{X_\Sigma}$ is ample, i.e., lies in $K^0(X_\Sigma)$. If $-K_{X_\Sigma} \in K(X_\Sigma)$, then X_Σ is called weak Fano. Notice that a Calabi-Yau manifold (i.e., $K_{X_\Sigma} = 0$) is obviously weak Fano, however, it is easy to see that in this case the defining fan can never be complete.
The projection $\mathbb{Z}^{\Sigma(1)} \twoheadrightarrow N$ is given by a matrix $(b_{ki})_{k=1,\ldots,n;i=1,\ldots,m}$ with respect to the basis (n_k) of N. Moreover, we will chose once and for all a basis $(p_a)_{a=1,\ldots,r}$ of $\mathbb{L}^\vee$ (with $r = m - n$ and $m = |\Sigma(1)|$) consisting of nef classes, i.e., lying inside of $K(X)$. Notice however that the Kähler cone (resp. the Mori cone) is not always simplicial, the simplest example being the toric del Pezzo surface obtained by blowing up three points in $\mathbb{P}^2$ (here rank $H^2(X_\Sigma, \mathbb{Z}) = 4$, and the Mori cone is generated by 6 classes). Then the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$ is given by a matrix $(m_{ia})_{i=1,\ldots,m;a=1,\ldots,r}$ with respect to the dual basis $(p_a^\vee)$.

Applying the functor $Hom_{\mathbb{Z}}(-, \mathbb{C}^*)$ (where $\mathbb{Z}$ acts on $\mathbb{C}^*$ by exponentiating) to the exact sequence 1 yields

$$1 \longrightarrow Hom_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \xrightarrow{\alpha} (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{\beta} Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \longrightarrow 1 \tag{2}$$

where $\alpha(y_1, \dots, y_k) = (w_i := \prod_{k=1}^n y_k^{b_{ki}})_{i=1,\dots,m}$ and $\beta(w_1, \dots, w_m) = (q_a := \prod_{i=1}^m w_i^{m_{ia}})_{a=1,\dots,r}$, here $(q_a)_{a=1,\dots,r}$ are the coordinates on $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$ corresponding to the basis (p_a) of $\mathbb{L}^\vee$, $(w_i)_{i=1,\dots,m}$ are the standard coordinates on $(\mathbb{C}^*)^{\Sigma(1)}$ and $(y_k)_{k=1,\dots,m}$ are the coordinates on $Hom_{\mathbb{Z}}(N, \mathbb{C}^*)$ corresponding to the basis $(n_k^\vee)$ of M.

Definition 1. *Let $W = \sum_{i=1}^m w_i$. The* Landau-Ginzburg *model of X_Σ is defined to be the restriction of W to the fibres of the map β.*

The following construction allows us to rewrite the restriction of W to the fibres of β as a family of Laurent polynomials. Chose a splitting $l : \mathbb{L}^\vee \to \mathbb{Z}^{\Sigma(1)}$ of the projection $l : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \mathbb{L}^\vee$, given, with respect to the above bases, by a matrix (l_{ia}). This yields a splitting (denoted abusively by the same letter) $l : Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \to (\mathbb{C}^*)^{\Sigma(1)}$ which sends $(q_1, \dots, q_r)$ to $(w_i := \prod_{a=1}^r q_a^{l_{ia}})$. Then putting $\Psi : Hom_{\mathbb{Z}}(N, \mathbb{C}^*) \times Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \to (\mathbb{C}^*)^{\Sigma(1)}$ where $\Psi(\underline{y}, \underline{q}) := \left(w_i := \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}}\right)_{i=1,\dots,r}$ yields a coordinate change on $(\mathbb{C}^*)^m$ such that β becomes the projection $p_2 : Hom_{\mathbb{Z}}(N, \mathbb{C}^*) \times Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \twoheadrightarrow Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$. Then we put

$$\begin{aligned} \widetilde{W} := W \circ \Psi : Hom_{\mathbb{Z}}(N, \mathbb{C}^*) \times Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) &\longrightarrow \mathbb{C} \\ (y_1, \dots, y_k, q_1, \dots, q_a) &\longmapsto \textstyle\sum_{i=1}^m \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}} \end{aligned}$$

which is a family of Laurent polynomials on $Hom_{\mathbb{Z}}(N, \mathbb{C}^*)$ parameterized by $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$.
Recall ([?]) that a single Laurent polynomial $\widetilde{W}_{\underline{q}} := \widetilde{W}(-, \underline{q}) \in \mathcal{O}_{Hom(N, \mathbb{C}^*)}$ is called convenient iff 0 lies in the interior of its Newton polyhedron, and non-degenerate iff for any proper face τ of its Newton polyhedron, the Laurent polynomial $(\widetilde{W}_{\underline{q}})_\tau = \sum_{b_i \in \tau} \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}}$ does not have any critical point on $Hom_{\mathbb{Z}}(N, \mathbb{C}^*)$. If we consider the whole family $\widetilde{W}$, the following holds

Proposition 2.

1. *$\widetilde{W}_{\underline{q}}$ is convenient for any $\underline{q} \in Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$*
2. *There is an algebraic subvariety $Z \subset Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$ such that $\widetilde{W}_{\underline{q}}$ is non-degenerate for all $\underline{q} \notin Z$. Write $\mathcal{M}^0 := Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \backslash Z$.*
3. *If X_Σ is Fano, then $Z = \emptyset$.*
4. *If X_Σ is weak Fano, then there exists an $\epsilon > 0$, such that for all $\underline{q} \in Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ with $|q| < \epsilon$, we have $\underline{q} \notin Z$. Here the inclusion $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ and the metric $|\cdot|$ refer to the chosen coordinates (q_a) on $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$.*

Proof.

1. Obvious, as X_Σ is compact.
2. Also clear, as the condition of being degenerate is algebraic.
3. This can be shown by studying the faces of the Newton polygon of $\widetilde{W}_{\underline{q}}$.
4. This is done in [Iri09, appendix 6.1].

□

2 Brieskorn lattice for families of Laurent polynomials

We are going to construct from the family $\widetilde{W}$ an object that will play the role of the quantum-$\mathcal{D}$-module. In order to do that, we will first describe the family of jacobian algebras of W resp. $\widetilde{W}$. Let

$$\Theta_\beta := \{\vartheta \in \Theta_{(\mathbb{C}^*)^{\Sigma(1)}} \mid d(\beta_a)(\vartheta) = 0\} = \bigoplus_{k=1}^n \mathcal{O}_{(\mathbb{C}^*)^{\Sigma(1)}} \left(\sum_{i=1}^m b_{ki} w_i \partial_{w_i}\right)$$

be the module of β-relative vector fields and call

$$J(W) = \mathcal{O}_{(\mathbb{C}^*)^{\Sigma(1)}}/dW(\Theta_\beta) = \frac{\mathbb{C}[w_1^\pm, \ldots, w_m^\pm]}{(\sum_{i=1}^m b_{ki} w_i)_{k=1,\ldots,n}}.$$

the Jacobian (or Milnor) algebra of the family W. This ring is naturally a $\mathcal{O}_{Hom_\mathbb{Z}(\mathbb{L},\mathbb{C}^*)}$-algebra via β, and can be written as such as

$$J(W) = \frac{\mathbb{C}[w_1^\pm, \ldots, w_m^\pm, q_1^\pm, \ldots, q_r^\pm]}{(\sum_{i=1}^m b_{ki} w_i)_{k=1,\ldots,n} + (q_a - \prod_{i=1}^m w_i^{m_{ia}})_{a=1,\ldots,r}}.$$

The latter ring (with a slightly different presentation which is easily seen to be equivalent to this one) is called *Batyrev ring* in [Iri09] because it appears in [Bat93], where it was already shown that it equals the small quantum cohomology ring of X_Σ. Notice that $J(W) \otimes \mathcal{O}_{\mathcal{M}^0}$ is a locally free $\mathcal{O}_{\mathcal{M}^0}$-module of rank $\mu := n! \cdot \mathrm{vol}(\Delta)$, where Δ is the convex hull of the rays $b_1, \ldots, b_m \in \Sigma(1)$ (in other words, the Newton polygon of any of the Laurent polynomials $\widetilde{W}_{\underline{q}}$), and where the volume of the hypercube $[0,1]^n \subset N \otimes \mathbb{R}$ is normalized to 1.
We can now start to define the family of Brieskorn lattices we are interested in. The result will be a locally free $\mathcal{O}_{\mathbb{C}\times\mathcal{M}^0}$-module, equipped with a connection with certain types of poles. The construction starts by the description of the topological part of this structure, which consists in a local system of abelian groups of finite rank.

Definition-Lemma 3. *Fix $z \in \mathbb{C}^*$ and $\underline{q} \in \mathcal{M}^0$, put $Y := Hom_\mathbb{Z}(N, \mathbb{C})$ and consider the following relative homology group*

$$(\mathcal{R}^\vee_\mathbb{Z})_{(z,\underline{q})} := H_n\left(Y, \left\{\underline{y} \,|\, \Re(z^{-1}\cdot \widetilde{W}_{\underline{q}}) \ll 0\right\}, \mathbb{Z}\right)$$

Then we have

1. *$(\mathcal{R}^\vee_\mathbb{Z})_{(z,\underline{q})}$ is a free abelian group of rank $\mu = n! \cdot \mathrm{vol}(\Delta)$*
2. *Varying z and q forms a local system $\mathcal{R}^\vee_\mathbb{Z}$ on $\mathbb{C}^* \times \mathcal{M}^0$.*
3. *Suppose that $\underline{q} \in \mathcal{M}^0$ is such that $\widetilde{W}_{\underline{q}}$ has only non-degenerate (Morse) critical points on Y. Then $(\mathcal{R}^\vee_\mathbb{Z})_{(z,\underline{q})} = \oplus_{i=1}^\mu \mathbb{Z}\Gamma_i$, where Γ_i is a Lefschetz thimble emanating from a fibre of $\widetilde{W}_{\underline{q}}$ over a point $t \in \mathbb{C}$ such that $\Re(t/z) \ll 0$ to the i'th critical point (i.e., a family of vanishing cycles over a path from t to the i'th critical value, supposing that the pathes to different critical values do not intersect).*
4. *There is a perfect pairing*
$$(-,-) : (\mathcal{R}^\vee_\mathbb{Z})_{(z,\underline{q})} \times (\mathcal{R}^\vee_\mathbb{Z})_{(-z,\underline{q})} \longrightarrow \mathbb{Z}$$
given by intersection Lefschetz thimbles going to opposite directions (so that the intersection take place in a compact subset of Y and is hence well-defined).

We write $\mathcal{R}_\mathbb{Z}$ for the dual local system, and $\mathcal{R} := \mathcal{R}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}^{an}_{\mathbb{C}^*\times\mathcal{M}^0}$ for the corresponding holomorphic vector bundle on $\mathbb{C}^* \times \mathcal{M}^0$, which is equipped with an integrable connection operator.

Concerning the proof of these statements, Iritani basically refers to older work of Pham ([Pha85]), where a similar construction for polynomials is done.
The next step is to extend the flat bundle to $\mathbb{C} \times \mathcal{M}^0$. This uses oscillating integrals. Namely, write $T := Y \times Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*) = Hom_\mathbb{Z}(N, \mathbb{C}^*) \times Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*)$ for short, consider for any $f \in \mathcal{O}^{an}_{\mathbb{C}^*\times T}$ the differential form

$$\varphi := f \cdot e^{-\widetilde{W}/z} \cdot \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n} \in \Omega^{n,an}_{T/\,Hom_\mathbb{Z}(\mathbb{L},\mathbb{C}^*)}$$

and put for any flat section $[\Gamma] \in \mathcal{R}^\vee_\mathbb{Z}$

$$\langle \varphi, \Gamma \rangle := \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma \varphi$$

One can show that this integral is convergent by choosing the representative Γ of the class $[\Gamma]$ sufficiently carefully, again here Iritani refers to Pham. Then we have the following result.

Theorem 4. *Define* $\phi : \mathcal{O}^{an}_{\mathbb{C}^*\times T} \to \mathcal{R}^{an}$ *by putting* $\phi(f) := [\Gamma \mapsto \langle \varphi, \Gamma\rangle]$ *and define* $\mathcal{R}^{(0),an} := \phi(\mathcal{O}_{\mathbb{C}\times T})$. *Then*

1. ϕ *is surjective.*
2. *The connection induced from* $\mathcal{R}_{\mathbb{Z}}$ *on sections* $\phi(f)$ *is written as*

$$\begin{aligned} \nabla_{\partial_z}\phi(f) &:= \phi\left(\partial_z f + \tfrac{1}{z^2}\widetilde{W} f - \tfrac{n}{2z} f\right) \\ \nabla_{\partial_{q_a}}\phi(f) &:= \phi\left(\partial_{a_q} f - \tfrac{\partial_{q_a}\widetilde{W}}{z} f\right) \end{aligned}$$

3. $\mathcal{R}^{(0),an}$ *is* $\mathcal{O}^{an}_{\mathbb{C}\times\mathcal{M}^0}$*-locally free of rank* μ.

The proof basically proceeds as follows.

- The space $\mathcal{M}^{00} := \{\underline{q} \in \mathcal{M}^0 \,|\, \widetilde{W}_{\underline{q}}$ has only non-degenerate critical points $\}$ is non-empty and open (however, its complement is in general not away from the limit point $\underline{q} = 0 \in \mathbb{C}^r$ as was the case for $Z = Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)\backslash\mathcal{M}^0$).
- For $\underline{q} \in \mathcal{M}^{00}$, we have the following formula expressing the oscillating integrals used to define the extension $\mathcal{R}^{(0),an}$. Write cr_i ($i \in \{1, \ldots, \mu\}$) for the non-degenerate critical points of $\widetilde{W}_{\underline{q}}$, and Γ_i for a Lefschetz thimble starting at cr_i.

$$\frac{1}{(-2\pi z)^{n/2}}\int_{\Gamma_i} f(z,\underline{y},\underline{q})e^{-\widetilde{W}_{\underline{q}}/z}\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_n}{y_n} = \frac{e^{-\widetilde{W}(\underline{q},cr_i)/z}}{\sqrt{\left(\det\frac{\partial^2\widetilde{W}_{\underline{q}}}{\partial y_k\partial y_l}\right)}}\cdot\left(f(0,cr_i,\underline{q})+O(z)\right)$$

- As a consequence of the last formula, we have that $Im(\phi)_{|\mathbb{C}\times\{q\}} = \mathcal{R}_{|\mathbb{C}\times\{q\}}$ (Let f run through the basis elements of the jacobian algebra $J(W)_{|\underline{q}}$ chosen to take the value one at a critical point cr_i and zero on the other critical points).
- $Im(\phi)$ is stable under ∇_{∂_z} and $\nabla_{\partial_{q_a}}$, hence, ϕ is surjective all over $\mathcal{M}^0$.
- $\mathcal{R}^{(0),an} = \oplus_{i=1}^{\mu}\mathcal{O}^{an}_{\mathbb{C}\times\mathcal{M}^0}\phi(g_i)$, where $(g_1,\ldots,g_\mu)$ is a $\mathcal{O}_{\mathcal{M}^0}$-basis of $J(W)$.

Let us summarize the result of the constructions sketched above in the following theorem.

Theorem 5. *Let* $\Sigma \subset N\otimes\mathbb{R}$ *be a smooth complete fan defining a weak Fano manifold* X_Σ*. Then the Landau-Ginzburg model of* X_Σ*, i.e., the restriction of the linear function* $W = \sum_{i=1}^m w_i$ *to the fibres of the torus fibration* $\beta : (\mathbb{C}^*)^{\Sigma(1)} \to \mathcal{M}^0$ *gives rise to a locally free* $\mathcal{O}^{an}_{\mathbb{C}\times\mathcal{M}^0}$*-module* $\mathcal{R}^{(0),an}$*, called (family of) Brieskorn lattice(s) of* (W,β)*, which is equipped with an integrable connection operator*

$$\nabla : \mathcal{R}^{(0),an} \longrightarrow \mathcal{R}^{(0),an}\otimes z^{-1}\Omega^{1,an}_{\mathbb{C}\times\mathcal{M}^0}(\log(\{0\}\times\mathcal{M}^0)),$$

a flat integer lattice $\mathcal{R}_{\mathbb{Z}} \subset \mathcal{R}^{(0),an}_{|\mathbb{C}^*\times\mathcal{M}^0}$ *and a non-degenerate pairing*

$$(-,-) : \mathcal{R}^{(0),an}\otimes(-)^*\mathcal{R}^{(0),an} \longrightarrow \mathcal{O}^{an}_{\mathbb{C}\times\mathcal{M}^0}$$

which is flat on $\mathbb{C}^*\times\mathcal{M}^0$ *and takes integer values on* $\mathcal{R}_{\mathbb{Z}}$.

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