# Landau-Ginzburg models for toric Fano manifolds

Plan of the talk:

1.) Toric varieties

2.) Families of Laurent polynomials

3.) Gauß-Manin Systems and GKZ-systems

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Very short reminder on (smooth) toric varieties

$N \simeq \bigoplus_{k=1}^{n} \mathbb{Z} \cdot e_k$ free abelian group, $\Sigma$ fan in $N_{\mathbb{R}}$, i.e.

$\Sigma = \{\sigma\}$, $\sigma$ $\underbrace{\text{strongly convex}}_{\sigma \cap (-\sigma) = \{0\}}$ $\underbrace{\text{polyhedral cone}}_{\sigma = \{\sum \lambda_i b_i, \lambda_i \geq 0\},\ b_i \in N}$

s.t. - $\sigma \in \Sigma \Rightarrow$ any face of $\sigma \in \Sigma$

- $\sigma, \tau \in \Sigma \Rightarrow \sigma \cap \tau$ is face of both $\sigma$ and $\tau$

Given fan $\Sigma \subset N_{\mathbb{R}} \Rightarrow X := X_\Sigma$ associated toric

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variety: $\forall \sigma \in \Sigma$ : $X_\sigma := \operatorname{Spec} \mathbb{C}[M \cap \check{\sigma}]$, where

$M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $\check{\sigma} := \{m \in M \otimes \mathbb{R} \mid m(v) \geq 0 \ \forall v \in \sigma\} \subset M \otimes \mathbb{R}$

dual cone, glue $X_\sigma$ & $X_\tau$ along $X_{\sigma \cap \tau}$.

Facts: - $X$ complete $\iff$ $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma = N \otimes \mathbb{R}$

- $X$ orbifold $\iff$ $\Sigma$ simplicial, i.e. $\forall \sigma \in \Sigma$,

its generators are linearly ind. $/\mathbb{R}$,

here generators: $\rho \in \Sigma(1) \rightsquigarrow v_\rho$ gen. of $N \cap \rho$

- $X$ smooth $\iff$ $\forall \sigma \in \Sigma$, generators of $\sigma$ can be

completed to a ($\mathbb{Z}$-)basis of $N$.

We have exact sequence:

$$0 \to \mathbb{L} \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{(e_{ki}) \in M(n \times m, \mathbb{Z})} N \simeq \bigoplus \mathbb{Z} e_k \to 0 \qquad (*)$$

$$f_\rho \longmapsto v_\rho$$

and dual:

$$0 \to M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathbb{L}^* \simeq A_{n-1}(X) \simeq \operatorname{Pic}(X) \simeq H^2(X, \mathbb{Z}) \to 0$$

Inside $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$, we have the Kähler cone

$K(X) \subset A^+_{n-1}(X) \subset A_{n-1}(X)$; $X$ Fano $\iff$ $\sum [f_i^*] \in K(X)^\circ$

$X$ weak F. $\iff$ $\sum [f_i^*] \in K(X)$

Chose a basis $(p_a)_{a\in\{1,\dots,r\}}$ of $\mathbb{L}^*$ generating $K(X)$, i.e., $p_a$ nef

consider $\mathrm{Hom}_{\mathbb{Z}}(*, \mathbb{C}^*)$:

$$1 \to \mathrm{Hom}(N, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{pr} \mathbb{L}^* \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^r \to 1$$

coordinates: $(w_i)_{i=1,\dots,m}$ on $(\mathbb{C}^*)^{\Sigma(1)} \leftrightarrow (f_i)$ basis of $\mathbb{Z}^{\Sigma(1)}$

$(q_a)_{a=1,\dots,r}$ on $\mathbb{L}^* \otimes \mathbb{C}^* \leftrightarrow (p_a)$ nef basis of $\mathbb{L}^*$

$(y_k)_{k=1,\dots,n}$ on $\mathrm{Hom}(N, \mathbb{C}^*) \leftrightarrow (e_k)$ basis of $N$

<u>Def</u>: We call the restriction of the linear function $W = w_1 + \dots + w_m$ to the fibres of the morphism $pr$ the mirror Landau-Ginzburg model of the manifold $X$.

<u>family of Laurent polynomials</u>: Choose morphism

$\ell : \mathbb{L}^\vee \longrightarrow \mathbb{Z}^{\Sigma(1)}$ which splits $\mathbb{L} \to \mathbb{Z}^{\Sigma(1)}$.

Dualizing gives section $\ell: \mathbb{L}^* \otimes \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{\Sigma(1)}$ of $pr$

i.e. $\ell(q_1, \dots, q_r) = \left(\prod_{a=1}^{r} q_a^{\ell_{ai}}\right)_{i\in\{1,\dots,m\}} =: (w_1, \dots, w_m)$.

For any $q \in \mathbb{L}^* \otimes \mathbb{C}$ fixed, this yields an isomorphism,

$$\mathrm{Hom}(N, \mathbb{C}^*) = Y_1 \longrightarrow Y_q$$

$$(w_1, \dots, w_m) \longmapsto \left(\prod_a q_a^{\ell_{ai}} \cdot w_i\right)_{i \in \{1..,m\}}$$

$$\left(\prod_{i=1}^m \left(\prod_b q_b^{\ell_{bi}}\right) w_i^{m_{ia}}\right)_a = \left(\prod_{i=1}^m \left(\prod_b q_b^{\overbrace{\ell_{bi} \cdot m_{ia}}^{\delta_{ab}}}\right) \cdot w_i^{m_{ia}}\right)_a$$

$$= \Big(q_a \cdot \underbrace{\prod_{i=1}^m w_i^{m_{ia}}}_{\substack{pr(\underline{w}) = 1 \\ \cap \\ Y_1}}\Big)_a = (q_a)_a = \underline{q}$$

<u>Hence:</u> $W_{|Y_q} = \sum_{i=1}^m \left(\prod_{a=1}^r q_a^{\ell_{ai}} \cdot \prod_{k=1}^n y_k^{b_{ki}}\right)$

## family of mirror Laurent polynomials of X.

<u>Fact</u>: (Kouchnirenko, ....)
$\exists \mathcal{M}^0 \subset \mathbb{L}^* \otimes \mathbb{C}^*$ open affine s.t. $\forall q \in \mathcal{M}^0$, $W_{|Y_q}$ is tame

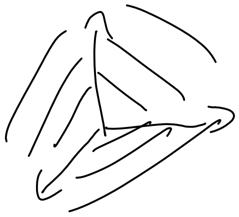
(more precisely, convenient, i.e., $0 \in \mathrm{Int}(\Delta)$ and non-degenerate)

Iritani: X weak Fano, $(p_a)$ choosen as above $\rightsquigarrow \exists \varepsilon > 0$ s.t.

$$\forall q \in \mathbb{L}^\vee \otimes \mathbb{C}^* = (\mathbb{C}^*)^r \subset \mathbb{C}^r : |q| < \varepsilon \Rightarrow q \in \mathcal{M}^0$$

exemples:

i) $X = \mathbb{P}^n \Rightarrow N \simeq \mathbb{Z}^n$, $b_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \dots, b_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}, b_{n+1} = \begin{pmatrix}-1\\\vdots\\-1\end{pmatrix}$

$\mathbb{P}^2$ 

$$0 \to \mathbb{L} \simeq \mathbb{Z}\cdot[H]^* \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix}} \mathbb{Z}^2 \to 0$$

$\mathrm{Hom}(-, \mathbb{C}^*)$

$$1 \to (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^3 \xrightarrow{pr} \mathbb{C}^* \to 0$$

$$(w_1, w_2, w_3) \mapsto w_1 \cdot w_2 \cdot w_3$$

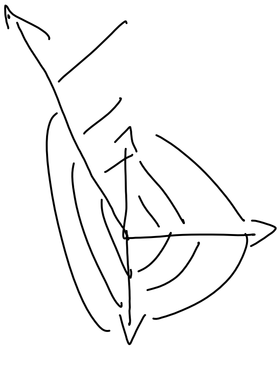
$$(y_1, y_2) \mapsto y_1, y_2, (y_1 y_2)^{-1}$$

choose $\ell : \mathbb{Z}^3 \to \mathbb{L}$; $f_1 \mapsto 0$, $f_2 \mapsto 0$, $f_3 \mapsto [H]^*$

$\Rightarrow W_{|Y_q} = y_1 + y_2 + \dfrac{q}{y_1 \cdot y_2}$; one checks: $\mathcal{M}_0 \simeq \mathbb{C}^*$

ii) $X = \mathbb{F}_2 = \underline{\mathrm{Proj}}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathbb{P}^1$; $-K_X$ nef

$N \simeq \mathbb{Z}^2$; $b_1 = \begin{pmatrix}1\\0\end{pmatrix}, b_2 = \begin{pmatrix}0\\1\end{pmatrix}, b_3 = \begin{pmatrix}-1\\2\end{pmatrix}, b_4 = \begin{pmatrix}0\\-1\end{pmatrix}$

$\Sigma$: 

$pr: (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2$

choice

$$(w_1, \dots, w_4) \mapsto \left(\frac{w_1 w_3}{w_2^2}, w_2 \cdot w_4\right) = (q_1, q_2)$$

$$W_{|Y_q} = w_1 + w_2 + \frac{q_1 w_2^2}{w_1} + \frac{q_2}{w_2}$$

same outside $q_1 = 1/4$

Aim: construct variation of ncHodge-structures (6

on $\mathbb{C} \times \mathcal{M}^0$,

first step: local system on $\mathbb{C}^* \times \mathcal{M}^0$

$$\mathcal{R}^\vee_{\mathbb{Z} \mid (q,z)} := H_n\left(Y_q, \mathrm{Re}\left(\tfrac{W_q}{z}\right) \ll 0, \mathbb{Z}\right)$$

$$\simeq \bigoplus_{i=0}^{\mu} \mathbb{Z}\,\Gamma_i \quad , \mu = n! \cdot \mathrm{vol}((\Delta)) \quad \text{if only non-deg. crit pts.}$$

$\Gamma_i$: Lefschetz thimble $\subset Y_q$

intersection pairing: $(\,,\,): \mathcal{R}^\vee_{\mathbb{Z} \mid (q,z)} \times \mathcal{R}^\vee_{\mathbb{Z},(q,-z)} \to \mathbb{Z}$

$$\mathcal{R}_{\mathbb{Z}} := \left(\mathcal{R}^\vee_{\mathbb{Z}}\right)^\vee, \quad \mathcal{R} := \mathcal{R}_{\mathbb{Z}} \otimes \mathcal{O}_{\mathbb{C}^* \times \mathcal{M}^0}$$

extension to $\mathbb{C} \times \mathcal{M}^0$: $\forall f \in \mathcal{O}_{\mathbb{C}^* \times Y^0}$ putting

$$\varphi := f \cdot e^{W_q/z} \cdot \underbrace{\frac{dy_1 \wedge \ldots \wedge dy_n}{y_1 \cdot \ldots \cdot y_n}}_{=:\,\mathrm{vol}}$$

and $\langle \varphi, \Gamma \rangle := \dfrac{1}{(-2\pi z)^{n/2}} \displaystyle\int_\Gamma \varphi$

gives section of $\mathcal{R}$ on $\mathbb{C}^* \times M^0$. Define $\mathcal{R}'$ to be be the $\mathcal{O}_{\mathbb{C}^* \times M^0}$-submod. generated by these sections. and $\mathcal{R}^{(0)}$ the $\mathcal{O}_{\mathbb{C} \times M^0}$-mod. generated by all such sections s.t. $f \in \mathcal{O}_{\mathbb{C} \times Y^0}$

Th: (Iritani):

1.) $\mathcal{R}' = \mathcal{R}$

2.) $\mathcal{R}^{(0)}$ is $\mathcal{O}_{\mathbb{C} \times M^0}$-locally free of rk $\mu$ and

$\mathcal{R}^{(0)}{}_{|\mathbb{C}^* \times M_0} = \mathcal{R}$

3.) Connection induced from $\mathcal{R}_{\mathbb{Z}}$ on sections given by $\varphi = f \cdot e^{W_q/z} \cdot \mathrm{vol}$;

$$\nabla_{\partial_z}(\varphi) = \left(\partial_z f - \frac{1}{z^2} W_q \cdot f - \frac{n}{2z} \cdot f\right) e^{W_q/z} \cdot \mathrm{vol}$$

$$\nabla_{\partial_{q_a}}(\varphi) = \left(\frac{\partial_{q_a} f}{q_a} + \frac{\partial_{q_a} W_q}{z} \cdot f\right) e^{W_q/z} \cdot \mathrm{vol}$$

Pole of type along $\{0\} \times M^0$

4.) The intersection pairing gives pairing

$( , ) : \mathcal{R}^{(0)} \otimes (-)^* \mathcal{R}^{(0)} \to \mathcal{O}_{\mathbb{C}^* \times M^0}$ which is

non-degenerate.

Main point in the proof: if $W_q$ has only Morse pts.

$$\frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_i} f \cdot e^{W_q/z} \, vol = \frac{e^{W_q(x_i)/z}}{\left(\det\left(\frac{\partial^2_y W_q}{\partial y_i \partial y_j}\right)(x_i)\right)^{1/2}} \cdot \left(f(q,0,x_i) + O(z)\right)$$

Lefschetz thimble from critical pts. $x_i$

Corollary: $\left(\mathcal{R}^{(0)}, \mathcal{R}_{\mathbb{Q}}, id : \mathcal{R} \to \mathcal{R}^{(0)}_{|\mathbb{C}^* \times M^0}\right)$

is a variation of nc Hodge-structures

on $M^0$

Rk: 1.) $\mathbb{Q}$-structure axiom hold as Stokes structure can be defined by Lefschetz thimbles,

2.) opposedness "should" hold by result of Claude ('08 Crelle paper); to do: understand relation of $\mathcal{R}^{(0)}$ with family of Brieskorn lattices $G_0$ in loc. cit.

## Cyclicity and Iritani's GKZ-system:

Put

$$\Theta(pr) = \{ v \in \Theta_Y \mid d(pr)(v) = 0 \}$$

and $J(W) := \mathcal{O}_Y \,/\, dW(\Theta(pr))$

$J(W)$ is $\mathcal{O}_M$ - algebra, and also $\mathcal{O}_{M_0}$ - free of rank $\mu$ when restricted to $M^0$:

Lemma: (Iritani) $\exists$ Iso of $\mathbb{C}[q_1^{\pm},\dots,q_r^{\pm}]$ -algebras

$$J(W) \simeq \mathbb{C}[q_1^{\pm},\dots,q_r^{\pm}][\tilde{q}_1,\dots,\tilde{q}_r] \,/\, I$$

where $I$ is generated by

$$\left( \prod_{a=1}^{r} q_a^{\langle p_a, \underline{d} \rangle} \cdot \prod_{i:\langle D_i, \underline{d}\rangle < 0} \left( \sum_{a=1}^{r} m_{ia} \tilde{q}_a \right)^{-\langle D_i, \underline{d} \rangle} - \prod_{i:\langle D_i, \underline{d}\rangle > 0} \left( \sum_{a=1}^{r} m_{ia} \tilde{q}_a \right)^{\langle D_i, \underline{d} \rangle} \right)$$

$\forall \underline{d} \in \mathbb{L}$

<u>Theorem</u>: Let $V_\varepsilon = \{q \in \mathcal{M} \mid |q| < \varepsilon\} \subset \mathcal{M}^\circ$

Then $\exists$ isomorphism of $\mathcal{O}_{\mathbb{C} \times V_\varepsilon}$-modules

$$\mathcal{M}_{GKZ} \otimes \mathcal{O}_{\mathbb{C}\times V_\varepsilon} \xrightarrow{\varphi} \mathcal{R}^{(0)} \otimes \mathcal{O}_{\mathbb{C}\times V_\varepsilon}$$

<u>where</u>:

$$\mathcal{M}_{GKZ} := \mathbb{C}[q^{\pm}, z]\langle z q_1 \partial_{q_1}, \dots, z q_r \partial_{q_r}\rangle / \widetilde{I}$$

$\widetilde{I}$ generated by

$$\left(\prod_{a=1}^{r} q_a^{\langle p_a, \underline{d}\rangle} \cdot \prod_{i : \langle D_i, \underline{d}\rangle < 0} \prod_{\nu=0}^{-\langle D_i, \underline{d}\rangle - 1} \left(\sum_{a=1}^{r} m_{ia} z q_a \partial_{q_a} - \nu z\right) - \prod_{i : \langle D_i, \underline{d}\rangle > 0} \prod_{\nu=0}^{\langle D_i, \underline{d}\rangle - 1} \left(\sum_{a=1}^{r} m_{ia} z q_a \partial_{q_a} - \nu \cdot z\right)\right)$$

$\forall \underline{d} \in \mathbb{L}$

and where $1 \in \mathcal{M}_{GKZ} \otimes \mathcal{O}_{\mathbb{C}\times V_\varepsilon}$ is sent to $[e^{W_q/z} \cdot \mathrm{vol}]$

and such that $\varphi$ is compatible with the connection operators $z q_a \partial_{q_a}$ resp. $z \nabla_{q_a \partial_{q_a}}$

(and also with $\nabla_{z\partial_z}$ and grading to be introd. on the LHS)