Integrable hierarchies and hypersurface singularities

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Integrable hierarchies

Algebraic point of view.

Space of differential polynomials

$$\mathbb{C}[U] = \mathbb{C}[u, \partial u, \partial^2 u, \dots],$$

where

$$\partial^k u = (\partial^k u_1, \dots, \partial^k u_N)$$

$\partial^k u_i$ is a single symbol!

$\mathbb{C}[U]$ is naturaly a differential algebra $\partial(\partial^k u) := \partial^{k+1} u$

Given

$$P_1^a, \dots, P_N^a \in \mathbb{C}[U]$$

we can define a differentiation via

$$\partial_a u_i = P_i^a(u, \partial u, \partial^2 u, \dots), \quad 1 \le i \le N \tag{1}$$

$$\partial_a \partial = \partial \partial_a,$$

and the Leibnitz rule.

An integrable hierarchy is a maximal infinite sequence of commuting differentiations.

Example: The disperseonless KdV hierarchy

$$\partial_n u = \frac{u^n}{n!}\,\partial u \quad n = 0, 1, 2, \ldots$$

$$\partial_m \partial_n (u) = \partial \left(\frac{u^{m+n}}{m!n!}\,\partial u \right)$$

Analytic point of view

H : vector space, $\dim_{\mathbb{C}} H = N$, points on LH have the form

$$u(x) = (u_1(x), \ldots, u_N(x)).$$

We alow the following class of functions on LH:

- For every $P(u, \partial u, \partial^2 u, \ldots) \in \mathbb{C}[U]$ we have a *local functional*

$$u \in LH \mapsto \int_{S^1} P(u(x), u'(x), u''(x), \ldots) dx$$

- For every $x \in S^1$, evaluation at x

$$P(u(x), u'(x), \ldots) = \int_{S^1} P(u(y), u'(y), u''(y), \ldots) \delta(x, y) dy$$

Using variational derivation, we can define vector fields on LH

$$\xi_a = \sum_{i=1}^{N} P_i^a(u, \partial u, \dots) \frac{\delta}{\delta u^i}$$

then (1) gives us a system of pairwise commuting flows on LH.

Cohomological Field Theories

Given a smooth projective manifold X, we define

$$\Lambda_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n) := \int_{\overline{\mathcal{M}}_{g,n}(X;\beta)}^{\overline{\mathcal{M}}_{g,n}} \mathrm{ev}^*(\gamma_1 \otimes \cdots \otimes \gamma_n) \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$$

where $\overline{\mathcal{M}}_{g,n}(X;\beta)$ is the moduli space of stable maps;

the integral is pushforward to $\overline{\mathcal{M}}_{g,n}$, and

$$\mathrm{ev} : \overline{\mathcal{M}}_{\mathrm{g,n}}(\mathrm{X};\beta) \to \mathrm{X}^{\mathrm{n}}$$

is the evanluation morphism.

Put $H = H^*(X; \mathbb{C})$ and fix $t \in H$. The maps

$$\Lambda_{g,n}(t) : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C}),$$

defined by

$$\Lambda_{g,n}(t)(\gamma) := \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{\mathcal{M}}_{g,n+k}}^{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}(\gamma \otimes t^{\otimes k})$$

satisfy some natural gluing properties:

(Tree gluing) Let

$$\rho_{tree} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$$

gluing the last marked point of the first curve and the first marked point of the second curve; then

$$\rho_{tree}^{*}\Big(\Lambda_{g,n}(\gamma_1, \cdots, \gamma_n)\Big) = \sum_{\alpha,\beta} \Lambda_{g_1,n_1+1}(\gamma_1, \cdots, \gamma_{n_1}, \alpha)\eta^{\alpha,\beta}\Lambda_{g_2,n_2+1}(\beta, \gamma_{n_1+1}, \cdots, \gamma_n).$$

(Loop gluing) Let

$$\rho_{loop} : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

be the morphism induced from gluing the last two marked points; then

$$\rho_{loop}^{*}\Big(\Lambda_{g,n}(\gamma_1, \cdots, \gamma_n)\Big) = \sum_{\alpha,\beta} \Lambda_{g-1,n+2}(\gamma_1, \cdots, \gamma_n, \alpha, \beta)\eta^{\alpha,\beta}.$$

Frobenius structure on H

(Pairing) same as Poincaré pairing

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(1, \gamma_1, \gamma_2) = \eta(\gamma_1, \gamma_2).$$

(Multiplication)

$$\eta(\gamma_1 \bullet_t \gamma_2, \gamma_3) = \int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(t)(\gamma_1, \gamma_2, \gamma_3)$$

$$\nabla = d + z^{-1} \sum_{a=1}^{N} (\partial/\partial t_a \bullet_t) dt_a$$

is a flat connection on TH.

Potential of the Cohomological Field Theory:

$$Z(\hbar;\mathbf{t}) = \exp\left(\sum_g \frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}(\mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_n))\right),$$

where

$$\mathbf{t}(\psi_i) = \sum_{a=1}^{N}\sum_{k=0}^{\infty} t_{a,k}\phi_a\psi_i^k,$$

where $\{\phi_a\}$ is a basis of H, ψ_i is the 1-st Chern class of the cotangent line at the i-th marked point.

Put $x := t_{1,0}$ and $u_a = \partial_{t_{a,0}}\partial_x \log Z$; then

Theorem 1[Eguchi–Xiong]. There are functions

$$\Omega_{k,a,b}(\hbar; u, \partial u, \dots) = \sum_{g=0}^{\infty} \Omega^{[g]}_{k,a,b}(u, \partial u, \dots)\hbar^g$$

such that

$$\partial_{k,a} u_b = \Omega_{k,a,b}(u, \partial u, \dots).\square$$

Theorem 2[Buryak–Posthuma–Shadrin]. If $\bullet_t$ is semi-simple for generic $t \in H$; then $\Omega^{[g]}_{k,a,b}$ are differential polynomials. $\square$

Theorem 3[Dubrovin–Zhang]. The above integrable hierarchy is bi-Hamiltonian and the corresponding brackets can be defined entirely in terms of genus-0 data. $\square$

Representation Theory

Lattice vertex algebra (after Borcherds)

$R, (\cdot|\cdot), Q$ ADE-root system, integral pairing, root lattice

$$V_Q = S^\bullet(\mathfrak{h}[t^{-1}]t^{-1}) \otimes \mathbb{C}_\epsilon[Q],$$

i.e. vector space spanned by

$$h^1_{-n_1-1} \dots h^r_{-n_r-1}\ e^\alpha,$$

where

$$h^i \in \mathfrak{h} := Q \otimes_{\mathbb{Z}} \mathbb{C}, \quad \alpha \in Q, \quad h^i_n := h^i t^n$$

State-field correspondence

We want to define

$$Y(\cdot,\zeta)\cdot : V_Q \otimes V_Q \to V_Q((\zeta)),$$

i.e., for all $a \in V_Q$ we will have

$$Y(a,\zeta) = \sum_{n\in\mathbb{Z}} a_{(n)}\,\zeta^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V_Q).$$

The most important properties are the following

Locality

For every $a, b \in V_Q$, we have

$$(\zeta - w)^N [Y(a,\zeta), Y(b,w)] = 0,$$

for some integer $N = N_{ab} \geq 0$.

Operator Product Expansion

$$Y(a_{(n)}b, \zeta) = \frac{1}{k!} \frac{d^k}{dw^k}\bigg|_{w=\zeta} (\zeta - w)^N Y(a,w) Y(b,\zeta),$$

where $k = N - n - 1$ and $N \gg 0$.

Heisenberg Lie algebra

$\mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}$ with commutator

$$[h'_m, h''_n] = m\,(h'|h'')\delta_{m,-n}, \quad \forall h', h'' \in \mathfrak{h}$$

acts naturally on V_Q; each

$$\mathcal{F}_\alpha = S^\bullet(\mathfrak{h}[t^{-1}]t^{-1}) \otimes e^\alpha$$

is an irreducible highest weight representation

$$h_n\,(1 \otimes e^\alpha) = \delta_{n,0}(\alpha|h)(1 \otimes e^\alpha).$$

For all $h \in \mathfrak{h}$ and $\alpha \in Q$, define n

$$Y(h_{-1}, \zeta) \ := \ \sum_n h_n \, \zeta^{-n-1},$$

$$Y(e^\alpha, \zeta) \ := \ e^\alpha \, \zeta^{\alpha_0} \exp\left(\sum_{n<0} \frac{\zeta^{-n}}{-n} \, \alpha_n \right) \exp\left(\sum_{n>0} \frac{\zeta^{-n}}{-n} \, \alpha_n \right).$$

Mutually local and induce uniquely a state-field correspondence on V_Q.

Theorem 4.[I. Frenkel–V. Kac] Let $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ be respectively the simple and the affine Lie algebra corresponding to R; then

$$\begin{aligned} \mathfrak{g} &\cong \text{span}_{\mathbb{C}} \left\{ e^{\alpha}_{(0)}, \quad h_0 \mid h \in \mathfrak{h}, \alpha \in R \right\} \\ \widehat{\mathfrak{g}} &\cong \text{span}_{\mathbb{C}} \left\{ e^{\alpha}_{(n)}, \quad h_n \mid h \in \mathfrak{h}, \alpha \in R, n \in \mathbb{Z} \right\} \oplus \mathbb{C}. \end{aligned}$$

Moreover,

$$V_Q \cong U(\widehat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t]+\mathbb{C}\,K} \mathbb{C}$$

<u>basic level 1 representation</u>.

Let $\mathcal{W}_Q := V_Q^{\mathfrak{g}}$, i.e., interesection of Ker($e^{\alpha}_{(0)}$), $\alpha \in R$.

Singularity Theory

Frobenius structure

$$W(x_0, x_1, \ldots, x_{2l}) \ \in \mathcal{O}_{\mathbb{C}^{2l+1},0}$$

isolated critical point at 0

$$f(s,x) = W(x) + \sum_{i=1}^{N} s_i\,\phi_i(x), \quad s \in B \subset \mathbb{C}^N$$

miniversal deformation.

Fix a primitive form in the sense of K. Saito

$$\omega \in \Omega^{2l+1}_{(B\times\mathbb{C}^{2l+1})/B},$$

so that B inherits a Frobenius structure (C. Hertling and Saito–Takahashi).

$\partial_a = \partial/\partial t_a$ flat vector fields on B

The period integrals

Milnor fibers

$$X_{t,\lambda} = \{x \mid f(t,x) = \lambda\}$$

For $\alpha \in H_{2l}(X_{0,1};\mathbb{C})$ we define

$$I_\alpha^{(n)}(t,\lambda) = -(2\pi)^{-l}\partial_\lambda^{n+l} d^B \int_{\alpha_{t,\lambda}} d^{-1}\omega \ \in \mathcal{T}_B^*,$$

where

$$\alpha_{t,\lambda} \in H_{2l}(X_{t,\lambda};\mathbb{C})$$

is a flat family of cycles.

Let $\sigma : H_{2l}(X_{0,1};\mathbb{C}) \to H_{2l}(X_{0,1};\mathbb{C})$ be the classical monodromy

σ-twisted representation of V_Q on the Fock space

$$M = \mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \dots]],$$

where $q_k = (q_k^1, \dots, q_k^N)$, consists of a state-field correspondence

$$Y^M(\cdot, \lambda)\cdot : V_Q \otimes M \to M((\lambda^{1/|\sigma|}))$$

satisfying locality and operator product expansion.

We are assuming that $Q = H_{2l}(X_{0,1}; \mathbb{Z})$ and R: vanishing cycles.

Define $Y(\alpha_{-1}, \lambda) =$

$$\sum_k (I_\alpha^{(-k)}(t,\lambda), \partial_i) q_k^i / \sqrt{\hbar} + \sum_k (-1)^{k+1} (I_\alpha^{(k+1)}(t,\lambda), \partial^i) \partial_{q_k^i} \sqrt{\hbar}$$

and

$$Y^M(e^\alpha, \lambda) := U_\alpha \Gamma_\alpha(\lambda)$$

These operators are mutually local ($\because$ the primitive form comes from the splitting of a mixed Hodge structure)

$\Rightarrow$ we can uniquely define $Y^M(a, \lambda)$ for all $a \in V_Q$.

Monodromy invariance

The operators $Y^M(a,\lambda)$ $(a \in V_Q)$ have the form

$$\sum_{I,J} P_a(t,\lambda)\mathbf{q}^I \partial_{\mathbf{q}}^J \ .$$

Analytical continuation agrees with the monodromy action iff the following holds:

$$\oint_C I_\alpha^{(0)}(t,\mu-\lambda) \bullet_t I_\beta^{(0)}(t,0) \ \in \ 2\pi\sqrt{-1}\,\mathbb{Z},$$

where C is a loop in $B-$discriminant, such that the cycles $\alpha, \beta \in Q$ are invariant.

Simple singularities

The monodromy invariance holds.

Theorem 5.[Fan–Jarvis–Ruan] The Frobenius structure for simple singularities comes from a CohFT (FJRW-invariants).

Y^M induces an isomorphism $V_Q \cong M$ as $\widehat{\mathfrak{g}}$-modules.

Theorem 6. [Givental-M] The FJRW potential belongs to the orbit of the affine Lie group, i.e., it is a tau-function of the principal Kac–Wakimoto hierarchy.

$\mathcal{W}$-constraints

Let $w \in \mathcal{W}_Q$; then w is monodromy invariant $\Rightarrow$

$$Y^M(w, \lambda) = \sum_{n \in \mathbb{Z}} w_n \lambda^{-n-1}$$

Theorem 7. [Bakalov-M] The FJRW potential is anihilated by w_n for all $w \in \mathcal{W}_Q$ and all $n \geq 0$.

Remark. The computation of $\mathcal{W}_Q$ is very complicated (Feigin–Frenkel, quantum BGG resolution and character formulas)