INTEGRAL STRUCTURE ON QUANTUM COHOMOLOGY (AFTER IRITANI)

by

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This talk was after the one of Samuel Boissière (on quantum D-module) and the one of Thierry Mignon (on GKZ-system).

PART I
INTEGRAL STRUCTURE AND MIRROR SYMMETRY

Recall that we have : For $\alpha \in H^*(X)$, we define

$$L(\tau, z)\alpha := e^{\tau_2/z}\alpha - \sum_{(d,\ell)\neq(0,0)} \sum_{k=0}^{N} \frac{\phi^k}{\ell!} \left\langle \phi_k, \tau', \dots, \tau', \frac{e^{-\tau_2/z}\alpha}{z+\psi} \right\rangle_{0,\ell+2,d} e^{\int_d \tau_2} = \alpha + O(z^{-1})$$

where $(z+\psi)^{-1} := \sum_{j\geq 0}(-1)^j z^{-j-1}\psi^j = z^{-1}(\dots)$.

On the trivial bundle $F := H^*(X, \mathbb{C}) \times M \times \mathbb{C} \to M \times \mathbb{C}$ where M is an open in $H^*(X)$ containing the *large radius limit*.

$$(\mathcal{O}(F), d_{U\times\mathbb{C}}) \xrightarrow{z^{-\mu}z^{\rho}} \left(\mathcal{O}(F), \overset{\nabla_X := d_X}{\nabla_{z\partial_z} := z\partial_z + \mu - z^{-1}\rho}\right) \xrightarrow{L(\tau,z)} \left(\mathcal{O}(F), \overset{\nabla_X := d_X + z^{-1}X\bullet}{\nabla_{z\partial_z} := z\partial_z - z^{-1}E\bullet + \mu}\right)$$

Recall that $\rho := c_1(TX)$ and $\mu(\phi_i) = \phi_i(deg(\phi_i) - n)/2$.

1. Integral structure

In this section, we define an integral structure on F which will be natural for K-theory and for mirror symmetry.

Let $\alpha \in H^*(X, \mathbb{Z})$ such that $\alpha\cup : H^*(X, \mathbb{Z}) \xrightarrow{\sim} H^*(X, \mathbb{Z})$. This induces a $\mathbb{Z}$-structure on the bundle F as follows. We have the following morphism of global (multivalued)-section

(1)
$$\begin{array}{ccccc} (\mathcal{O}(F), d_{U\times\mathbb{C}}) & \xrightarrow{z^{-\mu}z^{\rho}} & \left(\mathcal{O}(F), \overset{\nabla_X := d_X}{\nabla_{z\partial_z} := z\partial_z + \mu - z^{-1}\rho}\right) & \xrightarrow{L(\tau,z)} & \left(\mathcal{O}(F), \overset{\nabla_X := d_X + z^{-1}X\bullet}{\nabla_{z\partial_z} := z\partial_z - z^{-1}E\bullet + \mu}\right) \\ \big\uparrow & & & & \\ H^*(X, \mathbb{Z}) & \xleftarrow{\quad\alpha\cup\quad} & H^*(X, \mathbb{Z}) & & \end{array}$$

We consider a very special $\mathbb{Z}$-structure induced by the cohomology class

$$\Gamma(TX) := \prod_i \Gamma(1 + \delta_i) = \exp(-\gamma\rho + \sum_{k\geq 2}(k-1)!\zeta(k)\,\mathrm{Ch}_k(TX))$$

where $\rho = c_1(TX)$, δ_i are the Chern root of TX and γ is the Euler constant.

Talk at CIRM in the workshop at CIRM from 9th to 11th March.

Definition 1.1. — We define the $\mathbb{Z}$-structure on the bundle (F, ∇) by the diagram (1) and the following morphism $\Gamma(TX) \cup (2\mathbf{i}\pi)^{\deg/2} : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{C})$. We call it the **$\Gamma$-structure**.

In the following, we will give two reasons why this Γ-structure is good. The first one is a nice behaviour with respect to K-theory. The second one uses mirror symmetry but we need to restrict to smooth toric Fano variety.

2. Γ-structure and K-theory

Recall that the Chern character $\mathrm{Ch} : K(X) \to H^*(X, \mathbb{Q})$ become an isomorphism tensoring by $\mathbb{C}$.

***Theorem 2.1* (Iritani)**. — *For $V_1, V_2 \in K(X)$. We have*

$$S(Z_K(V_1), Z_K(V_2)) = (V_1, V_2)_{K(X)} (:= \chi(V_2^\vee \otimes V_1)).$$

Where Z_K is defined by the following commutative diagram

(2)
$$\begin{array}{ccccc} K(X) & & \xrightarrow{\qquad\qquad Z_K \qquad\qquad} & & \\ \downarrow \mathrm{Ch} & & & & \\ H^*(X, \mathbb{Q}) & \xrightarrow{\Gamma(TX)(2\mathbf{i}\pi)^{\deg/2}} & (\mathcal{O}(F), d_X + z\partial_z) & \xrightarrow[L(\tau,z)z^{-\mu}z^{\rho}]{} & (\mathcal{O}(F), \nabla) \end{array}$$

The reader which is familiar with K-theory would think that $\Gamma(TX)$ is a square root of the Todd class. Indeed, the relation is $\Gamma(TX)\Gamma(TX^\vee) = e^{-\rho/2} Td(TX)$.

3. Γ-function and mirror symmetry

In this section we assume that X is a **smooth toric Fano variety**. Recall that $\mathbf{1} \in H^*(X)$ was the unit. Put $\mathbf{J}(\tau, z) := L(\tau, z)^{-1}\mathbf{1} (= L(\tau, -z)^*\mathbf{1})$. Consider the following diagram

(3)
$$\begin{array}{ccccccc} H^*(X, \mathbb{Z}) & \xrightarrow{\Gamma(TX)(2\mathbf{i}\pi)^{\deg/2}} & (\mathcal{O}(F), d_X + z\partial_z) & \xrightarrow{z^{-\mu}z^{\rho}} & (\mathcal{O}(F), d_X + \nabla_{z\partial_z}) & \xrightarrow{L(\tau,z)} & (\mathcal{O}(F), \nabla) \\ & \searrow & \downarrow & \nearrow_{\mathbf{J}} \; \swarrow & & \nwarrow \nearrow_{\mathbf{1}} & \\ & & U \times \widetilde{\mathbb{C}^*} & & & & \end{array}$$

Remark 3.1. — The $\mathbf{J}$-function is a very important function in the work of Givental. For example, we can recover the quantum product via the $\mathbf{J}$-function as follows : We have $\nabla_{\partial_{t_k}} \mathbf{1} = \phi_k/z$. The previous diagram implies that $\partial_{t_k} \mathbf{J} = L(\tau, z)^{-1}\phi_k/z$. So we deduce that $z^2\partial_{t_i}\partial_{t_j}\mathbf{J} = L(\tau, z)^{-1}\phi_i \bullet_\tau \phi_j$. To compute the quantum product, one should expand $z^2\partial_{t_i}\partial_{t_j}\mathbf{J}$ with respect to the power of z.

Let us restrict the $\mathbf{J}$-function to $H^2(X, \mathbb{C})$ (where the divisor axiom holds) ie. $\tau = \tau_2 + \tau'$ where $\tau' = 0$. Put $\mathbb{J}(\tau_2, z) := \mathbf{J}(\tau_2 + 0, z)$. We also restrict the bundle to $U_2 := U \mid_{\tau'=0}$. Let $\phi_1, \ldots, \phi_r$ the basis of $H^2(X, \mathbb{Z})$ which are in the closure of the Kähler cone of X.

Definition 3.2. — We denote $\Sigma(1)$ the 1-dimensional cone of the fan Σ of X. For any ray ρ, we denote D_ρ the associate toric divisor. We define the I-function which is a cohomological valued function by

$$I(\tau_2, z) := e^{\tau_2/z} \sum_{d \in H^2(X,\mathbb{Z})} e^{\int_d \tau_2} \prod_{\rho \in \Sigma(1)} \frac{\prod_{\nu=D_\rho(d)}^{+\infty}(D_\rho + (D_\rho(d) - \nu)z)}{\prod_{\nu=0}^{+\infty}(D_\rho + (D_\rho(d) - \nu)z)}$$

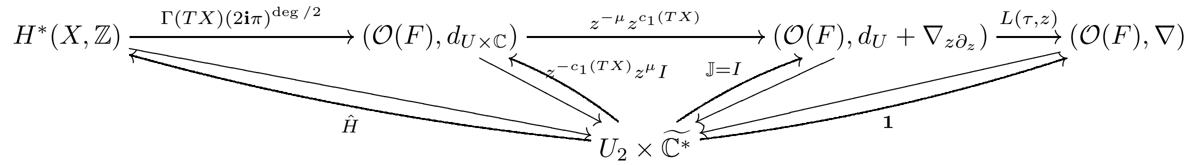
***Theorem 3.3* (Givental)**. — *If X is a smooth toric Fano variety then $I(\tau_2, z) = \mathbb{J}(\tau_2, z)$.*

Proposition 3.4. — *We have $\Gamma(TX) = \prod_\rho (1 + D_\rho)$ and*

$$z^{-c_1(TX)} z^\mu I(\tau_2, z) = \Gamma(TX) z^{-n/2} e^{\tau_2} z^{-c_1(TX)} \sum_{d \in H^2(X,\mathbb{Z})} \frac{e^{\int_d \tau_2} z^{-\int_d c_1(TX)}}{\prod_{\rho \in \Sigma(1)} \Gamma(D_\rho + D_\rho(d) + 1)}$$

$$\hat{H}(\tau_2, z) := z^{-n/2} e^{\tau_2/2\mathbf{i}\pi} z^{-c_1(TX)/2\mathbf{i}\pi} \sum_{d \in H^2(X,\mathbb{Z})} \frac{e^{\int_d \tau_2} z^{-\int_d c_1(TX)}}{\prod_{\rho \in \Sigma(1)} \Gamma(D_\rho/2\mathbf{i}\pi + D_\rho(d) + 1)}$$

where $\hat{H}$ is defined by the following diagram



We can now state the main result of Iritani that is that the integral structure given by the $\Gamma(TX)(2\mathbf{i}\pi)^{\deg/2}$ is related to the integral structure of its mirror. More precisely, we have the following result.

***Theorem 3.5* (Iritani).** — *Put $H(\tau_2, z) := \frac{(2\pi z)^{n/2}}{(-2\pi)^n}\hat{H}(\tau_2, z)$. We have*

$$\int_X H(\tau_2, -z) \cup \mathrm{Td}(TX) = \frac{1}{(2\mathbf{i}\pi)^n}\int_{\Gamma_\mathbb{R}} e^{-W_q/z}\omega_q.$$

where $W_q : Y_q \to \mathbb{C}$ is the mirror of X with $Y_q \simeq (\mathbb{C}^*)^{\#\Sigma(1)-\dim H_2(X,\mathbb{C})}$ and $\Gamma_\mathbb{R} = \{\underline{y} \in Y_q \mid y_\rho > 0\}$.

To see this Theroem in K-theory, we put $H_K(\tau_2, z) := \frac{(2\pi z)^n}{(-2\pi)^n} Z_K^{-1}(\mathbf{1})$.

Corollary 3.6. —

$$S(\mathbf{1}, Z_K(\mathcal{O}_X)) = \frac{1}{(-2\pi z)^{n/2}}\int_{\Gamma_\mathbb{R}} e^{-W_q/z}\omega_q$$

4. z-GKZ system and A-side

Here, we will suppose that X is Fano. There is a generalization of GKZ system where, we introduce an additional variable denoted by z. To do so, we should replace in the formulas of Definition, ∂_{λ_i} by $z\partial_{\lambda_i}$ in the definition of the classical GKZ system (see Thierry's talk).

With the same discussion as in the classical GKZ, for $d \in H_2(X,\mathbb{Z})$, we just look at the operators

$$(\Box'_{d,z} :=)\mathcal{P}_d := \prod_{i:D_i(d)>0} z\delta_i(z\delta_i - z)\cdots(z\delta_i - (D_i(d)-1)z) - q^d \prod_{i:D_i(d)<0} z\delta_i(z\delta_i - z)\cdots(z\delta_i - (-D_i(d)-1)z)$$

Recall that we have

$$\delta_i = \sum_{a=1}^{r} D_i(\beta_a) q_a \partial_{q_a}$$

We define the differential module

$$M_{GKZ} := \mathbb{C}[z, q^{\pm}]\langle z q_a \partial_{q_a}\rangle / \langle \mathcal{P}_d, d \in H_2(X,\mathbb{Z})\rangle.$$

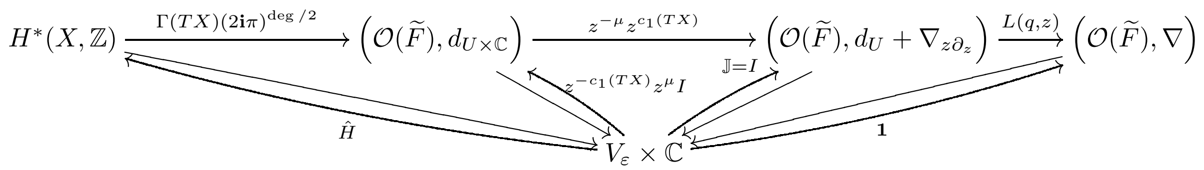
We define the associated sheaf

$$\mathcal{M}_{GKZ} := M_{GKZ} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\varepsilon\times\mathbb{C}}$$

where $V_\varepsilon := \{0 < |q_a| < \varepsilon\}$ is an open in $H^2(X,\mathbb{C})/\operatorname{Pic}(X) \simeq (\mathbb{C}^*)^r$.

Proposition 4.1. — *The sheaf $\mathcal{M}_{GKZ}$ is a finitely generated $\mathcal{O}_{V_\varepsilon\times\mathbb{C}}$-module. The fiber at any point $(q,z) \in V_\varepsilon \times \mathbb{C}$ is less than $\dim_\mathbb{C} H^*(X,\mathbb{C})$.*

In Section 1, we used the variables τ_2, but here we use the variables $q_a = e^{t_a}$. To make this precise, one should quotient the bundle $(\mathcal{O}(F), \nabla)$ with an action of the Picard group of X. The quotient bundle is denoted by $(\mathcal{O}(\widetilde{F}), \nabla)$. With the q_a's variable the large limit point is $q_a = 0$.



Lemma 4.2. — *For any $d \in H_2(X,\mathbb{Z})$, we have*

$$\mathcal{P}_d(\hat{H}(q,z)) = \mathcal{P}_d(I(q,z)) = 0 \text{ and } \mathcal{P}_d\left(\int_\Gamma e^{W_q/z}\omega_q\right) = 0$$

Proposition 4.3. — *The following morphism is an isomorphism*

$$\begin{aligned} M_{GKZ} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\varepsilon\times\mathbb{C}} &\longrightarrow (\mathcal{O}(\widetilde{F}), \nabla) \\ P(z,q,z\partial) &\longmapsto P(z,q,z\nabla)\mathbf{1} \end{aligned}$$

Sketch of proof. — The morphism is well-defined because of Lemma 4.2 and

$$P(z,q,z\nabla)\mathbf{1} = L(q,z)P(z,q,zq_a\partial_{q_a})I(q,z).$$

For $i \in \{1,\dots,m\}$, we have

$$I(q,z) = e^{\sum_{a=1}^r T_a \log q_a/z}(1+O(q,z^{-1}))$$
$$z\delta_i I(q,z) = e^{\sum_{a=1}^r T_a \log q_a/z}(D_i+O(q,z^{-1}))$$

As $L(q,z)\alpha = e^{\sum_{a=1}^r -T_a \log q_a/z}\alpha + O(q)$ and the cohomology of X is generated by the classes D_i, there exist operators $P_j(z,q,z\nabla)$ such that

$$P_j(z,q,z\nabla)\mathbf{1} = \phi_j + O(q)$$

where ϕ_j is a basis of $H^*(X,\mathbb{C})$. This implies the morphism of the proposition is onto. By rank consideration, we conclude. □

4.1. z-GKZ and B-side. — Let X be a smooth toric variety. Denote by $\Sigma(1)$ the set a rays of the fan Σ. Put $m := \#\Sigma(1)$. Denote by $D_1,\dots,D_m$ the toric divisors associated to the rays. We have the following exact sequence

$$0 \longrightarrow H_2(X,\mathbb{Z}) \xrightarrow{\underline{D}} \mathbb{Z}^m \xrightarrow{\beta} N \longrightarrow 0 \tag{4}$$

where $\underline{D} : d \mapsto \sum_{i=1}^m D_i(d)e_i$ and $\beta : e_i \mapsto v_i$ which are the generators of the rays.

The B-side is construct as follows. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to the exact sequence (4), we get

$$0 \longrightarrow \mathrm{Hom}(N,\mathbb{C}^*) \longrightarrow Y := (\mathbb{C}^*)^m \xrightarrow{\mathrm{pr}} \mathcal{M} := \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{C}^*) \longrightarrow 0$$

The Landau-Ginzburg model associated to the toric variety X is

$$\begin{array}{ccc} Y & \xrightarrow{W} & \mathbb{C} \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & \\ \mathcal{M} & & \end{array}$$

where $W = w_1 + \dots + w_m$. For $q \in \mathcal{M}$, we denote $Y_q := \mathrm{pr}^{-1}(q)$ and $W_q := W\,|_{Y_q}$. Notice that Y_q is isomorphic to $(\mathbb{C}^*)^n$ where $n = \mathrm{rk}\, N$. Let $\mathcal{M}^0$ be a Zariski open set of $\mathcal{M}$ where W_q is convenient and non-degenerated. For (q,z) in $\mathcal{M}^0 \times \mathbb{C}^*$, define

$$\mathcal{R}^{\vee}_{\mathbb{Z},(q,z)} := H_n(Y_q, y \in Y_q : \Re e(W_q(y)/z) \ll 0\}, \mathbb{Z})$$

Lemma 4.4. — *The relative homology group $\mathcal{R}^{\vee}_{\mathbb{Z},(q,z)}$ are a local system of rank* $\dim H^*(X,\mathbb{C})$.

We can also define a intersection pairing

$$\mathcal{R}^{\vee}_{\mathbb{Z},(q,-z)} \times \mathcal{R}^{\vee}_{\mathbb{Z},(q,z)} \to \mathbb{Z}.$$

Denote by $R_{\mathbb{Z}}$ the dual local system. Denote by $\mathcal{R} := \mathcal{R}_{\mathbb{Z}} \otimes \mathcal{O}_{\mathcal{M}^0\times\mathbb{C}^*}$. The associated locally free sheaf endowed with a flat connection and a pairing. Identifying Y_q with $(\mathbb{C}^*)^n$, we denote

$$\omega_q = \frac{dy_1 \wedge \dots \wedge dy_n}{y_1 \cdots y_n}.$$

A relative n-differential form

$$\varphi(q,z,y) := f(q,z,y)e^{W_q(y)/z}\omega_q \text{ where } f(q,z,y) \in \mathcal{O}_{\mathcal{M}^0\times\mathbb{C}^*\times Y_q}$$

defines a section of $\mathcal{R}$ via integration over Lefschetz thimbles $\Gamma \in \mathcal{R}^{\vee}_{\mathbb{Z},(q,z)}$:

$$[\varphi](q,z) := \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma f(q,z,y)e^{W_q(y)/z}\omega_q \in \mathcal{O}_{\mathcal{M}^0\times\mathbb{C}^*}.$$

Now we extend the bundle $\mathcal{R}$ over $\mathcal{M}^0 \times \mathbb{C}$ by relative n-form that are regular at $z = 0$. We denote this extension by $\mathcal{R}^{(0)}$.

Proposition 4.5. — *The following morphism is an isomorphism*

$$M_{GKZ} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\varepsilon\times\mathbb{C}} \longrightarrow (\mathcal{R}^{(0)}\,|_{V_\varepsilon\times\mathbb{C}}, \nabla)$$
$$P(z,q,z\partial) \longmapsto P(z,q,z\nabla)[e^{W_q(y)/z}\omega_q]$$

5. Integral structures and Mirror symmetry

In this section, we state the main result of Iritani that is the integral structure defined on both side are isomorphic.

Theorem 5.1. — *We have an isomorphism of between the locally free sheaves* $(\mathcal{O}(\widetilde{F}), \nabla, S(\cdot,\cdot))$ *and* $(\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_R)$ *such that the section* $\mathbf{1}$ *maps to* $[e^{W_q(y)/z}\omega_q]$ *i.e.*

$$\begin{array}{ccc} (\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_R) & \xrightarrow{\quad \text{Mir} \quad} & (\mathcal{O}(\widetilde{F}), \nabla, S(\cdot,\cdot)) \\ & {\scriptstyle [e^{W_q(y)/z}\omega_q]} \nwarrow \quad \nearrow {\scriptstyle \mathbf{1}} & \\ & V_\varepsilon \times \mathbb{C} & \end{array}$$

Moreover, the integral structures coincide via the morphism Mir.

Sketch of proof. — Denote by $\mathcal{O}(\widetilde{F})^\nabla$ the flat section of $\mathcal{O}(\widetilde{F})$. Consider the morphism

$$\begin{aligned} \psi : \mathcal{R}^\vee_{\mathbb{Z},(q,z)} := H_n(Y_q, y \in Y_q : \Re e(W_q(y)/z) \ll 0\}, \mathbb{Z}) &\longrightarrow \mathcal{O}(\widetilde{F})^\nabla \\ \Gamma &\longmapsto s_\Gamma(q,z) \end{aligned}$$

such that for any section $[\varphi]$ of $\mathcal{R}^{(0)}$

$$S(\text{Mir}([\varphi])), s_\Gamma(q,z)) = \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma f(q,z,y) e^{W_q(y)/z}\omega_q$$

where $\varphi = f(q,z,y)e^{W_q(y)/z}\omega_q$.

We have to show that $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$ is equal to $Z_K(K(X))$ which is the $\mathbb{Z}$-structure defined on the A-side.

Firstly, let us show that $s_{\Gamma_\mathbb{R}} = Z_K(\mathcal{O}_X)$ (see diagram (2) for the definition of Z_K). As $\text{Mir}(e^{W_q(y)/z}\omega_q) = \mathbf{1}$, the Corollary 3.6 implies that

$$S(\text{Mir}(e^{W_q(y)/z}\omega_q), Z_K(\mathcal{O}_X)) = \frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_\mathbb{R}} e^{-W_q/z}\omega_q.$$

Let $P_i(q, z, z\partial_{q_a})$ be an differential operator such that $P_i(q,z,z\nabla)\mathbf{1} = \phi_i + O(q)$. Applying this operator to the identity above, we get

$$S(\phi_i + O(q), Z_K(\mathcal{O}_X)) = \frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_\mathbb{R}} P_i \cdot (e^{-W_q/z}\omega_q).$$

We deduce that $s_{\Gamma_\mathbb{R}} = Z_K(\mathcal{O}_X)$.

Secondly, show that $Z_K(K(X)) \subset \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$. For any $L \in \text{Pic}(X)$, we have $Z_K(L) = L \cdot Z_K(\mathcal{O}_X)$. Moreover, the image $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$ is stable by the action of line bundles. So $Z_K(L)$ belongs to $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$. As $K(X)$ is generated by line bundles, we deduce that $Z_K(K(X)) \subset \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$.

Finally, as the pairings coincide and they are unimodular, we conclude that $Z_K(K(X)) = \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$. □

PART II
WHAT ARE THE CHANGES FOR TORIC ORBIFOLDS

The two main theorems 2.1 and 5.1 will also be true for weak Fano toric orbifold. They are many changes when one wants to prove this results for toric orbifolds. We will focus on the "Picard action". Hence, our quantum D-module will be the quotient of (F, ∇, S) by this action.

5.1. Orbifold quantum model D-module. — First, we recall some basic facts about orbifold cohomology. The *inertia stack*, denoted by $\mathcal{IX} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$, is the fiber product over the two diagonal morphisms $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$. The inertia stack is a smooth Deligne-Mumford stack but different components will in general have different dimensions. The identity section gives an irreducible component which is canonically isomorphic to $\mathcal{X}$. This component is called *the untwisted sector*. All the other components are called *twisted sectors*. We thus have

$$\mathcal{IX} = \mathcal{X} \sqcup \bigsqcup_{v\in T} \mathcal{X}_v$$

where T parametrizes the set of components of the twisted sectors of $\mathcal{IX}$.

The orbifold cohomology of $\mathcal{X}$ is defined, as vector space, by $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C}) := H^*(\mathcal{IX},\mathbb{C})$. We have

$$H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C}) = H^*(\mathcal{X},\mathbb{C}) \oplus \bigoplus_{v\in T} H^*(\mathcal{X}_v,\mathbb{C}).$$

We will put $M := H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C})$ in what follows.

To define a grading on M, we associate to any $v \in T$ a rational number called the *age* of $\mathcal{X}_v$. A geometric point (x, g) in $\mathcal{IX}$ is a point x of $\mathcal{X}$ and $g \in Aut(x)$. Fix a point $(x, g) \in \mathcal{X}_v$. As g acts on the tangent space $T_x\mathcal{X}$, we have an eigenvalue decomposition of $T_x\mathcal{X}$. For any $f \in [0, 1[$, we denote $(T_x\mathcal{X})_f$ the sub-vector space where g acts by multiplication by $\exp(2\sqrt{-1}\pi f)$. We define

$$\mathrm{age}(v) := \sum_{f\in[0,1[} f.\dim_{\mathbb{C}}(T_x\mathcal{X})_f.$$

This rational number only depends on v. Let α_v be a homogeneous cohomology class of $\mathcal{X}_v$. We define the *orbifold degree* of α_v by

$$\deg^{\mathrm{orb}}(\alpha_v) := \deg(\alpha_v) + 2\,\mathrm{age}(v).$$

Let $\phi_0, \dots, \phi_N$ be a graded homogeneous basis of $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{Q})$ such that $\phi_0 \in H^0(\mathcal{X},\mathbb{Q})$ and $\phi_1, \dots, \phi_s \in H^2(\mathcal{X},\mathbb{Q})$. Notice that the cohomology classes $\phi_1, \dots, \phi_s$ are in the cohomology of $\mathcal{X}$ *i.e* in the cohomology of the untwisted sector. We denote also by $\phi_0, \dots, \phi_N$ the image of these classes in $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C})$. We will denote by $\underline{t} := (t_0, \dots, t_N)$ the coordinates of M associated to this basis.

6. The trivial bundle and the flat meromorphic connection

Let F be the trivial vector bundle over $\mathbb{C} \times M$ whose fibers are $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C})$. For $i \in \{0, \dots, N\}$, we see ϕ_i as a global section of the bundle F.

Define the vector field, called *the Euler vector field*,

$$\mathfrak{E} := \sum_{i=0}^{N} \left(1 - \frac{\deg^{\mathrm{orb}}(\phi_i)}{2}\right) t_i\partial_i + \sum_{i=1}^{s} r_i\partial_i.$$

where the r_i are rational numbers determined by the equality $c_1(T\mathcal{X}) = \sum_{i=1}^{s} r_i\phi_i$ and ∂_i the vector field $\frac{\partial}{\partial t_i}$.

The big quantum product[(1)] endows the vector bundle F with a product. We define a $\mathcal{O}_M$-linear homomorphism which will turn out to be an Higgs field (*ie.* $\Phi \wedge \Phi = 0$.

$$\Phi : TM \to End\,(F) \;\text{ by }\; \Phi(\partial_i) = \phi_i \bullet_{\underline{t}}\,.$$

In coordinates, we have

$$\Phi = \sum_{i=0}^{N} \Phi^{(i)}(\underline{t})dt_i$$

where $\Phi^{(i)}(\underline{t})$ is the endomorphism $\phi_i\bullet_{\underline{t}}$.

As in the manifold case, define, on the trivial bundle F, the connection

$$\nabla := d_{M\times\mathbb{C}} + \frac{1}{z}\pi^*\Phi + \left(-\frac{1}{z}\Phi(\mathfrak{E}) + \mu\right)\frac{dz}{z}$$

where $\pi : \mathbb{C} \times M \to M$ is the projection and R_∞ is the semi-simple endomorphism whose matrix in the basis (ϕ_i) is

$$\mu(\phi_i) = \phi_i(\deg^{\mathrm{orb}}(\phi_i) - n)/2.$$

As for manifolds, we have that

Proposition 6.1. — *The meromorphic connection ∇ is flat.*

Now, we define the pairing on F. The vector space $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{C})$ is endowed with a nondegenerate pairing which is called the orbifold Poincaré pairing. We denote it by $\langle\cdot,\cdot\rangle$. It satisfies the following homogeneity property:

$$\text{if } \langle\phi_i, \phi_j\rangle \neq 0 \;\text{ then }\; \deg^{\mathrm{orb}}(\phi_i) + \deg^{\mathrm{orb}}(\phi_j) = 2n. \tag{5}$$

We define a pairing S on the global sections $\phi_0, \dots, \phi_N$ of F by

$$S(\phi_i, \phi_j) := \langle\phi_i, \phi_j\rangle.$$

and we extend it by linearity using the rules

$$a(z, \underline{t})S(\cdot,\cdot) = S(a(z, \underline{t})\cdot,\cdot) = S(\cdot, a(-z, \underline{t})\cdot) \tag{6}$$

(1) Usually, working on quantum cohomology, one has either to add the Novikov ring (see section 8.1.3 of Cox-Katz) or to assume that the quantum product converges on some open of M (see Assumption 2.1 in Iritani).

for any $a(z, \underline{t}) \in \mathcal{O}_{\mathbb{C}\times M}$.

Proposition 6.2. — *The pairing $S(\cdot,\cdot)$ is nondegenerate, symmetric and ∇-flat.*

We deduce that the tuple (M, F, ∇, S, n) is a quantum D-module on $\mathbb{C} \times M$.

7. Action of the Picard group on the quantum D-module for a toric orbifold

Here we simplify the exposition by assuming $\mathcal{X}$ is a toric orbifold.

If $\mathcal{X}$ is a toric orbifold, then we have $\mathcal{X} = [Z/G]$ where $G := \mathrm{Hom}(\mathrm{Pic}(\mathcal{X}), \mathbb{C}^*)$ and Z is a quasi-affine variety in some $\mathbb{C}^m$. The inertia stack is parametrized by a finite subset T of G. A line bundle L on $\mathcal{X}$ is given by a character χ_L of G.

We define the action of $\mathrm{Pic}(\mathcal{X})$ on the trivial bundle $F \to \mathbb{C} \times M$ as follows:

1. on the fibers of the bundle we define, for any $L \in \mathrm{Pic}(\mathcal{X})$ and for $\alpha_v \in H^*(\mathcal{X}_v, \mathbb{C})$,

$$L \cdot \alpha_v := \chi_L(v)\alpha_v$$

2. on $M \subset H^*_{\mathrm{orb}}(\mathcal{X}, \mathbb{C})$ we define

$$L \cdot \left(\alpha \oplus \bigoplus_{v\in T/\{1\}} \alpha_v\right) := (\alpha - 2\pi\sqrt{-1}d.c_1(L)) \oplus \bigoplus_{v\in T/\{1\}} \chi_L(v)\alpha_v$$

Proposition 7.1. — *(1) The big quantum product is equivariant with respect to this action: for any classes $\alpha, \beta \in H^*_{\mathrm{orb}}(\mathcal{X}, \mathbb{C})$, for any point $\underline{t} \in M$ and for any $L \in \mathrm{Pic}(\mathcal{X})$, we have*

$$(L \cdot \alpha) \bullet_{L\cdot\underline{t}} (L \cdot \beta) = L \cdot (\alpha \bullet_{\underline{t}} \beta).$$

(2) The pairing $S^{A,\mathrm{sm}}(\cdot,\cdot)$ is invariant with respect to this action.

Hence, we can take the quotient and this will be the quantum D-module.

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