INTEGRAL STRUCTURE ON QUANTUM COHOMOLOGY (AFTER IRITANI)

by

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In these two talks, we will explain a part of the paper of Iritani, titled "An integral structure in quantum cohomology and mirror symmetry for toric orbifolds" Arxiv 0903.1463v3 To simplify the exposition, we restrict to the manifold case.

PART I
FIRST TALK

1. Gromov-Witten invariants with gravitational descendants

Let X be a smooth proper manifold over $\mathbb{C}$. Let $d \in H_2(X,\mathbb{Z})$. We define the moduli space of stable maps to X, denoted by $\overline{\mathcal{M}}_{0,n}(X,d)$. To simplify the exposition, we only consider the geometric point of it.

$$\overline{\mathcal{M}}_{0,n}(X,d) := \left\{ \begin{array}{c} (C,f,(x_1,\ldots,x_n)) \text{ where } C \text{ is a nodal curve of genus } 0, \\ x_i \text{ are distincts marked points on the smooth part of } C \\ \text{and } f \to X \text{ such that } f_*[C]=d \text{ and the automorphism} \\ \text{group of } (C,f,\overline{x}) \text{ is finite} \end{array} \right\} / \sim$$

The moduli space $\overline{\mathcal{M}}_{0,n}(X,d)$ is a smooth proper orbifold of finite type over $\mathbb{C}$. We define the i-th evaluation map

$$\begin{aligned} \mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) &\to X \\ (C,f,\underline{x}) &\mapsto f(x_i) \end{aligned}$$

On $\overline{\mathcal{M}}_{0,n}(X,d)$, we have $\mathcal{L}_1,\ldots,\mathcal{L}_n$ line bundles which are the cotangent bundle of the curve at the marked point x_i i.e.

$$\mathcal{L}_i \mid_{(C,f,\underline{x})} := T^*_{x_i}C.$$

Definition 1.1. — Put $\psi_i := c_1(\mathcal{L}_i)$. Let $\gamma_1,\ldots,\gamma_n$ be in $H^*(X,\mathbb{C})$. We define the Gromov-Witten invariants with gravitational descendants by the formula

$$\left\langle \psi_1^{k_1}\gamma_1,\ldots,\psi_n^{k_n}\gamma_n \right\rangle_{0,n,d} := \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \psi_1^{k_1}\gamma_1 \cup \ldots \cup \psi_n^{k_n}\gamma_n$$

The Gromov-invariant satisfies some properties, we will just give "the divisor axiom". This axiom expresses the special role that plays the classes in $H^2(X,\mathbb{C})$.

Proposition 1.2. — *For $\gamma \in H^2(X,\mathbb{C})$, we have*

$$\begin{aligned} \left\langle \psi_1^{k_1}\gamma_1,\ldots,\psi_n^{k_n}\gamma_n,\gamma \right\rangle_{0,n+1,d} &= \left(\int_d \gamma\right) \left\langle \psi_1^{k_1}\gamma_1,\ldots,\psi_n^{k_n}\gamma_n,\gamma \right\rangle_{0,n,d} \\ &\quad + \sum_i \left\langle \psi_1^{k_1}\gamma_1,\ldots,\psi_i^{k_i-1}\gamma_i\cup\gamma,\ldots,\psi_n^{k_n}\gamma_n,\gamma \right\rangle_{0,n,d} \end{aligned}$$

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Fix $(\phi_0 = \mathbf{1}, \phi_1, \dots, \phi_N)$ a homogeneous basis of $H^*(X, \mathbb{C})$. Denote by $(t_0, \dots, t_N)$ the associated coordinates on $H^*(X)$. Put $\tau := \sum_i t_i \phi_i$. Denote by $(\phi^1, \dots, \phi^N)$ the dual basis with respect to the Poincaré duality.

Definition 1.3. — Let $\alpha, \beta \in H^*(X)$.

$$\alpha \bullet_\tau \beta := \sum_{d \in H_2(X,\mathbb{Z})} \sum_{\ell \geq 0} \sum_{k=1}^{N} \frac{e^{\int_d \tau_2}}{\ell!} \phi^k \langle \alpha, \beta, \tau', \dots, \tau', \phi_k \rangle_{0,\ell+3,d}$$

where $\tau = \tau' + \tau_2$ with $\tau_2 \in H^2(X)$ and $\tau' \in \oplus_{k \neq 1} H^{2k}(X)$.

The neutral element for this product is $\mathbf{1}$.

Assumption 1.4. — We assume that the quantum product is convergent over an open set $U \subset H^*(X)$ such that U contains the following directions :

1. $\tau' \to 0$
2. $\Re e(\int_d \tau_2) \to -\infty$ for any $d \neq 0 \in H_2(X, \mathbb{Z})$.

The limit point is called the **large radius limit**. At this large radius limit, the quantum product become the usual cup product.

2. An integrable connection

Definition 2.1. —
– We define a trivial holomorphic bundle F over $U \times \mathbb{C}$ with fibers $H^*(X)$ ie. $F := H^*(X) \times (U \times \mathbb{C}) \to U \times \mathbb{C}$. We denote z the coordinate on $\mathbb{C}$.
– We define the following meromorphic connection :

$$\nabla_{\partial_{t_i}} := \partial_{t_i} + \frac{1}{z} \phi_k \bullet_\tau \qquad \nabla_{z\partial_z} := z\partial_z - \frac{1}{z} E \bullet_\tau + \mu$$

where

$$E := c_1(TX) + \sum_{k=1}^{N} \left(1 - \frac{\deg \phi_k}{2}\right) t_k \phi_k \qquad \mu(\phi_k) := \frac{1}{2}(\deg \phi_k - n)\phi_k$$

– Denote by $\langle \cdot, \cdot \rangle$ the Poincaré duality on $H^*(X)$. Denote by $\iota : U \times \mathbb{C} \to U \times \mathbb{C}$ sending $(\tau, z) \mapsto (\tau, -z)$. On (F, ∇), we define a pairing

$$S : \iota^* \mathcal{O}(F) \times \mathcal{O}(F) \to \mathcal{O}_{U \times \mathbb{C}}$$

by $S(\phi_i, \phi_j) := \langle \phi_i, \phi_j \rangle$ and $S(a(\tau, -z) \cdot, \cdot) = S(\cdot, a(\tau, z) \cdot)$.

To have a variation of a nc-Hodge structure, we need to define a $\mathbb{Z}$-structure and to check that

– the $\mathbb{Z}$-structure is compatible with the Stokes data
– the opposedness axiom.

In what follows, we will define a $\mathbb{Z}$-structure which is natural from the point of view of mirror symmetry. I do not know is this $\mathbb{Z}$ structure is compatible with the stokes data. The opposedness axiom is true at the large radius limit (cf the paper of Iritani tt^*...)

Remark 2.2. —
1. The global section ϕ_k of F are not flat. Indeed, we have

$$\nabla_{\partial_{t_k}} \mathbf{1} := \frac{1}{z} \phi_k \text{ and } \nabla_{z\partial_z} \mathbf{1} := -\frac{1}{z} E - \frac{n}{2} \mathbf{1}$$

2. The Euler field $\mathfrak{E}$ is defined by

$$\mathfrak{E} := \sum_k r_k \partial_{t_k} + \sum_k \left(1 - \frac{\deg \phi_k}{2}\right) t_k \partial_{t_k}$$

where $c_1(TX) = \sum_k r_k \phi_k$. Put $\mathrm{Gr} := \nabla_{z\partial_z} + \nabla_{\mathfrak{E}} + n/2$. We have

$$\mathrm{Gr} = z\partial_z + d_{\mathfrak{E}} + \mu + n/2 \text{ and } \mathrm{Gr}(\mathbf{1}) = 0.$$

The data $(F, \nabla_{\partial_{t_k}}, \mathrm{Gr})$ is called a graded semi-infinite VHS defined by Serguei Barannikov.

The properties of the Gromov-Witten invariants implies that

Proposition 2.3. — *The connection ∇ is flat and the pairing $S(\cdot, \cdot)$ is ∇-flat.*

For $\alpha \in H^*(X)$, we define

$$L(\tau,z)\alpha := e^{\tau_2/z}\alpha - \sum_{(d,\ell)\neq(0,0)} \sum_{k=0}^{N} \frac{\phi^k}{\ell!} \left\langle \phi_k, \tau', \dots, \tau', \frac{e^{-\tau_2/z}\alpha}{z+\psi} \right\rangle_{0,\ell+2,d} e^{\int_d \tau_2} = \alpha + O(z^{-1})$$

where $(z+\psi)^{-1} := \sum_{j\geq 0}(-1)^j z^{-j-1}\psi^j = z^{-1}(...)$.

Proposition 2.4. — *Put $\rho := c_1(TX)$.*

1. *For $\alpha \in H^*(X)$, we have :*
$$\nabla_k L(\tau,z)\alpha = 0 \qquad\qquad \nabla_{z\partial_z} L(\tau,z)\alpha = L(\tau,z)\left(\mu - \frac{\rho}{z}\right)\alpha.$$
2. *The multi-valued section $L(\tau,z)z^{-\mu}z^{\rho}\alpha$ is ∇-flat.*
3. *Denote $\langle\cdot,\cdot\rangle$ the Poincaré duality. For any $\alpha,\beta \in H^*(X)$, we have $\langle L(\tau,z)\alpha, L(\tau,z)\beta\rangle = \langle\alpha,\beta\rangle$.*
4. *We have $L(\tau,z)^{-1} = L(\tau,-z)$.*
5. *The section $L(\tau,z)$ is characterized by its asymptotic at the large radius limit ie. $L : U\times\mathbb{C} \to \mathrm{GL}_n(H^*(X))$ is the unique application such that for any $\alpha \in H^*(X)$, we have $\nabla_X L(\tau,z)\alpha = 0$ for any vector field X and $L(\tau,z)\alpha \sim e^{-\tau_2/z}\alpha$ at the large radius limit.*

3. Integral structure

Let $\alpha \in H^*(X,\mathbb{Z})$ such that $\alpha\cup : H^*(X,\mathbb{Z}) \hookrightarrow H^*(X,\mathbb{Z})$. This induces a $\mathbb{Z}$-structure on the bundle F as follows. We have the following morphism of global (multivalued)-section

(1)
$$\begin{array}{ccccc} (\mathcal{O}(F), d_{U\times\mathbb{C}}) & \xrightarrow{z^{-\mu}z^{\rho}} & \left(\mathcal{O}(F), \overset{\nabla_X := d_X}{\nabla_{z\partial_z} := z\partial_z + \mu - z^{-1}\rho}\right) & \xrightarrow{L(\tau,z)} & \left(\mathcal{O}(F), \overset{\nabla_X := d_X + z^{-1}X\bullet}{\nabla_{z\partial_z} := z\partial_z - z^{-1}E\bullet + \mu}\right) \\ \uparrow & & & & \\ H^*(X,\mathbb{Z}) & \xleftarrow{\alpha\cup} & H^*(X,\mathbb{Z}) & & \end{array}$$

We consider a very special $\mathbb{Z}$-structure induced by the cohomology class

$$\Gamma(TX) := \prod_i \Gamma(1+\delta_i) = \exp(-\gamma\rho + \sum_{k\geq 2}(k-1)!\zeta(k)\,\mathrm{Ch}_k(TX))$$

where $\rho = c_1(TX)$, δ_i are the Chern root of TX and γ is the Euler constant.

Definition 3.1. — We define the $\mathbb{Z}$-structure on the bundle (F,∇) by the diagram (1) and the following morphism $\Gamma(TX)\cup(2\mathbf{i}\pi)^{\deg/2} : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{C})$. We call it the **$\Gamma$-structure**.

In the following, we will give two reasons why this Γ-structure is good. The first one is a nice behaviour with respect to K-theory. The second one uses mirror symmetry but we need to restrict to toric Fano smooth variety.

3.1. Γ-structure and K-theory.

— Recall that the Chern character $\mathrm{Ch} : K(X) \to H^*(X,\mathbb{Z})$ become an isomorphism tensoring by $\mathbb{C}$.

***Theorem 3.2* (Iritani).** — *For $V_1, V_2 \in K(X)$. We have*

$$S(Z_K(V_1), Z_K(V_2)) = (V_1,V_2)_{K(X)} (:= \chi(V_2^{\vee}\otimes V_1)).$$

Where Z_K is defined by the following commutative diagram

(2)
$$\begin{array}{ccccc} K(X) & & \xrightarrow{\qquad\qquad Z_K \qquad\qquad} & & (\mathcal{O}(F),\nabla) \\ \downarrow{\scriptstyle \mathrm{Ch}} & & & \nearrow{\scriptstyle L(\tau,z)z^{-\mu}z^{\rho}} & \\ H^*(X,\mathbb{Z}) & \xrightarrow{\Gamma(TX)(2\mathbf{i}\pi)^{\deg/2}} & (\mathcal{O}(F), d_X + z\partial_z) & & \end{array}$$

3.2. Γ-function and mirror symmetry. — In this section we assume that X is a **smooth toric Fano variety**. Recall that $\mathbf{1}\in H^*(X)$ was the unit. Put $\mathbf{J}(\tau,z):=L(\tau,z)^{-1}\mathbf{1}(=L(\tau,-z)\mathbf{1})$. Consider the following diagram

Remark 3.3. — The **J**-function is a very important function in the work of Givental. For example, we can recover the quantum product via the **J**-function as follows : We have $\nabla_{\partial_{t_k}}\mathbf{1}=\phi_i/z$. The previous diagram implies that $\partial_{t_k}\mathbf{J}=L(\tau,-z)\phi_i/z$. So we deduce that $z^2\partial_{t_i}\partial_{t_j}\mathbf{J}=L(\tau,-z)\phi_i\bullet_\tau\phi_j$. To compute the quantum product, one should expand $z^2\partial_{t_i}\partial_{t_j}\mathbf{J}$ with respect to the power of z.

Let us restrict the **J**-function to $H^2(X,\mathbb{C})$ (where the divisor axiom holds) ie. $\tau=\tau_2+\tau'$ where $\tau'=0$. Put $\mathbb{J}(\tau_2,z):=\mathbf{J}(\tau_2+0,z)$. We also restrict the bundle to $U_2:=U\mid_{\tau'=0}$. Let $\phi_1,\ldots,\phi_r$ the basis of $H^2(X,\mathbb{Z})$ which are in the closure of the Kähler cone of X.

Definition 3.4. — We denote $\Sigma(1)$ the 1-dimensional cone of the fan Σ of X. For any ray ρ, we denote D_ρ the associate toric divisor. We define the I-function which is a cohomological valued function by

$$I(\tau_2,z):=e^{\tau_2/z}\sum_{d\in H^2(X,\mathbb{Z})}e^{\int_d\tau_2}\prod_{\rho\in\Sigma(1)}\frac{\prod_{\nu=D_\rho(d)}^{+\infty}(D_\rho+(D_\rho(d)-\nu)z)}{\prod_{\nu=0}^{+\infty}(D_\rho+(D_\rho(d)-\nu)z)}$$

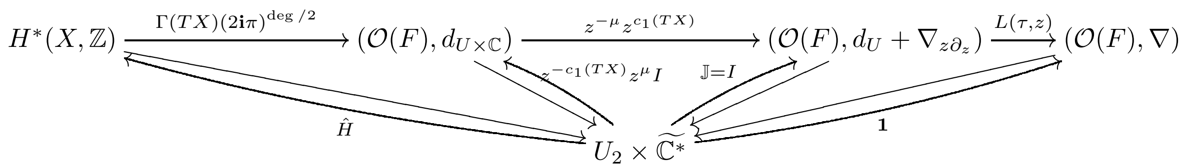
Theorem 3.5 **(Givental).** — *If X is a smooth toric Fano variety then $I(\tau_2,z)=\mathbb{J}(\tau_2,z)$.*

Proposition 3.6. — *We have $\Gamma(TX)=\prod_\rho(1+D_\rho)$ and*

$$z^{-c_1(TX)}z^{\mu}I(\tau_2,z)=\Gamma(TX)z^{-n/2}e^{\tau_2}z^{-c_1(TX)}\sum_{d\in H^2(X,\mathbb{Z})}\frac{e^{\int_d\tau_2}z^{-\int_d c_1(TX)}}{\prod_{\rho\in\Sigma(1)}\Gamma(D_\rho+D_\rho(d)+1)}$$

$$\hat{H}(\tau_2,z):=z^{-n/2}e^{\tau_2/2\mathbf{i}\pi}z^{-c_1(TX)/2\mathbf{i}\pi}\sum_{d\in H^2(X,\mathbb{Z})}\frac{e^{\int_d\tau_2}z^{-\int_d c_1(TX)}}{\prod_{\rho\in\Sigma(1)}\Gamma(D_\rho/2\mathbf{i}\pi+D_\rho(d)+1)}$$

where $\hat{H}$ is defined by the following diagram



We can now state the main result of Iritani that is that the integral structure given by the $\Gamma(TX)(2\mathbf{i}\pi)^{\deg/2}$ is related to the integral structure of its mirror. More precisely, we have the following result.

Theorem 3.7 **(Iritani).** — *Put $H(\tau_2,z):=\frac{(2\pi z)^{n/2}}{(-2\pi)^n}\hat{H}(\tau_2,z)$. We have*

$$\int_X H(\tau_2,-z)\cup\mathrm{Td}(TX)=\frac{1}{(2\mathbf{i}\pi)^n}\int_{\Gamma_\mathbb{R}}e^{-W_q/z}\omega_q.$$

where $W_q:Y_q\to\mathbb{C}$ is the mirror of X with $Y_q\simeq(\mathbb{C}^*)^{\#\Sigma(1)-\dim H_2(X,\mathbb{C})}$ and $\Gamma_\mathbb{R}=\{\underline{y}\in Y_q\mid y_\rho>0\}$.

To see this Theroem in K-theory, we put $H_K(\tau_2,z):=\frac{(2\pi z)^n}{(-2\pi)^n}Z_K^{-1}(\mathbf{1})$.

Corollary 3.8. —

$$S(\mathbf{1},Z_K(\mathcal{O}_X))=\frac{1}{(-2\pi z)^{n/2}}\int_{\Gamma_\mathbb{R}}e^{-W_q/z}\omega_q$$

PART II
SECOND TALK

4. GKZ-system

Definition 4.1. — Let $\{v_1,\ldots,v_m\}\in\mathbb{Z}^n(:=N)$ be a set where $m\geq n$ and $\{v_1,\ldots,v_m\}$ generates $N\otimes\mathbb{R}$. Let $a\in\mathbb{C}^n$. A GKZ-system associated to these data is definied by the following operators:

– for $j \in \{1,\ldots,n\}$, put

$$Z_{j,a} := \sum_{i=1}^{m} v_{ij}\lambda_i \partial_{\lambda_i} + a_j$$

– Let $\Lambda := \{\ell \in \mathbb{Z}^m \mid \sum_{i=1}^m \ell_i v_i = 0\}$. For any $\ell \in \Lambda$, put

$$\Box_\ell := \prod_{\ell_i>0} (\partial_{\lambda_i})^{\ell_i} - \prod_{\ell_i<0} (\partial_{\lambda_i})^{-\ell_i}$$

4.1. GKZ-system associated to a smooth toric variety. — Let X be a smooth toric variety. Denote by $\Sigma(1)$ the set a rays of the fan Σ. Put $m := \#\Sigma(1)$. Denote by $D_1,\ldots,D_m$ the toric divisors associated to the rays. We have the following exact sequence

$$0 \longrightarrow H_2(X,\mathbb{Z}) \xrightarrow{\underline{D}} \mathbb{Z}^m \xrightarrow{\beta} N \longrightarrow 0 \tag{3}$$

where $\underline{D} : d \mapsto \sum_{i=1}^m D_i(d)e_i$ and $\beta : e_i \mapsto v_i$ which are the generators of the rays. Applying the functor $\mathrm{Hom}(-,\mathbb{Z})$ to this exact sequence, we get

$$0 \longrightarrow M \xrightarrow{\beta^*} (\mathbb{Z}^m)^* \xrightarrow{\underline{D}^*} H^2(X,\mathbb{Z}) \longrightarrow 0 \tag{4}$$

where $\beta^* : m \mapsto \sum_{i=1}^m m(v_i)e_i^*$ and $\underline{D}^* : e_i^* \mapsto D_i$.

So the deduce the following equalities

$$\forall d \in H_2(X,\mathbb{Z}),\ \sum_{i=1} D_i(d)v_i = 0 \text{ in } N$$

$$\forall m \in M,\ \sum_{i=1} m(v_i)D_i = 0 \text{ in } H^2(X,\mathbb{Z})$$

$$\sum_{i=1}^m v_i D_i = 0 : \text{ as a map } H_2(X,\mathbb{Z}) \to N \tag{5}$$

To define the GKZ-system associated to X, we put

– $v_1,\ldots,v_m$ are the generators of the rays,
– $a := 0$.

Lemma 4.2. — *We have $\Lambda = H_2(X,\mathbb{Z})$.*

Using notation of Definition 4.1, we have for any $d \in H_2(X,\mathbb{Z})$,

$$\Box_d := \prod_{i:D_i(d)>0} (\partial_{\lambda_i})^{D_i(d)} - \prod_{i:D_i(d)<0} (\partial_{\lambda_i})^{-D_i(d)}$$

Let $\beta_1,\ldots,\beta_r$ be a basis of the Mori cone i.e. cone of effective classes in $H_2(X,\mathbb{Z})$. Let $T_1,\ldots,T_r$ be the Poincaré dual basis in $H^2(X,\mathbb{Z})$. For $a \in \{1,\ldots,r\}$, put

$$q_a := \prod_{i=1}^m \lambda_i^{D_i(\beta_a)}$$

$$q^d := \prod_{a=1}^r q_a^{T_j(d)} = \prod_{i=1}^m \lambda_i^{D_i(d)} \text{ for } d \in H_2(X,\mathbb{Z}). \tag{6}$$

Notice that with this notation, puting $q_a := e^{t_a}$, we have $e^{\tau_2} = \prod_{a=1}^r q_a^{T_a}$.

Lemma 4.3. — *For $i \in \{1,\ldots,n\}$, we have $Z_{i,0}(q^d) = 0$. Moreover, if for all $i \in \{1,\ldots,n\}$, we have $Z_{i,0}(\prod_{j=1}^m \lambda_j^{\ell_j}) = 0$ then $(\ell_1,\ldots,\ell_m) \in \Lambda = H_2(X,\mathbb{Z})$.*

So to solve the GKZ-system, we look for functions that depends on the q_a's variables such that $\Box_d\Phi = 0$.

In the literature, solutions of GKZ-system are

$$\Phi(\lambda_1,\ldots,\lambda_m,\alpha_1,\ldots,\alpha_m) := \sum_{d\in H_2(X,\mathbb{Z})} \prod_{i=1}^m \frac{\lambda_i^{D_i(d)+\alpha_i}}{\Gamma(D_i(d)+1+\alpha_i)}$$

where $\sum_{i=1}^m \alpha_i v_i = a(=0)$ and α_i are parameters.

As we have seen before in (5), we have $\sum_{i=1}^m D_i v_i = 0$, so we deduce a cohomological valued function

$$\begin{aligned}\Phi(\lambda_1,\dots,\lambda_m,D_1,\dots,D_m) &:= \sum_{d\in H_2(X,\mathbb{Z})}\prod_{i=1}^m \frac{\lambda_i^{D_i(d)+D_i}}{\Gamma(D_i(d)+1+D_i)}\\ &= \sum_d q^d \frac{\prod_{a=1}^r q_a^{T_a}}{\prod_{i=1}^m \Gamma(D_i(d)+1+D_i)}\\ &= e^{\tau_2}\sum_d q^d \frac{1}{\prod_{i=1}^m \Gamma(D_i(d)+1+D_i)} \text{ with the notation of (6)}\end{aligned}$$

Compare with Proposition 3.6, the last expression is almost the expression $\hat{H}(2\mathbf{i}\pi\tau_2, z=1)$.

If we want to use the logarithmic derivative in $\Box_d$ i.e. $\delta_i := \lambda_i\partial_{\lambda_i}$ we put for any $d\in H_2(X,\mathbb{Z})$

$$\Box'_d := \prod_{i:D_i(d)>0} \lambda_i^{D_i(d)}\Box_d.$$

We deduce

$$\Box'_d = \prod_{i:D_i(d)>0}\delta_i(\delta_i-1)\cdots(\delta_i-(D_i(d)-1)) - q^d \prod_{i:D_i(d)<0}\delta_i(\delta_i-1)\cdots(\delta_i-(-D_i(d)-1))$$

Notice that we can express the differential operator $\Box_d$ with the q_a's coordinates, namely we have

$$\delta_i = \lambda_i\partial_{\lambda_i} = \sum_{a=1}^r D_i(\beta_a)q_a\partial_{q_a}.$$

4.2. z-GKZ system and A-side. — Here, we will suppose that X is Fano. There is a generalization of GKZ system where, we introduce an additional variable denoted by z. To do so, we should replace in the formulas of Definition 4.1, ∂_{λ_i} by $z\partial_{\lambda_i}$.

With the same discussion as before, for $d\in H_2(X,\mathbb{Z})$, we just look at the operators

$$(\Box'_{d,z} :=)\mathcal{P}_d := \prod_{i:D_i(d)>0} z\delta_i(z\delta_i-z)\cdots(z\delta_i-(D_i(d)-1)z) - q^d\prod_{i:D_i(d)<0} z\delta_i(z\delta_i-z)\cdots(z\delta_i-(-D_i(d)-1)z)$$

Recall that we have

$$\delta_i = \sum_{a=1}^r D_i(\beta_a)q_a\partial_{q_a} = \sum_{a=1}^r \rho_a q_a\partial_{q_a}$$

where $c_1(TX) = D_1+\cdots+D_m = \sum_{a=1}^r \rho_a T_a$.

We define the differential module

$$M_{GKZ} := \mathbb{C}[z,q^{\pm}]\langle zq_a\partial_{q_a}\rangle/\langle \mathcal{P}_d, d\in H_2(X,\mathbb{Z})\rangle.$$

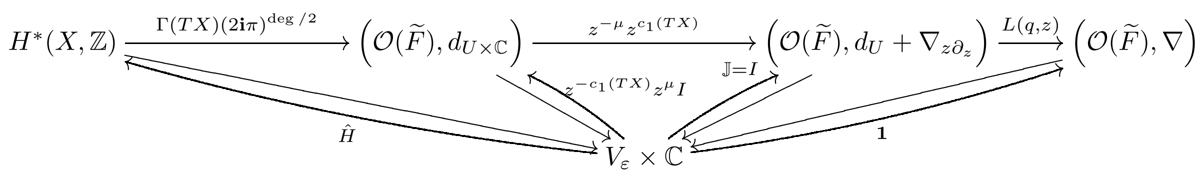
We define the associated sheaf

$$\mathcal{M}_{GKZ} := M_{GKZ}\otimes_{\mathbb{C}[z,q^{\pm}]}\mathcal{O}_{V_\varepsilon\times\mathbb{C}}$$

where $V_\varepsilon := \{0<|q_a|<\varepsilon\}$ is an open in $H^2(X,\mathbb{C})/\operatorname{Pic}(X)\simeq(\mathbb{C}^*)^r$.

Proposition 4.4. — *The sheaf $\mathcal{M}_{GKZ}$ is a finitely generated $\mathcal{O}_{V_\varepsilon\times\mathbb{C}}$-module. The fiber at any point $(q,z)\in V_\varepsilon\times\mathbb{C}$ is less than $\dim_{\mathbb{C}} H^*(X,\mathbb{C})$.*

In Section 3, we used the variables τ_2, but here we use the variables $q_a = e^{t_a}$. To make this precise, one should quotient the bundle $(\mathcal{O}(F),\nabla)$ with an action of the Picard group of X. The quotient bundle is denoted by $(\mathcal{O}(\widetilde{F}),\nabla)$. With the q_a's variable the large limit point is $q_a=0$.



Lemma 4.5. — *For any $d\in H_2(X,\mathbb{Z})$, we have*

$$\mathcal{P}_d(\hat{H}(q,z)) = \mathcal{P}_d(I(q,z)) = 0 \text{ and } \mathcal{P}_d\left(\int_\Gamma e^{W_q/z}\omega_q\right) = 0$$

Proposition 4.6. — *The following morphism is an isomorphism*

$$\begin{aligned} M_{GKZ} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\varepsilon \times \mathbb{C}} &\longrightarrow (\mathcal{O}(\widetilde{F}), \nabla) \\ P(z,q,z\partial) &\longmapsto P(z,q,z\nabla)\mathbf{1} \end{aligned}$$

Sketch of proof. — The morphism is well-defined because of Lemma 4.5 and

$$P(z,q,z\nabla)\mathbf{1} = L(q,z)P(z,q,zq_a\partial_{q_a})I(q,z).$$

For $i \in \{1,\dots,m\}$, we have

$$\begin{aligned} I(q,z) &= e^{\sum_{a=1}^r T_a \log q_a/z}(1 + O(q,z^{-1})) \\ z\delta_i I(q,z) &= e^{\sum_{a=1}^r T_a \log q_a/z}(D_i + O(q,z^{-1})) \end{aligned}$$

As $L(q,z)\alpha = e^{\sum_{a=1}^r -T_a \log q_a/z}\alpha + O(q)$ and the cohomology of X is generated by the classes D_i, there exist operators $P_j(z,q,z\nabla)$ such that

$$P_j(z,q,z\nabla)\mathbf{1} = \phi_j + O(q)$$

where ϕ_j is a basis of $H^*(X,\mathbb{C})$. This implies the morphism of the proposition is onto. By rank consideration, we conclude. ☐

4.3. z-GKZ and B-side. — The B-side is construct as follows. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to the exact sequence (3), we get

$$0 \longrightarrow \mathrm{Hom}(N,\mathbb{C}^*) \longrightarrow Y := (\mathbb{C}^*)^m \xrightarrow{\mathrm{pr}} \mathcal{M} := \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{C}^*) \longrightarrow 0$$

The Landau-Ginzburg model associated to the toric variety X is

$$\begin{array}{ccc} Y & \xrightarrow{W} & \mathbb{C} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \\ \mathcal{M} & & \end{array}$$

where $W = w_1 + \cdots + w_m$. For $q \in \mathcal{M}$, we denote $Y_q := \mathrm{pr}^{-1}(q)$ and $W_q := W\mid_{Y_q}$. Notice that Y_q is isomorphic to $(\mathbb{C}^*)^n$ where $n = \mathrm{rk}\, N$. Let $\mathcal{M}^0$ be a Zariski open set of $\mathcal{M}$ where W_q is convenient and non-degenerated. For (q,z) in $\mathcal{M}^0 \times \mathbb{C}^*$, define

$$\mathcal{R}^\vee_{\mathbb{Z},(q,z)} := H_n(Y_q, y \in Y_q : \Re e(W_q(y)/z) \ll 0\}, \mathbb{Z})$$

Lemma 4.7. — *The relative homology group $\mathcal{R}^\vee_{\mathbb{Z},(q,z)}$ are a local system of rank* $\dim H^*(X,\mathbb{C})$.

We can also define a intersection pairing

$$\mathcal{R}^\vee_{\mathbb{Z},(q,-z)} \times \mathcal{R}^\vee_{(q,z)} \to \mathbb{Z}.$$

Denote by $R_{\mathbb{Z}}$ the dual local system. Denote by $\mathcal{R} := \mathcal{R}_{\mathbb{Z}} \otimes \mathcal{O}_{\mathcal{M}^0 \times \mathbb{C}^*}$. The associated locally free sheaf endowed with a flat connection and a pairing. Identifying Y_q with $(\mathbb{C}^*)^n$, we denote

$$\omega_q = \frac{dy_1 \wedge \cdots \wedge dy_n}{y_1 \cdots y_n}.$$

A relative n-differential form

$$\varphi(q,z,y) := f(q,z,y)e^{W_q(y)/z}\omega_q \text{ where } f(q,z,y) \in \mathcal{O}_{\mathcal{M}^0 \times \mathbb{C}^* \times Y_q}$$

defines a section of $\mathcal{R}$ via integration over Lefschetz thimbles $\Gamma \in \mathcal{R}^\vee_{\mathbb{Z},(q,z)}$:

$$[\varphi](q,z) := \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma f(q,z,y)e^{W_q(y)/z}\omega_q \in \mathcal{O}_{\mathcal{O}^0 \times \mathbb{C}^*}.$$

Now we extend the bundle $\mathcal{R}$ over $\mathcal{M}^0 \times \mathbb{C}$ by relative n-form that are regular at $z = 0$. We denote this extension by $\mathcal{R}^{(0)}$.

Proposition 4.8. — *The following morphism is an isomorphism*

$$\begin{aligned} M_{GKZ} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\varepsilon \times \mathbb{C}} &\longrightarrow (\mathcal{R}^{(0)}\mid_{V_\varepsilon \times \mathbb{C}}, \nabla) \\ P(z,q,z\partial) &\longmapsto P(z,q,z\nabla)[e^{W_q(y)/z}\omega_q] \end{aligned}$$

5. Integral structures and Mirror symmetry

In this section, we state the main result of Iritani that is the integra structure defined on both side are isomorphic.

Theorem 5.1. — *We have an isomorphism of between the locally free sheaves* $(\mathcal{O}(\widetilde{F}), \nabla, S(\cdot,\cdot))$ *and* $(\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_R)$ *such that the section* $\mathbf{1}$ *maps to* $[e^{W_q(y)/z}\omega_q]$ *i.e.*

$$\begin{array}{ccc} (\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_R) & \xrightarrow{\quad \mathrm{Mir} \quad} & (\mathcal{O}(\widetilde{F}), \nabla, S(\cdot,\cdot)) \\ {\scriptstyle [e^{W_q(y)/z}\omega_q]} \nwarrow \searrow & & \swarrow \nearrow {\scriptstyle \mathbf{1}} \\ & V_\varepsilon \times \mathbb{C} & \end{array}$$

Moreover, the integral structures coincide via the morphism Mir.

Sketch of proof. — Denote by $\mathcal{O}(\widetilde{F})^\nabla$ the flat section of $\mathcal{O}(\widetilde{F})$. Consider the morphism

$$\begin{aligned} \psi : \mathcal{R}^\vee_{\mathbb{Z},(q,z)} := H_n(Y_q, y \in Y_q : \Re e(W_q(y)/z) \ll 0\}, \mathbb{Z}) &\longrightarrow \mathcal{O}(\widetilde{F})^\nabla \\ \Gamma &\longmapsto s_\Gamma(q,z) \end{aligned}$$

such that for any section $[\varphi]$ of $\mathcal{R}^{(0)}$

$$S(\mathrm{Mir}([\varphi])), s_\Gamma(q,z)) = \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma f(q,z,y) e^{W_q(y)/z}\omega_q$$

where $\varphi = f(q,z,y)e^{W_q(y)/z}\omega_q$.

We have to show that $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$ is equal to $Z_K(K(X))$ which is the $\mathbb{Z}$-structure defined on the A-side.

Firstly, let us show that $s_{\Gamma_\mathbb{R}} = Z_K(\mathcal{O}_X)$ (see diagram (2) for the definition of Z_K). As $\mathrm{Mir}(e^{W_q(y)/z}\omega_q) = \mathbf{1}$, the Corollary 3.8 implies that

$$(\mathrm{Mir}(e^{W_q(y)/z}\omega_q), Z_K(\mathcal{O}_X)) = \frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_\mathbb{R}} e^{-W_q/z}\omega_q.$$

Let $P_i(q,z,z\partial_{q_a})$ be an differential operator such that $P_i(q,z,z\nabla)\mathbf{1} = \phi_i + O(q)$. Applying this operator to the identity above, we get

$$(\phi_i + O(q), Z_K(\mathcal{O}_X)) = \frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_\mathbb{R}} P_i \cdot (e^{-W_q/z}\omega_q).$$

We deduce that $s_{\Gamma_\mathbb{R}} = Z_K(\mathcal{O}_X)$.

Secondly, show that $Z_K(K(X)) \subset \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$. For any $L \in \mathrm{Pic}(X)$, we have $Z_K(L) = L \cdot Z_K(\mathcal{O}_X)$. Moreover, the image $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$ is stable by the action of line bundles. So $Z_K(L)$ belongs to $\psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$. As $K(X)$ is generated by line bundles, we deduce that $Z_K(K(X)) \subset \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$.

Finally, as the pairings coincide and they are unimodular, we conclude that $Z_K(K(X)) = \psi(\mathcal{R}^\vee_{\mathbb{Z},(q,z)})$.

□

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