A B-model quantum differential system and application to mirror symmetry

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1 Introduction

The aim of this talk is to define a correspondance between quantum cohomology (A-side) and singularity theory (B-side) : this is an aspect of mirror symmetry. The main idea is to attach to each A-model (*resp.* B-model) a *quantum differential system* (and not only a *quantum D-module* in the sense of [9]), that is a *trivial* bundle on $\mathbb{P}^1 \times M$ (M depends on the situation : it can be the germ $(\mathbb{C}^\mu, 0)$ or $\mathbb{C}^*$...), equipped with a meromorphic connection with prescribed poles and a flat "metric" : two models will be mirror partners if their quantum differential systems are isomorphic. On the A-side, such quantum differential systems arise canonically (see Samuel Boissière's talk). On the B-side, the situation is *a priori* less clear : starting with a B-model (a regular, tame function on an affine manifold), it is indeed possible to construct several quantum differential systems, which can be difficult to compare. A canonical construction, which fits very well with mirror symmetry (at least on the known examples!), is given by Hodge theory ("M. Saito's method").

These notes are organized as follows : we first define quantum differential systems (for which Frobenius manifolds are a good motivation). We then explain how to attach (canonically) such a system to a regular, tame, function on an affine manifold. The last part is devoted mirror symmetry : a good test is provided by the small quantum cohomology of weighted projective spaces. We end with few words about the "J-function" of a quantum differential system.

2 Quantum differential systems (or Saito structures)

2.1 A motivation : Frobenius manifolds

Let M be a complex manifold. A *Frobenius structure* on M (see for instance [10], [13]) is defined by the following data (Θ_M denotes the sheaf of holomorphic vector fields on M) :

1. a commutative and associative product $*$ on Θ_M (a Higgs field *i.e* $\Phi : \Theta_M \to \Theta_M \otimes \Omega^1_M$, $\mathcal{O}_M$-linear, such that $\Phi \wedge \Phi = 0$),

2. a nondegenerate, symmetric, flat bilinear form g on Θ_M (its Levi-Civita connection $\bigtriangledown$ is *flat*),
3. two vector fields : e ($\bigtriangledown$-flat, the identity for the product) and the Euler vector field E which gives the homogeneity of the structure[1]. Often (semi-simple adE and in flat coordinates),

$$E = \sum_{i=1}^{\mu} (d_i t_i + c_i)\partial_{t_i}$$

The numbers d_i define the *spectrum* of the Frobenius structure.

These objects satisfy the following compatibility relation : if $\pi : \mathbb{P}^1 \times M \to M$, the connection ∇ which is defined on the trivial bundle $\pi^*\Theta_M$ by

$$\nabla = \pi^* \bigtriangledown + \frac{\pi^*\Phi}{\theta} + (\frac{\Phi(E)}{\theta} + \bigtriangledown E)\frac{d\theta}{\theta}$$

is *flat*[2]. Notice that this connection has a pole of Poincaré rank less or equal to 1 at $\theta = 0$ and a logarithmic pole at $\tau := \theta^{-1} = 0$.

The bilinear form g satisfy the following conditions :

1. *Frobenius property* : $g(\Phi_X(Y), Z) = g(Y, \Phi_X(Z))$, $X, Y, Z \in \Theta_M$,
2. *homogeneity* : there exists $D \in \mathbb{C}$ such that $Eg(X,Y) - g([E,X],Y) - g(X,[E,Y]) = Dg(X,Y)$, $X, Y \in \Theta_M$.

2.2 Quantum differential systems : definition

Conversely, the data

1. $\mathcal{G}$ a trivial[3] vector bundle on $\mathbb{P}^1 \times M$, equipped with a *flat* meromorphic connection ∇, with poles of Poincaré rank less or equal to 1 along $\{0\} \times M$ and logarithmic ones along $\{\infty\} \times M$,
2. S a ∇-flat pairing $\mathcal{O}(\mathcal{G}) \otimes j^*\mathcal{O}(\mathcal{G}) \to \theta^d \mathcal{O}_{\mathbb{P}^1 \times M}$ where $d \in \mathbb{Z}$ and

$$j : \mathbb{P}^1 \times M \to \mathbb{P}^1 \times M$$

is defined by $j(\theta, x) = (-\theta, x)$ (θ coordinate on $\mathbb{P}^1 - \{\infty\}$),
3. φ an isomorphism ("period map") from $i^*_{\{\theta=0\}}\mathcal{G}$ onto Θ_M,

allow to define a Frobenius structure on M. A word of explanation : in a basis of global sections the connection matrix is (because of the order of the poles)

$$(\frac{A_0(\underline{x})}{\theta} + A_\infty(\underline{x}))\frac{d\theta}{\theta} + C(\underline{x}) + \frac{D(\underline{x})}{\theta}$$

[1]For instance, $\bigtriangledown_e E = re$, $r \in \mathbb{C}$
[2]By definition $\Phi(E)$ is the multiplication $E*$
[3]$\mathcal{G} \simeq \pi^*\pi_*\mathcal{G}$ where $\pi : \mathbb{P}^1 \times M \to M$

where $\underline{x} = (x_1, \cdots, x_\mu) \in M$. The *flat* connection $\bigtriangledown$ is first defined on $i^*_{\{\theta=\infty\}}\mathcal{G}$ as the restriction of the flat connection ∇ at $\tau := \theta^{-1} = 0$, (its matrix is $C(\underline{x})$ in the obvious basis) and we then use the identification between $i^*_{\{\theta=\infty\}}\mathcal{G}$ and $i^*_{\{\theta=0\}}\mathcal{G}$, *via* the global sections of $\mathcal{G}$, to define $\bigtriangledown$ on $i^*_{\{\theta=0\}}\mathcal{G}$ *etc...* The connection ∇ can thus be written as

$$\nabla = \bigtriangledown + \frac{\Phi}{\theta} + (\frac{R_0}{\theta} + R_\infty)\frac{d\theta}{\theta}$$

where $\bigtriangledown$ is a connection on $E = i^*_{\{\theta=0\}}\mathcal{G}$ and $\phi : E \to E \otimes \Omega^1 M$ is $\mathcal{O}_M$-linear. We have $\bigtriangledown^2 = 0$, $\Phi \wedge \Phi = 0$ and $\bigtriangledown\Phi = 0$[4]. Notice that S sends the global sections onto $\theta^d \mathcal{O}_M$ and its flatness gives the expected Frobenius and homogeneity properties.

Last, the period map shifts all these structures to Θ_M and the flatness of ∇ gives the expected relations.

Definition 2.2.1 *I will call the tuple* $(M, \mathcal{G}, \nabla, S, d)$ *a* quantum differential system *(or a* Saito structure[5]*) on* M.

Example 2.2.2 (Quantum differential systems associated with the (small, orbifold...) quantum cohomology, A-side, see also S. Boissière's talk) *One can attach a Saito structure to the (small) quantum cohomology of a projective manifold* X *(we assume here that the quantum product* $\circ$ *is everywhere convergent). The trivial bundle* $\mathcal{G}$ *is the one with fibers* $H^*(X, \mathbb{C})$, *that is*

$$\pi : \mathbb{P}^1 \times M \times H^*(X, \mathbb{C}) \to \mathbb{P}^1 \times M$$

where $M = H^*(X, \mathbb{C})$[6]. *Let* $\{\phi_k\}_{k=1}^N$ *be a homogeneous basis of* $H^*(X)$, $\{t_k\}_{k=1}^N$ *be a system of dual coordinates on* $H^*(X)$. *The connection* ∇ *is defined by*

$$\nabla_{\partial_{t_k}} = \partial_{t_k} + \frac{1}{\theta}\phi_k \circ_\tau \ \text{ and } \nabla_{\partial_\theta} = \partial_\theta - \frac{1}{\theta^2} E \circ_\tau + \frac{1}{\theta} R$$

where $\circ_\tau$ *denotes the quantum product, parametrized by* $\tau \in H^*(X, \mathbb{C})$,

$$E := c_1(TX) + \sum_{k=1}^{N} (1 - \frac{1}{2} deg\phi_k) t^k \phi_k$$

and

$$R(\phi_k) := \frac{deg\phi_k}{2}\phi_k.$$

Notice that, by definition, the ϕ_k*'s are flat sections of* $\bigtriangledown$. *The flatness of* ∇ *follows from the associativity and the commutativity of the quantum product* etc... *The pairing* S *(with values in* $\theta^n \mathcal{O}_{\mathbb{P}^1 \times M}$, $d = n$ *here) is induced by* $(a, b) = \int_X a \cup b$, $a, b \in H^*(X)$ *: the homogeneity property follows from the fact that* $(a, b) \neq 0$ *only if* $deg a + deg b = 2n$. *Same thing for the small quantum cohomology, in which case we work over* $M = H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C})$. *Analogous construction for orbifolds.*

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[4]The (flat) connection $\bigtriangledown^{\theta^{-1}} := \bigtriangledown + \frac{\Phi}{\theta}$ is called the *Dubrovin connection*

[5]After K. Saito. It looks like the *quantum D-modules* defined in [9] but is in fact finer because only bundles on $\mathbb{C}$ are considered in *loc. cit*

[6]Replace M by U if the quantum product converges on $U \subset M$

3 Side B : the quantum differential system attached to a regular function

The (Fourier-Laplace) transform of the Gauss-Manin connection and the Brieskorn lattice of a tame regular function on an affine manifold yield the previous data. The idea is to work with the differential system (rather than its solutions, we are not interested in real structures here) satisfied by the Laplace integrals

$$\int_{\Gamma} e^{-\frac{f}{\theta}} \omega$$

where f is a regular function on U (in practise, $U = (\mathbb{C}^*)^n$ or $U = (\mathbb{C})^n$) and $\omega \in \Omega^n(U)$[7]. This system is in fact a meromorphic connection on $\mathbb{P}^1$, with poles at $\theta = 0$ and $\theta = \infty$, *i.e* a free $\mathbb{C}[\theta, \theta^{-1}]$-module G, of finite rank μ, equipped with a (flat) connection ∇ : G is the Laplace transform of the usual Gauss-Manin system[8]. We have

$$G = \frac{\Omega^n(U)[\theta, \theta^{-1}]}{(d - \theta^{-1} df) \wedge \Omega^{n-1}(U)[\theta, \theta^{-1}]}$$

(with words : we work modulo the exacts forms $d(e^{-\tau f}\omega)$). The connection ∇ is defined by

$$\theta^2 \nabla_{\partial_\theta} (\sum_i \omega_i \theta^i) = \sum_i f \omega_i \theta^i + \sum_i i \omega_i \theta^{i+1}$$

(because of the kernel $e^{-\frac{f}{\theta}}$).

Step 1 : construction of a trivial (algebraic) bundle on $\mathbb{P}^1$. Let U_0 (*resp.* U_∞) be the chart of $\mathbb{P}^1$ with coordinate θ (*resp.* τ), centered at 0 (*resp.* ∞). We need

1. a free $\mathbb{C}[\theta]$-submodule in G of maximal rank (in other words, a *lattice* in G, which gives an extension of G at $\theta = 0$), denote it by G_0,
2. an *opposite* to G_0, that is a free $\mathbb{C}[\tau]$-submodule G_∞ (an extension of G at $\theta = \infty$) such that[9]
$$G_0 = G_0 \cap G_\infty \oplus \theta G_0 \tag{1}$$
or/and $G_\infty = G_0 \cap G_\infty \oplus \tau G_\infty$. If $G_0 = \sum_i \mathbb{C}[\theta]\omega_i$, we can take for instance $G_\infty = \sum_i \mathbb{C}[\tau]\omega_i$. The choice of a basis ω thus defines a trivial extension $\mathcal{G}$ of G_0. Notice that the restrictions of $\mathcal{G}$ at $\theta = 0$ and $\theta = \infty$ are isomorphic *via* the global sections $G_0 \cap G_\infty = \sum_i \mathbb{C}\omega_i$, as it follows from equation (1).

[7] the cycle Γ is of course defined in order to give a meaning to the previous integral, see [12].
[8] *via* $\partial_t \to \theta^{-1}$ and $t \to \theta^2 \nabla_{\partial_\theta}$
[9] We have $G = G_0[\tau] = G_\infty[\theta]$ so G_0 and G_∞ glue on $\mathbb{C}^*$ and give a bundle $\mathcal{G}$ on $\mathbb{P}^1$; the decomposition (1) shows that this bundle is trivial.

An natural candidate for G_0 : the *Brieskorn lattice*, the image in G of $\Omega^n(U)[\theta]$ (*i.e* the forms that do not depend on θ^{-1}). We have obviously

$$\theta^2\nabla_{\partial_\theta}G_0 \subset G_0$$

By definition, we have also

$$G_0/\theta G_0 = \Omega^n(U)/df\wedge\Omega^{n-1}(U).$$

The main question : is G_0 a free $\mathbb{C}[\theta]$-module ? Not always, yes if f is moreover *tame* (a necessary, but not sufficient, condition is that $\mu := \dim_{\mathbb{C}}\Omega^n(U)/df\wedge\Omega^{n-1}(U) < +\infty$; in particular, f must have at most isolated critical points). [10]
A basic example : the (Laurent) polynomials which are convenient and nondegenerate with respect to their Newton polygons at infinity[11], for which the freeness result is obtained using a division theorem "with weights", allowing to shift suitably some basis of $G_0/\theta G_0$.

Step 2 : adding a connection with prescribed poles. We want more, a connection on the trivial bundle $\mathcal{G}$ with poles of Poincaré rank less or equal to 1 at $\theta = 0$ and logarithmic poles at $\theta = \infty$. In other words, we want that the matrix of this connection in the basis ω takes the form

$$\left(\frac{A_0}{\theta} + A_\infty\right)\frac{d\theta}{\theta} \tag{2}$$

(*a priori*, this matrix is $(\frac{A_0}{\theta} + A_\infty + \sum_{i=1}^{r} A_i\theta^i)\frac{d\theta}{\theta}$). This is the so-called *Birkhoff problem* for G_0, the most difficult piece (compare with [9]) : the main tool to solve this problem is Hodge theory.[12].
A major difficulty : two different solutions of the Birkhoff problem yield two different bundles (and thus two different quantum differential systems) and this is why we have to define a "canonical " one. It is built with the (global version of the) solution of the Birkhoff problem given by M. Saito's method : the good opposite filtration is built with Deligne's I^{pq}.

Step 3 : the metric. The Gauss-Manin system G of a *tame*, regular function, is self-dual (microlocal Poincaré duality [13]) : if

$$G^* = Hom_{\mathbb{C}[\theta,\theta^{-1}]}(G, \mathbb{C}[\theta,\theta^{-1}]) \text{ and } G_0^* = Hom_{\mathbb{C}[\theta]}(G_0, \mathbb{C}[\theta])$$

[10] Assume that $U = \mathbb{C}^2$: then the rank of G is equal to the one of the (classical) Gauss-Manin system which $\dim_{\mathbb{C}} H^1(F,\mathbb{C}) = \mu + \nu$ where F is the Milnor fiber, μ is the global Milnor number and ν the number of "vanishing cycles at ∞". If G_0 is free then $\nu = 0$. The converse is true.

[11] Let $f = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha \underline{u}^\alpha$ and Γ be the convex hull of the a_α's. Then f is nondegenerate "at infinity" if, for every face Δ of Γ, $f_{|\Delta} = \sum_{\alpha\in\Delta} a_\alpha\underline{u}^\alpha$ does not have critical points on $(\mathbb{C}^*)^n$. f is convenient if 0 is in the interior of Γ.

[12] Here is the statement : *The solutions of the Birkhoff problem are in one-to-one correspondance with the opposite filtrations, stable under the action of the monodromy, to the Hodge filtration defined on the nearby cycles.* Roughly speaking, the oppositness gives decomposition (1) and the stability with respect to the monodromy gives formula (2).

[13] Poincaré duality for the fibers of f extends to the classical Gauss-Manin system $M : 0 \to ker \to M \to M_! \to coker \to 0$ where *ker* and *coker* are, thanks to the tameness assumption, two free $\mathbb{C}[t]$-modules, of finite rank.

we have an isomorphism of connections $G^* \to j^*G$ which sends G_0^* onto $\theta^n j^* G_0$ or equivalently

$$S : G \times j^*G \to \mathbb{C}[\theta, \theta^{-1}]$$

such that

$$S : G_0 \times j^*G_0 \to \theta^n \mathbb{C}[\theta]$$

so that (on G_0)

$$S = S_n\theta^n + S_{n+1}\theta^{n+1} + \cdots +$$

The pairings S_k are called *higher residue pairings* and S_n is precisely the Grothendieck residue defined on $G_0/\theta G_0$. This gives the expected "metric" S.

Last, the form S extends to $\mathcal{G}$ if ω is *adapted* to S, *i.e* $S(\omega_i, \omega_j) \in \mathbb{C}\theta^n$ for all i, j.

Résumé of steps 1-3 : we get a (canonical) quantum differential system (∇, S, n) on a point.

Step 4 **: adding parameters.** A trivial bundle on $\mathbb{P}^1$ is not enough, we need a trivial bundle on $\mathbb{P}^1 \times M$: in other words, we have to extend the previous situation "with parameters". Several technics :

1. to mimic the previous construction, starting with an unfolding F of f (see for instance [6]) : one has to take into account now the covariant derivative of the Gauss-Manin connection with respect to the parameters. For instance, if

$$F(u, \underline{x}) = f(u) + \sum_{i=1}^{r} x_i g_i(u),$$

($\underline{x} = (x_1, \cdots, x_r) \in M$) we have

$$\nabla_{\partial_{x_i}}(\omega\theta^i) = L_{\partial_{x_i}}(\omega)\theta^i - \frac{\partial F}{\partial x_i}\omega\theta^{i-1}$$

(to get this formula take the derivative under $\int$) and, in a basis of G_0^F (the Brieskorn lattice of F, it should be shown that it is free), the matrix of $\nabla_{\partial_{x_i}}$ is *a priori*

$$\frac{C^{(i)}(x)}{\theta} + D^{(i)}(x) + \sum_{r=1}^{p} D_r^{(i)}(x)\theta^r \tag{3}$$

We want more precisely the formula[14]

$$D^{(i)}(x) + \frac{C^{(i)}(x)}{\theta} \tag{4}$$

If it happens to be the case, the expected trivial bundle is $\mathcal{G} = \pi^* E$ where $E := G_0/\theta G_0$. In this setting, notice that one can solve the Birkhoff problem in family if one can solve

[14]And this is the key point : if we do not have formula (4), we loose for instance the flatness of the residual connection.

it for a special value of the prarameter. Even if it seems natural, this method is finally 'highly transcendental', because of the 'critical points vanishing at infinity', see [6].

2. use the fact that the expected bundle on $\mathbb{P}^1 \times M$ is in fact uniquely determined by *initial data* (Dubrovin, Malgrange, Hertling-Manin, see for instance [8] and the references therein). If, for instance, R_0 is *regular* (*i.e* its characteristic polynomial is equal to its minimal polynomial), a *universal* unfolding of our quantum differential system (on a point) exists, and gives a quantum differential system on $M = (\mathbb{C}^\mu, 0)$[15]

Résumé of steps 1-4 : summarizing, we get, see [6], [4]

Theorem 3.0.3 *One can attach a canonical quantum differential system on $M = (\mathbb{C}^\mu, 0)$ to any regular, tame function on an affine manifold.*

Step 5 : the period map In order to get a Frobenius structure, it remains to shift the previous structures onto Θ_M : this is done with the help of a period map, associated with a *primitive* and *homogeneous* section ω,

$$\varphi : \Theta \to G_0/\theta G_0$$

defined by $\varphi_\omega(\xi) := \Phi_\xi(\omega)$[16] The main problem now is that we do not know if such a section always exists (for instance if f is a polynomial function). However, one can find such a section if f is a convenient and nondegenerate Laurent polynomial. The associated Euler vector field is

$$E = \sum_{i=1}^{\mu}(1-\alpha_i)t_i\partial_{t_i} + \sum_j c_j\partial_{t_j}$$

where the α_i's run through the *spectrum* (at infinity) of f and the c_i's are determined by the multiplication by f (on a graded module $gr^V E$).

4 Examples : MPWPS Laurent polynomials

4.1

A major source of examples, linked with mirror symmetry (see below) is the following : let $U = (\mathbb{C}^*)^n$ and

$$f(u_1, \cdots, u_n) = u_1 + \cdots + u_n + \frac{1}{u_1^{w_1}\cdots u_n^{w_n}}$$

[15]Precisely : under the previous assumption, there exists a unique triple of matrices $(A_0(\underline{x}), A_\infty, C(\underline{x}))$, $\underline{x} \in M$, satisfying the expected integrability relations. The trivial bundle on M is the one with basis $1 \otimes \omega$.

[16]ω *primitive* means that φ_ω is an isomorphism (and thus shifts objects from the right to the left) and homogeneous means that it is an eigenvector of R_∞. If R_0 is regular, a primitive section is a cyclic vector of R_0.

where $w_1, \cdots, w_n$ are integers, greater or equal to 1. We will call such functions MPWPS[17] Laurent polynomials. Then $\mu = 1 + w_1 + \cdots + w_n$. The duality and the canonical basis

$$\omega = (\omega_1, \cdots, \omega_\mu)$$

of G_0^f are computed as in [7] [18]. For instance, we have $\omega_1 = \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$, $\omega_2 = \frac{1}{u_1^{w_1} \cdots u_n^{w_n}} \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$, and, for $i \geq 2$, $\omega_i = u_1^{a_1(i)} \cdots u_n^{a_n(i)} \omega_2$ (the class of, of course) for suitable $(a_1(i), \cdots, a_n(i)) \in \mathbb{N}^n$. The matrix of ∇_{∂_θ} in the basis ω is

$$(\frac{A_0^f}{\theta} + A_\infty)\frac{d\theta}{\theta}$$

where

$$A_0^f = \mu \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/w_1^{w_1} \cdots w_n^{w_n} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $A_\infty = diag(\alpha_1, \cdots, \alpha_\mu)$, the α_i's running through the *spectrum at infinity* of the function f. The expected bilinear form (unique up to the multiplication of a non zero constant) is given by

$$S^f(\omega_k, \omega_\ell) = \begin{cases} S^f(\omega_1, \omega_{n+1}) \in \mathbb{C}^*\theta^n & \text{if } 1 \leq k \leq n+1 \text{ and } k + \ell = n+2, \\ \frac{1}{w^w} S^f(\omega_1, \omega_{n+1}) & \text{if } n+2 \leq k \leq \mu \text{ and } k + \ell = \mu + n + 2, \\ 0 & \text{otherwise} \end{cases}$$

where $w^w = w_1^{w_1} \cdots w_n^{w_n}$.[19] The matrix A_0^f is regular, so the previous results give a quantum differential system on $M = (\mathbb{C}^\mu, 0)$.

4.2

Consider now, for $x \in \mathbb{C}^*$,

$$F(u_1, \cdots, u_n, x) = u_1 + \cdots + u_n + \frac{x}{u_1^{w_1} \cdots u_n^{w_n}}.$$

Its Brieskorn lattice G_0^F is free of rank μ over $\mathbb{C}[x, x^{-1}, \theta]$ and we get a basis of G_0^F in which the matrix of the connection is

$$(\frac{A_0^F(x)}{\theta} + A_\infty)\frac{d\theta}{\theta} + (R(x) - \frac{A_0^F(x)}{\mu\theta})\frac{dx}{x}$$

[17] for Mirror Partner of Weighted Projective Spaces

[18] The Laurent polynomial considered in *loc. cit.* is $w_1 u_1 + \cdots + w_n u_n + \frac{1}{u_1^{w_1} \cdots u_n^{w_n}}$, but the computations are analogous

[19] In order to stick with quantum cohomology, it is convenient to choose the normalization $S^f(\omega_1, \omega_{n+1}) = 1/w_1 \cdots w_n \theta^n$.

where

$$A_0^F(x) = \mu \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x/w_1^{w_1}\cdots w_n^{w_n} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $R(x) = diag(c_1, \cdots, c_\mu)$, $c_i \in [0,1[$. We have now

$$S^F(\omega_k, \omega_\ell) = \begin{cases} S^f(\omega_1, \omega_{n+1}) \in \mathbb{C}^*\theta^n & \text{if } 1 \leq k \leq n+1 \text{ and } k+\ell = n+2, \\ \frac{x}{w^w} S^f(\omega_1, \omega_{n+1}) & \text{if } n+2 \leq k \leq \mu \text{ and } k+\ell = \mu+n+2, \\ 0 & \text{otherwise} \end{cases}$$

The point is that we get a quantum differential system *on the whole* $\mathbb{C}^*$ and not only, as before, on a neighborhood of $x = 1$.

Problem. Perform this kind of computations with the B-models considered in [9] :

$$u_0 + \cdots + u_n \text{ restricted at } u_0^{w_0^1} \cdots u_n^{w_n^1} = q_1, \cdots, u_0^{w_0^r} \cdots u_n^{w_n^1 r} = q_r$$

5 Mirror symmetry

5.1 Mirror symmetry via Frobenius manifolds

One can decide that two objects are mirror partners if they produce the same Frobenius manifold. In this sense,

- the cohomology of the projective space $\mathbb{P}^n$ and the Laurent polynomial $u_1 + \cdots + u_n + 1/u_1 \cdots u_n$ are mirror partners, see [1] : it is enough to show that the initial data on both sides are equal ;
- the orbifold cohomology of the weighted projective spaces $\mathbb{P}(w_0, \cdots, w_n)$ and the function $u_0 + \cdots + u_n$, restricted at $u_0^{w_0} \cdots u_n^{w_n} = 1$ are mirror partners (same method, see [11]).

5.2 Mirror symmetry via quantum differential systems

One can more generally say that *two models are mirror partners if the associated quantum differential systems are isomorphic*[20]. For instance, it is the good way to proceed if we are interested in the (small) quantum cohomology of weighted projective spaces : we have a mirror theorem which is much more precise than the one given by Iritani in [9, proposition 4.8] :

[20]Two quantum differential systems $(M_1, H_1, \nabla_1, S_1, n_1)$ and $(M_2, H_2, \nabla_2, S_2, n_2)$ are isomorphic if there exists an isomorphism $(id, \nu) : \mathbb{P}^1 \times M_1 \to \mathbb{P}^1 \times M_2$ and an isomorphism of vector bundles $\gamma : H_1 \to (id, \nu)^* H_2$ compatible with the connections and the metrics

Theorem 5.2.1 *(See [5], compare with [9]) The quantum differential system associated with F above is isomorphic to the one associated with the small orbifold quantum cohomology of the weighted projective spaces $\mathbb{P}(1, w_1, \cdots, w_n)$.*

Correspondance. Let $p = c_1(\mathcal{O}(1)) \in H^2_{orb}(\mathbb{P}(w), \mathbb{C})$ and

$$(p^{\circ_{tp}})^i = p \circ_{tp} \cdots \circ_{tp} p$$

(i times). Then

1. $(p^{\circ_{tp}})^i$ corresponds to ω_i,
2. the matrix of the small quantum multiplication $p\circ_{tp}$ in the basis $((p^{\circ_{tp}})^i)$ is equal to $A_0^F(e^t)$ which is the matrix of multiplication by ω_2 on $G_0^F/\theta G_0^F$ (in the basis induced by ω). This gives a correpondance of the products,
3. the α_i's correspond to half of the orbifold degrees.

The other advantage of this point of view is that we can then compute 'limits' using classical tools in the theory of meromorphic connections : V-filtration, nearby cycles ...This gives a(nother) precise meaning/interpretation to the process "put $q = 0$" in quantum cohomology. More generally, it should be the natural way to produce *logarithmic Frobenius manifolds* in the sense of [12].

5.3 The J-functions of a quantum differential system (?)

Let $\mathcal{S} = (M, \mathcal{G}, \nabla, S, d)$ be a quantum differential system on M : we thus have a flat connection

$$\nabla = \bigtriangledown + \frac{\Phi}{\theta} + (\frac{R_0}{\theta} + R_\infty)\frac{d\theta}{\theta}$$

on the trivial bundle $\mathcal{G}$. Recall the (flat) Dubrovin connection $\bigtriangledown^\tau := \bigtriangledown + \tau\Phi$ where $\tau = \theta^{-1}$. Let ω be a $\bigtriangledown$-flat basis of $\mathcal{G}$. There exists a matrix $P(\underline{x}, \tau)$, holomorphic in x and formal in τ, such that the matrix of $\bigtriangledown^\tau$, in the basis $e = \omega P$, is the zero matrix. We can normalize P such that $P(x, 0) = I$, because ω is $\bigtriangledown$-flat. We will call P *fundamental solution.* We will call it *conformal* if moreover the matrix ∇_{∂_τ} has the Levelt normal and *symmetric* if in addition $P^*P = I$ where * denotes the adjoint with respect to S. A fundamental and conformal solution is convergent in τ. There always exist fundamental, conformal and symmetric solutions (the two first points follow standard technics in the theory of differential systems ; the last one is a little bit more involved). Let P be a fundamental, conformal and symmetric solution, α be a primitive and homogeneous section of our quantum differential system (in the previous examples, α is the volume form $\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_n}{u_n}$). The function $J = P^{-1}\alpha$ is called *a* J- function of the quantum differential system. Notice that this function depends on P. This J-function determines the product (the Higgs bundle) of the quantum differential system. On the A-side, there is a kind of "canonical" J-function (see for instance [3] and [9] where the corresponding canonical P is denoted by L). *Question* : give an interpretation of this canonical J-function on the B-side.

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