Irregular Connections & Hitchin systems and Kac Moody Root systems

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Basic Aim

Initial steps in classification of

- meromorphic Hitchin systems
- moduli spaces of meromorphic connections on curves
- Painlevé type differential equations
- certain hyperkähler manifolds

i.e. "wild nonabelian Hodge structures"

§1 McKay correspondence & quiver varieties

graphs + other data $\longrightarrow$ hyperkähler manifolds

$\Gamma \subset SU(2)$ finite group

Define the McKay graph of Γ as follows:

$$I = \{\text{nodes}\} = \{\text{irreducible representations of } \Gamma\}$$
$$= \{V_0 = \mathbb{C}, V_1, \ldots, V_r\} \text{ say}$$

Now compute the decomposition

$$\mathbb{C}^2 \otimes V_i = \bigoplus_{j \in I} a_{ij} V_j$$

defining representation of Γ

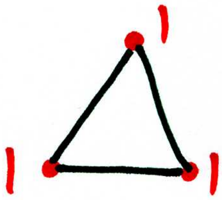
integer multiplicities

Define edges Q s.t. there are a_{ij} edges between i & j

McKay observed the graphs which arise are the simply laced affine Dynkin diagrams

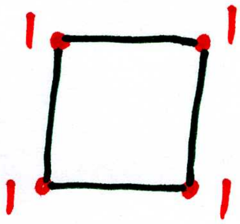
A_n , D_n , E_6, E_7, E_8

E.g.:



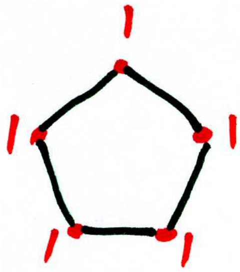
$\mathbb{Z}/3$

$\hat{A}_2$



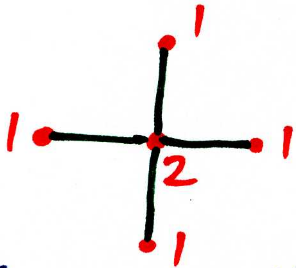
$\mathbb{Z}/4$

$\hat{A}_3$



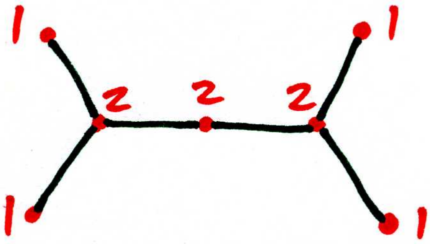
$\mathbb{Z}/5$

$\hat{A}_4$



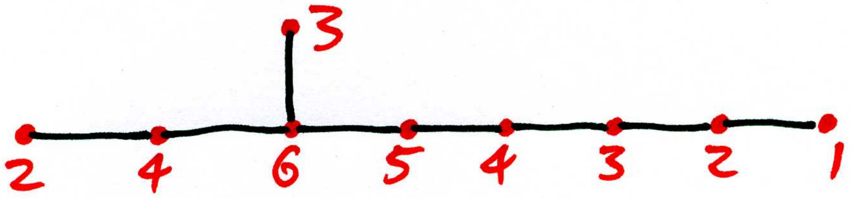
$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$\hat{D}_4$



binary dihedral

$\hat{D}_6$



binary icosahedral

$\hat{E}_8$

Now consider $\mathbb{C}^2/\Gamma$ (Kleinian singularity/rational double point)

and its minimal resolution $\mathcal{N} = \widetilde{\mathbb{C}^2/\Gamma}$

Theorem (Kronheimer)

- $\mathcal{N}$ admits a family of complete hyperkähler metrics
- They may be constructed in terms of the McKay graph of Γ
- All such metrics asymptotic to $\mathbb{C}^2/\Gamma$ arise this way

Some cases found earlier by physicists ("gravitational instantons")

$\hat{A}_1$ $T^*\mathbb{P}^1$ Eguchi-Hanson

$\hat{A}_n$ Gibbons-Hawking

Basic idea of construction

Graph Q, nodes I

Let $\underset{\sim}{d} = (d_i)$ be vector of dimensions

Put vector space $\mathbb{C}^{d_i}$ at node i for all nodes i

Consider • vector space $\mathbb{W}$ of linear maps in both directions along each edge

[If $\mathbb{C}^{d_i}$ given standard Hermitian form & Q oriented, then $\mathbb{W}$ becomes a (flat) hyperkähler vector space]

• group $U(\underset{\sim}{d}) = \prod U(d_i)$ product of unitary groups

Now use standard process to get new hyperkähler manifolds:

Perform the hyperkähler quotient of $\mathbb{W}$ by $U(\underset{\sim}{d})$ at a generic central value of the hyperkähler moment map

In fact this process works for any graph and dimension vector $\underset{\sim}{d}$

$\rightsquigarrow$ family of hyperkähler manifolds $\mathcal{N}$

(if $\underset{\sim}{d}$ indivisible & generic value used)

If nonempty

$$\dim_{\mathbb{C}} \mathcal{N} = 2 - (\underset{\sim}{d}, \underset{\sim}{d})$$

$$(\underset{\sim}{d}, \underset{\sim}{d}) = \sum_{i,j} d_i C_{ij} d_j$$

$$C = 2 - A$$

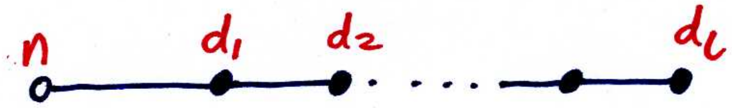
$A = (a_{ij})$ adjacency matrix of graph

"Nakajima quiver varieties"

— big impact in geometric representation theory

Minor modification:

Label each node as 'open' or 'closed' and only quotient by subgroup of $U(\underset{\sim}{d})$ at closed nodes

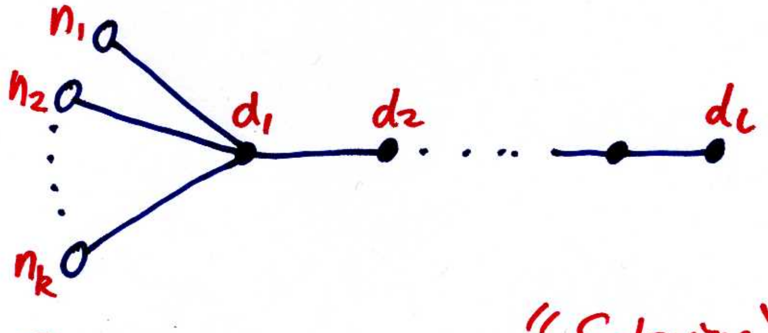
E.g. 

Q

$\rightsquigarrow \mathcal{N} \cong \overline{\mathcal{O}} \subset \mathfrak{gl}_n(\mathbb{C})$ or (partial) resolution

[Kraft – Procesi, Nakajima, Crawley-Boevey ...]

Note: $\mathcal{N}$ unchanged if Q replaced by



$\Sigma n_i = n$

"Splaying"

§2 Hitchin moduli spaces and generalisations

Riemann surface + other data $\longrightarrow$ hyperkähler manifold

Simplest case:

Σ compact Riemann surface

$G = \mathbb{C}^*$

Take $\mathcal{M} = H^1(\Sigma, G)$

Three descriptions:

De Rham • $\{$line bundles with holomorphic connections on $\Sigma\}/\sim$

Betti • $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$

Dolbeault • $T^* \mathrm{Jac}(\Sigma) \cong H^1(\Sigma, \mathbb{C}) / H^1(\Sigma, \mathbb{Z})$
by Hodge theory

Three algebraic structures, two inequivalent complex structures
– flat hyperkähler manifold

Hitchin spaces (usual picture with punctures)

Choose
- complex reductive group $G = K_{\mathbb{C}}$
- smooth projective curve Σ
- distinct points $a_1, \ldots, a_m \in \Sigma$
- conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_m \subset G$

(+ parabolic str.)

$$\Downarrow$$

Hyperkähler manifold $\mathcal{M}$

$\mathcal{M}_{Dol}$ $\qquad$ $\mathcal{M}_{DR}$ $\qquad$ $\mathcal{M}_{Betti}$

Hitchin, Donaldson, Corlette, Simpson, Nakajima, ...

For $G = GL_n(\mathbb{C})$, ignoring stability conditions

$\mathcal{M}_{Betti}$ is a space of representations of

$\pi_1(\Sigma \setminus \{a_i\})$ in G

Riemann Hilbert isomorphism $\cong$

[loop around $a_i \longmapsto$ conjugacy class $\mathcal{C}_i$]

$\mathcal{M}_{DR}$ is a space of rank n vector bundles with meromorphic connections having simple poles at $\{a_i\}$

nonabelian Hodge theory $\cong$

$\sim$ linear systems of differential equations on Σ with regular singularities

$\mathcal{M}_{Dol}$ is a space of meromorphic Higgs pairs

(V, Φ) $\quad \Phi \in H^0(\Omega^1(\mathrm{End}\,V)(\Sigma a_i))$

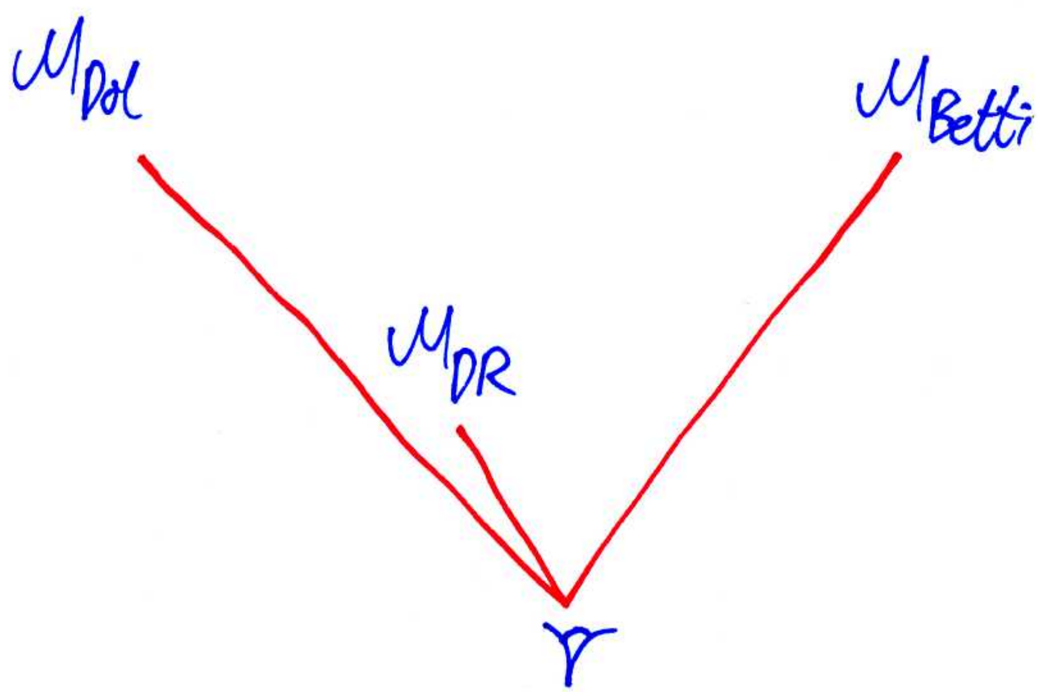
– fibred by Lagrangian abelian varieties (Hitchin's integrable system)

$\mathcal{M}_{Dol}$

$\mathcal{M}_{DR} \underset{\text{RH isom.}}{\overset{\sim}{\text{———}}} \mathcal{M}_{Betti}$

complex analytic geometer

two spaces!



complex algebraic geometer

three spaces, two of which are very close (deformation)

E_9 ("E_8 ALG space") — GL_6, $\mathbb{P}^1$, $m=3$, special orbits

Blow up 9 points on the smooth locus of a cuspidal cubic in $\mathbb{P}^2$ & remove strict transform of cubic

1) Get $\mathcal{M}_{Dol}$ if 9 points sum to zero
(elliptically fibred — Hitchin fibration)

2) Else get $\mathcal{M}_{DR}$ — (deformation) (cf. PB arxiv 0706)

3) $\mathcal{M}_{Betti}$ got by blowing up $\mathbb{P}^2$ in 8 points
& removing a nodal $\mathbb{P}^1$ (Etingof–Oblomkov–Rains 0

Qn: Can this story be extended to spaces of meromorphic connections with higher order poles, i.e. irregular singularities?

Some motivation:

- Appearance in classification of certain 2d quantum field theories (Cecotti-Vafa, Dubrovin)

- Lots of important irregular singular differential equations studied classically

- More examples of integrable systems, unifying lots of classical examples

- Natural arena for Painlevé equations / isomonodromic deformations

Wild Hitchin spaces

Basically fixing conjugacy class of monodromy around puncture $\Leftrightarrow$ connection with simple pole & residue in fixed adjoint orbit

$$\frac{A}{z}\,dz \qquad \begin{array}{l} A \in \mathcal{O} \subset \mathfrak{g} = Lie(G) \\ \exp(2\pi i \mathcal{O}) = \mathcal{C} \subset G \end{array}$$

$\Leftrightarrow$ fixing $G[\![z]\!]$ isomorphism class of connection

Generalisation — allow higher order poles in fixed formal isomorphism class

$$\left(\frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \dots + \frac{A_2}{z^2} + \frac{A_1}{z}\right) dz + \dots$$

Here: assume $A_1, \dots, A_k \in \mathfrak{h}$ (Cartan subalg. $\mathfrak{g}$)

- generic condition (follows e.g if $A_k \in \mathfrak{h}_{reg}$)

Meromorphic Higgs bundles ($\mathcal{M}_{Dol}$) much studied

[Fix $G[\![z]\!]$ orbit of principal part of Higgs field Φ at each pole, similarly]

<u>Theorem</u> (Bottacin, Markman) ~ 1993 & Beauville, Adams-Harnad-Hurtubise, Reiman-Semenov Tian Shansky, Adler-van Moerbeke if genus = 0

$\mathcal{M}_{Dol}$ is an algebraically completely integrable system

(~ fibred by Lagrangian abelian varieties)

Theorem (Biquard – PB)
2004

- wild nonabelian Hodge correspondence

$$\mathcal{M}_{Dol} \cong \mathcal{M}_{DR}$$

[map ← earlier by Sabbah]

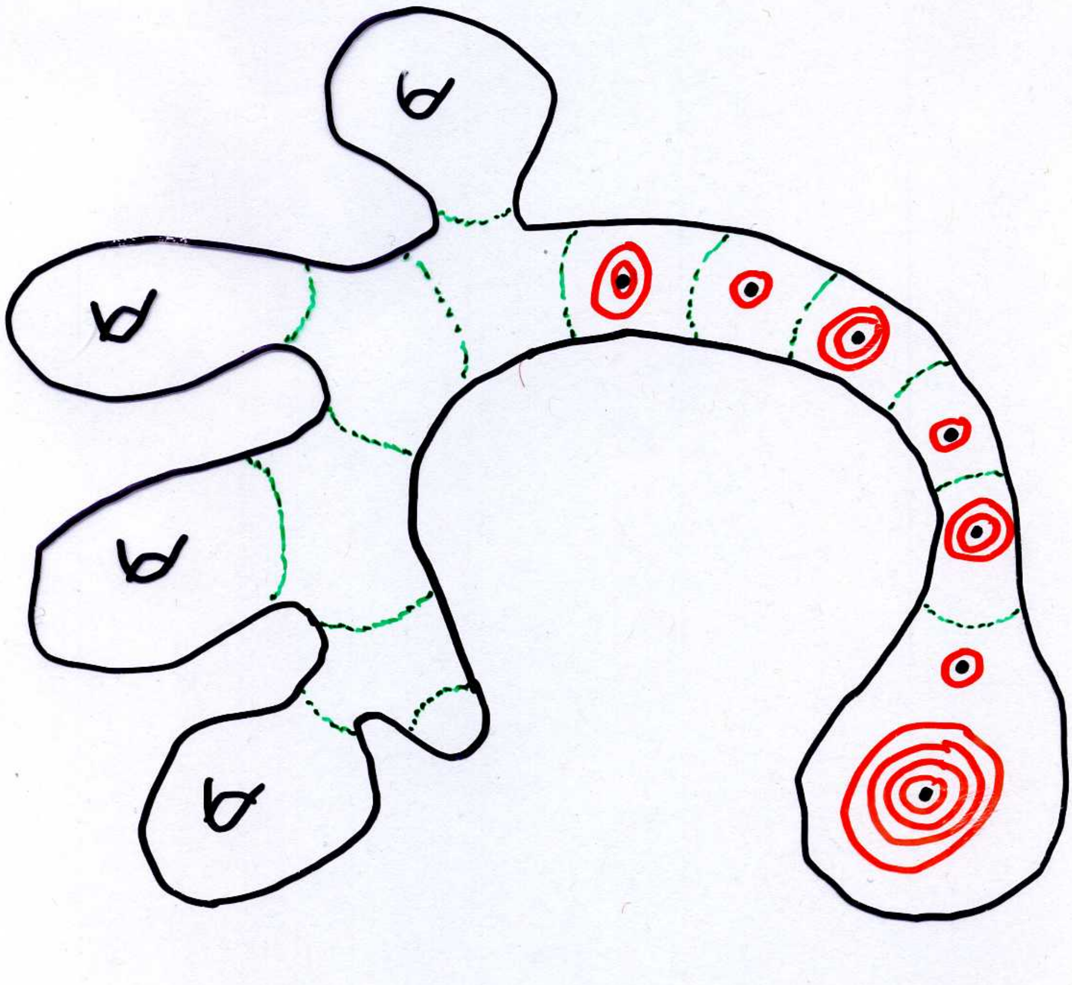
- Hyperkähler metrics (complete if spaces are smooth)

[generic residues $\{A_i\}$ ⇒ smoothness]

$\mathcal{M}_{\mathrm{Betti}}$ – can be described as space of certain representations of the "wild fundamental group" of Martinet–Ramis (Tannakian viewpoint)

– or more directly via Stokes multipliers

[Cf. PB Adv. Math '01, Duke '07]



Qn What set of hyperkähler manifolds $\mathcal{M}$ arise in this way?

Are they well parameterised by the input data? (Torelli type question)

i.e. is the map

$$G, \text{curve}, \text{points}, \text{formal types}, \ldots \longmapsto \mathcal{M}$$

from input data to hyperkähler manifolds injective?

Yes $g>1$, no poles, SL_n (Biswas–Gomez '01)

No In general:

Can do Fourier-Laplace / Nahm transform of meromorphic connections on $\mathbb{P}^1$

(Hyperkähler isometry by S. Szabo '05)

Lets fix $\Sigma = \mathbb{P}^1$ & try to construct invariants

§3 **Isomonodromic deformations**

[[Nonabelian Gauss-Manin connections]]

Usual cohomology:

variety X $\Longrightarrow$ vector space $H^*(X,\mathbb{C})$

family of varieties $\Longrightarrow$ vector bundle V with flat connection

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ & & \downarrow \\ & & B \end{array} \qquad \Longrightarrow \qquad \begin{array}{ccc} H^*(X,\mathbb{C}) & \hookrightarrow & V \\ & & \downarrow \\ & & B \end{array}$$

Explicitly get "Picard-Fuchs equations"
- linear differential equations coming from geometry

⊛ Same story works replacing $\mathbb{C}$ by G ⊛
[at least for H^1, & also in "wild" version]

Simplest nontrivial case

$$G = SL_2(\mathbb{C}), \quad X_t = \mathbb{P}^1 \setminus \{0, t, 1, \infty\}$$

[regular singularities, fixed monodromy classes]

$$\begin{array}{ccc} X_t & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{t\} & \subset & \mathbb{P}^1\setminus\{0,1,\infty\} \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{M}_t & \hookrightarrow & \widetilde{\mathcal{M}} \\ & & \downarrow \\ & & \mathbb{P}^1\setminus\{0,1,\infty\} \end{array}$$

nonlinear fibre bundle with flat connection

$\dim_{\mathbb{C}} \mathcal{M}_t = 2$ here, so in explicit coordinates flat connection is a 2nd order nonlinear differential equation (Painlevé VI equation)

"nonlinear differential equations coming from geometry"

—nowadays arise throughout mathematics and physics (Einstein manifolds, Frobenius manifolds, geometry of the string equation, ...)

The Painlevé Equations

$$\text{PI:}\quad y'' = 6y^2 + t$$

$$\text{PII:}\quad y'' = 2y^3 + ty + \alpha$$

$$\text{PIII:}\quad y'' = \frac{(y')^2}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

$$\text{PIV:}\quad y'' = \frac{(y')^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$$\text{PV:}\quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}$$

$$\begin{aligned}\text{PVI:}\quad y'' = {} & \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\ & + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters.

Qn What set of systems of nonlinear (overdetermined partial) differential equations arise in this way?

E.g. (Harnad '94)

Painlevé $\underline{VI}$ also arises as isomonodromy equation for case $G = GL_3$ on $\mathbb{P}^1$ with

- one order two pole &
- one order one pole

$$\left[D = 2(0) + (\infty) \right]$$

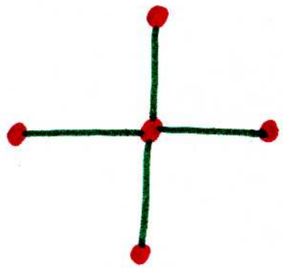
- isomorphic moduli spaces
- view as different "representations" or "realizations" of same nonlinear differential equation

Lets fix $\Sigma = \mathbb{P}^1$ and try to construct invariants ...

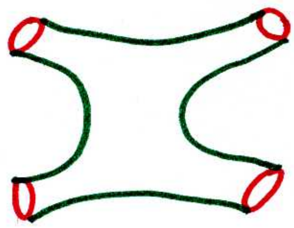
§4 Graphs and Hitchin spaces

Rough summary

Graph Q $\qquad\qquad$ Quiver variety $\mathcal{N}(Q)$

 $\Longrightarrow$ $\mathcal{N}(\hat{D}_4)$

Riemann surface $\qquad\qquad$ wild Hitchin space

 $\Longrightarrow$ $\mathcal{M}$

and some of these Hitchin spaces (with $\dim_{\mathbb{C}} \mathcal{M} = 2$) are "spaces of initial conditions" of Painlevé equations

Basic examples

	Approximations $\mathcal{M}^*$	$\mathcal{M}$
1	$\mathcal{O}/\!\!/_{\Lambda} H$ $\mathcal{O} \subset \mathfrak{g}^*$	$\mathcal{L}/\!\!/_{\Lambda} H$ $\mathcal{L} \subset G^*$ dual Poisson Lie gp
2	$H {}_{\Lambda_2}\backslash\!\!\backslash T^*G /\!\!/_{\Lambda_1} H$	$H {}_{\Lambda_2}\backslash\!\!\backslash \mathcal{D} /\!\!/_{\Lambda_1} H$ $\mathcal{D} \subset (G \times G^*)^2$ Lu-Weinstein sympl. double groupoid
3	special ALE spaces e.g. A_{1-3}, D_4, E_{6-8} $\sim \widetilde{\mathbb{C}^2/\Gamma}$	Okamoto Painlevé spaces "2d Hitchin systems"

Now if $\Sigma = \mathbb{P}^1$ $\quad \mathcal{M} = \mathcal{M}_{DR} = \{(V, \nabla)\}/\sim$

is well approximated by moduli space $\mathcal{M}^*$

where vector bundle V is holomorphically trivial

Typically $\mathcal{M}^* \subset \mathcal{M}$ open subset

Easy observation 1

In the case of Painlevé VI $\quad (SL_2, 4$ simple poles$)$

$$\mathcal{M}^* \cong \mathcal{N}(\widehat{D}_4)$$

Similarly (if $G = GL_n, SL_n$) for any number of simple poles

$$\mathcal{M}^* \cong \mathcal{N}(Q)$$

for some star-shaped graph Q

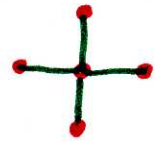
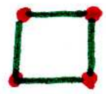
$\#$legs $= \#$ simple poles

– used by Crawley-Boevey in work on Deligne-Simpson problem

Other cases where $\mathcal{M}^*$ is a quiver variety?

In the 1980's K. Okamoto constructed and studied "spaces of initial conditions" of the Painlevé equations and he computed their symmetry groups

E.g.

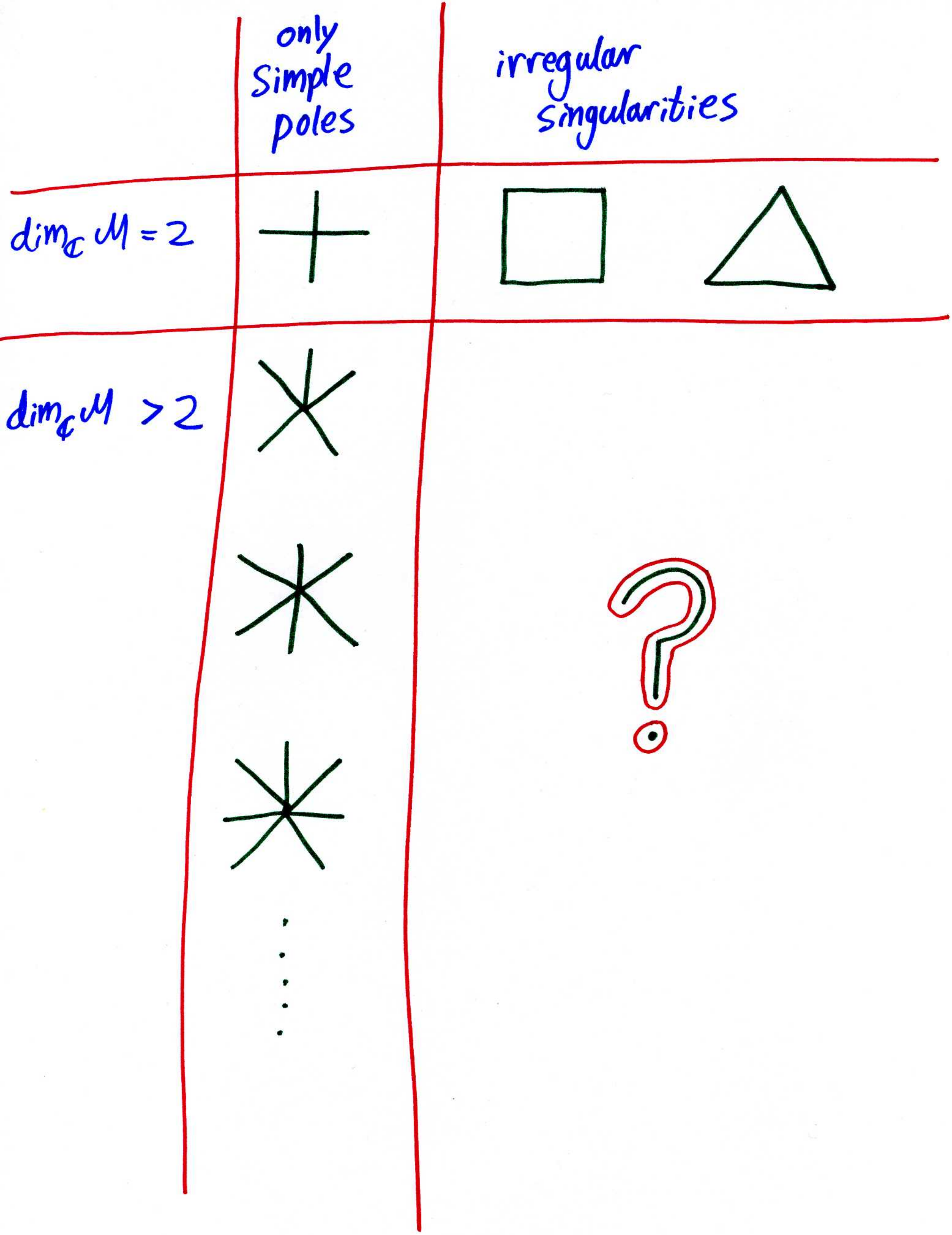
Painlevé equation		
VI	$W_{aff}(D_4)$	
V	$W_{aff}(A_3)$	
IV	$W_{aff}(A_2)$	

(Easy) observations ② & ③ (PB 0706.2634)

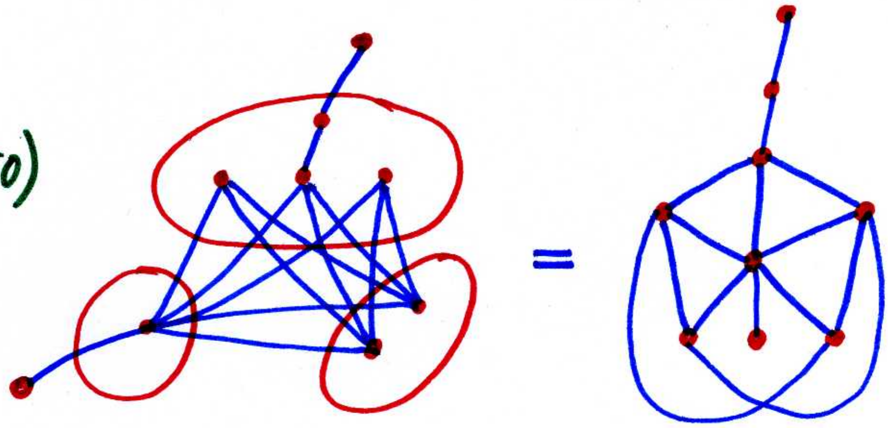
For Painlevé V $(GL_2, D = (0) + (1) + 2(\infty))$ $\mathcal{M}^* \cong \mathcal{N}(\hat{A}_3)$

For Painlevé IV $(GL_2, D = (0) + 3(\infty))$ $\mathcal{M}^* \cong \mathcal{N}(\hat{A}_2)$

- start to suspect spaces $\mathcal{M}$ (not just $\mathcal{N}$) intrinsically attached to certain special graphs
  - will call them "Hitchin graphs"
- need to see how to "read" connection data from such graphs


only simple poles
irregular singularities
$\dim_{\mathbb{C}} \mathcal{M} = 2$
$\dim_{\mathbb{C}} \mathcal{M} > 2$

Theorem (PB 0806.1050)



① Any complete k-partite graph is a Hitchin graph

② So is any graph obtained by gluing a 'leg' on to each node of a complete k-partite graph

③ Each such graph may be 'read' (in terms of meromorphic connections) in $k+1$ ways

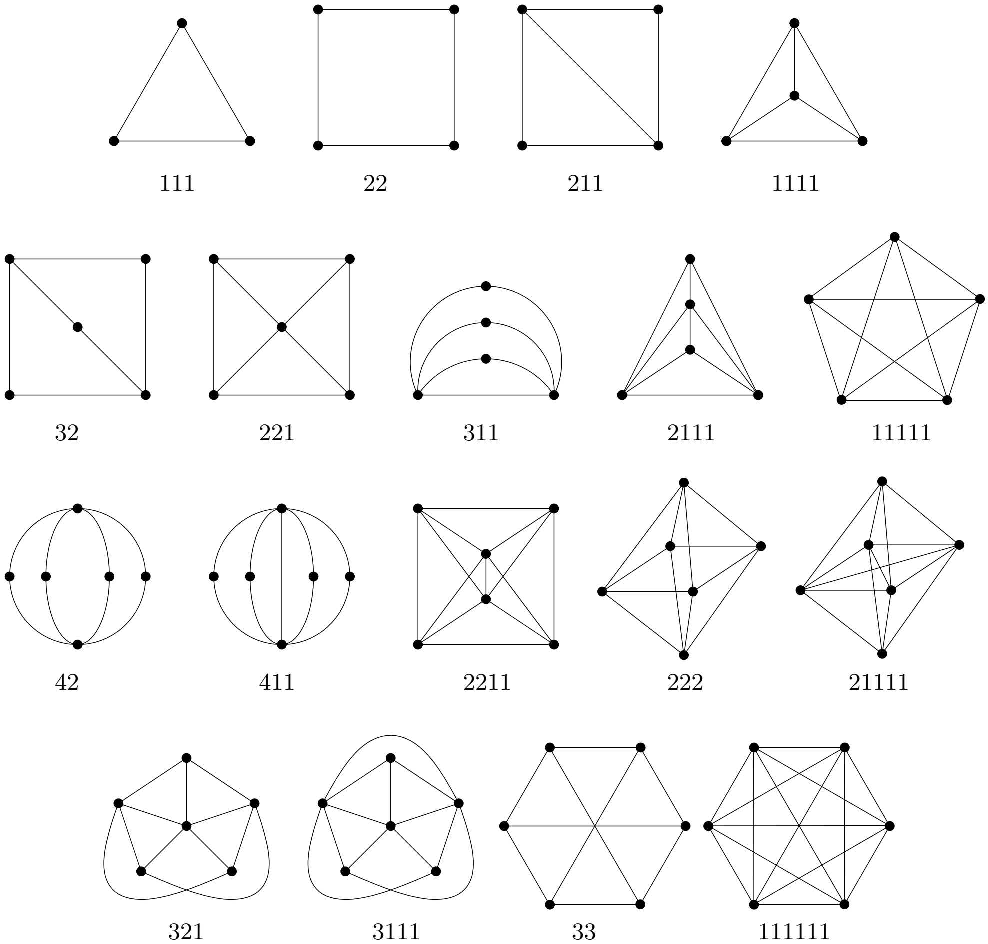


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)

General Pattern

Quiver Q, nodes I

$N \subset I$ nodes of central (k-partite) 'nucleus'

$N = P_1 \amalg P_2 \amalg \cdots \amalg P_k$

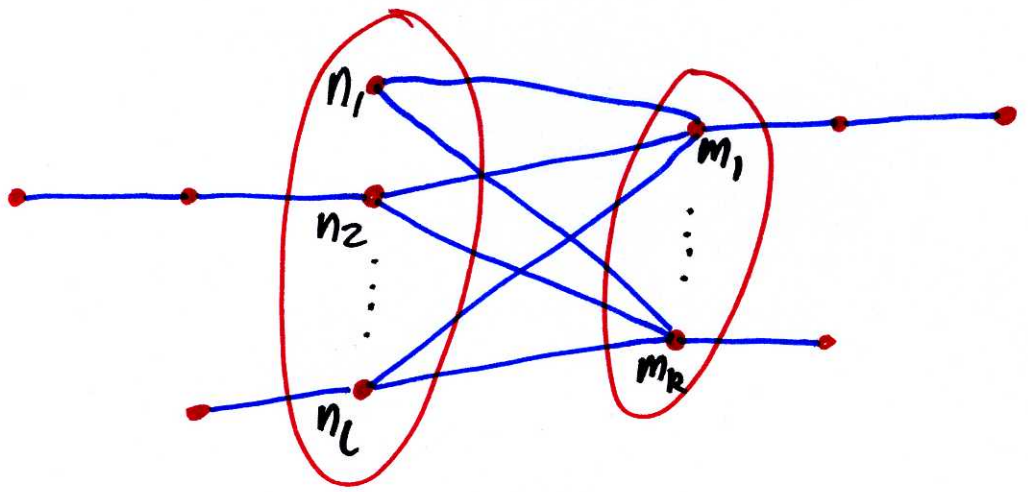
Let $P \subset N$ be a part, or empty

– then can 'read' Q as moduli space of meromorphic connections with a pole of order 3 and $(\#P)$-simple poles, on vector bundles of rank $\left(\sum_{i \in N \setminus P} d_i\right)$ = sum of dimensions of nodes in other parts

Degenerate cases:

If bipartite ($k=2$)/star shaped can sometimes reduce order 3 pole to 2nd or 1st order

E.g. $k=2$ (bipartite case)

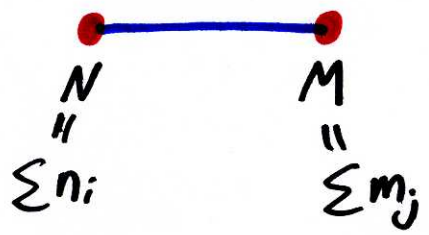


Reading 1 rank $= \sum_1^l n_i$, k simple poles
1 double pole

Reading 2 rank $= \sum_1^k m_i$, l simple poles
1 double pole

Reading 3 rank $= \sum m_i + \sum n_j$, 1 triple pole

(compare with: Harnad, Jimbo–Miwa–Mori–Sato)



E.g.

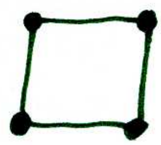
Rank 2, 2 simple poles + 1 double pole

or Rank 4, 1 triple pole

Basic example

$A, B, C \in \mathfrak{gl}_2$ generic

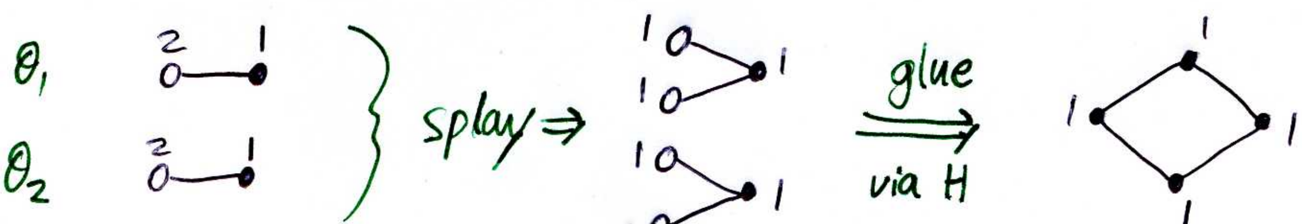
$$
\begin{tikzcd}
 & \text{Painlevé V eqn} \arrow[dr, "\text{Okamoto symmetry}"] & \\
\left(\frac{A}{z} + \frac{B}{z-1} + C\right)dz \arrow[ur, "\text{isomonodromy}"] \arrow[d, "\text{moduli space}"] & & \text{Affine } A_3 \arrow[d, "QV"] \\
\mathcal{M}^* \arrow[rr, phantom, "\cong"] & & \mathcal{N}
\end{tikzcd}
$$



Qn: How to "read" the connection from the square?

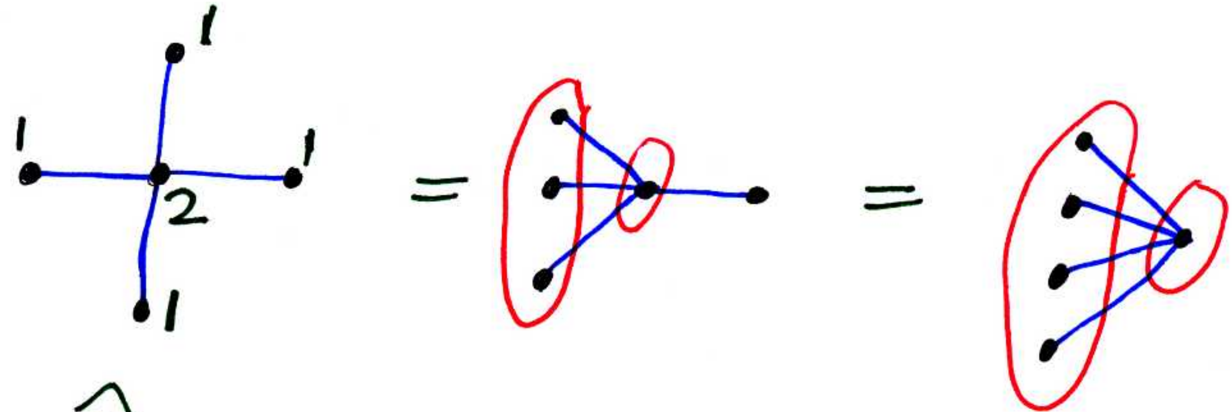
– tensor so A, B rank 1 & suppose C regular semisimple

$$\mathcal{M}^* \cong (\mathcal{O}_1 \times \mathcal{O}_2) /\!\!/_{\lambda} H \qquad \begin{cases} \mathcal{O}_i \subset \mathfrak{gl}_2 \text{ rk 1 orbits} \cong T^*\mathbb{C}^2 /\!\!/_{\lambda} \mathbb{C}^* \\ H = \mathrm{stab}(C) \cong \mathbb{C}^* \times \mathbb{C}^* \end{cases}$$



Stars: $\Gamma(n,1)$ (with legs)

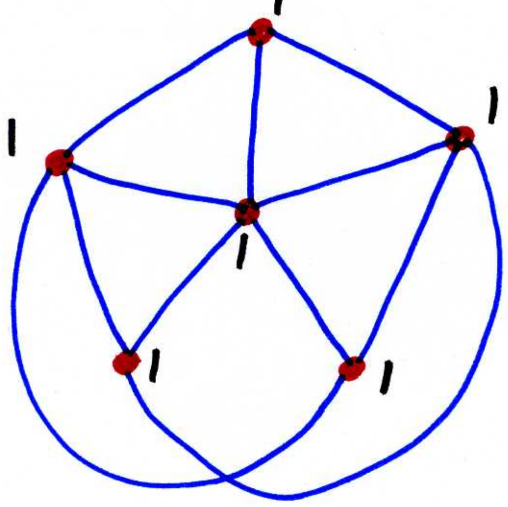
e.g



$\widehat{D}_4$

Rank	poles
6	3
4	2+1
3	2+1
2	1+1+1+1

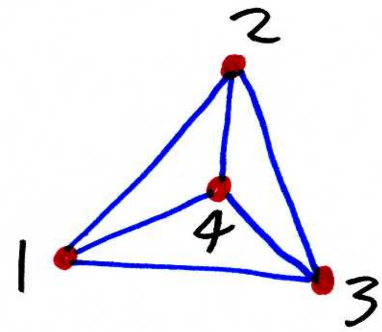
[Additive / $\mathcal{M}^*$ version of isom. $2 \cong 4$ for $\Gamma(n,1)$ is complexification of "Gelfand MacPherson duality" ~ dilogarithm]



Tripartite ⇒ 4 readings

Rank	poles
6	3
5	3+1
4	3+1+1
3	3+1+1+1

E.g. Tetrahedron



$\dim_{\mathbb{C}} \mathcal{M} = 12$, four partite graph

$\Rightarrow$ 5 readings

rank of vector bundles	pole orders
10	3
9	3+1
8	3+1
7	3+1
6	3+1

Ulterior motive for attaching a graph to $\mathcal{M}$

$\rightsquigarrow$ get a Kac-Moody root system

1. Get precise criteria for existence of stable connections in $\mathcal{M}^*$ (phrased in terms of roots)

   - extending work of Crawley-Boevey on (additive) Deligne-Simpson problem (simple pole case)

2. Get "reflection functors" - action of KM Weyl group on auxiliary data

Claim These induce more isomorphisms between $\mathcal{M}$'s

[Typical orbits infinite]

Given graph Γ, nodes I, $n = \# I$

- Cartan matrix $C = 2 - A$ $(n \times n)$

  $A_{ij} = \#$ edges node $i \leftrightarrow$ node j

- Root lattice $\mathbb{Z}^I = \bigoplus_{i \in I} \mathbb{Z} \varepsilon_i$ has bil. form $(\,,\,)$

$$(\varepsilon_i, \varepsilon_j) = C_{ij}$$

- Weyl group $W \circlearrowleft \mathbb{Z}^I$ generated by $\{s_i\}_{i \in I}$

$$s_i(x) = x - (x, \varepsilon_i)\varepsilon_i$$

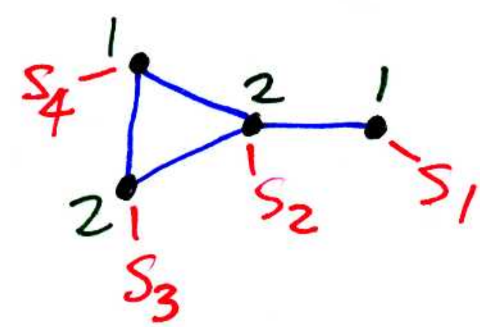
  (& dual reflections $r_i \circlearrowleft \mathbb{C}^I$ s.t $r_i(\lambda) \cdot s_i(x) = \lambda \cdot x$)

- Root system $\subset \mathbb{Z}^I$ (real & imaginary roots)

View dimension vector $\underset{\sim}{d} \in \mathbb{Z}^I$

Example of W action

A_2^{++}



Read e.g as
Rk 3, poles 3+1

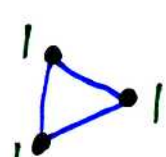
$\underset{\sim}{d} = (1,2,2,1)$

Here $W \underset{\text{index 2}}{\supset} W^+ \cong PSL_2(E)$ $\qquad E = \mathbb{Z}[\omega]$ (Eisenstein integers)

(cf. Feingold–Kleinschmidt–Nicolai '08)

① Let $w = s_1 s_4 s_1 s_2 s_4 s_1 s_3 s_1$

Compute $w^n(1,2,2,1) \rightsquigarrow$ Read as connections on bdles of rank $n^2+(n-1)+(n-2)^2$

② $s_1 s_2 s_3 (1221) = (0111) \Rightarrow$ 

So $\mathcal{M}^* \cong A_2$ ALE space $\quad (\dim_{\mathbb{C}} = 2)$

Extensions

1. Higher order pole ✓ (need multiple edges)

2. $\geqslant 2$ irregular singularities

   $\mathcal{M}^*$ not a quiver variety

   $\rightsquigarrow$ more general picture (~bows)

(e.g. Rk 2, 2+2 $\mathcal{M}^* \cong D_2$ ALF space)

Other directions

① **Stokes algebras** ~ quiver description of corresponding monodromy and Stokes data

$\cup$

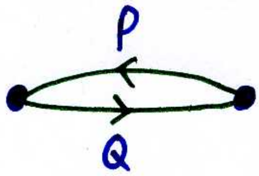
multiplicative preprojective algebras of Crawley-Boevey & Shaw

$\cup$

generalised DAHA of Etingof-Oblomkov-Rains

② **Isomonodromy**

- generalise viewpoint of Jimbo-Miwa-Mori-Sato & Harnad on the JMMS equations ($\neq$ Schlesinger equations)



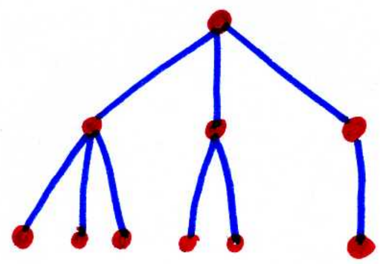
Double of complete graph with 2 nodes

Double of complete graph with k nodes (for any k)

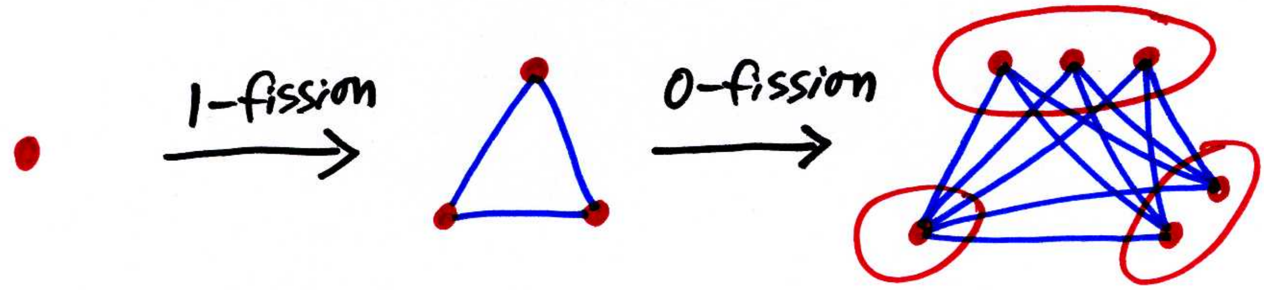
Fission picture

Partitions $\longleftrightarrow$ Height 3 rooted trees

$3+2+1$



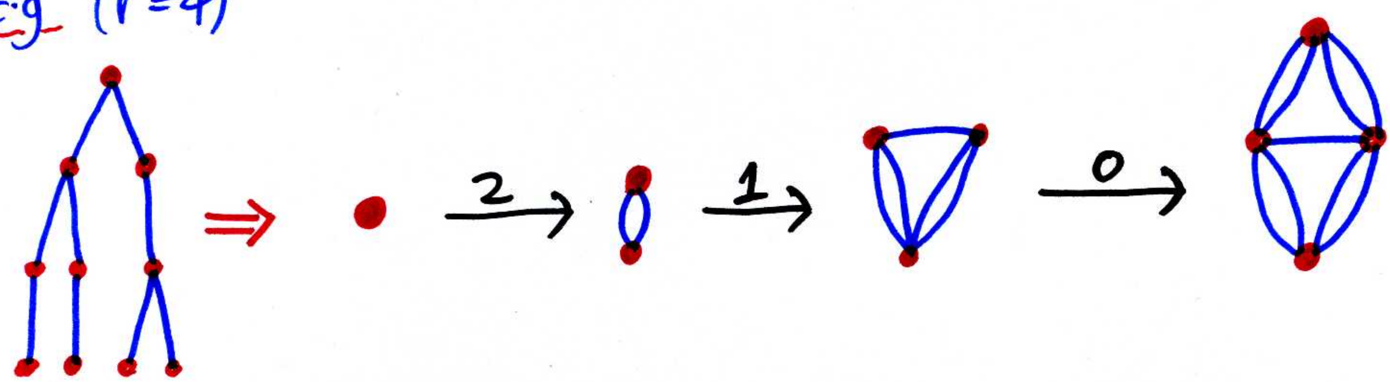
$\longleftrightarrow$ complete k-partite graphs



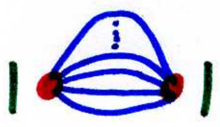
$\sim$ pole of order 3

For pole of order r, do $(r-2)$-fission, ..., 0-fission determined by height r rooted tree

E.g. $(r=4)$



E.g.



n edges

$\mathcal{M}^* \cong T^*\mathbb{CP}^{n-1}$

Calabi's examples