Irregular connections

Dynkin diagrams
and
Fission

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SEDIGA Luminy 2012

Themes:

- (wild) non-abelian Hodge correspondence on curves & hyperkähler moduli spaces
- nonlinear symplectic braid group actions
- example moduli spaces on $\mathbb{P}^1$ $\left(\mathcal{M}^*_{DR} \subset \mathcal{M}_{DR}\right)$
- symplectic geometry of wild character varieties
- (Logahoric connections & Grothendieck-Brieskorn-Springer)

Wild nonabelian Hodge theory on curves

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- Σ compact smooth complex algebraic curve
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Definition If $a \in \Sigma$, an irregular type Q at a is an element $Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\mathcal{O}})$

If z is a local coordinate vanishing at a

$$\hat{\mathcal{O}} = \mathbb{C}[\![z]\!], \quad \hat{K} = \mathbb{C}(\!(z)\!)$$

$$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t} = \mathrm{Lie}(T)$$

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"irregular curve" $\left\{ \begin{array}{l} \bullet\ \Sigma \text{ compact smooth complex algebraic curve} \\ \bullet\ a_1, \dots, a_m \in \Sigma \text{ distinct points} \\ \bullet \text{ irregular types } Q_i \text{ at } a_i,\ i = 1, \dots, m \end{array} \right.$

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  $\left((\theta_i)_{jj} \in [0,1) \quad j = 1, \dots, n \right)$

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Let $\mathfrak{h}_i = C_{\mathfrak{g}}(Q_i) \subset \mathfrak{g}$ (centraliser)

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Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $\mathfrak{p}_\theta \subset \mathfrak{g}$

$$\mathfrak{p}_\theta(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \;\middle|\; \lim_{z \to 0} z^{\theta} X z^{-\theta} \text{ along any ray exists} \right\}$$

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$$P_\theta(\mathfrak{g}) = \mathrm{Stab}(\mathcal{F}_\theta), \quad (\mathcal{F}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \begin{pmatrix} \text{eigenspaces} \\ \text{of } \theta \end{pmatrix}$$

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& similarly $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & $\mathfrak{l}_i$ is Levi of $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \longrightarrow \Sigma$ rank n holom. vector bundle
- $\nabla : V \longrightarrow V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \dots, V_{a_m}$

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Near a_i V has a local trivialization in which

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- $\pi(\Lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ $(\pi : \mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \twoheadrightarrow \mathfrak{l}_i)$

Thm (Biquard-B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Sabbah, ...)

The moduli space $\mathcal{M}_{DR}(\Sigma, \underset{\sim}{\theta}, \underset{\sim}{\alpha})$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
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- Higgs fields should look like $-\frac{1}{2} dQ_i + \Lambda_i \frac{dz}{z} +$ holom. near a_i
- same 'rotation' of the weights/eigenvalues as in Simpson 1990

Irregular curve

$$\Downarrow$$

Hyperkahler manifold

$\mathcal{M}$

Irregular curve (+ weighted conjugacy classes)

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(wild nonabelian Hodge structure)

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$\swarrow \qquad \Downarrow \qquad \searrow$

$\mathcal{M}_{Dol}$

Higgs bundles

algebraic integrable systems (mero. Hitchin systems)

$\mathcal{M}_{DR}$

mero. connections

isomonodromy systems

$\mathcal{M}_B$

monodromy & Stokes data

symplectic braid & (irregular) mapping class group actions

Braiding/Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Braiding / Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(PB 2001)

(extended to irregular case)

1900's (higher) Painlevé equations

~ 1980 Sato Miwa Jimbo Ueno ...

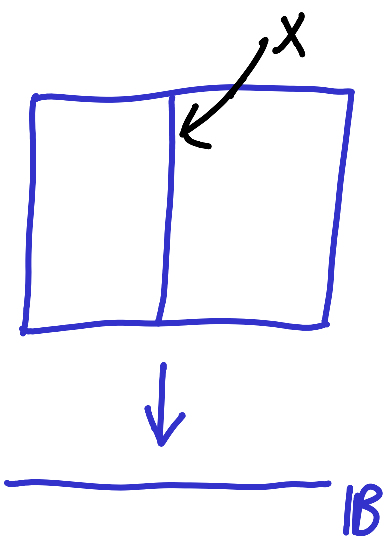
Simpson 1994

Braiding/Mapping class group actions

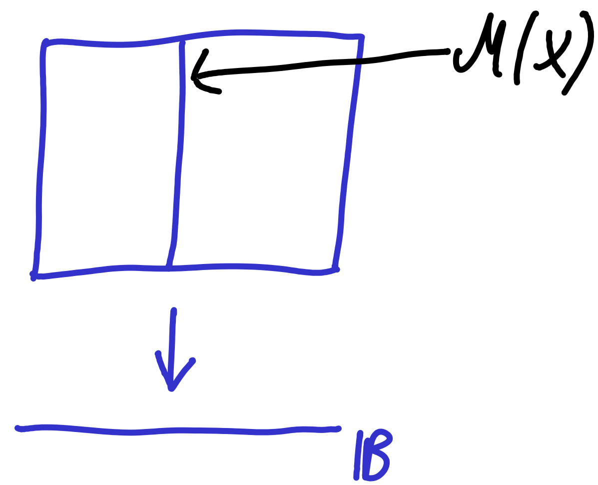
"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Reg. case:



$\Longrightarrow$



family of curves with marked points

'Family' of moduli spaces

- nonlinear fibre bundle with flat algebraic connection
- Betti spaces form a local system of varieties

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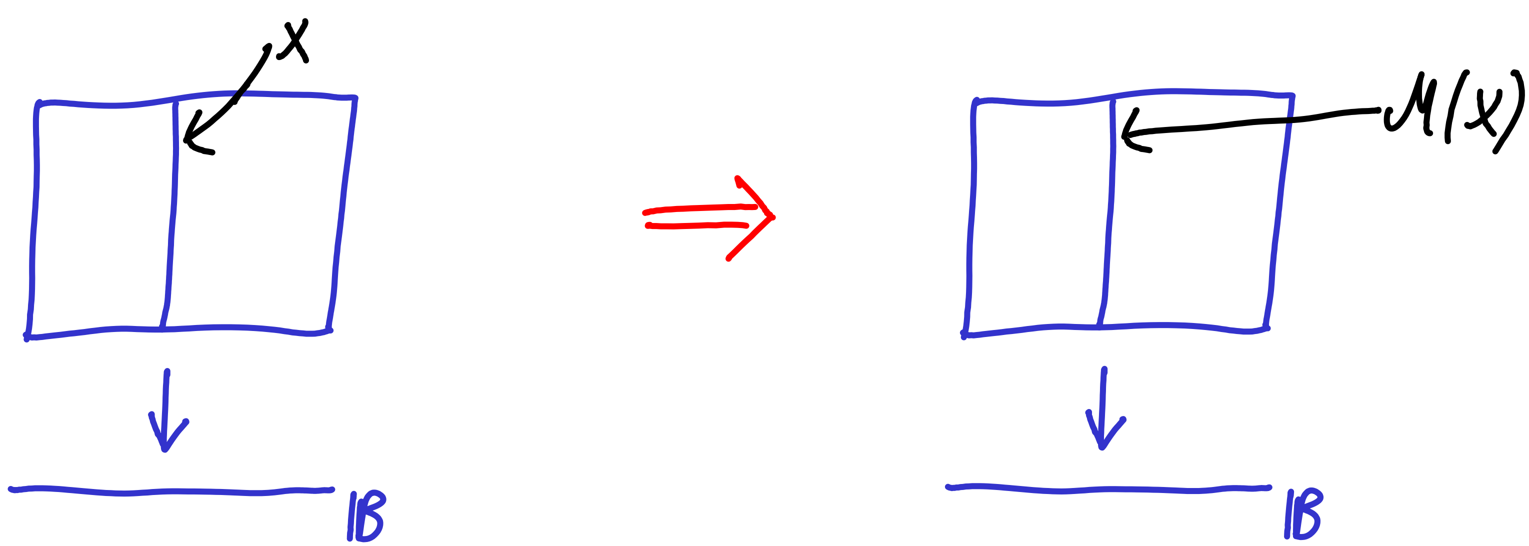

family of curves with marked points

$\boxed{\pi_1(\mathbb{B}) \circlearrowright \mathcal{M}(X)}$

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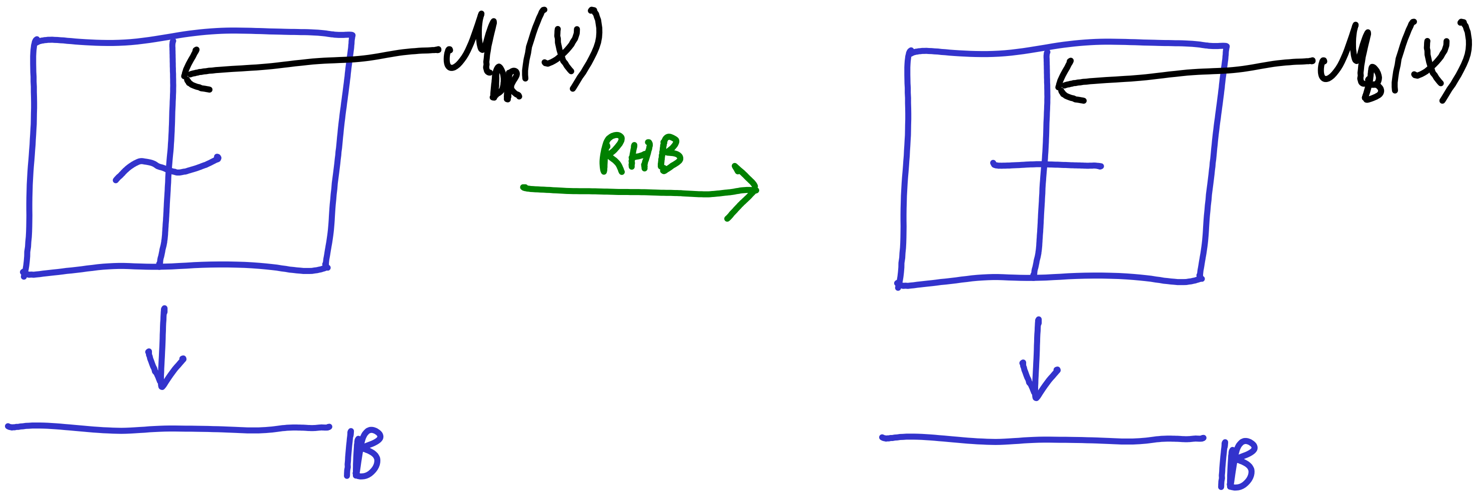

Isomonodromic Deformations

(picture from '01)

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Look at admissible deformations of irreg. curve, such that:

- Σ remains smooth,
- points a_i remain distinct
- Pole Order $(\alpha \circ Q_i)$ does not change $\left(\forall \text{ roots } \alpha \in R \subset \mathfrak{t}^*\right)$

e.g. if $A_r \in \mathfrak{t}_{reg}$ $\qquad \mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right)$

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  e.g. if $A_r \in \mathfrak{t}_{reg}$

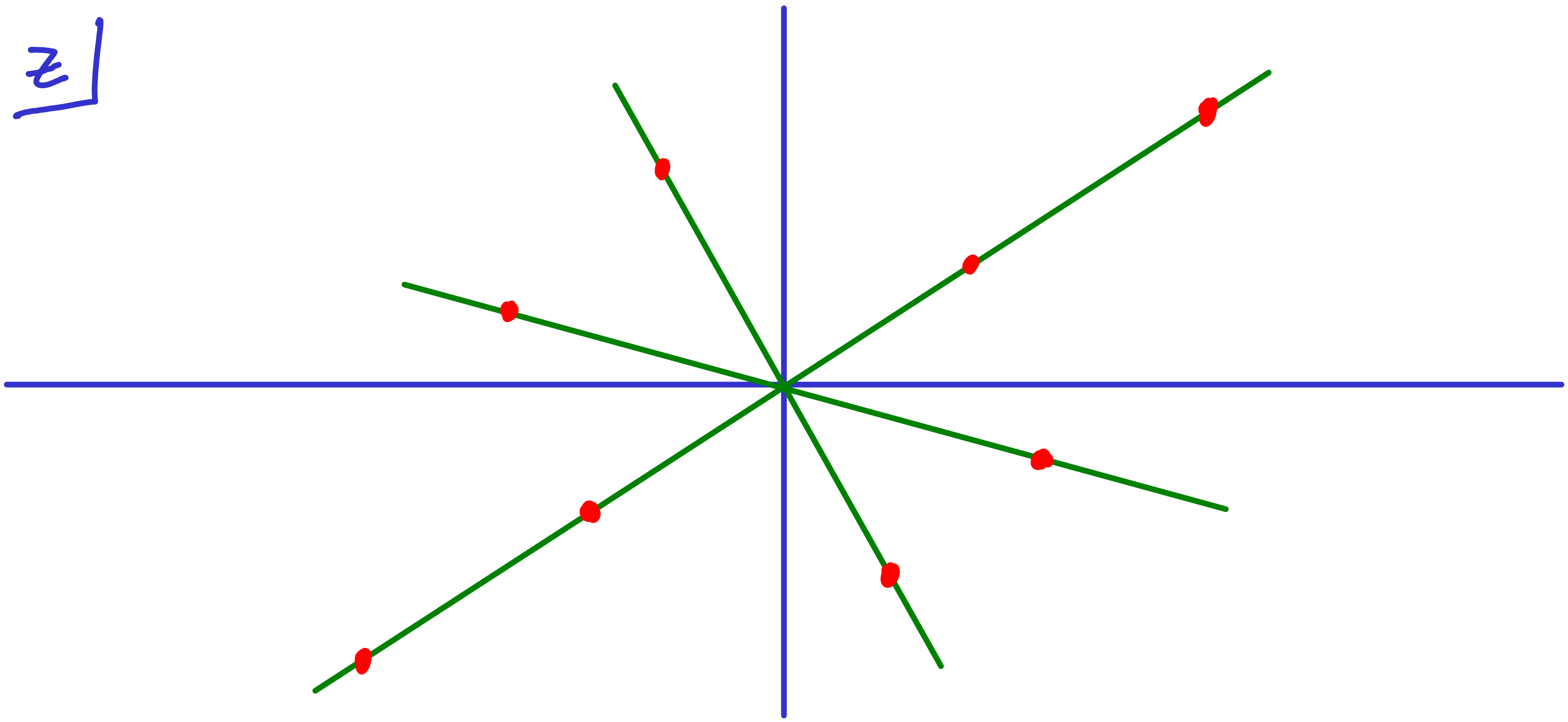
Then again the Betti spaces form a local system of varieties

and get notion of isomonodromic deformations of mero. connections

(cf. Jimbo-Miwa-Ueno '81 (GL_n), PB '02, '11 (other G, $A_r \notin \mathfrak{t}_{reg}$))

Simplest example (PB '02) $\quad r=1, \quad Q = \dfrac{-A_1}{z}, \quad A_1 \in t_{reg}$

Plot roots on z-plane: $\langle A_1, R \rangle \subset \mathbb{C}^*$



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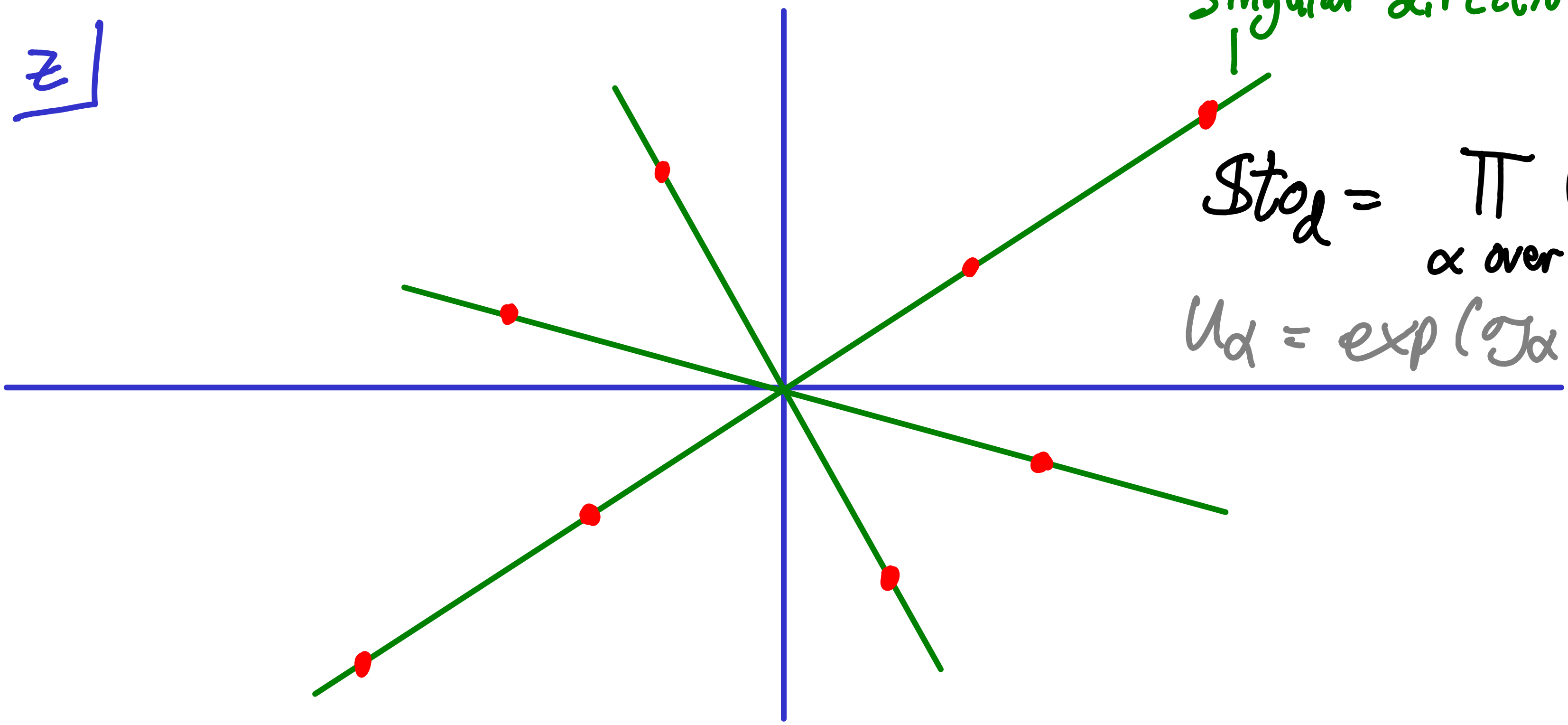

singular directions d

$$\mathrm{Sto}_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$$

$U_\alpha = \exp(\mathfrak{g}_\alpha)$ root gp

$$\{\text{Stokes data}\} = \prod_d \mathrm{Sto}_d$$

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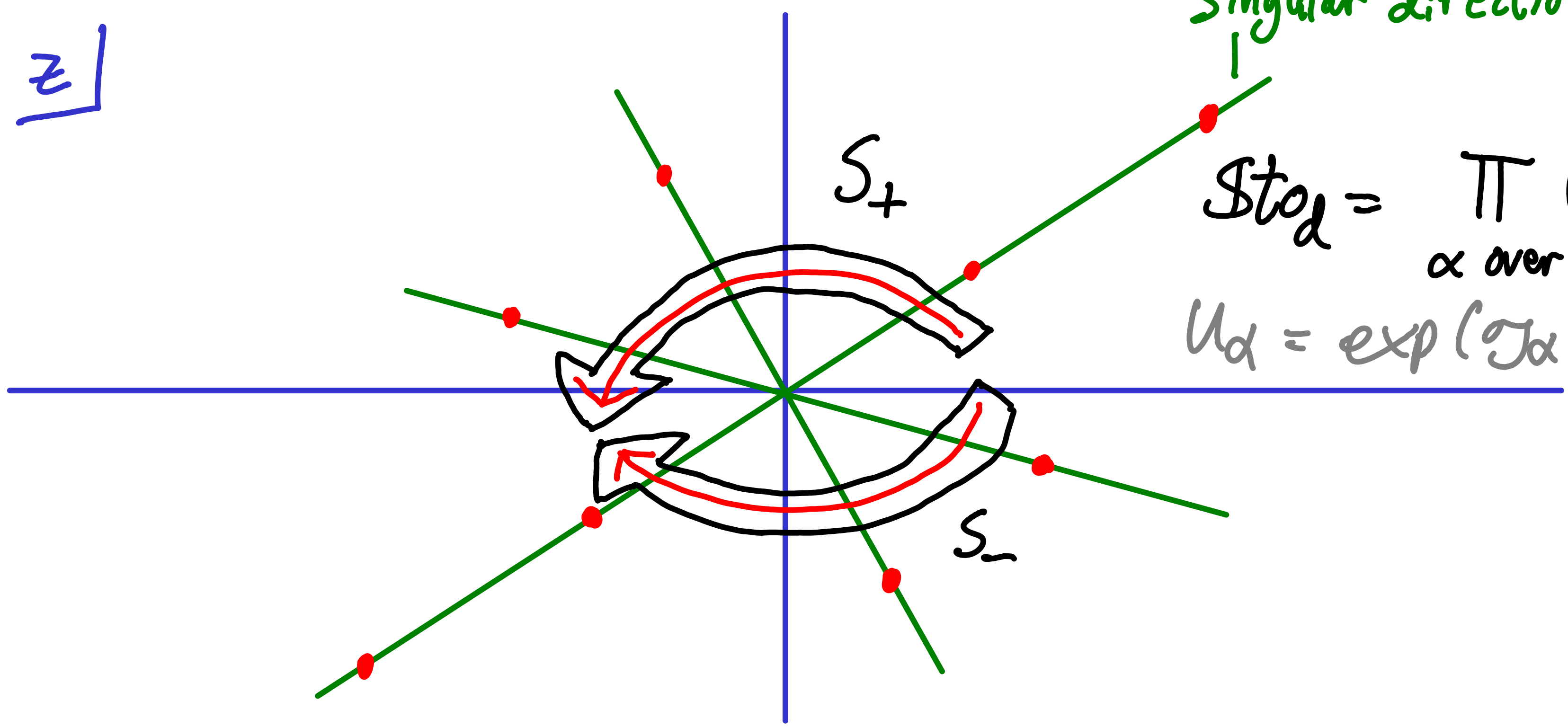

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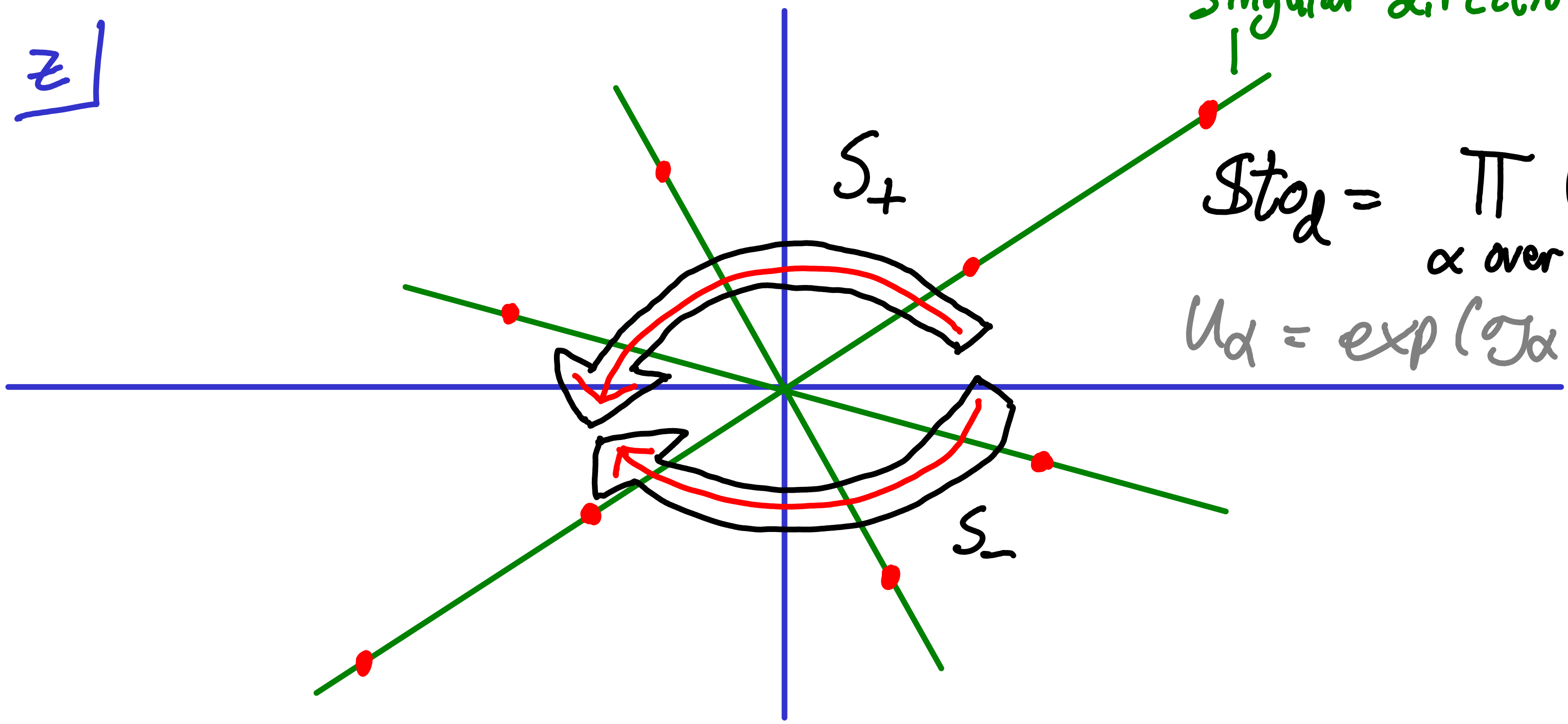

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Isomonodromy: Vary $A_1 \in \mathfrak{t}_{reg}$ & keep $S_\pm$ const. (locally)

In this example the resulting braid gp action had been previously seen:

$$\begin{tikzcd}
& U_q \mathfrak{g} & \\
\mathrm{Fun}(G^*) \arrow[ur] & & U\mathfrak{g} \arrow[ul] \\
& \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) \arrow[ul] \arrow[ur] &
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Quantum Weyl group action
(Lusztig, Soibelman ...)

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Thm (- '02)

The DKP action arises from isomonodromy ($U_+ \times U_-$ = Stokes data)

- Purely geometric origin (not just explicit generators)
- $U_q\mathfrak{g}$ thus quantizes a moduli space of meromorphic connections

Example (cont.)

$$B \in \mathfrak{g}^*$$

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Thm (PB '01–'02)

ν_{A_1} is Poisson & is generically a local analytic isomorphism

($\Rightarrow$ new direct proofs certain results of Duistermaat, Ginzburg–Weinstein, Kostant)

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Isomonodromy equations: $\quad dB = [B, \mathrm{ad}_{A_1}^{-1}[dA_1, B]]$ $\quad$ (*)

($\mathfrak{t}_{reg} = \{A_1\}$ = 'times')

Formula for (part of) ν_{A_1} by Bridgeland–Toledano ~2008

Expand (*) in root spaces:

$$B = \sum_{\alpha \in R} b_\alpha \quad , \qquad d b_\alpha = \sum_{\beta+\gamma=\alpha} [b_\beta, b_\gamma] \, d\log\gamma \quad (*)$$

$$\left(\pi_{\mathfrak{t}}(B) = 0\right)$$

Expand ⊛ in root spaces:

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- ⊛ arises in Frobenius manifolds / GW invts if $B \in \mathfrak{gl}_n(\mathbb{C})$, $B^T = -B$ (Dubrovin)
- relaxed to $B \in \mathfrak{gl}_n(\mathbb{C})$ and then $B \in \mathfrak{g}$ to understand geometry / braiding and defined G-valued Stokes data to integrate ⊛ (PB '01, '02)

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① no physical interpretation of b_α's in DT context (yet)

② KS, Gaiotto-Moore-Neitzke interpret certain DT invariants as giving "formulae" for Hitchin type hyperkahler metrics on $\mathcal{M}$

General case is similar; space of Stokes data more complicated:

$$\{\text{Stokes data}\} = \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d$$

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- singular directions $\mathbb{A} = \left\{ d \in S^1 \;\middle|\; \begin{array}{l} e^{\alpha \circ Q(z)} \text{ has max. decay} \\ \text{as } z \to 0 \text{ along } d \text{ for some root } \alpha \end{array} \right\}$ $\circledast$

- $\mathbb{S}to_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) < G$ (unipotent subgroup)

- $R(d) = \{\text{roots s.t } \circledast \text{ holds in direction } d\} < R$

Guide to moduli spaces on $\mathbb{P}^1$

Typically $\qquad \mathcal{M}^* \subset \mathcal{M}$

open part where bundle holom. trivial $/\mathbb{P}^1$

& $\mathcal{M}^*$ again a complete hyperkahler manifold

"approximation" to more transcendental metric on $\mathcal{M}$

Remark ($G = GL_n$, $A_r \in \mathfrak{t}_r$)

In effect Jimbo–Miwa–Ueno considered $\mathcal{M}^*$ in 1981 & defined precise global space $\mathcal{M}_B$ of monodromy & Stokes data & showed

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& extended to arbitrary g, G, topological type in '07

Classical hyperkahler mfds

1. Complex coadjoint orbits $\quad \mathcal{O} \subset \mathfrak{g}^*$

(Kronheimer, Biquard, Kovalev)

If pole divisor $2(0) + (\infty) \subset \mathbb{P}^1$

have examples where $\quad \mathcal{M}^* \cong \mathcal{O} /\!/\!/_{\lambda} T_K$

$$\left[\mathcal{M}_{Betti} = \mathcal{L} /\!/_{\lambda} T \ , \quad \mathcal{L} \subset G^* \text{ symplectic leaf} \right]$$

($T_K \subset T$ compact torus)

2 T^*G (Kronheimer)

If pole divisor $2(0) + 2(\infty) \subset \mathbb{P}^1$ have examples where $\mathcal{M}^* \cong T_K \backslash\!\backslash\!\backslash_{\lambda_1} T^*G /\!/\!/_{\lambda_2} T_K$

$$\mathcal{M}_{Betti} = T \backslash\!\backslash_{\lambda_1} \mathbb{D} /\!/_{\lambda_2} T$$

$\mathbb{D} \subset (G \times G^*)^2$ Lu-Weinstein double sympl. groupoid

③ ALE spaces deformations of $\widetilde{\mathbb{C}^2/\Gamma}$

(Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SU_2$ finite $\longleftrightarrow$ ADE affine Dynkin graph

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Fact In cases E_8, E_7, E_6, D_4 (logarithmic realisations), A_3, A_2, A_1 (only irregular realisations)

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	Pole orders
A_3	$2+1+1$
A_2	$3+1$
A_1	4

– Okamoto found in 1987 the corresponding affine Weyl groups are the sym gps of the corresponding Painlevé equations

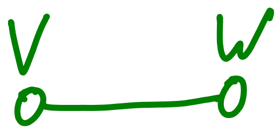
Rough classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

E_8 $\quad E_7$ $\quad E_6$ $\quad D_4$ $\quad A_3 = D_3$ $\quad D_2$ $\quad (D_1)$ $\quad (D_0)$

A_2 $\quad A_1$ $\quad (A_0)$

reg $\leftarrow | \rightarrow$ irreg

4 (Nakajima) Quiver varieties



$\mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V)$ is hyperkahler $U(V) \times U(W)$ space

Graph = ADE dynkin graph $\Rightarrow$ ALE space (Kronheimer)

else in general get higher dim^n hyperkahler mfd (or empty)

– lets consider simply-laced cases

E.g. Fuchsian case $\quad G = GL_n(\mathbb{C})$

$$\mathcal{M}^* \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G$$

$(\mathcal{O}_i \subset \mathfrak{g}^*$ coadjoint orbits$)$

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point of $\mathcal{M}^*$ $\sim$ Fuchsian system $\displaystyle\sum_1^m \frac{A_i}{z-a_i}\,dz$ $\quad A_i \in \mathcal{O}_i$, $\sum A_i = 0$

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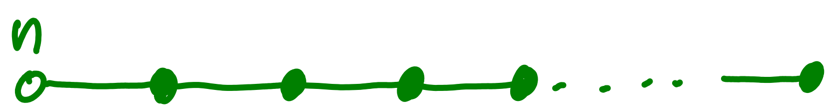
Relation to quivers (Kraft-Procesi, Nakajima, ...)

$\mathcal{O}_i \cong$



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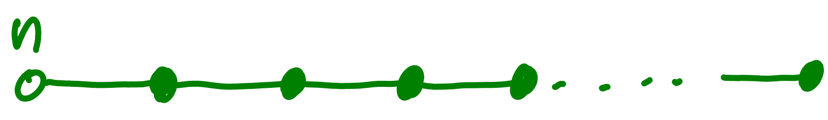

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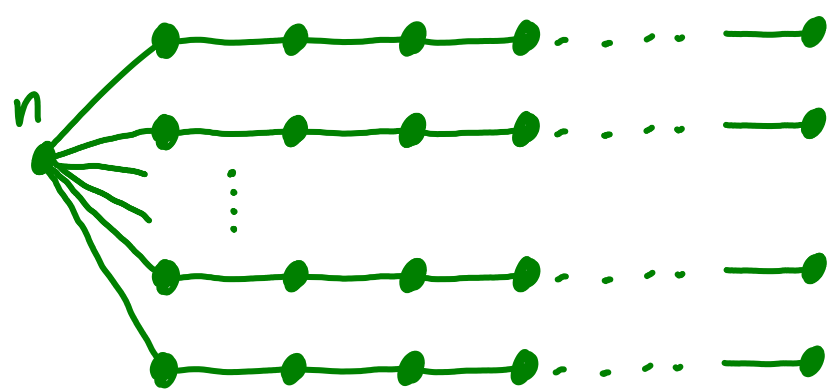
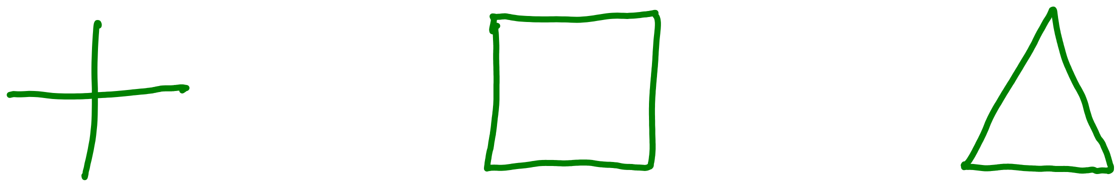
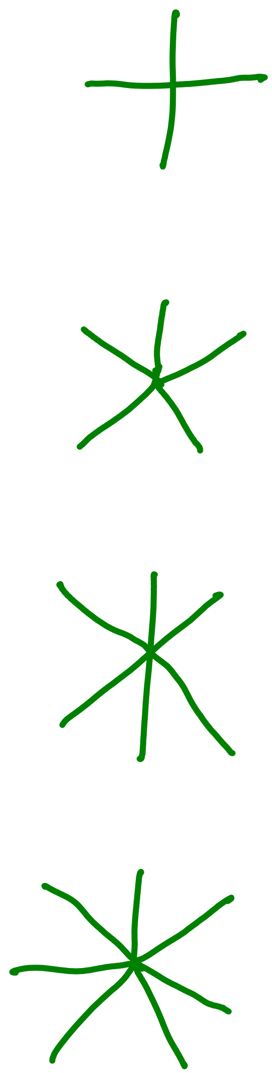


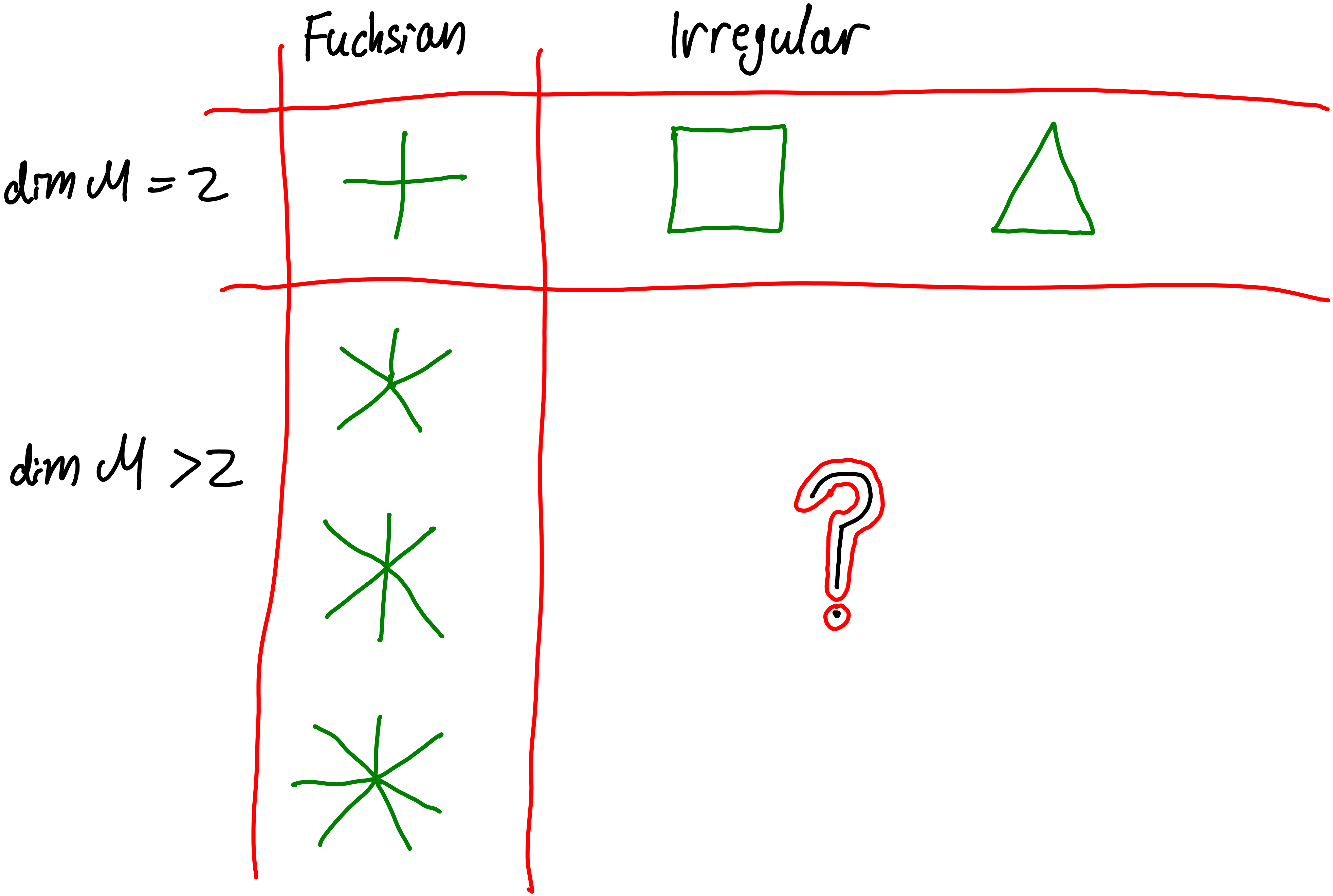
"Starshaped" quivers used by Crawley-Boevey in Deligne-Simpson problem

Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type A_2, A_3, D_4 resp.



Recall Crawley-Boevey related moduli spaces of Fuchsian systems to star-shaped quivers (building on Kraft-Procesi, Nakajima, ...)

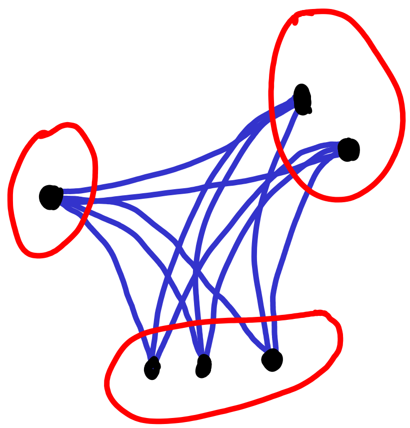



Fuchsian
Irregular
dim M = 2
dim M > 2

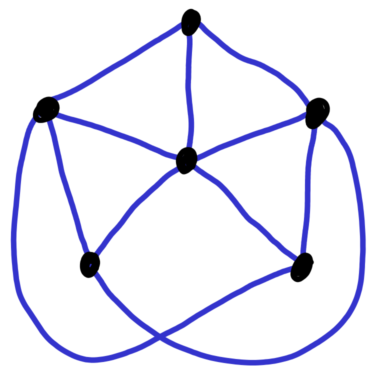
Thm

Can take any complete k-partite graph (for any k)

E.g.



$\cong$



$\Gamma(3,2,1)$

– get action of corresponding (not necessarily affine) Kac-Moody Weyl group

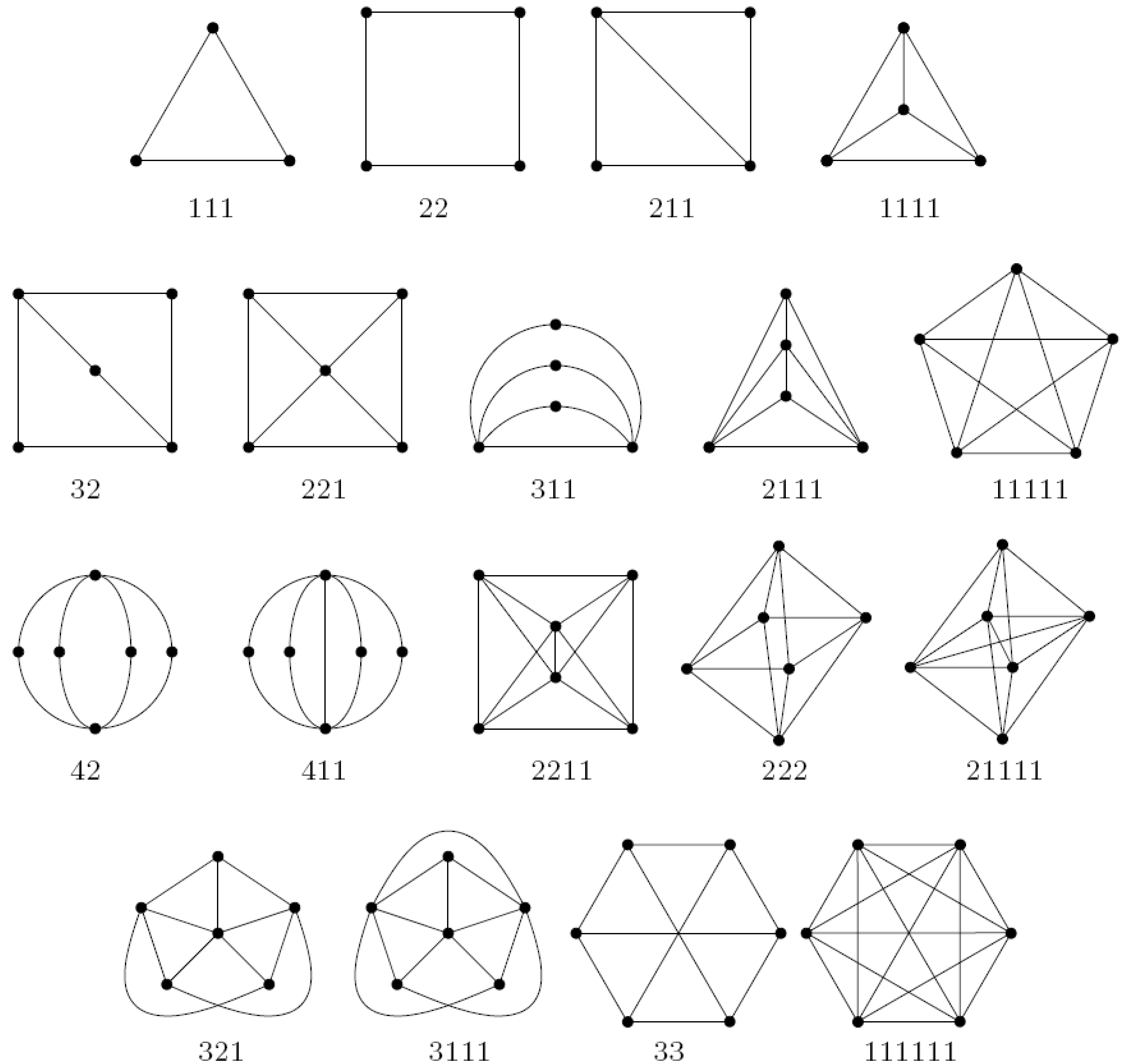


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)

More general and precise statement:

Definition A graph Γ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve Σ

s.t. $\mathcal{M}^*(\Sigma) \cong$ a quiver variety attached to Γ

$$\begin{array}{c}\mathcal{M}^*(\Sigma)\\ \cap\\ \mathcal{M}(\Sigma)\end{array}$$

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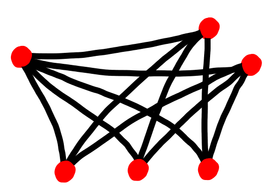

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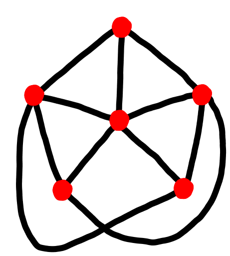

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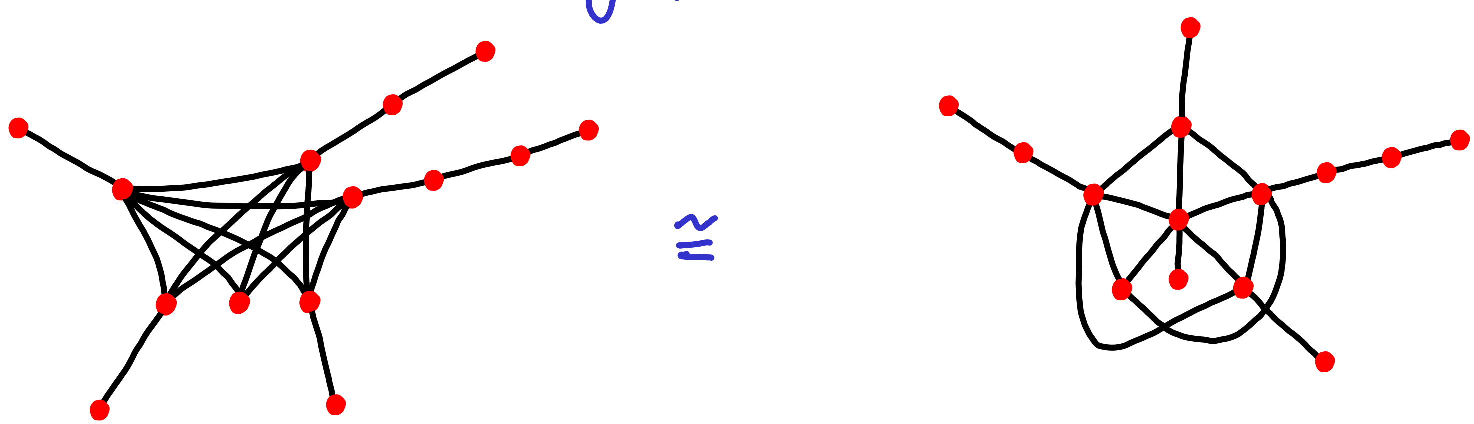

— generalising the star-shaped graphs

Thm

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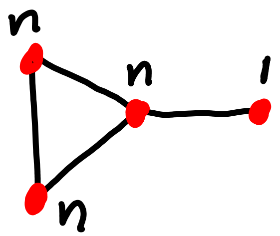
Moreover Γ determines a (symmetric) Kac-Moody root system & Weyl group, and Weyl group elements lift to give isomorphisms between such systems

E.g. Higher/hyperbolic/Hilbert Painlevé systems

$\Gamma_n =$ 

$\Longrightarrow \quad hP_{IV}^{n} := \mathcal{M}(\Gamma_n)$ dimension $2n$

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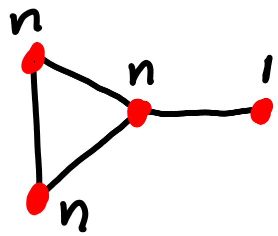
$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{IV} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$n=1 \qquad hP^1_{IV} \cong P_{IV}$ dim 2

$$\mathcal{M}^*(\Gamma_n) \underset{\text{diffeo}}{\cong} \mathrm{Hilb}^n(\mathcal{M}^*(\Gamma_1))$$

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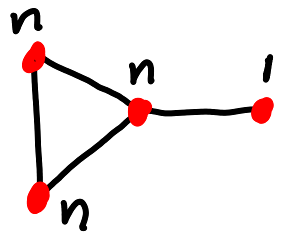
$\Gamma_n =$  $\Longrightarrow$ $hP_{IV}^{n} := \mathcal{M}(\Gamma_n)$ dimension $2n$

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Question: $\mathcal{M}(\Gamma_n) \overset{?}{\cong} \mathrm{Hilb}^n(\mathcal{M}(\Gamma_1))$ (for generic parameters)

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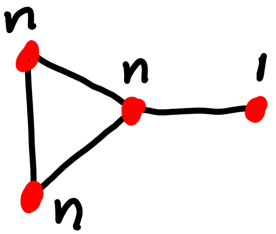
$\Gamma_n =$ 

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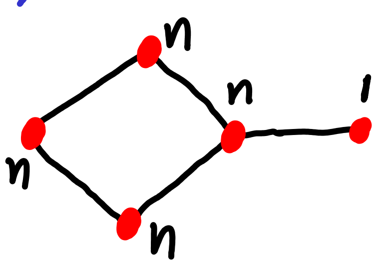
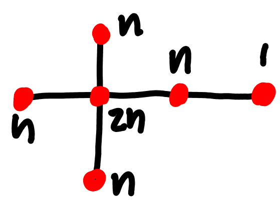
similarly for any 2d Hitchin system e.g:

$\Gamma_n =$ 

$\Longrightarrow$ $hP^n_{V} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$\Gamma_n =$ 

$\Longrightarrow$ $hP^n_{VI} := \mathcal{M}(\Gamma_n)$ dimension $2n$

Complex character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

Riemann surface — Poisson variety

Atiyah-Bott, Goldman, Karshon, Fock-Rosly, Weinstein, Guruprasad-Huebschmann-Jeffrey-Weinstein, Andersen-Mattes-Reshetikhin ...

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \sqcup \dots \sqcup \partial_m \qquad (\partial_i \cong S^1)$

Choose basepoints $b_i \in \partial_i$

Let $\Pi = \pi_1(\Sigma, \{b_1, \dots, b_m\})$

Thm (Alekseev et al) $\quad \mathrm{Hom}(\Pi, G)$ is a smooth affine variety which is naturally a quasi-Hamiltonian G^m-space

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& symplectic leaves are $\mu^{-1}(C)/G^m$ $\quad (C = (C_1, \dots, C_m) \subset G^m)$

Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = GL_n(\mathbb{C})$

(Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

– will give explicit as possible approach using groupoids (for any reductive G)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points $a_1, \dots, a_m$, irreg. types $Q_1, \dots, Q_m$)

Let $\hat{\Sigma} \to \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial\hat{\Sigma} = \partial_1 \sqcup \dots \sqcup \partial_m$)

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(each a_i replaced by a circle ∂_i, so $\partial\hat{\Sigma} = \partial_1 \sqcup \ldots \sqcup \partial_m$)

Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

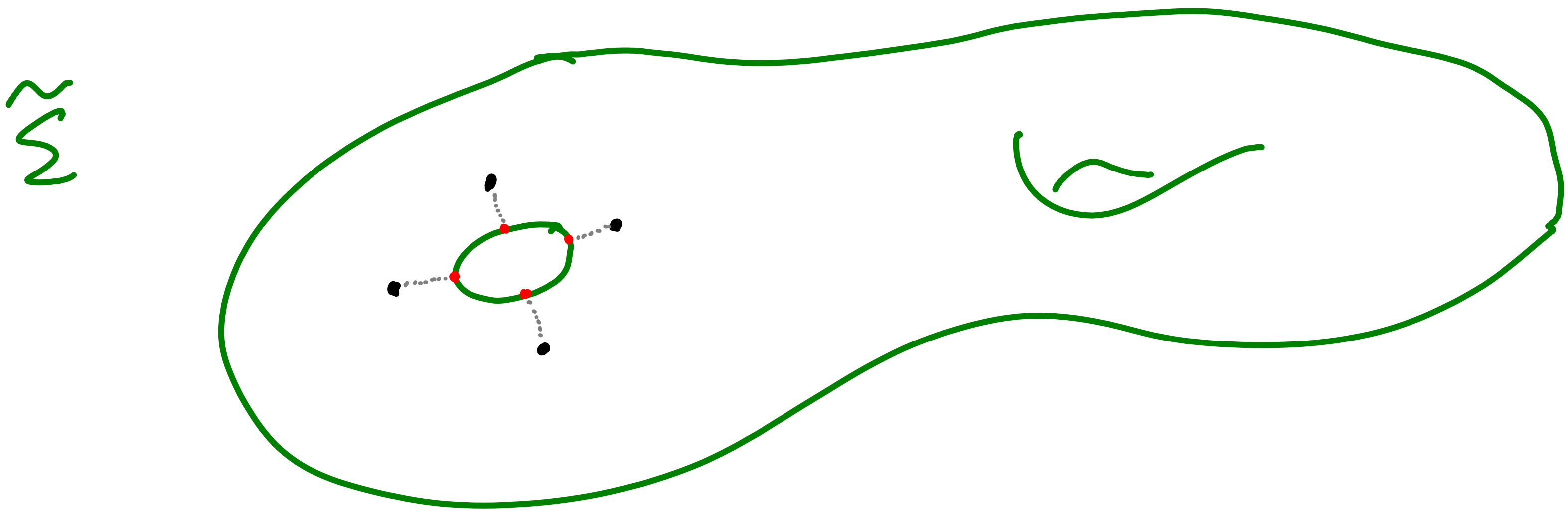
2) A finite set $\mathbb{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathbb{A}_i$

3) A unipotent group $\mathrm{Sto}_d(Q_i) \subset G$ normalised by H_i

Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i$, $i=1,\dots,m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface



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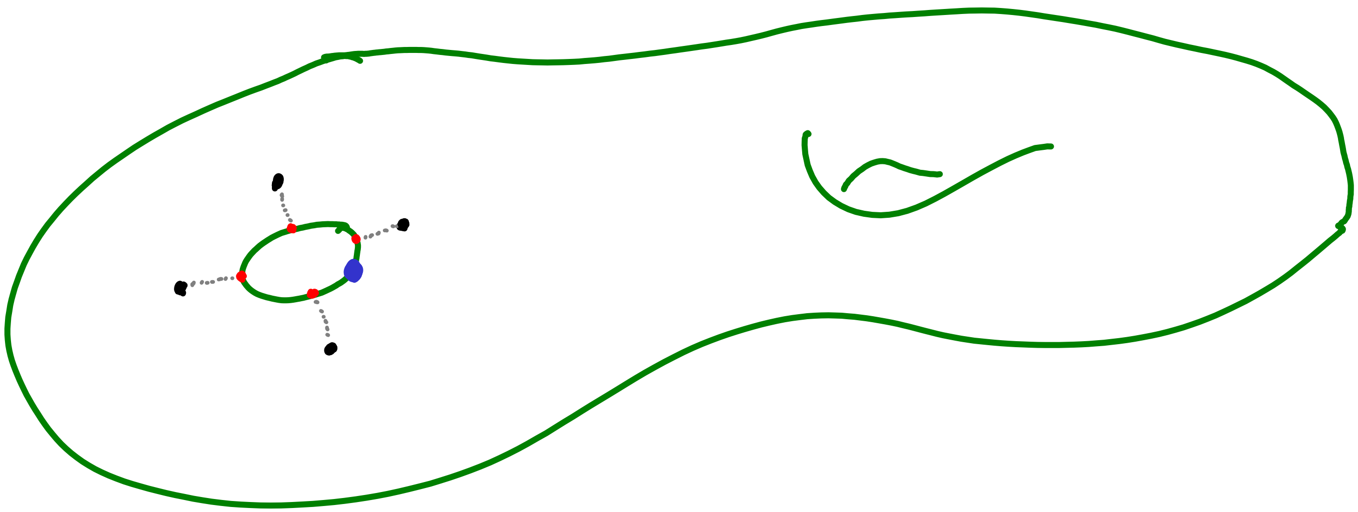

Choose a base point $b_i \in \partial_i$ in each boundary circle

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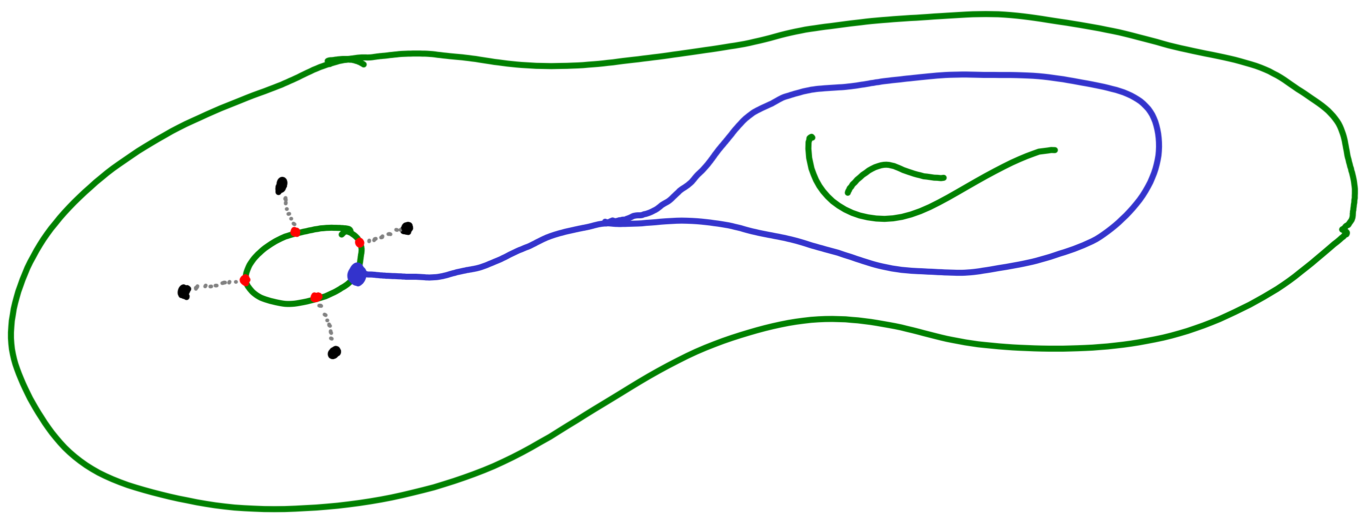

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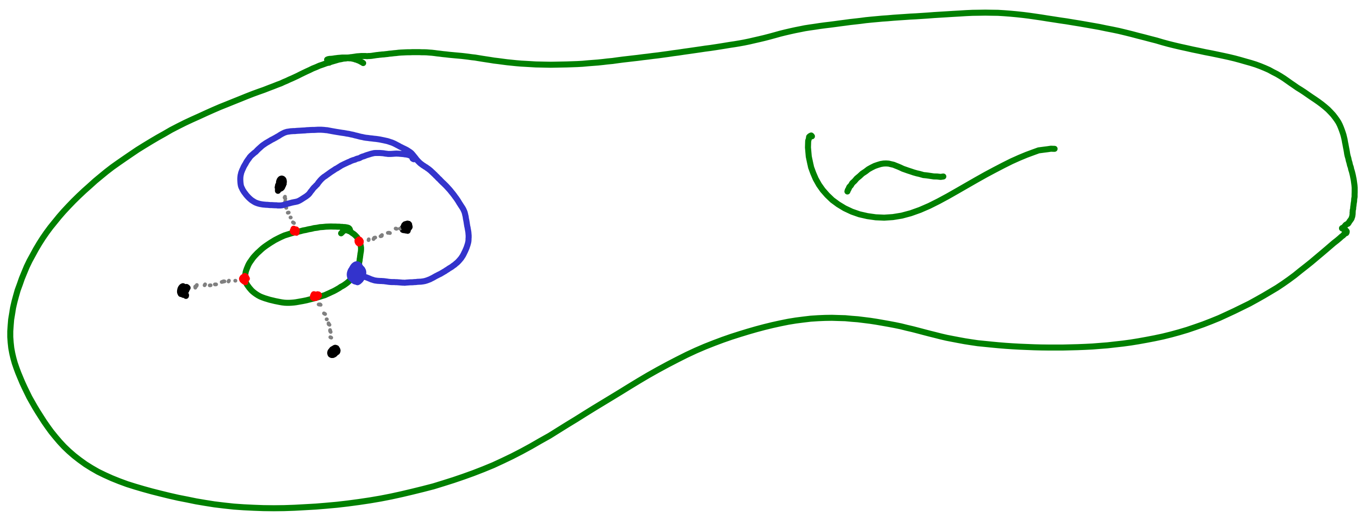

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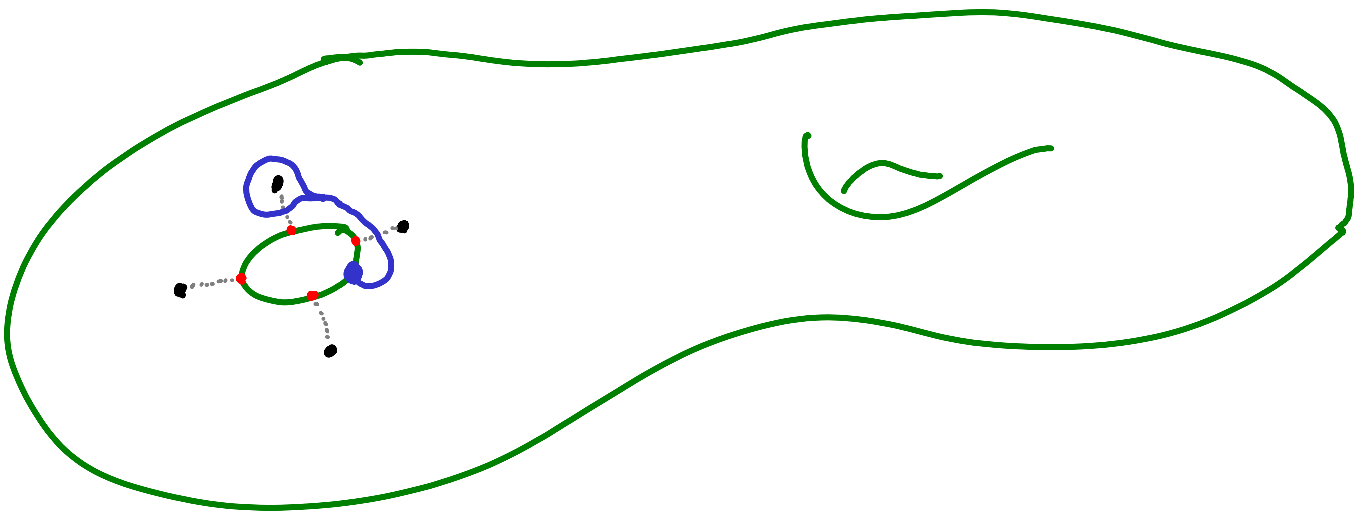

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$\cup$

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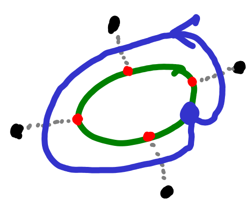
and the subset $\mathrm{Hom}_{\mathbb{S}}(\pi, G)$ of "Stokes representations" satisfying:

1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



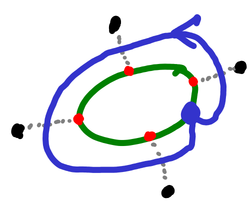
Now consider $\quad \mathrm{Hom}(\pi, G)$

$$\cup$$

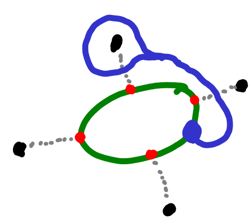
and the subset $\mathrm{Hom}_{\mathbb{S}}(\pi, G)$ of "Stokes representations"

satisfying:

1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



2) If γ goes around ∂_i from b_i until $d \in \mathbb{A}_i$ then loops around the corresponding puncture before returning to b_i, then $\rho(\gamma) \in \mathbb{S}\mathrm{to}_d$



Thm

The space of Stokes representations $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\underset{\sim}{H}$-space $(\underset{\sim}{H} = H_1 \times \cdots \times H_m)$

Corollary $\mathcal{M}_B(\Sigma, G) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$

inherits an intrinsic Poisson structure (algebraically) with

symplectic leaves $\mu^{-1}(e) / \underset{\sim}{H}$ for $e = (e_1, \ldots, e_m) \subset \underset{\sim}{H}$

[$\mathcal{M}_B$ classifies irreg. connections with the given irreg. types

& Betti weights zero (else use $\hat{e}$)]

Corollary $\mathcal{M}_B(\Sigma, G) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$

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[$\mathcal{M}_B$ classifies irreg. connections with the given irreg. types & Betti weights zero (else use $\hat{\mathcal{C}}$)]

Also studied stability for $\underset{\sim}{H} \circlearrowright \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$:

Hilbert-Mumford + general quasi-Hamiltonian properties $\Longrightarrow$

Thm If $\mathcal{C}$ sufficiently generic semisimple conjugacy class then $\mu^{-1}(\mathcal{C}) / \underset{\sim}{H}$ symplectic orbifold (smooth if $G = GL_n(\mathbb{C})$)

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$$

Irregular curve $\qquad$ Poisson variety

Thm

If $\Sigma \longrightarrow \mathbb{B}$ is an admissible family of irregular curves
$\Sigma_p = \pi^{-1}(p)$, $p \in \mathbb{B}$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowright \mathrm{Hom}_{\mathbb{S}}(\Pi(p), G)/\underset{\sim}{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, G)$ form a local system of (Poisson) varieties"

Definition A holomorphic quasi-Hamiltonian G-space is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu : M \longrightarrow G$ (G acts on G by conjugation) such that

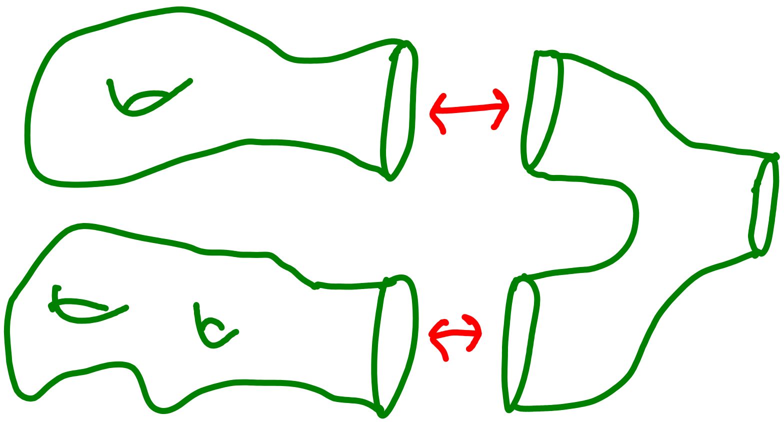
1. $d\omega = \mu^*(\eta)$
2. $\forall X \in \mathfrak{g} \quad \omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X)$
3. $\forall m \in M \quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$

where η = biinvariant 3-form on G, $\theta, \bar{\theta}$ Maurer-Cartan forms on G

- These axioms are 'what we get from ∞-d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with $\mathfrak{g}^*$-valued moment map)

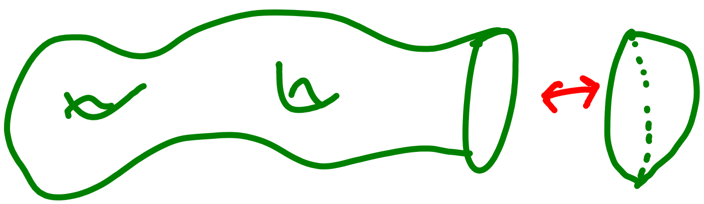
Operations

① Can 'fuse' 2 q Hamiltonian G-spaces:



⇛ New qH G-space $M_1 \circledast M_2$

② & reduce:



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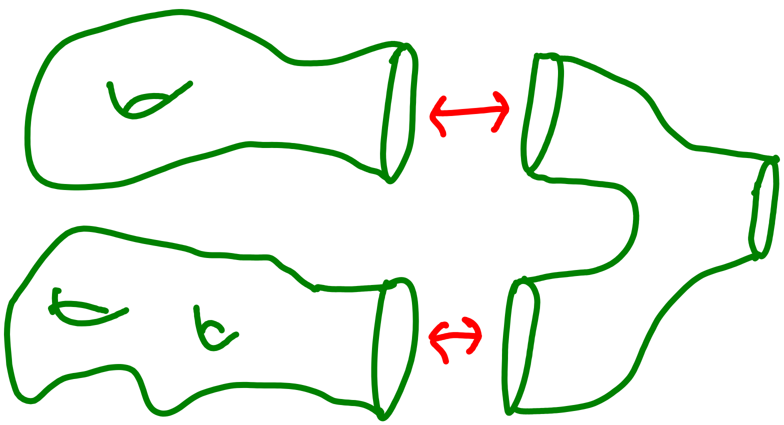

$\Rightarrow$ New qH G-space $M_1 \circledast M_2$

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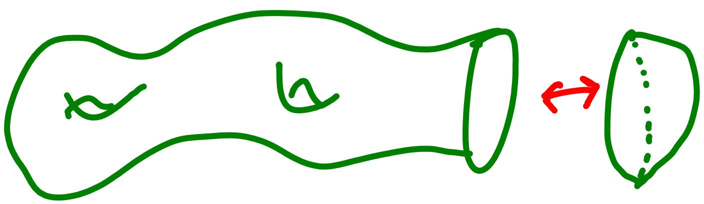
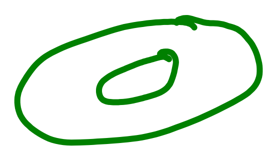


$\Rightarrow$ qH $\{\cdot\}$-space = a symplectic manifold

Basic examples

① Conjugacy classes $\mathcal{C} \subset G$

② $D = G \times G$ qH $G \times G$ space (double)

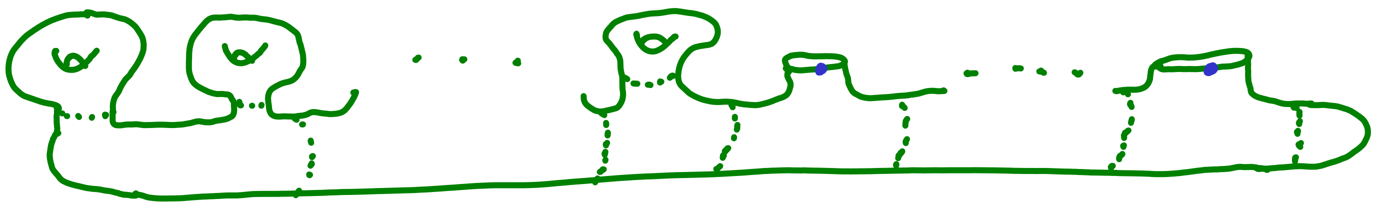


Annulus

③ $\mathbb{D} = G \times G$ qH G-space (internally fused double)

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \underbrace{D \circledast \cdots \circledast D}_{m} /\!/ G \cong \mathrm{Hom}(\pi, G)$$



$$\mu^{-1}(e)/G^m \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1 \,,\; M_i \in C_i \right\} / G$$

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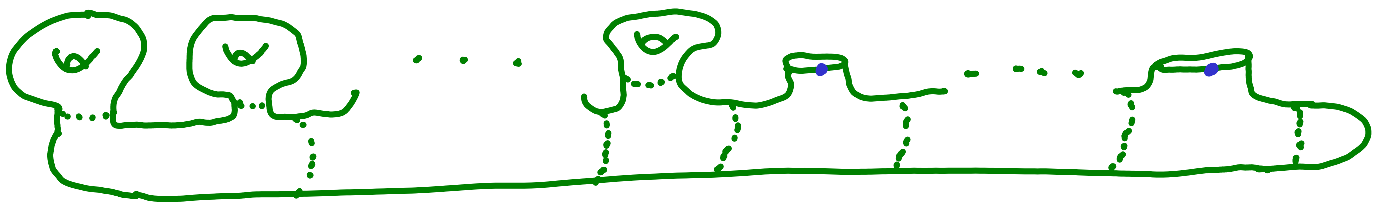

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Aim: New pieces to construct irregular Betti spaces?

(have "irreg. Atiyah-Bott" from 1999)

Fission spaces

Choose $P_\pm \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_\pm \subset P_\pm$ unipotent radicals

Thm (– '02, '09, '11)

The "fission space" ${}_G\mathcal{A}^r_H := G \times (U_+ \times U_-)^r \times H$

is a quasi-Hamiltonian $G \times H$ space

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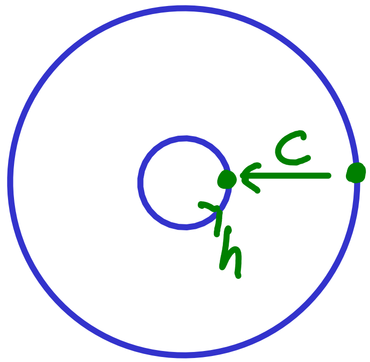
is a quasi-Hamiltonian $G \times H$ space

- moment map $\mu(C, S_1, \ldots, S_{2r}, h) = (C^{-1} h S_{2r} \cdots S_1 C, h^{-1})$
- $(U_+ \times U_-)^r \cong$ Stokes data of connections with $Q = \frac{A}{z^r}$, $C_G(A) = H$

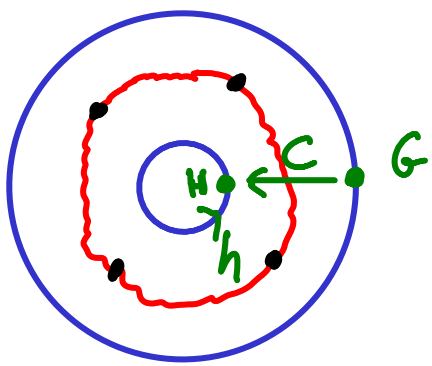
Picture If $P_\pm = G = H$ $\quad {}_G\mathcal{A}_H = G \times G$ is the double

$$\begin{array}{c} \Downarrow \\ (c, h) \end{array}$$



$$\mu = (c^{-1}hc,\ h^{-1})$$

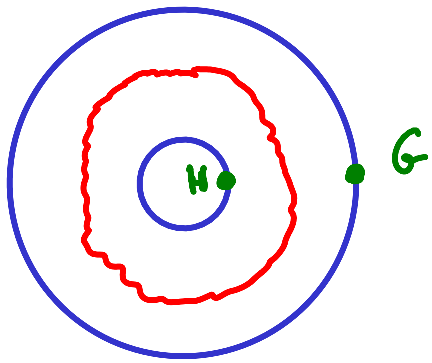
General case can be pictured similarly (breaking group from G to H)



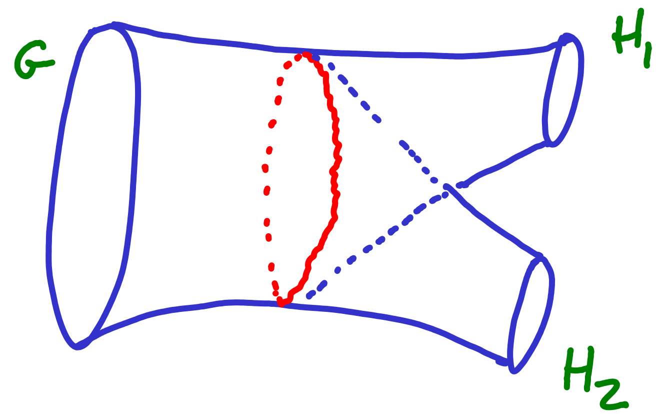
$$\mu = (c^{-1}h s_{2r} \cdots s_1 c,\ h^{-1})$$

Typically H is a product e.g. $H = H_1 \times H_2$

- can glue on both a qH H_1-space & a qH H_2-space



$\cong$



"fission" operation ($\neq$ fusion)

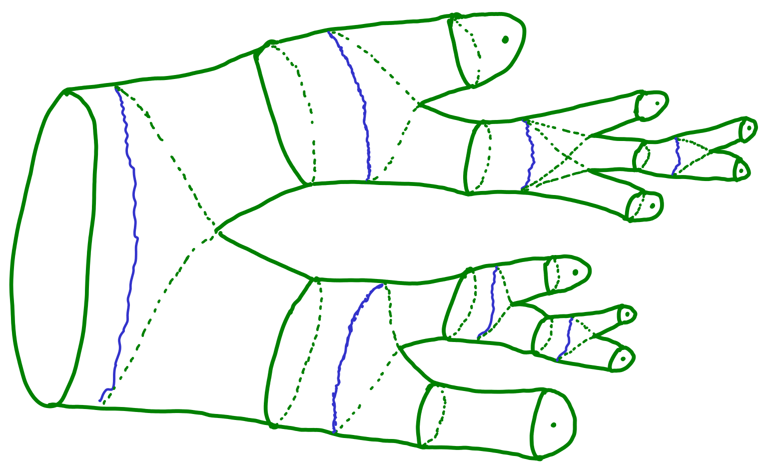
If $Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$

Define $G = H_r \supset H_{r-1} \supset \cdots \supset H_0 = H \supset T$

via $H_{i-1} = C_{H_i}(A_i)$

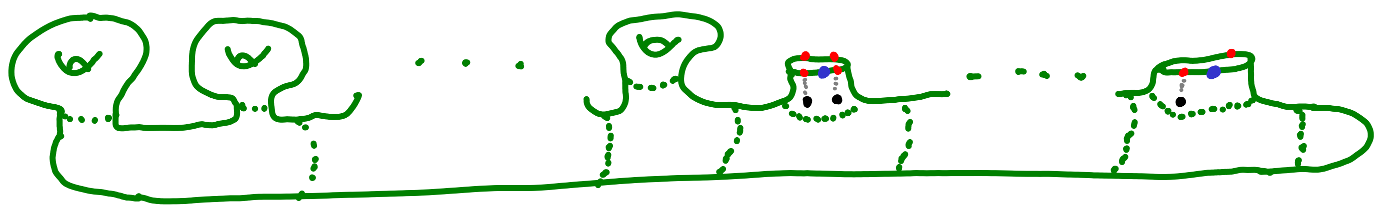
Then $\mathcal{A}(Q) := G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$\mathcal{A}(Q) \cong {}_G\mathcal{A}^r_{H_{r-1}} \circledast {}_{H_{r-1}}\mathcal{A}_{H_{r-2}} \circledast \cdots \circledast {}_{H_1}\mathcal{A}_H$$



If Σ an irregular curve :

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \cong \underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{A}(Q_1) \circledast \cdots \circledast \mathcal{A}(Q_m) /\!/ G$$



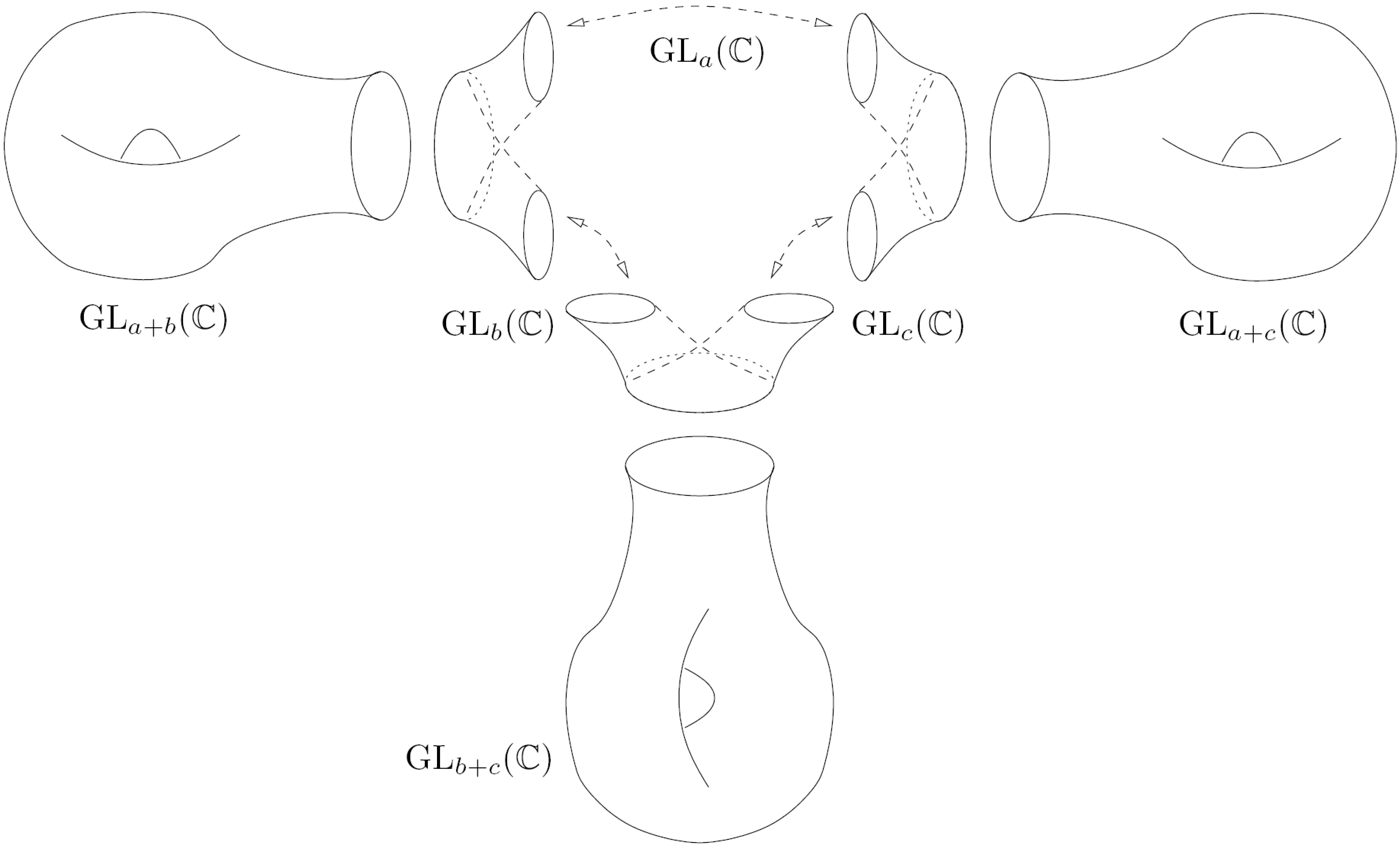
$$\mu^{-1}(e)/\underset{\sim}{H} \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{h}, \underset{\sim}{S}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} \mu_i = 1,\ h_i^{-1} \in \mathcal{C}_i \right\} / \underset{\sim}{H}$$

$$\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i$$

But there are many other examples of fission varieties

- e.g. can glue surfaces $\widetilde{\Sigma}$ along their boundaries (provided the groups H_i match up)

- can obtain all the so-called multiplicative quiver varieties



$GL_a(\mathbb{C})$
$GL_{a+b}(\mathbb{C})$
$GL_b(\mathbb{C})$
$GL_c(\mathbb{C})$
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$GL_{b+c}(\mathbb{C})$

Revisit (resonant) logarithmic/tame case (arxiv 16/3/10)

G connected complex reductive group

$P_0 \subset G$ parabolic, $\mathbb{P} \cong G/P_0$ parabolics conjugate to P_0

$\pi : P_0 \longrightarrow L$ projection onto Levi factor

Choose $\mathcal{C} \subset L$ a conjugacy class

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Let $\hat{\mathcal{C}} = \{ (M, P) \in G \times \mathbb{P} \mid M \in P, \ \pi(M) \in \mathcal{C} \}$

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Thm $\hat{\mathcal{C}}$ is a q-Hamiltonian G-space with moment map $(M,P) \mapsto M$

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- Lie algebra version well-known & GL_n case is due to D. Yamakawa
- If P_0 a Borel (& G ss, sc) $\hat{\mathcal{C}}$ appears in Bries.-Groth.-Springer resolution
- $\hat{\mathcal{C}} = (\mathbb{M} \circledast \mathcal{C}) /\!/ L$ where $\mathbb{M} = G \times P_0 / U$ qH $G \times L$ space

  $\dim \mathbb{M} = 2 \dim P_0$ "tame fission"

Want to understand all these spaces moduli theoretically:

1. Log. connections on vector bundles (Levelt filtrations)
2. Log. connections on G-bundles
3. Log. connections on parabolic G-bundles
4. Log. connections on parahoric torsors ("Logahoric")

(cf. Simpson 1990 for GL_n — can stop at 3)

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(cf. Simpson 1990 for GL_n — can stop at 3)

$K = \mathbb{C}\{z\}[z^{-1}]$, $LG = G(K)$

$\exists$ canonical bijection:

$$\{(A,p) \mid p \in \mathcal{B}(LG),\ A \in \mathcal{A}_p\}/LG \cong \{(M,b) \mid b \in \mathbb{B}(G),\ M \in P_b\}/G$$

weighted parahorics = points of BT building

weighted parabolics

$\mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R}$ so $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C}$

Fix $\theta \in \mathfrak{t}_{\mathbb{R}}$ (weight), $\mathfrak{g}_\lambda = \lambda$ eigenspace of $\mathrm{ad}_\theta \subset \mathfrak{g}$ $(\lambda \in \mathbb{R})$

$t_{\mathbb{R}} = X_*(T) \otimes \mathbb{R}$ so $t = t_{\mathbb{R}} \otimes \mathbb{C}$

Fix $\theta \in t_{\mathbb{R}}$ (weight), $\mathfrak{g}_\lambda = \lambda$ eigenspace of $ad_\theta \subset \mathfrak{g}$ $(\lambda \in \mathbb{R})$

$$P_\theta = \left\{ X = \sum_{\substack{i \in \mathbb{Z} \\ \lambda \in \mathbb{R}}} X_{i\lambda} z^i \;\middle|\; X_{i\lambda} \in \mathfrak{g}_\lambda,\ i+\lambda \geq 0 \right\} \subset \mathfrak{g}(K)$$

$$\mathfrak{n}_\theta = \{ \underline{\hspace{8cm}},\ i+\lambda > 0 \}$$

$$L_\theta = \{ \underline{\hspace{8cm}},\ i+\lambda = 0 \}$$ "Levi" of P_θ

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$$\mathfrak{U}_\theta = \{ \underline{\qquad\qquad\qquad\qquad\qquad\qquad}, \ i+\lambda > 0 \}$$

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$\hat{H}_\theta = C_G(e^{2\pi i \theta}) \subset G$, $\mathfrak{h}_\theta = \mathrm{Lie}(\hat{H}_\theta) \cong \mathfrak{l}_\theta$

[not nec. Levi of G
cf. $SL_3 \subset G_2$

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$$\hat{P}_\theta = \{ g \in G(K) \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ on any ray} \}$$

$$= \hat{L}_\theta \cdot U_\theta \quad \left[\hat{L}_\theta = \{ z^{-\theta} h z^\theta \mid h \in \hat{H}_\theta \},\ U_\theta = \exp(\mathfrak{u}_\theta) \right]$$

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$= \hat{L}_\theta \cdot U_\theta$ $\left[\hat{L}_\theta = \{ z^{-\theta} h z^\theta \mid h \in \hat{H}_\theta \},\ U_\theta = \exp(\mathfrak{u}_\theta) \right]$

$$\boxed{A_\theta = P_\theta \frac{dz}{z}}$$ — $\hat{P}_\theta$ acts on A_θ by gauge transformations

Lemma $\widehat{\mathcal{L}}_\theta$ gauge orbits in L_θ $\longleftrightarrow$ $\widehat{H}_\theta$ adjoint orbits in H_θ

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$$\left(\tau + \sigma + \sum a_i z^i\right)\frac{dz}{z} \in \mathcal{O} \subset L_\theta \qquad\qquad \phi + \sigma + n$$

$\theta, \tau, \phi \in \mathfrak{t}_{\mathbb{R}}$ $\qquad \phi = \tau + \theta$

$\sigma \in \sqrt{-1}\,\mathfrak{t}_{\mathbb{R}}$

$n \in H_\theta$ nilpotent, $[\phi, n] = [\sigma, n] = 0$, $n = \sum a_i$, $[\tau, a_i] = i a_i = [a_i, \theta]$

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Let $P_\phi \subset G$ be parabolic attached to ϕ

$L = C_G(\phi)$ Levi of P_ϕ

$\mathcal{C} \subset L$ conjugacy class of $\exp_L(2\pi i(\tau + \sigma))\exp_L(2\pi i n)$

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Thm There is a canonical bijection: $\{A\in \mathcal{A}_\theta \mid \pi(A)\in \mathcal{O}\}/\hat{\mathcal{P}}_\theta \cong \hat{\mathcal{C}}/G$

and all the spaces $\hat{\mathcal{C}}$ appear in this way