Seminaire: Poincare-Hopf Theorem

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Poincare Hopf Theorem Let $M$ be a compact manifold and $w$ a smooth vector field on $M$ with isolated zeros. If $M$ has a boundary then $w$ is required to point outward at all boundary points. The sum $\sum \iota$ of the indices at the zeros of such a vector field is equal to the Euler number

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \text{rank} H_i(M).$$

In particular this index sum is a topological invariant of $M$: it does not depend on the particular choice of vector field.

1 Some definitions

Critical value A smooth map $f : M \to N$, $M, N$ manifolds dimension $m, n$. Set of critical points $C = \{x \in M | df_x : TM_x \to TN_{f(x)}, \text{rank } df_x < n\}$. $f(C) := \text{set of critical values}$. $N - f(c) = \text{set of regular values}$.

Sard Theorem (admit) Let $f : U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^m$, and let $C = \{x \subset U | \text{rank } df_x < n\}$. Then $f(c) \subset \mathbb{R}^n$ has Lebesgue measure zero.

If $f : M \to N$, and $\text{dim } M = \text{dim } N$. Regular values exist and dense.

Oriented manifolds An orientation on an $n$–dimensional manifold is given by a nowhere vanishing differential $n$–form. i.e. $\exists \{U, \psi_U\}$, a chart of manifold $M$, and $\forall U \cap V \neq 0$, $\psi_V \circ \psi_U^{-1} : \mathbb{R}^n \to \mathbb{R}^n, |d(\psi_V \circ \psi_U^{-1})| > 0$.

Orientation for boundary Each orientation for $M$ determines an orientation for $\partial M$ : For $\forall x \in \partial M$, $(v_1, ..., v_m)$ positively oriented basis for $TM_x$, s.t. $v_1, ..., v_m \in T\partial M_x$ and $v_1$ is an outward vector.

Let $M, N$ be oriented $n$–dimensional manifolds without boundary and let $f : M \to N$ be a smooth map. $M$ is compact and $N$ is connected. For any regular value $y \in N$ define:

$$\text{deg}(f; y) = \sum_{x \in f^{-1}(y)} \text{sign } df_x.$$ 

$\text{deg}(f; y)$ is a locally constant function of $y$.

It is defined on a dense open subset of $N$. 

1
2 Brower degree

Theorem A : The integer \( \deg(f; y) \) does not depend on the choice of regular value \( y \).

Theorem B : If \( y \) is smoothly homotopic to \( g \). Then \( \deg f = \deg g \).

Lemma 2.1 \( M = \partial X \), \( X \) compact oriented. \( M \) oriented as \( \partial \). If \( f : M \to N \) extends to a smooth map \( F : X \to N \), then \( \deg(f; y) = 0 \) for every regular value \( y \).

Proof: First suppose \( y \) is a regular value for \( F \). Then the compact 1-manifold \( F^{-1}(y) \) is a finite union of arcs and circles(Implicit function theorem), with only boundary points of the arcs lying on \( M = \partial X \). Let \( A \subset F^{-1}(y) \) be one of these arcs. \( \partial A = \{x\} \cup \{y\} \).

We will show that \( \text{sign } df_a + \text{sign } df_b = 0 \).

The idea is to show that orientations for \( X \) and \( N \) determine an orientation for \( A \) (while its two boundary points lie on \( M \)), then \( A \) goes inward at one boundary point, and outward at the other point. As \( M \) is oriented as boundary related to an outward vector, we have \( \text{sign } df_a + \text{sign } df_b = 0 \).

Here we define orientation for \( A \) as follows:

\( v_1(x) \) denote the positively oriented unit vector tangent to \( A \) at \( x \) if:

1). \( dF_x(v_1) = 0 \) (\( v_1 \) tangent to \( A \).

2). \( dF_x(v_2, ..., v_{m+1}) \) is a positively oriented basis for \( TN_y \) and \( v_1, v_2, ..., v_{m+1} \) is a positively oriented basis for \( TX_x \).

\( v_1(x) \) is a smooth function and points out at one boundary point (say \( b \)), and inward at the other boundary point (say \( a \)). Thus sign \( df_a = -1 \), sign \( df_b = 1 \). Adding up over all such arcs \( A \), we have \( \deg(f; y) = 0 \).

More generally suppose \( y_0 \) is a regular value for \( f \), but not for \( F \), as \( \deg(f; y) \) is constant within some neighborhood \( U \) of \( y_0 \), choose a regular value \( y \) for \( F \) within \( U \), then \( \deg(f; y_0) = \deg(f; y) = 0 \).\( \sharp \)

Lemma 2.2 The degree \( \deg(g; y) \) is equal to \( \deg(f; y) \) for any common regular value \( y \).

Proof Consider a smoothly homotopy: \( F = [0,1] \times M \to N \). \( f(x) = F(0, x) \), \( g(x) = F(1, x) \). \( [0,1] \times M \) can be oriented as a product. \( 1 \times M \) with the correct orientation, \( 0 \times M \) with the wrong orientation. Then \( \deg(g; y) - \deg(f; y) = 0 \).\( \sharp \)

If \( y \) and \( z \) are both regular values for \( f : M \to N \). Choose a diffeomorphism(existence of \( h \) will be talked about later for those who have interest) \( h : N \to N \) that carries \( y \) to \( z \) and is isotopic(connected 'through homeomorphisms') to the identity. By definition, \( \deg(f; y) = \deg(h \circ f, h(y)) = \deg(h \circ f, z) \), and by isotopy and Lemma 2.2, \( \deg(h \circ f; z) = \deg(f; z) \). This completes the proof of Theorem A and Theorem B.\( \sharp \)

In order to proof the existence of \( h \), we need the following:

Homogeneity Lemma Let \( y \) and \( z \) be arbitrary interior points of the smooth, connected manifold \( N \). Then there exists a diffeomorphism \( h : M \to N \) that is smoothly isotopic to the identity and carries \( y \) into \( z \).
Proof As $N$ is connected, we could consider locally, it is enough to proof this lemma for $N = \mathbb{R}^n$. The idea is to fix the area out of unit ball $B^n$, and construct a vector field whose flow sends 0 to an arbitrary point in $B^n$, and tends to zero near $S^{n-1}$.

$\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ who satisfies

1) $\psi(x) > 0$ for $\|x\| < 1$.
2) $\psi(x) = 0$ for $\|x\| \geq 1$.

(ex: $\psi(x) = \exp(- (1 - \|x\|^2)^{-1})$).

For $\forall \bar{x} \in B^n$, $\exists! x(t), x(0) = \bar{x}, F_t(\bar{x}) := x(t)$.

1). $F_t(\bar{x})$ is defined for all $t$ and $\bar{x}$ in $B^n$, and depends smoothly on $t$ and $\bar{x}$.
2). $F_0(\bar{x}) = \bar{x}$.
3). $F_{t+s} = F_s \circ F_t(\bar{x})$.

$\forall y \in B^n$, let $c = \frac{y}{\|y\|}$. As $\psi(x) > 0$ for $\|x\| < 1$, flow $F_t$ travels by direction $c$ with a nonzero speed in $B$, so for some time $t$, $F_t(0) = y$.

3 Vector fields and the Euler number

Index of a vector field

Consider an open set $U \subset \mathbb{R}^m$ and a smooth vector field $v : U \rightarrow \mathbb{R}^m$ with an isolated zero at the point $z \in U$. $\bar{v}(x) := \frac{v(x)}{\|v(x)\|}$ maps a small sphere centered at $z$ into the unit sphere. The degree of this mapping is called the index $\tau$ of $v$ at the zero $z$, note $\text{Ind}_z(v)$, and $\text{Ind}(v) = \sum_{z,v(z)=0} \text{Ind}_z(v)$.

Remark: Index of a vector field is counted only at its zeros, because at nonzero points, $\bar{v}(x)$ could extends to the whole ball, and Lemma 2.1 tells us $\deg(\bar{v}(x), z) = 0$.

To define the concept of index for a vector field $w$ on an arbitrary manifold, we need the following lemma:

Lemma 3.1 Suppose that the vector field $v$ on $U$ corresponds to $v' = df \circ v \circ f^{-1}$ on $U'$ under a diffeomorphism $f : U \rightarrow U'$. Then the index of $v$ at an isolated zero $z$ is equal to the index of $v'$ at $f(z)$.

According to definition, index of a vector field at one zero is a local property. So it’s enough to consider the case in $\mathbb{R}^m$.

Lemma 3.2 Any orientation preserving diffeomorphism $f$ of $\mathbb{R}^m$ is smoothly isotopic to the identity.

Proof suppose $f(0) = 0$.

$f(x) = \sum x_i g_i(x)$ for $g_i(x) = \int_0^1 \frac{df}{dt}(tx) dt$.

Let $F(x,t) = f(xe^t)$, therefore $F(x,0) = df_0(x)$.

Thus $F(x,t)$ gives a isotopy between $f$ and linear map $df_0(x)$, as $SL(R,m)$ is path connected, $df_0(x)$ is isotopic to id.
For \(|df_y| > 0\), use Lemma 3.2. If \(|df_y| < 0\), it’s enough to consider a reflection \(\rho\), and \(v' = \rho \circ \rho^{-1}\), and observe that \(\text{Ind}_{y}(v) = \text{Ind}_{\rho(y)}(v')\).

Let \(M\) be a compact manifold and \(w_0, w_1\) smooth vector fields on \(M\) with isolated zeros \((w_0 \neq \lambda w_1)\). If \(M\) has a boundary, then \(w_i\) is required to point outward at all boundary points.

**Lemma 3.3** \(\text{Ind}(w_0) = \text{Ind}(w_1)\).

Consider smooth vector fields \(w_t = tw_0 + (1 - t)w_1, t \in [0,1]\). Let \(S = \{t| \text{Ind}(w_t = \text{Ind}(w_0)\},\) we will show that \(S\) is open and closed.

\(S\) is open: for \(w_{t_0}\) with finite isolated zeros \(z_i\), and for each \(z_i\) choose a sufficient small sphere \(S_i\) to calculate \(\text{Ind}_{z_i}(w_{t_0})\). \(\exists \epsilon \text{ s.t. } \forall t, |t - t_0| < \epsilon,\) the zeros of \(w_t\) (maybe more than zeros of \(w_{t_0}\) lie in \(S_i\). According to Lemma 2.1, \(\text{Ind}(w_{t_0}) = \text{Ind}(w_t)\).

\(S\) is closed: if \(\{t_k\} \subset S,\) and \(t\) is the limit, then the same argument shows that \(\text{Ind}(w_{t_k}) = \text{Ind}(w_t)\) for some \(k\) great.

Thus \(S = [0,1]\) and specially, \(\text{Ind}(v_0) = \text{Ind}(v_1)\).

**Vector Bundle** A real vector bundle consists of:

1) topological spaces \(X\) (base space and \(E\) total space
2) a continuous subjection \(\pi E \rightarrow X\) (bundle projection)
3) \(\forall x \in X, \pi^{-1}(x)\) has a finite-dimensional real vector space structure
   where the following compatibility condition is satisfied: \(\forall x \in X, \exists U \subset X, k\) a natural number, and a homeomorphism: \(\psi : U \times R^k \rightarrow \pi^{-1}(U)\) such that for all \(x \in U\).
   a) \((\pi \circ \psi)(x, v) = x\) for all vectors \(v\) in \(R^k\), and
   b) \(v \rightarrow \psi(x, v)\) is an isomorphism between \(R^k\) and \(\pi^{-1}(x)\).

\((U, \psi)\) is called local trivialisation of the vector bundle, and \(x \rightarrow k_x\) is a function locally constant, so if \(E\) is connected, \(k_x\) is equal to a constant \(k\), and \(E\) is said to be a vector bundle of rank \(k\).

**Generalization on vector bundle** A smooth vector field \(w\) on a compact manifold \(M\), could be seen as a section of its tangent bundle \(TM\). More generally, consider a rank \(n\) vector bundle over a compact oriented manifold without boundary \(M\), \(\text{dim} M = \text{rank} E\), and a smooth section \(s : M \rightarrow E\). In the same way, we could define index of a smooth section with isolated zeros, named \(\text{Ind}_{E}(s)\). Here \(\text{Ind}_{E}(s)\) is defined on zeros of \(s\), as \(T_{o_0}E \cong T_{o_0}M \oplus E_{o_0}\). And the same argument as in Lemma 3.3, we have \(\text{Ind}_{E}(s_1)=\text{Ind}_{E}(s_2), \forall s_1, s_2\) smooth sections with isolated zeros.

**Property** Given a smooth vector bundle \(F \rightarrow Y\), and a smooth map \(f : X \rightarrow Y, \text{dim} X = \text{dim} Y = \text{rank} F\), one could define a ”pullback” vector bundle \(f^*F\) on \(X\), that is the fiber over a point \(x \in X\) is essentially just the fiber over \(f(x) \in Y\). And for any smooth section \(s : Y \rightarrow F\), with isolated zeros, we have \(\text{Ind}_{f^*F}(s \circ f) = \text{deg}(f) * \text{Ind}_{F}(s)\).