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Chow Rings, Decomposition of the Diagonal, and the Topology of Families

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Contents

Preface

These notes grew out of lectures I delivered in Philadelphia (the Rademacher lectures) and in Princeton (the Hermann Weyl lectures). The central objects are the diagonal of a variety X and the small diagonal in X^3. Topologically, we have the Künneth decomposition of the diagonal, which has as a consequence, for example, the Poincaré–Hopf formula, but such a Künneth decomposition does not exist in the context of Chow groups, unless the variety has trivial Chow groups. The Bloch–Srinivas principle and its generalizations provide the beginning of such a decomposition in the Chow group of $X \times X$, under the assumption that Chow groups in small dimension are trivial (that is, parametrized by the cohomological cycle class). The study of the diagonal thus allows us to study Chow groups $\mathrm{CH}(X)$ of X seen additively, but not the ring structure of $\mathrm{CH}(X)$. The latter is governed by the small diagonal, which, seen as a correspondence between $X \times X$ and X, induces the cup-product in cohomology and the intersection product on Chow groups.

The second central topic of the book is the spread of cycles and rational equivalence, which appeared first in Nori's work and which has become very important to relate Chow groups and topology in a refined way.

I first considered the small diagonal in joint work with Beauville where we proved that the small diagonal of a $K3$ surface has a very special Chow-theoretic decomposition. I then realized that this partially extends to some Calabi–Yau varieties, and furthermore that this decomposition, when spread up over a family, implies very special multiplicative properties of the Leray spectral sequence.

Concerning the diagonal itself, I proved recently, by a spreading argument applied to a cohomological decomposition of the diagonal, that for varieties like complete intersections, admitting large families of deformations with very simple total space, the generalized Hodge conjecture predicting equality between the Hodge coniveau and the geometric coniveau is equivalent to the generalized Bloch conjecture saying that the Hodge coniveau governs the triviality of Chow groups of small dimension.

This book also reflects my interest in recent years in questions involving cycles with $\mathbb{Z}$-coefficients rather than $\mathbb{Q}$-coefficients. The diagonal decomposition and, more generally, the spreading principle for rational equivalence, become wrong with $\mathbb{Z}$-coefficients, and this is a source of interesting torsion invariants. I have also included a discussion of the defect of the Hodge conjecture with integral coefficients, as the recent proof of the Bloch–Kato conjecture gave an important new impulse to the subject.

Needless to say, even though I have tried to include some background material and also to present an overall view of some more advanced results, so that the notes can be used by students, this book presents a very personal and very incomplete view of the subject of algebraic cycles. In particular, a number of topics are missing and a very geometric point of view has been adopted, which does not reflect the general abstract theory of algebraic cycles well, particularly algebraic K-theory and motivic cohomology. I apologize to the many people whose work should have been quoted and discussed here, and is missing from these notes.

Thanks. *I thank the Institute for Advanced Study for inviting me to deliver the Hermann Weyl lectures and for giving me the opportunity to write up and publish these notes. I also thank Lie Fu for his careful reading and corrections.*

Claire Voisin
Paris, 28 March 2013

Chapter One

Introduction

These lectures are devoted to the interplay between cohomology and Chow groups, and also to the consequences, for the topology of a family of smooth projective varieties, of statements concerning Chow groups of the general or very general fiber.

A crucial notion is that of the *coniveau* of a cohomology. A Betti cohomology class has geometric coniveau $\geq c$ if it is supported on a closed algebraic subset of codimension $\geq c$. The coniveau of a class of degree k is $\leq \frac{k}{2}$. As a smooth projective variety X has nonzero cohomology in degrees $0, 2, 4, \ldots$ obtained by taking $c_1(L)$, with L an ample line bundle on X, and its powers $c_1(L)^i$, it is not expected that the whole cohomology of X has large coniveau. But it is quite possible that the "transcendental" cohomology $H^*_B(X)^{\perp \mathrm{alg}}$, consisting of classes orthogonal (with respect to the Poincaré pairing) to cycle classes on X, has large coniveau.

There is another notion of coniveau: the *Hodge coniveau*, which is computed by looking at the shape of the Hodge structures on $H^*_B(X, \mathbb{Q})$. Classes of algebraic cycles are conjecturally detected by Hodge theory as Hodge classes, which are the degree $2k$ rational cohomology classes of Hodge coniveau k. The generalized Hodge conjecture due to Grothendieck [50] more generally identifies the coniveau above (or geometric coniveau) to the Hodge coniveau.

The next crucial idea goes back to Mumford [71], who observed that for a smooth projective surface S, there is a strong correlation between the structure of the group $\mathrm{CH}_0(S)$ of 0-cycles on S modulo rational equivalence and the spaces of holomorphic forms on S. The degree 1 holomorphic forms govern the Albanese map, which itself provides us with a certain natural quotient of the group $\mathrm{CH}_0(S)_{\mathrm{hom}}$ of 0-cycles homologous to 0 (that is, of degree 0 if S is connected), which is in fact an abelian variety. This part of $\mathrm{CH}_0(S)_{\mathrm{hom}}$ is small in different (but equivalent) senses, first of all because it is parametrized by an algebraic group, and second because, for any ample curve $C \subset S$, the composite map

$$\mathrm{CH}_0(C)_{\mathrm{hom}} \to \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$$

is surjective. Thus 0-cycles supported on a given ample curve are sufficient to exhaust this part of $\mathrm{CH}_0(S)_{\mathrm{hom}}$.

Mumford's theorem [71] says the following.

Theorem 1.1 (Mumford 1968). *If $H^{2,0}(S) \neq 0$, no curve $C \stackrel{j}{\hookrightarrow} S$ satisfies the property that $j_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective.*

The parallel with geometric coniveau in cohomology is obvious in this case; indeed, the assumption that $H^{2,0}(S) \neq 0$ is equivalent (by the Lefschetz theorem on $(1,1)$-classes) to the fact that the cohomology $H^2_B(S,\mathbb{Q})$ is not supported on a divisor of S. Thus Mumford's theorem exactly says that if the degree 2 cohomology of S is not supported on any divisor, then its Chow group $\mathrm{CH}_0(S)$ is not supported on any divisor.

The converse to such a statement is the famous Bloch conjecture [13]. The Bloch conjecture has been generalized in various forms, one involving filtrations on Chow groups, the graded pieces of the filtration being governed by the coniveau of Hodge structures of adequate degree (see [58], [89], and Section 2.1.4). The crucial properties of this conjectural filtration are functoriality under correspondences, finiteness, and the fact that correspondences homologous to 0 shift the filtration.

We will focus in these notes on a more specific higher-dimensional generalization of the Bloch conjecture, "the generalized Bloch conjecture," which says that if the cohomology $H^*_B(X,\mathbb{Q})^{\perp \mathrm{alg}}$ has coniveau $\geq c$, then the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2n-2i}_B(X,\mathbb{Q})$ is injective for $i \leq c-1$. In fact, if the variety X has dimension > 2, there are two versions of this conjecture, according to whether we consider the geometric or the Hodge coniveau. Of course, the two versions are equivalent assuming the generalized Hodge conjecture. In Section 4.3 we will prove this conjecture, following [114], for the geometric coniveau and for very general complete intersections of ample hypersurfaces in a smooth projective variety X with "trivial" Chow groups, that is, having the property that the cycle class map

$$\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}_B(X,\mathbb{Q})$$

is injective (hence an isomorphism according to [67]).

A completely different approach to such statements was initiated by Kimura [59], and it works concretely for those varieties that are dominated by products of curves. It should be mentioned here that all we have said before works as well in the case of motives (see Section 2.1.3). In the above-mentioned work of Kimura, one can replace "varieties that are dominated by products of curves" by "motives that are a direct summand of the motive of a product of curves." In our paper [114], we can work with a variety X endowed with the action of a finite group G and consider the submotives of G-invariant complete intersections obtained by considering the projectors $\Gamma_\pi \in \mathrm{CH}(Y \times Y)_{\mathbb{Q}}$ associated via the action of G on Y to projectors $\pi \in \mathbb{Q}[G]$.

An important tool introduced by Bloch and Srinivas in [15] is the so-called decomposition of the diagonal. It relates information concerning Chow groups $\mathrm{CH}_i(X)$, for small i, to the geometric coniveau of X. Bloch and Srinivas initially considered the decomposition of the diagonal in its simplest form, starting from information on $\mathrm{CH}_0(X)$, and this has subsequently been generalized in [66], [80] to a generalized decomposition of the diagonal. This leads to an elegant proof of the so-called generalized Mumford–Roitman theorem, stating that if the cycle class map

$$\mathrm{cl}:\mathrm{CH}_i(X)_{\mathbb{Q}}\to H_B^{2n-2i}(X,\mathbb{Q})$$

is injective for $i\leq c-1$, then the transcendental cohomology $H_B^*(X,\mathbb{Q})^{\perp\mathrm{alg}}$ has geometric coniveau $\geq c$. (The generalized Bloch conjecture is thus the converse to this statement.)

The study of the diagonal will play a crucial role in our proof of the generalized Bloch conjecture for very general complete intersections. The diagonal will appear in a rather different context in Chapter 5, where we will describe our joint work with Beauville and further developments concerning the Chow rings of $K3$ surfaces and hyper-Kähler manifolds. Here we will be concerned not with the diagonal $\Delta_X\subset X\times X$ but with the small diagonal $\Delta\cong X\subset X\times X\times X$. The reason is that if we consider Δ as a correspondence from $X\times X$ to X, we immediately see that it governs, among other things, the ring structure of $\mathrm{CH}^*(X)$. In [11] we obtained for $K3$ surfaces X a decomposition of Δ involving the large diagonals, and a certain canonical 0-cycle o canonically attached to X.

We will show in Section 5.3 an unexpected consequence, obtained in [110], of this study combined with the basic spreading principle described in Section 3.1, concerning the topology of families of $K3$ surfaces.

In a rather different direction, in the final chapter we present recent results concerning Chow groups and Hodge classes with integral coefficients. Playing on the defect of the Hodge conjecture for integral Hodge classes (see [5]), we exhibit a number of birational invariants which vanish for rational projective varieties and are of torsion for unirational varieties. Among them is precisely the failure of the Bloch–Srinivas diagonal decomposition with integral coefficients: in general, under the assumption that $\mathrm{CH}_0(X)$ is small, only a multiple of the diagonal of X can be decomposed as a cycle in $X\times X$. The minimal such multiple appears to be an interesting birational invariant of X.

In the rest of this introduction, we survey the main ideas and results presented in this monograph a little more precisely. Background material is to be found in Chapter 2.

1.1 DECOMPOSITION OF THE DIAGONAL AND SPREAD

1.1.1 Spread

The notion of the spread of a cycle is very important in the geometric study of algebraic cycles. The first place where it appears explicitly is Nori's paper [76], where it is shown that the cohomology class of the spread cycle governs many invariants of the cycle restricted to general fibers. The idea is the following (see also [47]): Assume that we have a family of smooth algebraic varieties, that is, a smooth surjective morphism

$$\pi:\mathcal{X}\to B,$$

with geometric generic fiber $\mathcal{X}_{\overline{\eta}}$ and closed fiber $\mathcal{X}_s$. If we have a cycle $Z \in \mathcal{Z}^k(\mathcal{X}_{\overline{\eta}})$, then we can find a finite cover $\widetilde{U} \to U$ of a Zariski open set U of B such that Z is the restriction to the geometric generic fiber of a cycle $\mathcal{Z}_{\widetilde{U}} \in \mathcal{Z}^k(\mathcal{X}_{\widetilde{U}})$.

If we are over $\mathbb{C}$, we can speak of the spread of a cycle $Z_s \in \mathcal{Z}^k(X_s)$, where $s \in B$ is a very general point. Indeed, we may assume that π is projective. We know that there are countably many relative Hilbert schemes $M_i \to B$ parametrizing all subschemes in fibers of π. Cycles $Z = \sum_i n_i Z_i$ in the fibers of π are similarly parametrized by countably many varieties $\pi_J : N_J \to B$, where the π_J's are proper, and the indices J also encode the multiplicities n_i.

Let $B' \subset B$ be the complement of the union $\cup_{J\in E} \operatorname{Im} \pi_J$, where E is the set of indices J for which π_J is not surjective. A point of B' is a very general point of B, and by construction of B', for any $s \in B'$, and any cycle $Z_s \in \mathcal{Z}^k(X_s)$, there exist an index J such that $M_J \to B$ is surjective, and a point $s' \in M_J$ such that $\pi_J(s') = s$, and the fiber $\mathcal{Z}_{M_J,s'}$ at s' of the universal cycle

$$\mathcal{Z}_{M_J} \subset \mathcal{X}_{M_J} = \mathcal{X} \times_B M_J,$$

parametrized by M_J, is the cycle Z_s. By taking linear sections, we can then find $M'_J \subset M_J$, with $s' \in M'_J$, such that the morphism $M'_J \to B$ is dominating and generically finite. The restriction $\mathcal{Z}_{M'_J}$ of the universal cycle $\mathcal{Z}_{M_J}$ to $\mathcal{X} \times_B M'_J$ is then a spread of Z_s.

1.1.2 Spreading out rational equivalence

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism, where B is smooth irreducible and quasi-projective, and let $\mathcal{Z} \subset \mathcal{X}$ be a codimension k cycle. Let us denote by $Z_t \subset X_t$ the restriction of $\mathcal{Z}$ to the fiber X_t. We refer to Chapter 2 for the basic notions concerning rational equivalence, Chow groups, and cycle classes.

An elementary but fundamental fact is the following result, proved in Section 3.1.

THEOREM 1.2 (See Theorem 3.1). *If for any $t \in B$ the cycle Z_t is rationally equivalent to 0, there exist a Zariski open set $U \subset B$ and a nonzero integer N such that $N\mathcal{Z}_{|\mathcal{X}_U}$ is rationally equivalent to 0, where $\mathcal{X}_U := \pi^{-1}(U)$.*

Note that the set of points $t \in B$ such that Z_t is rationally equivalent to 0 is a countable union of closed algebraic subsets of B, so that we could in the above statement, by a Baire category argument, make the a priori weaker (but in fact equivalent) assumption that Z_t is rationally equivalent to 0 for a very general point of B.

This statement is what we call the spreading-out phenomenon for rational equivalence. This phenomenon does not occur for weaker equivalence relations such as algebraic equivalence.

An immediate but quite important corollary is the following.

COROLLARY 1.3. *In the situation of Theorem 1.2, there exists a dense Zariski open set $U \subset B$ such that the Betti cycle class $[\mathcal{Z}] \in H^{2k}_B(\mathcal{X}, \mathbb{Q})$ vanishes on the open set $\mathcal{X}_U$.*

The general principle above applied to the case where the family $\mathcal{X} \to B$ is trivial, that is, $\mathcal{X} \cong X \times B$, leads to the so-called decomposition principle due to Bloch and Srinivas [15]. In this case, the cycle $\mathcal{Z} \subset B \times X$ can be seen as a family of cycles on X parametrized by B or as a correspondence between B and X. Then Theorem 1.2 says that if a correspondence $\mathcal{Z} \subset B \times X$ induces the trivial map

$$\mathrm{CH}_0(B) \to \mathrm{CH}^k(X), \quad b \mapsto Z_b,$$

then the cycle $\mathcal{Z}$ vanishes up to torsion on some open set of the form $U \times X$, where U is a dense Zariski open set of B.

The first instance of the diagonal decomposition principle appears in [15]. This is the case where $X = Y \setminus W$, with Y smooth and projective, and $W \subset Y$ is a closed algebraic subset, $B = Y$, and $\mathcal{Z}$ is the restriction to $Y \times (Y \setminus W)$ of the diagonal of Y. In this case, to say that the map

$$\mathrm{CH}_0(B) \to \mathrm{CH}_0(X), \quad b \mapsto Z_b,$$

is trivial is equivalent to saying, by the localization exact sequence (2.2), that any point of Y is rationally equivalent to a 0-cycle supported on W. The conclusion is then the fact that the restriction of the diagonal cycle Δ to a Zariski open set $U \times (Y \setminus W)$ of $Y \times Y$ is of torsion, for some dense Zariski open set $U \subset Y$. Using the localization exact sequence, one concludes that a multiple of the diagonal is rationally equivalent in $Y \times Y$ to the sum of a cycle supported on $Y \times W$ and a cycle supported on $D \times Y$, where $D := Y \setminus U$. Passing to cohomology, we get the following consequence.

Corollary 1.4. *If Y is smooth projective of dimension n and $\mathrm{CH}_0(Y)$ is supported on $W \subset Y$, the class $[\Delta_Y] \in H_B^{2n}(Y \times Y, \mathbb{Q})$ decomposes as*

$$[\Delta_Y] = [Z_1] + [Z_2],$$

where the cycles Z_i are cycles with $\mathbb{Q}$-coefficients on $Y \times Y$, Z_1 is supported on $D \times Y$ for some proper closed algebraic subset $D \subsetneq Y$, and Z_2 is supported on $Y \times W$.

1.1.3 Applications of Mumford-type theorems

In the paper [15] by Bloch and Srinivas, an elegant proof of Mumford's theorem (Theorem 1.1) is provided, together with the following important generalization.

Theorem 1.5 (Roitman 1980, Bloch and Srinivas 1983; see Theorem 3.13). *Let X be a smooth projective variety and $W \subset X$ be a closed algebraic subset of dimension $\leq k$ such that any point of X is rationally equivalent to a 0-cycle supported on W. Then $H^0(X, \Omega_X^l) = 0$ for $l > k$.*

This theorem, together with other very important precisions concerning the coniveau (see Section 2.2.5) of the cohomology of X, is obtained using only the cohomological decomposition of the diagonal of X, that is, Corollary 1.4.

Theorem 1.2 is also the only ingredient in the proof of the generalized decomposition of the diagonal (see [66], [80], [101, II, 10.3], and Section 3.2.1). The Bloch–Srinivas decomposition of the diagonal described in the previous subsection is a decomposition (up to torsion and modulo rational equivalence) involving two pieces, one supported via the first projection over a divisor of X, the other supported via the second projection over a subset $W \subset X$. It is obtained under the condition that $\mathrm{CH}_0(X)$ is supported on W. The generalized decomposition of the diagonal subsequently obtained independently by Laterveer and Paranjape is the following statement, where X is smooth projective of dimension n.

THEOREM 1.6 (See Theorem 3.18). *Assume that for $k < c$, the cycle class maps*

$$\mathrm{cl} : \mathrm{CH}_k(X) \otimes \mathbb{Q} \to H_B^{2n-2k}(X, \mathbb{Q})$$

are injective. Then there exists a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-1} + Z' \in \mathrm{CH}^n(X \times X), \tag{1.1}$$

where $m \neq 0$ is an integer, Z_i is supported in $W_i' \times W_i$ with $\dim W_i = i, \dim W_i' = n - i$, and Z' is supported in $T \times X$, where $T \subset X$ is a closed algebraic subset of codimension $\geq c$.

Note that a version of this theorem involving the Deligne cycle class instead of the Betti cycle class was established in [37]. We refer to Section 3.2.1 for applications of this theorem. In fact, the main application involves only the corresponding cohomological version of the decomposition (1.1), that is, the generalization of Corollary 1.4, which concerned the case $c = 1$. It implies that under the same assumptions, the transcendental cohomology $H_B^*(X, \mathbb{Q})^{\perp \mathrm{alg}}$ has geometric coniveau $\geq c$.

Other applications involve the decomposition (1.1) in the group $\mathrm{CH}(X \times X)/\mathrm{alg}$ of cycles modulo algebraic equivalence. This is the case of applications to the vanishing of positive degree unramified cohomology with $\mathbb{Q}$-coefficients (and in fact with $\mathbb{Z}$-coefficients (see [6], [15], [24]) thanks to the Bloch–Kato conjecture proved by Rost and Voevodsky; see [97]).

1.1.4 Another spreading principle

Although elementary, the following spreading result proved in Section 4.3.3 is crucial for our proof of the equivalence of the generalized Bloch and Hodge conjectures for very general complete intersections in varieties with trivial Chow groups (see [114] and Section 4.3). Its proof is based as usual on the countability of the relative Hilbert schemes for a smooth projective family $\mathcal{Y} \to B$.

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism and let $(\pi, \pi) : \mathcal{X} \times_B \mathcal{X} \to B$ be the fibered self-product of $\mathcal{X}$ over B. Let $\mathcal{Z} \subset \mathcal{X} \times_B \mathcal{X}$ be a codimension k algebraic cycle. We denote the fibers $\mathcal{X}_b := \pi^{-1}(b)$, $\mathcal{Z}_b := \mathcal{Z}_{|\mathcal{X}_b \times \mathcal{X}_b}$.

THEOREM 1.7 (See Proposition 4.25). *Assume that for a very general point $b \in B$, there exist a closed algebraic subset $Y_b \subset \mathcal{X}_b \times \mathcal{X}_b$ of codimension c, and an algebraic cycle $Z'_b \subset Y_b \times Y_b$ with $\mathbb{Q}$-coefficients, such that*

$$[Z'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

Then there exist a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c, and a codimension k algebraic cycle $\mathcal{Z}'$ with $\mathbb{Q}$-coefficients on $\mathcal{X} \times_B \mathcal{X}$, such that $\mathcal{Z}'$ is supported on $\mathcal{Y} \times_B \mathcal{Y}$, and for any $b \in B$,

$$[\mathcal{Z}'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

1.2 THE GENERALIZED BLOCH CONJECTURE

As we will explain in Section 3.2.1, the generalized decomposition of the diagonal (Theorem 1.6, or rather its cohomological version) leads to the following result.

THEOREM 1.8 (See Theorem 3.20). *Let X be a smooth projective variety of dimension m. Assume that the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c - 1$. Then we have $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$).

The Hodge structures on $H_B^k(X, \mathbb{Q})^{\perp \mathrm{alg}}$ are thus all of Hodge coniveau $\geq c$; in fact they are even of geometric coniveau $\geq c$, that is, these Hodge structures satisfy the generalized Hodge conjecture (Conjecture 2.40) for coniveau c.

The generalized Bloch conjecture is the converse to this statement (it can also be generalized to motives). It generalizes the Bloch conjecture which concerned the case of 0-cycles on surfaces. One way to state it is the following.

CONJECTURE 1.9. *Assume conversely that $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$). Then the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c - 1$.

However, as a consequence of Theorem 1.8 above, this formulation also contains a positive solution to the generalized Hodge conjecture, which predicts, under the above vanishing assumptions, that the transcendental part of the cohomology of X is supported on a closed algebraic subset of codimension c.

A slightly restricted version of the generalized Bloch conjecture (which is equivalent for surfaces or motives of surfaces) is thus the following (see [58]).

CONJECTURE 1.10. *Let X be a smooth projective complex variety of dimension m. Assume that the transcendental cohomology $H_B^*(X, \mathbb{Q})^{\perp \mathrm{alg}}$ is supported on a closed algebraic subset of codimension c. Then the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c-1$.

Note that Conjecture 1.11 below is also an important generalization of the Bloch conjecture, concerning only 0-cycles and starting with rather different cohomological assumptions. There is no direct relation between Conjectures 1.9 and 1.11 except for the fact that they coincide for varieties X with $H^{i,0}(X) = 0$ for all $i > 0$. Of course they both fit within the general Bloch–Beilinson conjecture (Conjecture 2.19) on filtrations on Chow groups, and can be considered as concrete consequences of them.

CONJECTURE 1.11. *Let X be a smooth projective variety such that $H^{i,0}(X) = 0$ for $i > r$. Then there exists a dimension r closed algebraic subset $Z \overset{j}{\hookrightarrow} X$ such that $j_* : \mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$ is surjective.*

Our main result in [114] presented in Section 4.3 is obtained as a consequence of the spreading principle, Theorem 1.7. It proves Conjecture 1.10 for very general complete intersections in ambient varieties with "trivial Chow groups," assuming the Lefschetz standard conjecture. The situation is the following: X is a smooth projective variety of dimension n that satisfies the property that the cycle class map

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2n-2i}(X, \mathbb{Q})$$

is injective for all i. Let L_i, $i = 1, \ldots, r$ be very ample line bundles on X. We consider smooth complete intersections $X_t \subset X$ of hypersurfaces $X_i \in |L_i|$. They are parametrized by a quasi-projective base B.

THEOREM 1.12 (Voisin 2011; see Theorem 4.16). *Assume that for the very general point $t \in B$, the vanishing cohomology $H_B^{n-r}(X_t, \mathbb{Q})_{\mathrm{van}}$ is supported on a codimension c closed algebraic subset of X_t. Assume also the Lefschetz standard conjecture. Then the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X_t)_{\mathbb{Q}} \to H_B^{2n-2r-2i}(X_t, \mathbb{Q})$$

is injective for $i \leq c-1$.

In dimension $(n-r) \geq 4$, the theorem above is conditional on the Lefschetz standard conjecture (or more precisely on Conjecture 2.29 for codimension $(n-r)$ cycles). It turns out that in dimensions $(n-r) \leq 3$, the precise instance of the conjecture we need will be satisfied, so that the result is unconditional for surfaces and threefolds.

In applications, this theorem is particularly interesting in the case where X is the projective space $\mathbb{P}^n$. In this case, the hypersurfaces X_i are characterized by their degrees d_i and we may assume $d_1 \leq \cdots \leq d_r$. When n is large compared to the d_i's, the Hodge coniveau (conjecturally, the geometric coniveau) of X_t is also large due to the following result established in [48] in the case of hypersurfaces (see [38] for the case of complete intersections), the proof of which will be sketched in Section 4.1.

THEOREM (See Theorem 4.1). *A smooth complete intersection $X_t \subset \mathbb{P}^n$ of r hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$ has Hodge coniveau $\geq c$ if and only if*

$$n \geq \sum_i^r d_i + (c-1)d_r.$$

If we consider motives (see Section 2.1.3) and in particular those that are obtained starting from a variety X with an action by finite group G, and looking at invariant complete intersections $X_t \subset X$ and motives associated to projectors of G, we get many new examples where the adequate variant of Theorem 1.12 holds, because a submotive often has a larger coniveau. A typical example is the case of Godeaux quintic surfaces, which are free quotients of quintic surfaces in $\mathbb{P}^3$ invariant under a certain action of $G \cong \mathbb{Z}/5\mathbb{Z}$. The G-invariant part of $H^{2,0}(S)$ is 0 although the quotient surface S/G is of general type; the Bloch conjecture has already been proved for the quotient surfaces S/G in [98] but the proof we give here is much simpler and has a much wider range of applications. In fact, a much softer version of Theorem 1.12 for surfaces is established in [109], and it gives a proof of the Bloch conjecture for other surfaces with $p_g = q = 0$.

In Section 3.2.3, we will also describe the ideas of Kimura, which lead to results of a similar shape, namely the implication from the generalized Hodge conjecture (geometric coniveau = Hodge coniveau) to the generalized Bloch conjecture (the Chow groups CH_i are "trivial" for i smaller than the Hodge coniveau), but for a completely different class of varieties. More precisely, Kimura's method applies to all motives that are direct summands in the motive of a product of curves.

1.3 DECOMPOSITION OF THE SMALL DIAGONAL AND APPLICATION TO THE TOPOLOGY OF FAMILIES

In Chapter 5, we will exploit the spreading principle, Theorem 1.2(ii), or rather its cohomological version, to exhibit a rather special phenomenon satisfied by families of abelian varieties, $K3$ surfaces, and conjecturally also by families of Calabi–Yau hypersurfaces in projective space. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. The decomposition theorem, proved by Deligne in [30] as a consequence of the hard Lefschetz theorem, is the following statement.

THEOREM (Deligne 1968). *In the derived category of sheaves of $\mathbb{Q}$-vector spaces on B, there is a decomposition*

$$R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]. \tag{1.2}$$

The question we consider in Chapter 5, following [110], is the following.

QUESTION 1.13. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a decomposition as above that is multiplicative, that is, compatible with the morphism $\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ given by the cup-product?*

As we will see quickly, the answer is generally negative, even for projective bundles $\mathbb{P}(E) \to B$. However, it is affirmative for families of abelian varieties (see Section 5.3) due to the group structure of the fibers.

The right formulation in order to get a larger range of families satisfying such a property is to ask whether there exists a decomposition isomorphism that is multiplicative over a Zariski dense open set U of the base B (or more optimistically, that is multiplicative locally on B for the Zariski topology).

One of our main results in this chapter is the following (see [110] and Section 5.3).

THEOREM 1.14 (Voisin 2011; see Theorem 5.35).

(i) For any smooth projective family $\pi : \mathcal{X} \to B$ of $K3$ surfaces, there exist a nonempty Zariski open subset B^0 of B and a multiplicative decomposition isomorphism as in (5.27) for the restricted family $\pi : \mathcal{X}^0 \to B^0$.

(ii) The class of the diagonal $[\Delta_{\mathcal{X}^0/B^0}] \in H^4_B(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ belongs to the direct summand $H^0(B^0, R^4(\pi,\pi)_*\mathbb{Q})$ of $H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q})$ for the induced decomposition of $R(\pi,\pi)_*\mathbb{Q}$.

(iii) For any line bundle L on $\mathcal{X}$, there is a Zariski dense open set $B^0 \subset B$ such that its topological first Chern class $c_1^{\mathrm{top}}(L) \in H^2_B(\mathcal{X}, \mathbb{Q})$ restricted to $\mathcal{X}^0$ belongs to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$.

In the second statement, $(\pi,\pi) : \mathcal{X}^0 \times_{B^0} \mathcal{X}^0 \to B^0$ denotes the natural map. A decomposition $R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i]$ induces a decomposition

$$R(\pi,\pi)_*\mathbb{Q} = \oplus_i R^i(\pi,\pi)_*\mathbb{Q}[-i]$$

by the relative Künneth isomorphism

$$R(\pi,\pi)_*\mathbb{Q} \cong R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q}.$$

In statements (ii) and (iii), we use the fact that a decomposition isomorphism as in (1.2) for $\pi : \mathcal{X} \to B$ induces a decomposition

$$H^k(\mathcal{X}, \mathbb{Q}) \cong \oplus_{p+q=k} H^p(B, R^q\pi_*\mathbb{Q})$$

(which is compatible with cup-product if the given decomposition isomorphism is).

This result is in fact a formal consequence of the spreading principle (Theorem 1.2) and of the following result proved in [11].

THEOREM (Beauville and Voisin 2004; see Theorem 5.3). *Let S be a smooth projective $K3$ surface. There exists a 0-cycle $o \in \mathrm{CH}_0(S)$ such that we have the following equality:*

$$\Delta = \Delta_{12} \cdot o_3 + (\text{perm.}) - (o_1 \cdot o_2 + (\text{perm.})) \text{ in } \mathrm{CH}^4(S \times S \times S)_{\mathbb{Q}}. \tag{1.3}$$

Here $\Delta \subset S \times S \times S$ is the small diagonal $\{(x,x,x),\ x \in S\}$ and the Δ_{ij}'s are the diagonals $x_i = x_j$. The class $o \in \mathrm{CH}_0(S)$ is the class of any point belonging to a rational curve in S and the o_i's are its pull-back in $\mathrm{CH}^2(S \times S \times S)_{\mathbb{Q}}$ via the various projections. The cycle $\Delta_{12} \cdot o_3$ is thus the algebraic subset $\{(x,x,o),\ x \in S\}$ of $S \times S \times S$. The term +(perm.) means that we sum over the permutations of $\{1,2,3\}$.

We will also prove in Section 5.2.1 a partial generalization of this result for Calabi–Yau hypersurfaces.

Our topological application (Theorem 1.14 above) involves the structure of the cup-product map

$$R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q},$$

which is not surprising since the small diagonal, seen as a correspondence from $S \times S$ to itself, governs the cup-product map on cohomology, as well as the intersection product on Chow groups. For the same reason, the decomposition (1.3) is related to properties of the Chow ring of $K3$ surfaces. These applications are described in Section 5.1. Our partial decomposition result concerning the small diagonal of Calabi–Yau hypersurfaces (Theorem 5.21) allows the following result to be proved in a similar way (see [110]).

THEOREM (Voisin 2011; see Theorem 5.25). *Let $X \subset \mathbb{P}^n$ be a Calabi–Yau hypersurface in projective space. Let Z_i, Z'_i be positive-dimensional cycles on X such that* $\operatorname{codim} Z_i + \operatorname{codim} Z'_i = n-1$. *Then if we have a cohomological relation*

$$\sum_i n_i[Z_i] \cup [Z'_i] = 0 \ \textit{in}\ H_B^{2n-2}(X,\mathbb{Q}),$$

this relation already holds at the level of Chow groups:

$$\sum_i n_i Z_i \cdot Z'_i = 0 \ \textit{in}\ \mathrm{CH}^{n-1}(X)_{\mathbb{Q}}.$$

1.4 INTEGRAL COEFFICIENTS AND BIRATIONAL INVARIANTS

Everything that has been said before was with rational coefficients. This concerns Chow groups and cohomology. In fact it is known that the Hodge conjecture is wrong with integral coefficients and also that the diagonal decomposition principle (Theorem 3.10 or Corollary 3.12) is wrong with integral coefficients (that is, with an integer N set equal to 1). But it turns out that some birational invariants can be constructed out of this. For example, assume for simplicity that $\mathrm{CH}_0(X)$ is trivial. Then the smallest positive integer N such that there is a Chow-theoretic (respectively cohomological) decomposition of N times the diagonal in $\mathrm{CH}^n(X \times X)$ (respectively $H^{2n}(X \times X, \mathbb{Z})$), $n = \dim X$, that is,

$$N\Delta_X = Z_1 + Z_2,\ \text{(respectively, } N[\Delta_X] = [Z_1] + [Z_2]\text{)}, \tag{1.4}$$

where Z_1, Z_2 are codimension n cycles in $X \times X$ with

$$\operatorname{Supp} Z_1 \subset T \times X, \quad T \subsetneq X, \qquad Z_2 = X \times x$$

for some $x \in X$, is a birational invariant of X. In the cohomological case, on which we will focus, we get in general a nontrivial invariant simply because, as we will show, it annihilates the torsion in $H^3_B(X, \mathbb{Z})$ and, at least when $\dim X \leq 3$, the torsion in the whole integral cohomology $H^*_B(X, \mathbb{Z})$. But we will see that it also controls much more subtle phenomena.

The groups $Z^{2i}(X)$ defined as

$$Z^{2i}(X) := \mathrm{Hdg}^{2i}(X, \mathbb{Z})/H^{2i}_B(X, \mathbb{Z})_{\mathrm{alg}}$$

themselves are birationally invariant for $i = 2$, $i = n-1$, $n = \dim X$, as remarked in [93], [105]. One part of Chapter 6 is devoted to describing recent results concerning the groups $Z^{2i}(X)$ for X a rationally connected smooth projective complex variety. On the one hand, it is proved in [24] that for such an X, the group $Z^4(X)$ is equal to the third unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z})$ with torsion coefficients. This result, also proved in [6], is a consequence of the Bloch–Kato conjecture, which has now been proved by Voevodsky and Rost. We will sketch in Section 6.2.2 the basic facts from Bloch–Ogus theory (see [14]) that are needed to establish this result.

It follows then from the work of Colliot-Thélène and Ojanguren [23] that there are rationally connected sixfolds X for which $Z^4(X) \neq 0$. Such examples are not known in smaller dimensions, and we proved that they do not exist in dimension 3. We even have the following result [103].

Theorem (See Theorem 6.5). *Let X be a smooth projective threefold that is either uniruled or Calabi–Yau (that is, with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$). Then $Z^4(X) = 0$.*

The Calabi–Yau case has been extended and used by Höring and Voisin in [53] to show the following (see also [41] for the generalization to Fano n-folds of index $n-3$, $n \geq 8$).

Theorem (See Section 6.2.1). *Let X be a Fano fourfold or a Fano fivefold of index 2. Then the group $Z^{2n-2}(X)$ is trivial, $n = \dim X$. Hence the cohomology $H^{2n-2}_B(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves.*

It is in fact very likely that the group $Z^{2n-2}(X)$ is trivial for rationally connected varieties, as follows from its invariance under deformations and also specialization to characteristic p (see Theorem 6.10 and [112]).

The final section is devoted to the study of the existence of an integral cohomological decomposition of the diagonal, particularly for a rationally connected threefold. This is in fact closely related in this case to the integral Hodge conjecture, in the following way: If X is a rationally connected threefold (or more generally, any smooth projective threefold with $h^{3,0}(X) = 0$), then the intermediate Jacobian $J(X)$ is an abelian variety. There is a degree 4 integral Hodge class $\alpha \in \mathrm{Hdg}^4(X \times J(X), \mathbb{Z})$ built using the canonical isomorphism

$$H_1(J(X), \mathbb{Z}) \cong H_B^3(X, \mathbb{Z})/\text{torsion}.$$

As we will show in Theorem 6.38, if X admits an integral cohomological decomposition of the diagonal, that is, a decomposition as in (1.4) with $N = 1$, the above class α is algebraic, or equivalently, there is a universal codimension 2 cycle on X parametrized by $J(X)$. The existence of such a universal codimension 2 cycle is not known in general even for rationally connected threefolds.

1.5 ORGANIZATION OF THE TEXT

Chapter 2 is introductory. We will review Chow groups, correspondences and their cohomological and Hodge-theoretic counterpart. The emphasis will be put on the notion of coniveau and the generalized Hodge conjecture which states the equality of geometric and Hodge coniveau.

Chapter 3 is devoted to a description of various forms of the "decomposition of the diagonal" and applications of it. This mainly leads to one implication that is well understood now, namely the fact that for a smooth projective variety X, having "trivial" Chow groups of dimension $\leq c - 1$ implies having (geometric) coniveau $\geq c$. We state the converse conjecture (generalized Bloch conjecture).

In Chapter 4, we will first describe how to compute the Hodge coniveau of complete intersections. We will then explain a strategy to attack the generalized Hodge conjecture for complete intersections of coniveau 2. The guiding idea is that although the powerful method of the decomposition of the diagonal suggests that computing Chow groups of small dimension is the right way to solve the generalized Hodge conjecture, it might be better to invert the logic and try to compute the geometric coniveau directly. And indeed, this chapter culminates with the proof of the fact that for very general complete intersections, assuming Conjecture 2.29 or the Lefschetz standard conjecture, the generalized Hodge conjecture implies the generalized Bloch conjecture; in other words, for a very general complete intersection, the fact that its geometric coniveau is $\geq c$ implies the triviality of its Chow groups of dimension $\leq c - 1$.

In Chapter 5, we turn to the study of the Chow rings of $K3$ surfaces and other K-trivial varieties. This study is related to a decomposition of the small diagonal of the triple self-product. We finally show the consequences of this study for the topology of certain K-trivial varieties: $K3$ surfaces, abelian varieties, and Calabi–Yau hypersurfaces.

The final chapter is devoted in part to the study of the groups $Z^{2i}(X)$ measuring the failure of the Hodge conjecture with integral coefficients. Some vanishing and nonvanishing results are presented, together with a comparison of the group $Z^4(X)$ with the so-called unramified cohomology of X with torsion coefficients. We also consider various forms of the existence of an integral cohomological decomposition of the diagonal (see (1.1)) of a threefold X with trivial

CH_0 group. We show that an affirmative answer to this question is equivalent to the vanishing of numerous birational invariants of X.

Convention. Most of the time, we will work over $\mathbb{C}$ and we will write X for $X(\mathbb{C})$. Unless otherwise specified, cohomology of X with constant coefficients will be Betti cohomology of $X(\mathbb{C})$ endowed with the Euclidean topology (only in Chapter 6 will we discuss other cohomology theories). A general point of X is a complex point $x \in U(\mathbb{C})$, where $U \subset X$ is a Zariski dense open set. When we say that a property is satisfied by a general point, we thus mean that there exists a Zariski open set $U \subset X$ such that the property is satisfied for any point of $U(\mathbb{C})$. This is not always equivalent to being satisfied at the geometric generic point (assuming X is connected), since some properties are not Zariski open. A typical example is the following: Let $\phi : Y \to X$ be a smooth projective morphism with fiber Y_t, $t \in X$. Let L be a line bundle on Y. The property that $\mathrm{Pic}\, Y_t = \mathbb{Z}L_{|Y_t}$ is not Zariski open.

A very general point of X is a complex point $x \in X' \subset X(\mathbb{C})$, where $X' \subset X(\mathbb{C})$ is the complement of a countable union of proper closed algebraic subsets of X. When we say that a property is satisfied by a very general point, we mean that there exists a countable collection of dense Zariski open sets U_i such that the property is satisfied by any element of $\cap_i U_i(\mathbb{C})$. For example, in the situation above, we find that the property $\mathrm{Pic}\, Y_t = \mathbb{Z}L_{|Y_t}$ is satisfied at a very general point of X if it is satisfied at the geometric generic point of X.

Chapter Two

Review of Hodge theory and algebraic cycles

This chapter is introductory. We review the necessary background material used in the rest of the notes. This includes Chow groups, correspondences and motives on the purely algebraic side, cycle classes, and (mixed) Hodge structures on the algebraic–topological side. The emphasis is put on the notion of Hodge versus geometric coniveau.

2.1 CHOW GROUPS

2.1.1 Construction

We follow [43] and [101, II]. Let X be a scheme over a field K (which in practice will always be a quasi-projective scheme, that is, a Zariski open set in a projective scheme). Let $\mathcal{Z}_k(X)$ be the group of k-dimensional algebraic cycles of X, that is, the free abelian group generated by the (reduced and irreducible) closed k-dimensional subvarieties of X defined over K. If $Y \subset X$ is a subscheme of dimension $\leq k$, we can associate a cycle $c(Y) \in \mathcal{Z}_k(X)$ to it as follows: Set

$$c(Y) = \sum_W n_W W, \tag{2.1}$$

where the sum is taken over the k-dimensional irreducible reduced components of Y, and the multiplicity n_W is equal to the length $l(\mathcal{O}_{Y,W})$ of the Artinian ring $\mathcal{O}_{Y,W}$, the localization of $\mathcal{O}_Y$ at the point W.

If $\phi : Y \to X$ is a proper morphism between quasi-projective schemes, we can define

$$\phi_* : \mathcal{Z}_k(Y) \to \mathcal{Z}_k(X)$$

by associating to a reduced irreducible subscheme $Z \subset Y$ the cycle $\deg[K(Z) : K(Z')]Z'$, $Z' = \phi(Z)$, if $\phi : Z \to Z'$ is generically finite, and 0 otherwise (in the latter case we have $\dim Z' < k$). The properness is already used here in order to guarantee that Z' is closed in X.

If W is a normal algebraic variety, the localized rings at the points of codimension 1 of W are discrete valuation rings, so we can define the divisor $\mathrm{div}(\phi)$ of a nonzero rational function $\phi \in K(W)^*$. This divisor $\mathrm{div}(\phi)$ is the $(w-1)$-cycle of W, $w := \dim W$ defined as $\sum_{\mathrm{codim}\, T=1} \nu_T(\phi)T$. If $W \subset X$ is a closed subvariety and $\tau : \widetilde{W} \to X$ is the normalization of W, we have the map $\tau_* : \mathcal{Z}_k(\widetilde{W}) \to \mathcal{Z}_k(X)$. This allows us to give the following definition.

DEFINITION 2.1. The subgroup $\mathcal{Z}_k(X)_{\mathrm{rat}}$ of cycles rationally equivalent to 0 is the subgroup of $\mathcal{Z}_k(X)$ generated by cycles of the form

$$\tau_* \operatorname{div}(\phi), \quad \dim W = k+1, \quad \phi \in K(W)^*,$$

where $\tau : \widetilde{W} \to W \subset X$ is the normalization of the closed $(k+1)$-dimensional subvariety W of X.

We write $\mathrm{CH}_k(X) = \mathcal{Z}_k(X)/\mathcal{Z}_k(X)_{\mathrm{rat}}$.

If X is n-dimensional and smooth, or more generally, locally factorial, then $\mathcal{Z}_{n-1}(X)$ is the group of Cartier divisors, and the group $\mathrm{CH}_{n-1}(X)$ can be identified with the group $\mathrm{Pic}(X)$ of algebraic line bundles on X modulo isomorphism. Indeed, this isomorphism makes the line bundle $\mathcal{O}_X(D) = \otimes_i \mathcal{I}_{D_i}^{\otimes - n_i}$ correspond to a divisor $D = \sum_i n_i D_i$, and D is rationally equivalent to D' if and only if there exists $\phi \in K(X)$ such that $\operatorname{div}(\phi) = D - D'$. Clearly, multiplication by ϕ then induces an isomorphism

$$\phi : \mathcal{O}_X(D) \cong \mathcal{O}_X(D').$$

Conversely, every algebraic line bundle L admits a meromorphic section, by trivialization on a Zariski dense open set, and is thus isomorphic to a sheaf of the form $\mathcal{O}_X(D)$.

In general, for X reduced and irreducible of dimension n, we have a map

$$s : \mathrm{Pic}\, X \to \mathrm{CH}_{n-1}(X)$$

that associates to a line bundle $\mathcal{L}$ the class of the cycle $\tau_* \operatorname{div}(\sigma)$, where $\tau : \widetilde{X} \to X$ is the normalization and σ is a nonzero meromorphic section of $\tau^*\mathcal{L}$.

2.1.2 Localization exact sequence

Let X be a quasi-projective scheme, and let $F \xrightarrow{l} X$ be the inclusion of a closed subscheme. Let $j : U = X - F \hookrightarrow X$ be the inclusion of the complement. The morphism l is proper since it is a closed immersion. The morphism j is an open immersion, and j^* will be defined by restricting cycles to the open set U. Thus, we have the inverse image morphism j^* and the direct image morphism l_*. Moreover, it is clear that $j^* \circ l_* = 0$, since the cycles supported on F do not intersect U.

LEMMA 2.2. *The following sequence, known as the localization sequence, is exact:*

$$\mathrm{CH}_k(F) \xrightarrow{l_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \to 0. \tag{2.2}$$

PROOF. The surjectivity on the right follows from the fact that if $Z \subset U$ is a k-dimensional subvariety, then its Zariski closure $\overline{Z} \subset X$ is a k-dimensional subvariety whose intersection with U is equal to Z.

Furthermore, let $Z \in \mathcal{Z}_k(X)$ be such that $j^*Z \in \mathcal{Z}_k(U)_{\mathrm{rat}}$. Then there exist $W_i \subset U$, $\dim W_i = k+1$, $\phi_i \in K(W_i)^*$, and integers n_i, such that

$$Z \cap U = \sum_i n_i (\tau_i)_*(\operatorname{div}(\phi_i)), \tag{2.3}$$

where $\tau_i : \widetilde{W_i} \to U$ is the normalization of W_i. Then, if $\overline{W_i}$ is the closure of W_i in X and $\overline{\tau}_i : \widetilde{\overline{W_i}} \to X$ is its normalization, we have $\phi_i \in K(\overline{W_i})^*$, and the equality (2.3) shows that

$$Z - \sum_i n_i (\overline{\tau}_i)_*(\operatorname{div}(\phi_i)) \in \mathcal{Z}_k(F).$$

This proves the exactness of the sequence (2.2) in the middle, since the cycle $Z - \sum_i n_i (\overline{\tau}_i)_*(\operatorname{div}(\phi_i))$ is rationally equivalent to Z in X. □

In [43], Fulton proposes an intersection theory,

$$\mathrm{CH}_k(X) \times \mathrm{CH}_l(X) \to \mathrm{CH}_{k+l-n}(X)$$

for a smooth n-dimensional variety X, which is particularly well adapted to computing the excess intersection formulas. If Z and Z' are two irreducible reduced subschemes of X, of dimensions k and l, respectively, which intersect properly and generically transversally, that is, such that $\dim Z \cap Z' = k + l - n$, and generically along the intersection $Z \cap Z'$, Z and Z' are smooth and the intersection is transverse, then one classically defines $Z \cdot Z'$ to be the cycle associated to the scheme $Z \cap Z'$ (which in fact has all its components of multiplicity 1 by assumption). By bilinearity, we can define the intersection $Z \cdot Z'$ this way for any pair of cycles whose supports intersect properly and generically transversally.

If Z and Z' do not intersect properly, the classical theory replaces Z by a cycle $\widetilde{Z}$ that is rationally equivalent to Z and intersects Z' properly (such a cycle exists by "Chow's moving lemma"), and defines $Z \cdot Z'$ to be the class of $\widetilde{Z} \cap Z'$ in $\mathrm{CH}_{k+l-n}(X)$.

Let $|Z| = \cup_i Z_i$ denote the support of the cycle $Z = \sum_i n_i Z_i$. Fulton's theory avoids the recourse to Chow's moving lemma, and gives a refined intersection, that is, a cycle

$$Z \cdot Z' \in \mathrm{CH}_{k+l-n}(|Z| \cap |Z'|)$$

for every pair of cycles Z and Z', whose image in $\mathrm{CH}_{k+l-n}(X)$ is $Z \cdot Z'$. Thus it provides an exact answer to the problems of excess, that is, the "explicit" computation of $Z \cdot Z'$ as a cycle supported on $\operatorname{Sup} Z \cap \operatorname{Sup} Z'$ when Z and Z' do not intersect properly.

2.1.3 Functoriality and motives

If $p : Y \to X$ is a proper morphism of quasi-projective schemes, we have the morphism
$$p_* : \mathcal{Z}_k(Y) \to \mathcal{Z}_k(X)$$
defined above.

LEMMA 2.3. *The group morphism p_* defined above sends $\mathcal{Z}_k(Y)_{\mathrm{rat}}$ to $\mathcal{Z}_k(X)_{\mathrm{rat}}$, and thus induces a morphism*
$$p_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X).$$

PROOF. Let $\tau : \widetilde{W} \to W \subset Y$ be the normalization of a closed $(k+1)$-dimensional subvariety of Y, and let $\phi \in K(W)^*$. Assume first that the composition $p \circ \tau : \widetilde{W} \to X$ is generically finite, and let $W' = \operatorname{Im} p \circ \tau \subset X$, $\tau' : \widetilde{W}' \to X$ be its normalization. We have a factorization,

$$\begin{array}{lccc} \tau : & \widetilde{W} & \longrightarrow & Y \\ & {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tau' : & \widetilde{W}' & \longrightarrow & X, \end{array}$$

so that $p_* \circ \tau_* = \tau'_* \circ \tilde{p}_* : \mathcal{Z}_k(\widetilde{W}) \to \mathcal{Z}_k(X)$.

The function field $K(\widetilde{W})$ is an algebraic extension of $K(\widetilde{W}')$. Consider the norm morphism
$$N : K(\widetilde{W})^* \to K(\widetilde{W}')^*$$
relative to this field extension.

LEMMA 2.4. *For every nonzero rational function on $\widetilde{W}$, we have the equality*
$$\tilde{p}_*(\mathrm{div}(\phi)) = \mathrm{div}(N(\phi)).$$

This formula and the definition of rational equivalence imply Lemma 2.3. □

We now assume that $p : Y \to X$ is a flat morphism of relative dimension $l = \dim Y - \dim X$. If $Z \subset X$ is a reduced irreducible k-dimensional subscheme, then $p^{-1}(Z)$ is a $(k+1)$-dimensional subscheme of Y, and thus it admits an associated cycle $p^*Z \in \mathcal{Z}_{k+l}(Y)$. Extending this definition by $\mathbb{Z}$-linearity, we thus obtain $p^* : \mathcal{Z}_k(X) \to \mathcal{Z}_{k+l}(Y)$.

LEMMA 2.5. *The map p^* defined above sends $\mathcal{Z}_k(X)_{\mathrm{rat}}$ to $\mathcal{Z}_{k+l}(Y)_{\mathrm{rat}}$. Thus, it induces a morphism*
$$p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+l}(Y).$$

REMARK 2.6. Let $\mathrm{CH}^k(X) := \mathrm{CH}_{n-k}(X)$, where X is irreducible of dimension n. Then flat pull-back sends $\mathrm{CH}^k(X)$ to $\mathrm{CH}^k(Y)$.

Lemma 2.5 shows that the pull-back morphism p^* is well defined when p is flat, but this assumption on p is much too restrictive. A crucial point now is the existence of a pull-back morphism

$$p^* : \mathrm{CH}^k(X) \to \mathrm{CH}^k(Y)$$

once X is smooth. For this one uses the fact that p factors as $\mathrm{pr}_2 \circ j_p$, where $\mathrm{pr}_2 : Y \times X \to X$ is the second projection (which is flat), and j_p is the inclusion $y \mapsto (y, p(y))$ of Y into $Y \times X$. By smoothness of X, the image of j_p is a local complete intersection in $X \times Y$, and restriction to local complete intersection subschemes is well defined inductively, starting from the divisor case (see [43, 6.6]). The pull-back map p^* is then defined as the composition of pr_2^* followed by the restriction map to $\mathrm{Im}\, j_p = Y$.

We have the following compatibility between the intersection product and the morphisms p_*, p^*, where X, Y are smooth and $p : Y \to X$ is a morphism.

Proposition 2.7.

(1) *(Projection formula, [43, 8.1].) If p is proper, then for $Z \in \mathrm{CH}(Y)$ and $Z' \in \mathrm{CH}(X)$, we have*

$$p_*(p^*Z \cdot Z') = Z \cdot p_*Z' \in \mathrm{CH}(X). \tag{2.4}$$

(2) *With no hypothesis of properness on p, for Z, $Z' \in \mathrm{CH}(X)$, we have*

$$p^*(Z \cdot Z') = p^*Z \cdot p^*Z' \text{ in } \mathrm{CH}(Y).$$

The following corollary is a consequence of the projection formula.

Corollary 2.8. *If $p : Y \to X$ is a proper morphism with $\dim X = \dim Y$, then*

$$p_* \circ p^* Z = \deg p\, Z$$

for $Z \in \mathrm{CH}(X)$, where $\deg p$ is defined to be equal to 0 if p is not dominant.

Proof. This follows from formula (2.4) with $Z' = c(Y) \in \mathrm{CH}_n(Y)$, $n = \dim Y$. Indeed, by the definition of p_*, we have $p_*(c(Y)) = \deg p\ c(X) \in \mathrm{CH}(X)$. □

Definition 2.9. A correspondence between two smooth varieties X and Y is a cycle $\Gamma \in \mathrm{CH}(X \times Y)$. A 0-correspondence between X and Y is an element of $\mathrm{CH}^n(X \times Y)$, $n = \dim X$.

If X is projective, the second projection $X \times Y \to Y$ is proper. A correspondence $\Gamma \in \mathrm{CH}^k(X \times Y)$ then defines a morphism

$$\Gamma_* : \mathrm{CH}^i(X) \to \mathrm{CH}^{i+k-\dim X}(Y)$$

given by

$$\Gamma_*(Z) = \mathrm{pr}_{2*}(\mathrm{pr}_1^*(Z) \cdot \Gamma), \quad Z \in \mathrm{CH}(X).$$

Note that if Γ is a 0-correspondence, $\Gamma_* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ preserves the degree.

If Y is projective, we can also consider the morphism

$$\Gamma^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$$

given by

$$\Gamma^*(Z) = \mathrm{pr}_{1*}(\mathrm{pr}_2^*(Z) \cdot \Gamma), \quad Z \in \mathrm{CH}(Y).$$

Now, assume that X, Y, and W are smooth varieties, with X and Y projective, and let $\Gamma \in \mathrm{CH}^k(X \times Y)$, $\Gamma' \in \mathrm{CH}^{k'}(Y \times W)$ be correspondences. We can then define the composition $\Gamma' \circ \Gamma \in \mathrm{CH}^{k''}(X \times W)$, where $k'' = k + k' - \dim Y$, by the formula

$$\Gamma' \circ \Gamma = p_{13*}(p_{12}^*\Gamma \cdot p_{23}^*\Gamma'), \tag{2.5}$$

where the p_{ij}, $i = 1, 2, 3$ are the projections of $X \times Y \times W$ onto the product of its ith and jth factors.

We can show that the composition of correspondences is associative. In particular, it equips the group $\mathrm{CH}(X\times X)$ with the structure of a (noncommutative) ring.

Finally, we have the following essential formula, which is a consequence of the projection formula (2.4) (see [101, II, 9.2.2]).

Proposition 2.10. *Let*

$$\Gamma_* : \mathrm{CH}(X) \to \mathrm{CH}(Y), \qquad \Gamma'_* : \mathrm{CH}(Y) \to \mathrm{CH}(W)$$

be the morphisms associated to the correspondences Γ, Γ'. *Then*

$$(\Gamma' \circ \Gamma)_* = \Gamma'_* \circ \Gamma_*.$$

2.1.3.1 Motives

Here we follow [75]; the reader can find a more precise and advanced presentation in [2]. The fact that correspondences can be composed suggests considering a category of smooth projective varieties, the morphisms being correspondences modulo a given equivalence relation. A better category is proposed by Grothendieck (see [2], [75]). First of all, consider the category of effective motives: Instead of considering the smooth projective varieties X as objects, one considers the pairs (X, p), where X is a smooth projective variety of dimension n, and $p \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ is a projector, namely

$$p \circ p = p \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}}.$$

The morphisms between (X, p) and (Y, q) are the 0-correspondences between X and Y of the form

$$q \circ \gamma \circ p, \quad \gamma \in \mathrm{CH}^n(X \times Y), \quad n = \dim X.$$

Next, general Chow motives are defined as follows: Objects are triple (X, p, m), where X is smooth projective, $p \in \mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is a projector, and m is an integer. Then the morphisms between (X, p, m) and (Y, q, m') are the elements of $\mathrm{CH}^{\dim X + m' - m}(X \times Y)_{\mathbb{Q}}$ of the form

$$q \circ \gamma \circ p, \quad \gamma \in \mathrm{CH}^{\dim X + m' - m}(X \times Y).$$

Of course, one can make the theory for other equivalence relations, for example homological equivalence, or modify it by introducing more morphisms as in [1], where André allows further morphisms by inverting certain Lefschetz operators. If the Hodge conjecture (or more precisely, the Lefschetz standard conjecture) holds, this is the same set of morphisms, but his theory works unconditionally and the resulting category has better semisimplicity properties, deduced from the corresponding semisimplicity properties of the category of polarized Hodge structures (see Theorem 2.22).

Example 2.11 (Lefschetz motive). Let 0 be any point of $\mathbb{P}^1$. One considers the pair $(\mathbb{P}^1, p)$, where p is the projector given by $\mathbb{P}^1 \times 0$. The independence of the choice of point is due to the fact that all points on $\mathbb{P}^1$ are rationally equivalent on any field, even nonalgebraically closed fields.

Note that we can consider sums and tensor products of motives. Sums are induced (at least for the effective motives $(X, p, 0)$) by disjoint unions, while tensor products are induced by the usual products. We can thus speak of the self-product of a motive (X, p): this is the motive $(X \times X, p \times p)$.

Example 2.12 (Powers of the Lefschetz motive). The motive $L^{\otimes n}$ is $((\mathbb{P}^1)^n, p_n)$ where p_n is the projector given by $(\mathbb{P}^1)^n \times (0, \ldots, 0)$, where 0 is any point of $\mathbb{P}^1$. As $(\mathbb{P}^1)^n$ is birationally equivalent to $\mathbb{P}^n$, $L^{\otimes n}$ is isomorphic to the motive $(\mathbb{P}^n, p_n)$, where p_n is the class of $\mathbb{P}^n \times \mathrm{pt}$ for any point of $\mathbb{P}^n$.

Let us give a few more examples.

Example 2.13 (Generically finite morphisms). Let X, Y be smooth varieties of dimension n and

$$\phi : X \to Y$$

be a generically finite morphism. Consider the correspondence $\Gamma_Y = \frac{(\phi,\phi)^*\Delta_Y}{\deg \phi} \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$, where Δ_Y is the diagonal of Y.

Lemma 2.14. *The correspondence Γ_Y is a projector, and (X, Γ_Y) is isomorphic to Y.*

Proof. We apply the definition of the composition of correspondences, the projection formula, and the fact that $\phi_* \Delta_X = \deg \phi \, \Delta_Y$. The isomorphism between Y and (X, Γ_Y) is given by the transpose of the graph of ϕ. □

We thus get another motive, $(X, \Delta_X - \Gamma_Y)$, which is the motive of X with the motive of Y removed.

In the above example, (X, Δ_X) is the full motive of X, with projector p induced by the diagonal of X, which acts as the identity on $\mathrm{CH}(X)$.

EXAMPLE 2.15 (Finite group actions). Let X be a smooth projective variety of dimension n and G a finite group acting on X. For any $g \in G$ we have the graph of g, which provides a correspondence $\Gamma_g \in \mathrm{CH}^n(X \times X)$. Now let $\pi = \sum_{g \in G} \alpha_g g \in \mathbb{Q}[G]$ be a projector, and consider the following correspondence:

$$\Gamma_\pi = \frac{1}{|G|} \sum_{g \in G} \alpha_g \Gamma_g \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}.$$

The cycle Γ_π is a projector due to the fact that $\Gamma_g \circ \Gamma_{g'} = \Gamma_{gg'}$.

2.1.4 Cycle class

Assume that X is a smooth complex quasi-projective variety over $\mathbb{C}$. Thus X can also be seen as a smooth complex manifold, usually denoted by X_{an} to emphasize the use of the holomorphic structural sheaf, or X_{cl} to emphasize the use of the classical topology. Given a subvariety Z in X, one defines the cycle class $[Z] \in H_B^{2n-2k}(X, \mathbb{Z})$ in the Betti cohomology groups of X (that is, the cohomology of $X(\mathbb{C})$ endowed with the classical topology). Let Z be a reduced irreducible subvariety of codimension k in X. By Hironaka's theorem, there is a desingularization

$$\tilde{i} : \widetilde{Z} \longrightarrow Z \subset X$$

of Z, and we may consider

$$H_{2n-2k,B}(\widetilde{Z}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\tilde{i}_*} H_{2n-2k,B}(X, \mathbb{Z}) \cong H_B^{2k}(X, \mathbb{Z}),$$

where the last isomorphism is given by Poincaré duality, and the first isomorphism comes from the fact that $\widetilde{Z}(\mathbb{C})$ is a connected compact complex manifold, hence canonically oriented. The image of 1 gives a class

$$[Z] \in H_B^{2k}(X, \mathbb{Z}).$$

This is the integral *Betti cycle class* of Z. In many places we will use the rational cycle class $[Z] \in H_B^{2k}(X, \mathbb{Q})$.

We extend the above cycle class by linearity to any cycle $Z = \sum_i n_i Z_i \in \mathcal{Z}^k(X)$.

LEMMA 2.16. *If Z is rationally equivalent to 0, then $[Z] = 0$ in $H_B^{2k}(X, \mathbb{Z})$. The map $Z \mapsto [Z]$ thus gives the "cycle class" map*

$$\mathrm{cl} : \mathrm{CH}_l(X) \to H_B^{2n-2l}(X, \mathbb{Z}).$$

The cycle class map is compatible with the intersection product

$$\cdot : \mathrm{CH}^l(X) \times \mathrm{CH}^k(X) \to \mathrm{CH}^{k+l}(X),$$

and the cup-product

$$\cup : H_B^{2l}(X,\mathbb{Z}) \times H_B^{2k}(X,\mathbb{Z}) \to H_B^{2k+2l}(X,\mathbb{Z}).$$

PROPOSITION 2.17. *For $Z \in \mathrm{CH}^l(X)$, $Z' \in \mathrm{CH}^k(X)$, we have*

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(Z) \cup \mathrm{cl}(Z') \in H_B^{2k+2l}(X,\mathbb{Z}).$$

The cycle class map satisfies the following functoriality properties.

PROPOSITION 2.18. *Let $i : Y \to X$ be a morphism between smooth varieties:*

(1) If $Z \in \mathrm{CH}^k(X)$, then

$$i^* \mathrm{cl}(Z) = \mathrm{cl}(i^* Z) \in H_B^{2k}(Y,\mathbb{Z}).$$

(2) If i is proper and $Z \in \mathrm{CH}^k(Y)$, then

$$\mathrm{cl}(i_* Z) = i_* \mathrm{cl}(Z) \in H_B^{2k-2\dim Y+2\dim X}(X,\mathbb{Z}).$$

It follows that the class map is compatible with correspondences. If X and Y are proper and smooth, and $\Gamma \in \mathrm{CH}^r(X \times Y)$, then for every $Z \in \mathrm{CH}^k(X)$, we have

$$\mathrm{cl}(\Gamma_*(Z)) = [\Gamma]_*(\mathrm{cl}(Z)),$$

where $[\Gamma]_* : H_B^{2k}(X,\mathbb{Z}) \to H_B^{2l}(Y,\mathbb{Z})$, $l = r + k - \dim X$ is defined by

$$[\Gamma]_*(\alpha) = \mathrm{pr}_{2*}(\mathrm{pr}_1^* \alpha \cup [\Gamma])$$

(see [101, II, 9.2.4]).

We will also need the cycle class for cycles on smooth quasi-projective complex algebraic varieties. Let $Z = \sum_i n_i Z_i$, $Z_i \subset X$ be such a cycle, with $\mathrm{codim}\, Z_i = k$. In order to define

$$[Z] \in H_B^{2k}(X,\mathbb{Z}),$$

we choose a smooth projective completion $\overline{X}$ of X. Then X is a Zariski open set of $\overline{X}$ and the localization exact sequence (2.2) shows that there exists a cycle $\overline{Z} \in \mathrm{CH}^k(\overline{X})$ such that

$$\overline{Z}_{|X} = Z$$

and that $\overline{Z}$ is well defined up to a cycle Z' of $\overline{X}$ supported on $\overline{X} \setminus X$. Looking at the definition of the cycle class for cycles on $\overline{X}$, we see that the class of such a cycle Z' vanishes in $H_B^{2k}(X,\mathbb{Z})$ since it vanishes in $H_B^{2k}(\overline{X} \setminus \mathrm{Supp}\, Z',\mathbb{Z})$ and $\mathrm{Supp}\, Z' \subset \overline{X} \setminus X$. We thus conclude that the class $[\overline{Z}]_{|X}$ does not depend on the choice of $\overline{Z}$. For similar reasons it also does not depend on the choice of compactification $\overline{X}$.

The cycle class $[Z] \in H_B^{2k}(X, \mathbb{Z})$ contains only a very small piece of information on the rational equivalence class of a cycle Z on X. This is already seen in the case of divisors, since the group $\mathrm{CH}^1(X)_{\mathrm{hom}}$ is the group $\mathrm{Pic}^0(X)$ of topologically trivial line bundles on X, which is an abelian variety. This is still more obvious with higher-codimensional cycles, since the group of 0-cycles can be infinite-dimensional as a consequence of Mumford's theorem (Theorem 1.1; see [71]), while the cohomology class of a 0-cycle only gives its degree.

On the other hand, there is the following challenging conjecture by Bloch and Beilinson which says that if $Z \in \mathrm{CH}^k(X \times Y)_{\mathbb{Q}}$ is a correspondence, the morphism $Z_* : \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(Y)_{\mathbb{Q}}$ is largely controlled by $[Z]$. We refer to [58] for an expanded exposition of the conjecture and to [89] for the construction of a candidate for such a filtration.

Conjecture 2.19. *For any smooth projective variety X, there exists a decreasing filtration F on $\mathrm{CH}^i(X)_{\mathbb{Q}}$, with the following properties:*

(1) $F^0 \mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{CH}^i(X)_{\mathbb{Q}}$ and $F^1 \mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$, the group of cycles cohomologous to 0.

(2) The filtration F is stable under correspondences: if $Z \in \mathrm{CH}^k(X \times Y)_{\mathbb{Q}}$, then $Z_*(F^i \mathrm{CH}^l(X)_{\mathbb{Q}}) \subset F^i \mathrm{CH}^{l+k-n}(Y)_{\mathbb{Q}}$, where $n = \dim X$.

(3) The induced map $Z_* : \mathrm{Gr}_F^i \mathrm{CH}^l(X)_{\mathbb{Q}} \to \mathrm{Gr}_F^i \mathrm{CH}^{l+k-n}(Y)_{\mathbb{Q}}$ vanishes if $[Z] = 0$ in $H^{2k}(X \times Y, \mathbb{Q})$.

(4) One has $F^{k+1} \mathrm{CH}^k(X)_{\mathbb{Q}} = 0$ for any X and k.

2.2 HODGE STRUCTURES

Definition 2.20. A weight k rational Hodge structure $(L, L^{p,q})$ consists of the data of a $\mathbb{Q}$-vector space L and a decomposition

$$L_{\mathbb{C}} := L \otimes \mathbb{C} = \oplus_{p+q=k} L^{p,q}$$

satisfying the Hodge symmetry condition

$$\overline{L^{p,q}} = L^{q,p}.$$

The Hodge filtration $F^* L_{\mathbb{C}}$ associated to such a Hodge structure is the decreasing filtration defined by

$$F^p L_{\mathbb{C}} = \oplus_{i \geq p} L^{i,k-i}.$$

If X is a smooth projective variety, the kth Betti cohomology group $H_B^k(X, \mathbb{Q})$ of X with rational coefficients carries a Hodge structure of weight k (see [101, I, 7.1]). The corresponding Hodge filtration on $H_B^k(X, \mathbb{C})$ is obtained using the isomorphism

$$H_B^k(X, \mathbb{C}) = \mathbb{H}^k(X_{\mathrm{cl}}, \Omega^\bullet_{X_{\mathrm{an}}}),$$

where $\Omega^i_{X_{\rm an}}$ is the sheaf of holomorphic i-forms on $X_{\rm cl}$, and is induced by the naive filtration on the complex $\Omega^\bullet_{X_{\rm an}}$,

$$F^p\Omega^\bullet_{X_{\rm an}} := \Omega^{\bullet \geq p}_{X_{\rm an}}.$$

Thus the Hodge filtration is easy to define. The hard part is to prove that, letting

$$H^{p,q}(X) = F^pH^k_B(X,\mathbb{C}) \cap \overline{F^qH^k_B(X,\mathbb{C})}, \quad k = p+q,$$

one has the decomposition

$$H^k_B(X,\mathbb{C}) = \oplus_{p+q=k}H^{p,q}(X).$$

2.2.1 Polarization

Definition 2.21. A polarization on a weight k Hodge structure $(L, L^{p,q})$ consists of a nondegenerate intersection pairing $(\, , \,)$ on L, which is skew symmetric if k is odd and symmetric if k is even, and satisfies the following conditions. Let H be the Hermitian intersection pairing on $L_{\mathbb{C}}$ defined by

$$H(a,b) = i^k(a,\bar{b}),$$

then we have the following:

(i) (First Hodge–Riemann bilinear relations). The Hodge decomposition of L is orthogonal with respect to H.

(ii) (Second Hodge–Riemann bilinear relations). The restriction $H_{|L^{p,q}}$ is definite, of sign $(-1)^p$.

The interest of polarized Hodge structures lies in the following semisimplicity result.

Theorem 2.22. *Let $(L, L^{p,q})$ be a rational polarized Hodge structure and $L' \subset L$ be a sub-Hodge structure. Then L decomposes as a direct sum of Hodge structures:*

$$L = L' \oplus L''.$$

Proof. The key point is to observe that the restriction of $(\, , \,)$ to L' is nondegenerate. This follows indeed from the fact that H remains nondegenerate on each $L'^{p,q}$ because it is definite on each $L^{p,q}$, and from the fact that the $L'^{p,q}$ are mutually perpendicular with respect to H. Having this, we observe that because of condition (i) above, the orthogonal complement $L'' := L'^{\perp}$ of L' in L is a sub-Hodge structure of L. This concludes the proof since we have just observed that L' and L'' are supplementary. ☐

One important application is the following Corollary 2.24 for which we need the notion of a *Hodge class* (that we will develop in Section 2.2.2).

DEFINITION 2.23. Let $(L, L^{p,q})$ be a Hodge structure of weight $2k$. A Hodge class of L is a class in $L \cap L^{k,k}$, the intersection being taken in $L_{\mathbb{C}}$.

COROLLARY 2.24. *Let $\phi : L \to M$ be a morphism of rational Hodge structure, where L is polarized. Then $\operatorname{Im}\phi$ is a sub-Hodge structure of M, and L contains a sub-Hodge structure L' such that $\phi_{|L'}$ is an isomorphism onto $\operatorname{Im}\phi$.*

In particular, if $\beta \in \operatorname{Im}\phi$ is a Hodge class, there exists a Hodge class $\beta' \in L$ such that $\beta = \phi(\beta')$.

PROOF. The first point is obvious and does not need the polarization. Let $L'' := \operatorname{Ker}\phi$. This is a sub-Hodge structure of L. By Theorem 2.22, there is a decomposition of L into a direct sum of Hodge structures: $L = L' \oplus L''$. It is then clear that $\phi_{|L'}$ is an isomorphism of Hodge structures onto its image. □

If X is a smooth projective variety, the Hodge structures on $H^k_B(X, \mathbb{Q})$ admit polarizations. This is a crucial difference between projective and Kähler geometry (see [102]). Such polarizations are obtained as follows (see [101, I, 7.1.2]): One chooses an ample line bundle $\mathcal{L}$ on X and considers the class $l := c_1(\mathcal{L}) \in H^2_B(X, \mathbb{Q})$. This class satisfies the hard Lefschetz property,

$$l^{n-k}\cup : H^k_B(X, \mathbb{Q}) \cong H^{2n-k}_B(X, \mathbb{Q}) \ \forall\, k \leq n := \dim X. \tag{2.6}$$

A formal consequence of this is the Lefschetz decomposition of $H^k_B(X, \mathbb{Q})$ into a direct sum of sub-Hodge structures,

$$H^k_B(X, \mathbb{Q}) = \oplus_{k-2r\geq 0} l^r \cup H^{k-2r}_B(X, \mathbb{Q})_{\mathrm{prim}},$$

where

$$H^{k-2r}_B(X, \mathbb{Q})_{\mathrm{prim}} := \operatorname{Ker}\left(H^{k-2r}_B(X, \mathbb{Q}) \overset{l^{n-k+2r+1}}{\to} H^{2n-k+2r+2}_B(X, \mathbb{Q})\right).$$

This decomposition is orthogonal with respect to the pairing

$$(a, b)_l := \int_X l^{n-k} \cup a \cup b$$

on $H^k_B(X, \mathbb{Q})$.

The final step is to prove that up to a sign, the pairing $(\ ,\ )_l$ polarizes each sub-Hodge structure $l^r \cup H^{k-2r}_B(X, \mathbb{Q})_{\mathrm{prim}}$, and this is exactly the content of the second Hodge-Riemann bilinear relations.

2.2.2 Hodge classes

Let $(L, L^{p,q})$ be a rational Hodge structure of weight $2k$. We introduced Hodge classes in L in Definition 2.23. Let X be a smooth projective complex variety. Cycle classes $[Z] \in H^{2k}_B(X, \mathbb{Q})$ are Hodge classes on X, that is, Hodge classes for the Hodge structure on $H^{2k}_B(X, \mathbb{Q})$ (see [101, I, 11.3]). We will denote by $\mathrm{Hdg}^{2k}(X)$ the $\mathbb{Q}$-vector space of degree $2k$ Hodge classes on X. The Hodge conjecture states the following.

CONJECTURE 2.25. *Any Hodge class* $\alpha \in \mathrm{Hdg}^{2k}(X)$ *is a linear combination with rational coefficients of Betti cycle classes of algebraic subvarieties of* X, *so*

$$\alpha = \textstyle\sum_{i=1}^{N} a_i\, [Z_i]_B, \quad a_i \in \mathbb{Q}.$$

A well-known fact is that the Hodge conjecture is true for $k = 1$. This is known as the Lefschetz theorem on $(1,1)$-classes (see [101, I, 11.3]). It is also true for degree $(2n-2)$ Hodge classes on smooth projective varieties of dimension n, by the degree 2 case combined with the hard Lefschetz isomorphism (5.24), which gives an isomorphism of Hodge structures

$$l^{n-2} : H_B^2(X, \mathbb{Q}) \cong H_B^{2n-2}(X, \mathbb{Q}).$$

Indeed, this isomorphism sends cycle classes to cycle classes, by compatibility of the cycle class map with the cup-product and the intersection product, and induces an isomorphism

$$\mathrm{Hdg}^2(X) \cong \mathrm{Hdg}^{2n-2}(X).$$

Hodge classes play a crucial role in the theory of motives because of the next lemma. Let X, Y be projective complex manifolds with $\dim X = n$. Suppose that $k + l = 2r$ is even. We apply the Künneth decomposition. Given

$$\alpha \in H_B^k(X,\, \mathbb{Q}) \otimes H_B^l(Y,\, \mathbb{Q}) \subset H_B^{2r}(X \times Y, \mathbb{Q}),$$

we can, by duality, see α as an element,

$$\tilde{\alpha} \in \mathrm{Hom}(H_B^{2n-k}(X,\, \mathbb{Q}), H_B^l(Y,\, \mathbb{Q})).$$

With this terminology, we have the following result (see [101, I, Lemma 11.41]).

LEMMA 2.26. α *is a Hodge class in* $X \times Y$ *if and only if* $\tilde{\alpha}$ *is a morphism of Hodge structures of bidegree* $(r-n, r-n)$.

2.2.3 Standard conjectures

Let X be a smooth projective variety of dimension n. The Künneth decomposition of $H_B^*(X \times X, \mathbb{Q})$ gives

$$H_B^m(X \times X, \mathbb{Q}) \cong \bigoplus_{p+q=m} H_B^p(X, \mathbb{Q}) \otimes H_B^q(X, \mathbb{Q}).$$

Poincaré duality on X allows this to be rewritten as

$$H_B^m(X \times X, \mathbb{Q}) \cong \bigoplus_{p+q=m} \mathrm{Hom}(H_B^{2n-p}(X, \mathbb{Q}), H_B^q(X, \mathbb{Q})). \tag{2.7}$$

There are two kinds of particularly interesting Hodge classes on $X \times X$ obtained from Lemma 2.26:

(a) *Künneth components of the diagonal.* Let $m = 2n$ and consider for each $0 \leq q \leq 2n$ the element $\mathrm{Id}_{H_B^q(X,\mathbb{Q})}$, which provides by (2.7) and Lemma 2.26 a Hodge class δ_q of degree $2n$ on X. This class is called the qth Künneth component of the diagonal of X. The first standard conjecture (the Künneth standard conjecture or Conjecture C in the terminology of [61]) is the following.

CONJECTURE 2.27. *The classes δ_i are algebraic, that is, are classes of algebraic cycles on $X \times X$ with rational coefficients.*

(b) *Lefschetz operators and their inverses.* Let $\mathcal{L}$ be an ample line bundle on X, and $l := c_1(\mathcal{L}) \in H_B^2(X, \mathbb{Q})$. For any integer $k \leq n$, the hard Lefschetz theorem [101, I, 6.2.3] says that the cup-product map

$$l^{n-k}\cup : H_B^k(X,\mathbb{Q}) \to H_B^{2n-k}(X,\mathbb{Q})$$

is an isomorphism. This is clearly an isomorphism of Hodge structure. Its inverse

$$(l^{n-k}\cup)^{-1} : H_B^{2n-k}(X,\mathbb{Q}) \to H_B^k(X,\mathbb{Q})$$

is also an isomorphism of Hodge structures, which by (2.7) and Lemma 2.26 provides a Hodge class λ_{n-k} of degree $2k$ on $X \times X$. The second standard conjecture we will consider (the Lefschetz conjecture or Conjecture B in the terminology of [61]) is the following.

CONJECTURE 2.28. *The classes λ_i are algebraic, that is, are classes of algebraic cycles on $X \times X$ with rational coefficients.*

The following conjecture (which could have been stated as a standard conjecture) is stated in [114].

CONJECTURE 2.29. *Let X be a smooth complex algebraic variety and let $Y \subset X$ be a closed algebraic subset. Let $Z \subset X$ be a codimension k algebraic cycle, and assume that the cohomology class $[Z] \in H_B^{2k}(X,\mathbb{Q})$ vanishes in $H_B^{2k}(X \setminus Y,\mathbb{Q})$. Then there exists a codimension k cycle Z' on X with $\mathbb{Q}$-coefficients, which is supported on Y and such that $[Z'] = [Z]$ in $H_B^{2k}(X,\mathbb{Q})$.*

REMARK 2.30. It is a nonobvious fact that, under our assumptions, there is a rational Hodge class β on a desingularization $\tau : \widetilde{Y} \to Y$ of Y, such that $(j \circ \tau)_*\beta = [Z]$, where j is the inclusion of Y in X (see the proof of Lemma 2.31 below). Thus Conjecture 2.29 is implied by the Hodge conjecture.

There is a particular numerical situation where this conjecture is proved.

LEMMA 2.31 (See [114]). *Conjecture 2.29 is satisfied by codimension k cycles Z of X whose cohomology class vanishes away from a codimension $(k-1)$ closed algebraic subset $Y \subset X$.*

In particular, Conjecture 2.29 is satisfied by codimension 2 cycles.

Proof. Indeed, if we have a codimension k cycle $Z \subset X$, whose cohomology class $[Z] \in H_B^{2k}(X, \mathbb{Q})$ vanishes on the open set $X \setminus Y$, where $\operatorname{codim} Y \geq k-1$, then we claim that there are Hodge classes $\alpha_i \in \mathrm{Hdg}^{2k-2c_i}(\widetilde{Y}_i, \mathbb{Q})$ such that

$$[Z] = \sum_i \tilde{j}_{i*}\alpha_i,$$

where $\tilde{j}_i : \widetilde{Y}_i \to X$ are projective desingularizations of the irreducible components Y_i of Y, and $c_i := \operatorname{codim} Y_i$. Indeed, here we use the fact that the "pure part" of the mixed Hodge structure on $H_{2n-2k,B}(Y, \mathbb{Q})$ (see Section 2.2.4 below) is equal to the image of $\oplus H_{2n-2k,B}(\widetilde{Y}_i, \mathbb{Q})$ by Theorem 2.36. The class $[Z]$ comes from a class in $H_{2n-2k,B}(Y, \mathbb{Q})$, hence by Theorem 2.36 from a class in the pure part of $H_{2n-2k,B}(Y, \mathbb{Q})$, thus from a class in $\oplus H_{2n-2k,B}(\widetilde{Y}_i, \mathbb{Q})$. We now use Corollary 2.24 to conclude that it comes from a Hodge class in $\oplus H_{2n-2k,B}(\widetilde{Y}_i, \mathbb{Q})$. The claim is proved. As $c_i \geq k-1$ for all i, the classes α_i are cycle classes on $\widetilde{Y}_i$ by the Lefschetz theorem on $(1,1)$-classes, which concludes the proof. □

The following is proved in [114].

Proposition 2.32 (Voisin 2011). *The Lefschetz conjecture for any X is equivalent to the conjunction of the Künneth standard conjecture (Conjecture 2.27) and of Conjecture 2.29 for any X.*

Proof. Let us assume that the Künneth standard conjecture holds for X and Conjecture 2.29 holds for any pair $Y \subset X'$. Let $i < n$. Consider the Künneth component δ_{2n-i} of Δ_X, so $\delta_{2n-i} \in H_B^i(X, \mathbb{Q}) \otimes H_B^{2n-i}(X, \mathbb{Q})$ is the class of an algebraic cycle Z on $X \times X$. Let $Y_i \stackrel{j_i}{\hookrightarrow} X$ be a smooth complete intersection of $(n-i)$ ample hypersurfaces in X. Then the Lefschetz theorem on hyperplane sections (see [101, II, 1.2.2]) says that

$$j_{i*} : H_B^i(Y_i, \mathbb{Q}) \to H_B^{2n-i}(X, \mathbb{Q})$$

is surjective. It follows that the class of the cycle Z vanishes on $X \times (X \setminus Y_i)$. By Conjecture 2.29, there is an n-cycle Z' supported on $X \times Y_i$ such that the class $(\mathrm{id}, j)_*[Z']$ is equal to $[Z]$. Consider the morphism of Hodge structures induced by $[Z']$:

$$[Z']_* : H_B^{2n-i}(X, \mathbb{Q}) \to H_B^i(Y_i, \mathbb{Q}).$$

Composing with the morphism $j_{i*} : H_B^i(Y_i, \mathbb{Q}) \to H_B^{2n-i}(X, \mathbb{Q})$, we get $j_{i*} \circ [Z']_* = \mathrm{Id}_{H_B^{2n-i}(X, \mathbb{Q})}$. It follows that $[Z']_*$ is injective, and that its transpose $[Z']^* : H_B^i(Y_i, \mathbb{Q}) \to H_B^i(X, \mathbb{Q})$ is surjective. We now apply [19, Proposition 8] and induction on dimension to conclude that the Lefschetz standard conjecture holds for X.

Conversely, assume the Lefschetz standard conjecture holds for any smooth projective variety. It obviously implies the Künneth standard conjecture. It is a well-known fact (see [61], [108, Theorem 4]) that the Lefschetz standard conjecture for a smooth complex projective variety M implies that homological and numerical equivalence coincide for cycles on M. Let us show that the conclusion of Conjecture 2.29 for X, Y, Z satisfying the given assumptions is already implied by the fact that homological and numerical equivalence coincide for cycles on X and a desingularization $\widetilde{Y}$ of Y. Set $n = \dim X$, $k = \operatorname{codim} Z$, $l = \operatorname{codim} Y$. Let $\tilde{j} : \widetilde{Y} \to X$ be a desingularization of Y. By the same arguments used in the proof of Lemma 2.31, we have that there exists a class $\beta \in H_B^{2k-2l}(\widetilde{Y}, \mathbb{Q})$ such that $\tilde{j}_*\beta = [Z]$. This class β gives by intersection a linear form η on the space $H_B^{2n-2k}(\widetilde{Y}, \mathbb{Q})_{\mathrm{alg}}$. If homological and numerical equivalence coincide for cycles on $\widetilde{Y}$, there is a codimension $(k-l)$ cycle Z' with $\mathbb{Q}$-coefficients on Y, such that η is given by intersecting with the class of Z'. The class $j_*[Z'] \in H_B^{2k}(X, \mathbb{Q})$ has the property that for any codimension $(n-k)$ cycle class γ on X,

$$\langle j_*[Z'], \gamma \rangle_X = \langle [Z'], j^*\gamma \rangle_{\widetilde{Y}} = \langle \beta, j^*\gamma \rangle_{\widetilde{Y}} = \langle j_*\beta, \gamma \rangle_X = \langle [Z], \gamma \rangle_X.$$

Thus, if homological and numerical equivalence coincide for cycles on X, we conclude that $[Z] = j_*[Z']$. □

Remark 2.33. All the (more-or-less) standard conjectures stated above are particular instances of the Hodge conjecture (Conjecture 2.25). The reason why they are stated separately is that the Hodge classes appearing there have a universal character that makes them much better candidates to be classes of algebraic cycles: they are *absolute Hodge classes* (see [33], [104]). This means that they have certain special properties satisfied by cycle classes, and the most striking one from the viewpoint adopted there is the following: Grothendieck's theorem [49] says that the cohomology with complex coefficients $H_B^{2k}(X, \mathbb{C})$ of a complex algebraic variety can be computed as algebraic de Rham cohomology, that is, via algebraic differential forms. It follows that if X is defined over a field $K \subset \mathbb{C}$ then this cohomology group is also defined over K. This K-structure has nothing to do with the Betti $\mathbb{Q}$-structure of $H_B^{2k}(X, \mathbb{C})$. One property of absolute Hodge classes of degree $2k$ is that, after multiplication by $(2i\pi)^k$, they become defined over a finite extension of K.

2.2.4 Mixed Hodge structures

Definition 2.34. A rational (real) mixed Hodge structure of weight n is given by a $\mathbb{Q}$-vector space ($\mathbb{R}$-vector space) H equipped with an increasing filtration W_iH called the weight filtration, and a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$, called the Hodge filtration $F^kH_{\mathbb{C}}$. The induced Hodge filtration on each $\mathrm{Gr}_i^W H$ is required to equip $\mathrm{Gr}_i^W H$ with a Hodge structure of weight $(n+i)$.

Naturally, these filtrations are also required to satisfy $F^iH = 0$ for sufficiently large i, $F^iH = H$ for sufficiently small i, and similarly $W_iH = 0$ for sufficiently small i, $W_iH = H$ for sufficiently large i.

Equivalently, for each i, k we must have

$$\mathrm{Gr}_i^W H_{\mathbb{C}} = F^k \,\mathrm{Gr}_i^W H_{\mathbb{C}} \oplus \overline{F^{n+i-k+1}\,\mathrm{Gr}_i^W H_{\mathbb{C}}},$$

where

$$F^k \,\mathrm{Gr}_i^W H_{\mathbb{C}} = \mathrm{Im}(F^k H_{\mathbb{C}} \cap W_i H_{\mathbb{C}} \to \mathrm{Gr}_i^W H_{\mathbb{C}}).$$

REMARK 2.35. Our conventions do not follow Deligne's conventions, according to which the Gr_i^W-part should have weight i. Our feeling is that the mixed Hodge structures coming from geometry (for example, $H^k(X, \mathbb{Q})$, where X is an algebraic variety) have a natural weight k indicated by geometry, which is the one we would like to attribute to the pure part. Thus there is in these notes a shift of the indices for the weight filtration with respect to standard terminology.

Note that with this notation, one can define the twist $L' = L(r)$ of a mixed Hodge structure L. It is the mixed Hodge structure of weight $(n-2r)$ with the same underlying rational vector space as L obtained by deciding that $W_i L' = W_{2r+i} L$ and $F^p L'_{\mathbb{C}} = F^{r+p} L_{\mathbb{C}}$.

We have the obvious notion of a morphism of mixed Hodge structures. A morphism α of filtered vector spaces (U, F) and (V, G) is said to be strict if

$$\mathrm{Im}\,\alpha \cap G^p V = \alpha(F^p U).$$

It is an elementary fact (see [101, II, 7.3.1]) that the morphisms of rational Hodge structures are strict for the Hodge filtration. This result extends to mixed Hodge structures; see [31].

THEOREM 2.36 (Deligne 1971). *The morphisms*

$$\alpha : (H, W, F) \to (H', W', F')$$

of (rational or real) mixed Hodge structures are strict for the filtrations W and F.

This result follows from the following fact, for which we refer to [31] or [101, II, 4.3.2].

LEMMA 2.37. *Let (H, W, F) be a mixed Hodge structure. There exists a decomposition as a direct sum,*

$$H_{\mathbb{C}} = \oplus_{p,q} H^{p,q}, \tag{2.8}$$

with $H^{p,q} \subset F^p H_{\mathbb{C}} \cap W_{p+q-n} H_{\mathbb{C}}$, such that under the projection $W_{p+q-n} H_{\mathbb{C}} \to \mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}$, $H^{p,q}$ can be identified with

$$H^{p,q}(\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}) := F^p \,\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}} \cap \overline{F^q \,\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}}.$$

More generally, we have

$$W_i H_{\mathbb{C}} = \oplus_{p+q \le n+i} H^{p,q}, \tag{2.9}$$

$$F^i H_{\mathbb{C}} = \oplus_{p \ge i} H^{p,q}. \tag{2.10}$$

This decomposition is respected by the morphisms of mixed Hodge structures.

Proof of Theorem 2.36. Indeed, if $l' \in \alpha(H_{\mathbb{C}}) \cap W'_i H'$, let us write $l' = \alpha(l)$ and decompose $l = \sum l^{p,q}$ as in (2.8). Then $\alpha(l)$ admits the decomposition $\alpha(l) = \sum \alpha(l^{p,q})$ with $\alpha(l^{p,q}) \in H'^{p,q}$. But as $l' \in W'_i H'_{\mathbb{C}}$, we have $\alpha(l^{p,q}) = 0$ for $p+q > n+i$. Thus,

$$l' = \alpha\left(\sum_{p+q\leq n+i} l^{p,q}\right) \in \alpha(W_i H_{\mathbb{C}}).$$

Therefore,

$$\operatorname{Im} \alpha_{\mathbb{C}} \cap W'_i H'_{\mathbb{C}} = \alpha_{\mathbb{C}}(W_i H_{\mathbb{C}}).$$

It is then easy to see that this still holds when $\mathbb{C}$ is replaced by $\mathbb{R}$ or $\mathbb{Q}$.

The same argument shows that α is also strict for the filtration F. ☐

2.2.5 Coniveau

Definition 2.38. A weight k Hodge structure $(L, L^{p,q})$ has (Hodge) coniveau $c \leq \frac{k}{2}$ if the Hodge decomposition of $L_{\mathbb{C}}$ takes the form

$$L_{\mathbb{C}} = L^{k-c,c} \oplus L^{k-c-1,c+1} \oplus \cdots \oplus L^{c,k-c}$$

with $L^{k-c,c} \neq 0$.

If L has coniveau $\geq r$, we can define a Hodge structure $L(r)$ of weight $(k-2r)$ with the same underlying $\mathbb{Q}$-vector space as L, and Hodge decomposition

$$L(r)^{p,q} = L^{p+r,q+r}. \tag{2.11}$$

A fundamental result is the following (see [50]).

Theorem 2.39. *If X is a smooth complex projective variety and $Y \subset X$ is a closed algebraic subset of codimension c, then* $\operatorname{Ker}(j^* : H^k_B(X,\mathbb{Q}) \to H^k_B(X \setminus Y, \mathbb{Q}))$*, where $j : X \setminus Y \hookrightarrow X$ is the inclusion map, is a sub-Hodge structure of coniveau $\geq c$ of $H^k_B(X,\mathbb{Q})$.*

Proof. Choose a desingularization $\tau : \widetilde{Y} \to Y$, and assume that $\widetilde{Y}$ has pure complex dimension $(n-c)$.

We need the fact that morphisms of mixed Hodge structures are strict for the weight filtration (see Theorem 2.36). We already know (via Poincaré duality for open varieties) that the kernel of j^* is the same as the image of

$$i_* : H_{2n-k,B}(Y,\mathbb{Q}) \to H_{2n-k,B}(X,\mathbb{Q}) \overset{D_X}{\cong} H^k_B(X,\mathbb{Q}),$$

where D_X is the Poincaré duality isomorphism. There is a mixed Hodge structure on both sides, of respective weights $k-2c$, k. The composition is a morphism of mixed Hodge structures of bidegree (c,c), with a *pure* Hodge structure on the right (see [31]). Thus by Theorem 2.36, its image is the same as the image of the map

$$i_* : W_0 H_{2n-k,B}(Y, \mathbb{Q}) \to H_{2n-k,B}(X, \mathbb{Q}) \cong H^k_B(X, \mathbb{Q}).$$

But by construction, the pure part $W_0 H_{2n-k,B}(Y, \mathbb{Q})$ of the mixed Hodge structure on $H_{2n-k,B}(Y, \mathbb{Q})$ coincides with the image of the map

$$\tau_* : H_{2n-k,B}(\widetilde{Y}, \mathbb{Q}) \to H_{2n-k,B}(Y, \mathbb{Q}),$$

where on the left we have a pure Hodge structure of weight $(k - 2c)$. That concludes the proof, because, applying Poincaré duality on $\widetilde{Y}$, we proved that $\operatorname{Ker} j^* = \operatorname{Im}(i \circ \tau)_* : H^{k-2c}_B(\widetilde{Y}, \mathbb{Q}) \to H^k_B(X, \mathbb{Q})$ and this morphism is a morphism of Hodge structures of bidegree (c, c). □

The conjecture below, due to Grothendieck [50], proposes a characterization of cohomology classes supported on a subvariety of codimension $\geq c$ (correcting the original Hodge conjecture).

Conjecture 2.40 (Generalized Hodge conjecture, Grothendieck 1969). *Let $L \subset H^k_B(X, \mathbb{Q})$ be a rational sub-Hodge structure of Hodge coniveau $\geq c$. Then there exists a closed algebraic subset $Z \subset X$ of codimension c such that L vanishes under the restriction map $H^k_B(X, \mathbb{Q}) \to H^k_B(U, \mathbb{Q})$, where $U := X \setminus Z$.*

Let us explain the link between the "standard" Hodge conjecture (Conjecture 2.25) and the generalized Hodge conjecture.

The Hodge conjecture implies the generalized Hodge conjecture in two particular cases. The most obvious case occurs when $k = 2c$. In this case we have $L_{\mathbb{C}} = L^{c,c}$ and L consists of Hodge classes. The Hodge conjecture provides codimension c cycles $Z_1, \ldots, Z_N$ of X such that

$$L = \big\langle [Z_1], \ldots, [Z_N] \big\rangle \otimes \mathbb{Q}.$$

But then L vanishes on $X \setminus (\operatorname{Sup} Z_1 \cup \cdots \cup \operatorname{Sup} Z_N)$, as required.

The next, much more sophisticated, case is that in which $k = 2c + 1$, so that

$$L_{\mathbb{C}} = L^{c+1,c} \oplus L^{c,c+1}. \tag{2.12}$$

Here we follow [83, 7.1.2]. In this case, referring to (2.11), $L' = L(c)$ is a polarized Hodge structure of weight 1. We get such Hodge structures on the first cohomology groups of curves, though not every Hodge structure of weight 1 is actually a Hodge structure of a curve. However, we have the following result.

Lemma 2.41. *Any polarized Hodge structure of weight 1 arises as $H^1_B(A, \mathbb{Q})$ for some abelian variety A.*

The key point is the existence of polarization (arising from the intersection form), but without polarizations, the lemma remains true with "abelian varieties" replaced by "complex tori." The lemma is a reformulation of the fact that the category of weight 1 rational polarized Hodge structures is the same as the category of abelian varieties up to isogeny (see [101, I, Section 7.2.2]).

Using this lemma, we shall prove the following theorem.

THEOREM 2.42 (See Peters and Steenbrink 2008, Observation 7.7). *The Hodge conjecture for degree $(2k+2)$ on products $C \times X$, where C is any smooth projective curve, implies Conjecture 2.40 for sub-Hodge structures of $H_B^{2k+1}(X, \mathbb{Q})$ of coniveau c, with $k = 2c+1$.*

PROOF. We start with a Hodge substructure $L \subset H_B^{2c+1}(X, \mathbb{Q})$ of coniveau c. It is polarized, because the Hodge structure on $H_B^{2c+1}(X, \mathbb{Q})$ is polarized (note however that the polarizations are not canonical). There exists then a polarized Hodge structure L' of weight 1 and an isomorphism of Hodge structures $\phi : L' \cong L$ of bidegree (c, c). By Lemma 2.41 we may assume $L' = H_B^1(A, \mathbb{Q})$ as Hodge structures. Having an abelian variety A and a morphism of Hodge structures $H_B^1(A, \mathbb{Q}) \to H_B^{2c+1}(X, \mathbb{Q})$, we can choose a curve C that is a complete intersection of ample hypersurfaces in A. Then, by the Lefschetz theorem on hyperplane sections [101, II, 1.2.2], there is a monomorphism

$$i^*: \ H_B^1(A, \mathbb{Q}) \to H_B^1(C, \mathbb{Q}), \tag{2.13}$$

which is a morphism of Hodge structures of pure weight 1. Since the category of polarized Hodge structures of weight 1 is semisimple, (2.13) always splits, so $H_B^1(A, \mathbb{Q})$ is a direct summand, as a Hodge substructure, of $H_B^1(C, \mathbb{Q})$. We thus get a morphism of Hodge structures $\psi : H_B^1(C, \mathbb{Q}) \to H_B^{2c+1}(X, \mathbb{Q})$ of bidegree (c, c). By Lemma 2.26, ψ determines a Hodge class $\tilde{\psi}$ of degree $(2c+2)$ on $C \times X$. If the usual Hodge conjecture is true on $C \times X$, then this class is the class of a codimension $(c+1)$ cycle $Z = \sum_i a_i Z_i$ of $C \times X$:

$$\tilde{\psi} = [Z] = \sum a_i [Z_i], \quad a_i \in \mathbb{Q}.$$

The resulting correspondence induces maps

$$\begin{array}{ccc} H_B^1(\mathrm{Sup}\, Z, \mathbb{Q}) & \stackrel{p_1^*}{\longleftarrow} & H_B^1(C, \mathbb{Q}) \\ & {\scriptstyle \sum_i a_i p_{2,i*}} \searrow & \Big\downarrow {\scriptstyle [Z]_*} \\ & & H_B^{2c+1}(X, \mathbb{Q}). \end{array}$$

(In fact, one should desingularize the components Z_i of Z and replace the support $\mathrm{Sup}\, Z$ of Z by $\bigsqcup \widetilde{Z}_i$.) Hence $\mathrm{Im}([Z]_*) = \mathrm{Im}\, \psi = L$ vanishes away from $p_2(\mathrm{Sup}\, Z)$, which is of codimension c. Thus, L satisfies the generalized Hodge conjecture. □

The above argument shows that the usual Hodge conjecture for varieties that are products with a curve implies the generalized Hodge conjecture for weight $n = 2c+1$ and coniveau c. In order to deduce, by a similar argument, the generalized Hodge conjecture from the standard Hodge conjecture, there is an important missing point, namely, the adequate generalization of Lemma 2.41. This leads to the following question.

QUESTION 2.43. *Given a weight k Hodge structure $L \subset H_B^k(X, \mathbb{Q})$ of coniveau r, so that*

$$L_{\mathbb{C}} = L^{k-r,r} \oplus \cdots \oplus L^{r,k-r},$$

consider the weight $(k-2r)$ Hodge structure $L' = L(r)$ which has the same underlying lattice as L, and Hodge decomposition

$$L'^{p,q} = L^{p+r,q+r}.$$

Does there exist a smooth projective variety Y admitting L' as a Hodge substructure of $H_B^{k-2r}(Y, \mathbb{Q})$?

Chapter Three

Decomposition of the diagonal

In this chapter, we explain the method initiated by Bloch and Srinivas, and later developed independently by Lewis, Schoen, Laterveer, and Paranjape, which leads to statements of the following type (see Theorem 3.20): if a smooth projective variety has trivial Chow groups of k-cycles homologous to 0 for $k \leq c-1$, then its transcendental cohomology has geometric coniveau $\geq c$.

This result is a vast generalization of Mumford's theorem (Theorem 1.1). A major open problem is the converse of this result.

It turns out that statements of this kind are a consequence of a general spreading principle for rational equivalence (see Theorem 3.1). Consider a smooth projective family $\mathcal{X} \to B$ and a cycle $\mathcal{Z} \to B$, everything defined over $\mathbb{C}$; then, if at the very general point $b \in B$, the restricted cycle $\mathcal{Z}_b \subset \mathcal{X}_b$ is rationally equivalent to 0, there exist a dense Zariski open set $U \subset B$ and an integer N such that $N\mathcal{Z}_U$ is rationally equivalent to 0 on $\mathcal{X}_U$.

We will spell out the consequences of this principle, such as the generalized decomposition of the diagonal, and explain other spreading principles that will be used in subsequent chapters.

3.1 A GENERAL PRINCIPLE

The following result is essentially due to Bloch and Srinivas [15]. Let $f : X \to Y$ be a smooth projective morphism, where X and Y are smooth (for simplicity), and let $Z \in \mathrm{CH}^k(X)$ be a cycle in X. Consider the following property.

($*$) *There exists a subvariety $X' \subset X$ such that for every $y \in Y$, the cycle $Z_y := Z_{|X_y} := j_y^* Z \in \mathrm{CH}^k(X_y)$ vanishes in $\mathrm{CH}^k(X_y - X'_y)$, where j_y is the inclusion of the fiber $X_y = f^{-1}(y)$ into X.*

Theorem 3.1. *If Z satisfies the property ($*$), then there exist an integer $m > 0$ and a cycle Z' supported in X', such that we have the equality*

$$mZ = Z' + Z'' \text{ in } \mathrm{CH}^k(X), \tag{3.1}$$

where Z'' is a cycle supported in $X_{Y'} := f^{-1}(Y')$ for some proper closed algebraic subset $Y' \subset Y$.

We refer to [15] or to [101, II, 10.2.1] for the proof of this theorem. Note that the statement given here is slightly different from the one given in [15], the reason being that we work over $\mathbb{C}$, which is very big, and contains, in particular, any finitely generated field over $\mathbb{Q}$ or the algebraic closure of such a field. Theorem 3.1 would be (expectedly) wrong if we were working over a countable field like $\overline{\mathbb{Q}}$ and considering only the closed points $y \in Y(\overline{\mathbb{Q}})$. Indeed, a conjecture of Beilinson says that looking at cycles defined over $\overline{\mathbb{Q}}$ on a variety defined over $\overline{\mathbb{Q}}$ does not allow us to conclude anything about the Chow groups of the corresponding variety over $\mathbb{C}$, except for the small part of them which is encoded in the (Deligne) cycle class. What makes this expectation possible is the following basic fact.

Lemma 3.2. *Let $f : X \to Y$ be a projective fibration, where X and Y are smooth, and let $Z \in \mathrm{CH}^k(X)$ be a cycle in X. Then the set of points $y \in Y$ such that the restricted cycle Z_y is rationally equivalent to 0 is a countable union of closed algebraic subsets of Y.*

Assuming everything is defined over $\overline{\mathbb{Q}}$, this countable union could exhaust $Y(\overline{\mathbb{Q}})$ without saying anything about what happens over the very general point or the geometric generic point.

Remark 3.3. Note also that Lemma 3.2 shows that in Theorem 3.1 we could have replaced our assumption that $Z_y := Z_{|X_y} := j_y^* Z \in \mathrm{CH}^k(X_y)$ vanishes in $\mathrm{CH}^k(X_y - X'_y)$ for all $y \in Y$ by the condition that $Z_y := Z_{|X_y} := j_y^* Z \in \mathrm{CH}^k(X_y)$ vanishes in $\mathrm{CH}^k(X_y - X'_y)$ for a very general point y of Y, that is, for any point of Y in the complement of a countable union of proper closed algebraic subsets of Y. Indeed, by a Baire category argument and Lemma 3.2, the two assumptions are equivalent.

We will apply this principle in Section 5.3, in the following rather simple form: First of all we assume the variety X' above is empty. Second, we only consider the cohomological version of (3.1).

Corollary 3.4. *Let $\pi : X \to Y$ be a smooth morphism, where X and Y are smooth, and let Z be a codimension k algebraic cycle on X. Assume that the restrictions $Z_{|X_y}$ vanish in $\mathrm{CH}^k(X_y)$ for any $y \in Y$. Then there is a nonzero integer m and a proper closed algebraic subset $Y' \subset Y$ such that*

$$m[Z] = 0 \text{ in } H_B^{2k}(X \setminus X_{Y'}, \mathbb{Z}).$$

Equivalently, $[Z] = 0$ in $H_B^{2k}(X \setminus X_{Y'}, \mathbb{Q})$.

It is important to point out that this corollary holds only for rational equivalence, and not for weaker equivalence relations like algebraic equivalence which is defined as follows.

Definition 3.5. A cycle $Z \in \mathcal{Z}^k(X)$ is said to be algebraically equivalent to 0 if there exists a smooth curve C, a 0-cycle $z \in \mathcal{Z}_0(C)$ homologous to 0, and a correspondence $\Gamma \in \mathcal{Z}^k(C \times X)$ such that $Z = \Gamma_*(z)$.

Example 3.6. A 0-cycle on a smooth projective variety X is algebraically equivalent to 0 if and only if it is homologous to 0 (or equivalently of degree

0 if X is connected). Indeed, such a 0-cycle is then homologous to 0 on any curve $C \subset X$ supporting it, assuming $C \cap X_i$ is connected for each connected component X_i of X.

EXAMPLE 3.7. A divisor $D \in \mathrm{CH}^1(X)$ is algebraically equivalent to 0 if and only if it is homologous to 0. This follows from the identification between $\mathrm{CH}^1(X)$ and $\mathrm{Pic}(X)$. The exponential exact sequence and the identification $\mathrm{Pic}(X) = \mathrm{Pic}(X_{an}) = H^1(X_{an}, \mathcal{O}^*_{X_{an}})$ imply that

$$\mathrm{Pic}^0(X) := \mathrm{Ker}(c_1 : H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \to H^2_B(X, \mathbb{Z})) = \mathrm{CH}^1(X)_{\mathrm{hom}}$$

is parametrized by the abelian variety which as a complex torus is the quotient $H^1(X_{an}, \mathcal{O}_{X_{an}})/H^1_B(X, \mathbb{Z})$. As this abelian variety is connected, divisors parametrized by $\mathrm{Pic}^0(X)$ are algebraically equivalent to 0 (see [28] or [101, I, 12.1, 12.2] for more details).

If one takes a smooth projective family of curves $\mathcal{C} \to Y$, and a family $\mathcal{Z} \subset \mathcal{C}$ of 0-cycles homologous to 0 in the fibers, the cycles $\mathcal{Z}_t$, $t \in Y$ are thus algebraically equivalent to 0 in the closed fibers (and also in the geometric generic fiber) by Example 3.6, but the class $[\mathcal{Z}] \in H^2_B(\mathcal{C}, \mathbb{Q})$ is nonzero in general, and in fact does not vanish over any Zariski open set Y^0 of Y; it is computed by the theory of the class of a normal function [101, II, 8.2.2].

One way to understand why the spreading principle holds for rational equivalence and not for algebraic equivalence is the following: In Corollary 3.4, if we replace rational equivalence by algebraic equivalence, our assumption is (by a countability argument) equivalent to the fact that the restriction of the cycle $\mathcal{Z}$ to the geometric generic fiber (which is a variety $\mathcal{X}_{\overline{\eta}}$ defined over $\overline{\mathbb{C}(Y)}$) is algebraically equivalent to 0. But this does not imply that there exists a Zariski open set Y^0 of Y such that, denoting $X^0 := \pi^{-1}(Y^0)$, the restriction $\mathcal{Z}_{|\mathcal{X}^0}$ is up to torsion, algebraically equivalent to 0 *over* $\mathbb{C}$, for the following reason: there is a curve $C_{\overline{\eta}}$ defined over $\overline{\mathbb{C}(Y)}$ and a 0-cycle z homologous to 0 on $C_{\overline{\eta}}$, such that $\mathcal{Z}_{\overline{\eta}}$ is, as above, the image of z under a codimension k correspondence Γ between $C_{\overline{\eta}}$ and $\mathcal{X}_{\overline{\eta}}$. This correspondence can be spread out over a finite cover of a Zariski open set of Y, and this can be done as well for the curve $C_{\overline{\eta}}$ and the cycle z, giving rise to families over a finite cover of a Zariski dense open set $\widetilde{Y}^0$ of the base. The problem is that the spread-out curve $\mathcal{C}$ is not in general isotrivial on $\widetilde{Y}^0$, that is, isomorphic to $C_0 \times \widetilde{Y}^0$, maybe up to passing to a finite étale cover of $\widetilde{Y}^0$. Even if it was isotrivial, the codimension 1 cycle z would spread to a codimension 1 cycle $\mathcal{Z}'$ on $C_0 \times \widetilde{Y}^0$, which might not be cohomologous to 0 over any Zariski open set of $\widetilde{Y}^0$, if $g(C_0) > 0$. Indeed, we only know that it is cohomologous to 0 on the fibers of $\mathrm{pr}_2 : C_0 \times \widetilde{Y}^0 \to \widetilde{Y}^0$, but when $g(C_0) > 0$, the Künneth decomposition of $H^2_B(C_0 \times \widetilde{Y}^0, \mathbb{Q})$ involves a nontrivial term $H^1_B(C_0, \mathbb{Q}) \otimes H^1_B(\widetilde{Y}^0, \mathbb{Q})$, which in general does not vanish on any Zariski open set of $\widetilde{Y}^0$.

On the contrary, when the curve C_0 is $\mathbb{P}^1$, we have $H^1_B(C_0, \mathbb{Q}) = 0$, hence

$$H^2_B(C_0 \times \widetilde{Y}^0, \mathbb{Q}) = H^2_B(C_0, \mathbb{Q}) \oplus H^2_B(\widetilde{Y}^0, \mathbb{Q}),$$

and thus a codimension 1 cycle $\mathcal{Z}'$ in $C_0 \times \widetilde{Y}^0$, which is of degree 0 on the fibers $C_0 \times t$, is homologous to (hence algebraically equivalent to) the pull-back of a codimension 1 cycle on $\widetilde{Y}^0$, hence its cohomology class (and in fact also its rational or algebraic equivalence class) vanishes over a dense Zariski open set of $\widetilde{Y}^0$.

In order to state different applications of Theorem 3.1, we consider now the case where $X = X_1 \times Y$, f is the second projection, and $X' = X_1' \times Y$. We then obtain the following result.

COROLLARY 3.8. *Let $\Gamma \in \mathrm{CH}^k(X_1 \times Y)$, and assume that for every $y \in Y$, the cycle $\Gamma^*(y) \in \mathrm{CH}^k(X_1)$ restricts to 0 in $\mathrm{CH}^k(X_1 - X_1')$. Then we have a decomposition*

$$mZ = Z' + Z'' \in \mathrm{CH}^k(X_1 \times Y),$$

where Z' is supported in $X_1' \times Y$ and Z'' is supported in $X_1 \times Y'$, for a proper closed $Y' \subset Y$.

REMARK 3.9. Our assumption is equivalent, by the localization exact sequence (2.2), to the fact that the cycle $\Gamma^*(y)$ is rationally equivalent to a cycle supported on X_1'.

Applying this result to the diagonal of a smooth projective variety Y, we get the famous Bloch–Srinivas decomposition of the diagonal (see [15]).

THEOREM 3.10 (Bloch and Srinivas 1983). *If Y is a smooth projective variety such that $\mathrm{CH}_0(Y)$ is supported on some closed algebraic subset $W \subset Y$, there is an equality in $\mathrm{CH}^d(Y \times Y)$, $d = \dim Y$:*

$$N\Delta_Y = Z_1 + Z_2, \tag{3.2}$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$\mathrm{Supp}\, Z_1 \subset D \times Y, \quad D \subsetneq Y, \qquad \mathrm{Supp}\, Z_2 \subset Y \times W.$$

REMARK 3.11. The meaning of the integer N appearing above is discussed in the paper [113] (see also Section 6.3). If $\dim Y = 3$ and $\dim W \leq 1$, then N annihilates, for example, the torsion in $H_B^*(Y, \mathbb{Z})$, hence it cannot in general be set equal to 1.

Although the cohomological version of the statement above may seem much weaker, it leads in practice to a number of applications. We thus state it separately as the cohomological Bloch–Srinivas decomposition of the diagonal.

COROLLARY 3.12 (Cohomological decomposition of the diagonal). *If Y is a smooth projective variety such that $\mathrm{CH}_0(Y)$ is supported on some closed algebraic subset $W \subset Y$, there is an equality of cycle classes in $H_B^{2d}(Y \times Y, \mathbb{Q})$, $d = \dim Y$:*

$$N[\Delta_Y] = [Z_1] + [Z_2], \tag{3.3}$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$\mathrm{Supp}\, Z_1 \subset T \times Y, \quad T \subsetneq Y, \qquad \mathrm{Supp}\, Z_2 \subset Y \times W.$$

3.1.1 Mumford's theorem

The following is a generalization of the original Mumford theorem in [71] (see Theorem 1.1) which concerned 0-cycles and holomorphic forms on surfaces (see also [86] for higher-dimensional generalizations). The proof we give below is due to Bloch and Srinivas.

THEOREM 3.13 (Generalized Mumford theorem). *Let X be smooth projective of dimension n. If there exists a closed algebraic subset $j : X' \hookrightarrow X$ such that $\dim X' < r$ and the map*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective, then $H^0(X, \Omega_X^k) = 0$ for $k \geq r$.

PROOF. We use the decomposition (3.3). We thus have equality of the corresponding cohomology classes:

$$m[\Delta_X] = [Z'] + [Z''] \in H_B^{2n}(X \times X, \mathbb{Z}),$$

where m is a nonzero integer and Z', Z'' are codimension n cycles of $X \times X$ with

$$\mathrm{Supp}\, Z' \subset T \times X, \quad T \subsetneq X, \qquad \mathrm{Supp}\, Z'' \subset X \times X'.$$

We can view these cohomology classes, or rather their Künneth components of adequate degree, as morphisms of Hodge structures

$$m[\Delta_X]^*, \quad [Z']^*, \quad [Z'']^* : H_B^k(X, \mathbb{Z}) \to H_B^k(X, \mathbb{Z})$$

(see [101, I, 11.3.3]). Then we have the equality

$$m[\Delta_X]^* = [Z']^* + [Z'']^* \in \mathrm{Hom}(H_B^k(X, \mathbb{Z}), H_B^k(X, \mathbb{Z})). \tag{3.4}$$

Now, the morphism $[\Delta_X]^*$ is equal to the identity, by the fact that Δ_X is the diagonal. Thus, (3.4) gives

$$m\,\mathrm{Id} = [Z']^* + [Z'']^* \in \mathrm{Hom}(H_B^k(X, \mathbb{Z}), H_B^k(X, \mathbb{Z})).$$

In particular, for $\eta \in H^0(X, \Omega_X^k) \subset H_B^k(X, \mathbb{C})$, we have

$$m\eta = [Z']^*\eta + [Z'']^*\eta \in H_B^k(X, \mathbb{C}). \tag{3.5}$$

Now let $l : \widetilde{T} \to X$ be a desingularization of T. As the cycle Z'' is supported in $T \times X$, it comes from a cycle $\widetilde{Z}''$ of $\widetilde{T} \times X$, $Z'' = (l, \mathrm{Id})_*(\widetilde{Z}'')$, so we get

$$[Z''] = (l, \mathrm{Id})_*([\widetilde{Z}'']) \text{ in } H_B^{2d}(X \times X, \mathbb{Z}). \tag{3.6}$$

Recall that if $\alpha \in H_B^{2n}(X \times X, \mathbb{Z})$, the corresponding morphism

$$\alpha_k^* : H_B^k(X, \mathbb{Z}) \to H_B^k(X, \mathbb{Z})$$

is defined by

$$\alpha_k^*(\beta) = p_{1*}(p_2^*(\beta) \cup \alpha). \tag{3.7}$$

Formulas (3.6) and (3.7) then show that

$$[Z'']^* = l_* \circ [\widetilde{Z}'']^*. \tag{3.8}$$

Similarly, let $\tilde{j} : \widetilde{X}' \to X$ be a desingularization of X', and let $\widetilde{Z}'$ be a cycle of $X \times \widetilde{X}'$ such that $(\mathrm{Id}, \tilde{j})_*(\widetilde{Z}') = Z'$. Then formula (3.7) shows that

$$[Z']^* = [\widetilde{Z}']^* \circ \tilde{j}^*. \tag{3.9}$$

By equations (3.8) and (3.9), (3.5) now gives

$$m\eta = [\widetilde{Z}']^* \circ \tilde{j}^*\eta + l_* \circ [\widetilde{Z}'']^*\eta. \tag{3.10}$$

But since $\dim X' < r$, for $k \geq r$ we have

$$\tilde{j}^*\eta = 0 \text{ in } H^0(\widetilde{X}', \Omega^k_{\widetilde{X}'}).$$

Thus, we have $[\widetilde{Z}']^* \circ \tilde{j}^*\eta = 0$ and

$$m\eta = l_* \circ [\widetilde{Z}'']^*\eta. \tag{3.11}$$

Moreover, since $\dim T < \dim X$, the morphism of Hodge structures l_* is of bidegree (s, s) with $s = \operatorname{codim} T > 0$, so the intersection of its image with $H^0(X, \Omega^k_X) = H^{k,0}(X)$ is reduced to 0. Thus, the equality (3.11) implies that $\eta = 0$ for $\eta \in H^0(X, \Omega^k_X), k \geq r$. □

3.1.2 Further applications

The following result is proved in [15].

Theorem 3.14 (Bloch and Srinivas 1983). *On a smooth projective variety X with $\mathrm{CH}_0(X)$ supported on a surface, homological equivalence and algebraic equivalence coincide for codimension 2 cycles.*

Let us make a few comments. First of all, the result is optimal, since Griffiths proved that there exist codimension 2 cycles on certain threefolds, which are homologous to 0, no multiple of them being algebraically equivalent to 0 (see [48]). Clemens even proved in [20] that the group $\mathrm{CH}^2(X)_{\mathbb{Q}}/\mathrm{alg}$ of cycles modulo algebraic equivalence tensored by $\mathbb{Q}$ can be infinitely generated. Next, this theorem has two distinct parts, namely proving that under the same assumptions, the group $\mathrm{CH}^2(X)/\mathrm{alg}$ is of torsion, and proving that the torsion does not appear. The second part is much more sophisticated, as it uses the Merkurjev–Suslin theorem and Bloch–Ogus formula for $\mathrm{CH}^2(X)/\mathrm{alg}$ (see Section 6.2.2). We will content ourselves with establishing the statement with $\mathbb{Q}$-coefficients.

Proof of Theorem 3.14 with $\mathbb{Q}$-coefficients. We apply Theorem 3.10. Letting $d = \dim X$, there is a surface $W \subset X$ and an equality in $\mathrm{CH}^d(X \times X)$:

$$N\Delta_X = Z_1 + Z_2,$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$\operatorname{Supp} Z_1 \subset D \times X, \quad D \subsetneq X, \qquad \operatorname{Supp} Z_2 \subset X \times W.$$

If we introduce desingularizations $\tilde{j}_D : \widetilde{D} \to X$ of $D \hookrightarrow X$ and $\tilde{j}_W : \widetilde{W} \to X$ of $W \hookrightarrow X$, the cycles Z_1 and Z_2 lift (maybe with $\mathbb{Q}$-coefficients) to $\widetilde{Z}_1 \subset \widetilde{D} \times X$, and $\widetilde{Z}_2 \subset X \times \widetilde{W}$, respectively. We then have the equality

$$N\Delta_X = (\tilde{j}_D, \mathrm{Id}_X)_*(\widetilde{Z}_1) + (\mathrm{Id}_X, \tilde{j}_W)_*(\widetilde{Z}_2) \text{ in } \mathrm{CH}^d(X \times X)_{\mathbb{Q}},$$

and deduce, by letting both sides act on $\mathrm{CH}^2(X)_{\mathbb{Q}}$, that for any codimension 2 cycle γ on X, we have

$$N\gamma = \tilde{j}_{D*}(\widetilde{Z}_1^*\gamma) + \widetilde{Z}_2^*(\tilde{j}_W^*\gamma) \text{ in } \mathrm{CH}^2(X)_{\mathbb{Q}}. \tag{3.12}$$

Assume now that γ is cohomologous to 0 on X. Then $\widetilde{Z}_1^*\gamma$ is a codimension 1 cycle cohomologous to 0 on $\widetilde{D}$, hence it is algebraically equivalent to 0 by Example 3.7. Similarly, $\tilde{j}_{\widetilde{W}}^*\gamma$ is a codimension 2 cycle (hence a 0-cycle) cohomologous to 0 on the surface $\widetilde{W}$, hence it is algebraically equivalent to 0. It follows from (3.12) that γ is algebraically equivalent to 0. □

Let us give two further applications also proved in [15], which use only the cohomological version of the decomposition of the diagonal (Corollary 3.12).

Theorem 3.15 (Bloch and Srinivas 1983). *Let X be a smooth complex projective variety such that there exists a closed algebraic subset $j : X' \hookrightarrow X$, of dimension ≤ 3, such that the map*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective. Then the Hodge conjecture holds for classes of degree 4.

This result was originally proven by Conte and Murre [26] in the case where X is a four-dimensional variety covered by rational curves. Such a variety X satisfies the hypothesis, since every point x is contained in a rational curve C_x whose normalization is isomorphic to $\mathbb{P}^1$. By the definition of rational equivalence, all the points $y \in C_x$ are rationally equivalent in C_x, so also in X, and if X' is an ample hypersurface of X, then C_x intersects X' and x is rationally equivalent in X to any point of $X' \cap C_x$.

Proof of Theorem 3.15. Applying Corollary 3.12, we see that there exists a proper subset $T \subset X$, which we may assume to be of codimension 1, and

n-dimensional cycles Z' and Z'' supported in $T \times X$ and $X \times X'$, respectively, such that

$$m[\Delta_X] = [Z'] + [Z''] \text{ in } H_B^{2n}(X \times X, \mathbb{Z}) \tag{3.13}$$

for some nonzero integer m. Let $k : \widetilde{T} \to X$ and $\tilde{j} : \widetilde{X}' \to X$ be desingularizations of T and X', respectively. The decomposition (3.13) then yields the equalities of the morphisms of Hodge structure associated to the Künneth components of type $(4, 2n-4)$ of these three Hodge classes:

$$m[\Delta_X]^* = [Z']^* + [Z'']^* : H_B^4(X, \mathbb{Z}) \to H_B^4(X, \mathbb{Z}).$$

Now, let $\widetilde{Z}' \subset \widetilde{T} \times X$ be a cycle of codimension n such that

$$(k, \mathrm{Id})_* \widetilde{Z}' = Z',$$

and let $\widetilde{Z}'' \subset X \times \widetilde{X}'$ be a cycle of codimension n such that

$$(\mathrm{Id}, \tilde{j})_* \widetilde{Z}'' = Z''.$$

We have

$$[Z'] = (k, \mathrm{Id})_*[\widetilde{Z}'], \qquad [Z''] = (\mathrm{Id}, \tilde{j})_*[\widetilde{Z}''] \text{ in } H_B^{2n}(X \times X, \mathbb{Z}),$$

and it follows that for every $\alpha \in H_B^4(X, \mathbb{Z})$, we have

$$[Z']^*\alpha = k_*([\widetilde{Z}']^*\alpha), \qquad [Z'']^*\alpha = [\widetilde{Z}'']^*(\tilde{j}^*\alpha).$$

But as X' is of dimension ≤ 3, the rational Hodge conjecture holds for $\widetilde{X}'$ in every degree. (Indeed, it holds for classes of degree 2 by the Lefschetz theorem on $(1,1)$-classes, and for classes of degree 4 by the argument given in Section 2.2.2. Moreover, the Hodge conjecture is satisfied for classes of degree 0 and $2n$ for every smooth projective variety of dimension n.)

If $\alpha \in H_B^4(X, \mathbb{Q}) \cap H^{2,2}(X)$ is a rational Hodge class, the classes $\tilde{j}^*\alpha \in H_B^4(\tilde{X}', \mathbb{Q}) \cap H^{2,2}(\tilde{X}')$ and $[Z']^*\alpha \in H_B^2(\tilde{T}, \mathbb{Q}) \cap H^{1,1}(\tilde{T})$ are thus classes of algebraic cycles of $\widetilde{X}'$ and $\widetilde{T}$, respectively. The relation

$$m[\Delta_X]^*\alpha = m\alpha = k_*([\widetilde{Z}']^*\alpha) + [\widetilde{Z}'']^*(\tilde{j}^*\alpha)$$

and the compatibility of the cycle class map with correspondences then show that α is also the class of an algebraic cycle with rational coefficients. □

The second application is an improvement of the generalized Mumford theorem (Theorem 3.13), where the Bloch–Srinivas method appears to be more powerful than the Mumford method. It is important enough to justify a separate statement. Using the notion of coniveau introduced in Section 2.2.5, we can restate Theorem 3.13 by saying that if X has its group $\mathrm{CH}_0(X)$ supported on some $X' \subset X$ of dimension $< r$, then the coniveau of the Hodge structures on $H_B^k(X, \mathbb{Q})$ is at least 1 for $k \geq r$. Recall that the generalized Hodge conjecture then predicts that $H_B^k(X, \mathbb{Q})$ is supported on a proper algebraic subset of X. This is indeed what the next theorem says.

THEOREM 3.16 (Bloch and Srinivas 1983). *Let X be smooth projective of dimension n. If there exists a closed algebraic subset $j : X' \hookrightarrow X$ such that $\dim X' < r$ and the map*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective, then for $k > r$, the cohomology groups $H^k_B(X, \mathbb{Q})$ vanish on a dense Zariski open set of X. In other words, $H^{*>r}_B(X, \mathbb{Q})$ has geometric coniveau ≥ 1.

PROOF. We start as in the proof of Theorem 3.13 and again write equality (3.10) for any $\eta \in H^k_B(X, \mathbb{Q})$:

$$m\eta = [\widetilde{Z}']^* \circ \tilde{j}^*(\eta) + l_* \circ [\widetilde{Z}'']^*(\eta),$$

where $l : \widetilde{T} \to X$ is a desingularization of T (which we may assume to be a divisor of X) and $\tilde{j} : \widetilde{X}' \to X$ is a desingularization of X'.

By definition, $l_* \circ [\widetilde{Z}'']^*(\eta)$ is of geometric coniveau ≥ 1. It thus suffices to prove that $[\widetilde{Z}']^* \circ \tilde{j}^*(\eta)$ is also of geometric coniveau ≥ 1 if $\deg \eta = k > r := \dim X'$. This is because the class $\tilde{j}^*(\eta)$ on $\widetilde{X}'$ is of degree $k > \dim X$, hence has geometric coniveau ≥ 1 on $\widetilde{X}'$ by the Lefschetz isomorphism (5.24),

$$\cup l^{k-r} : H^{2r-k}(\widetilde{X}', \mathbb{Q}) \cong H^k(\widetilde{X}', \mathbb{Q}),$$

where l is the class of an ample divisor on $\widetilde{X}'$. □

REMARK 3.17. Because the integer m cannot in general be set equal to 1 (see [113]), the proof above only works for cohomology with rational coefficients. However, the Bloch–Kato conjecture, proved by Voevodsky ([97]) and Rost, implies that the result holds also for cohomology with integral coefficients. Indeed, the Bloch–Kato conjecture implies (see [6], [15], [24], and Section 6.2.2) that the Bloch–Ogus sheaves of $\mathbb{Z}$-modules $\mathcal{H}^k(\mathbb{Z})$ on X_{Zar} associated to the presheaves $U \mapsto H^k_B(U, \mathbb{Z})$ have no torsion.

3.2 VARIETIES WITH SMALL CHOW GROUPS

3.2.1 Generalized decomposition of the diagonal

The Bloch–Srinivas decomposition of the diagonal has been generalized by Paranjape [80] and Laterveer [66] under triviality assumptions on Chow groups of small dimension.

THEOREM 3.18. *Assume that for $k < c$, the cycle class maps*

$$\mathrm{cl} : \mathrm{CH}_k(X) \otimes \mathbb{Q} \to H^{2n-2k}_B(X, \mathbb{Q})$$

are injective. Then there exists a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-1} + Z' \in \mathrm{CH}^n(X \times X), \tag{3.14}$$

where $m \neq 0$ is an integer, Z_i is supported in $W'_i \times W_i$ with $\dim W_i = i$ and $\dim W'_i = n-i$, and Z' is supported in $T \times X$, where $T \subset X$ is a closed algebraic subset of codimension $\geq c$.

Theorem 3.10 is in fact the case $c = 1$ of this theorem.

Proof of Theorem 3.18. We use induction on c. The case $c = 1$ has already been considered. We may thus assume that we have a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-2} + Z' \in \mathrm{CH}^n(X \times X), \tag{3.15}$$

where $m \neq 0$ is an integer, Z_i is supported in $W'_i \times W_i$ with $\dim W_i = i$ and $\dim W'_i = n-i$, and Z' is supported in $T \times X$, where $T \subset X$ is a closed algebraic subset of codimension $\geq c-1$. We may assume that T is of codimension $(c-1)$. Let $\tau : \widetilde{T} \to X$ be a desingularization. Let $\widetilde{Z}' \subset \widetilde{T} \times X$ be a cycle such that $(\tau, \mathrm{Id})_*\widetilde{Z}' = Z'$. The cycle $\widetilde{Z}' \subset \widetilde{T} \times X$ is of codimension $(n-c+1)$ and induces a morphism $\widetilde{Z}'_* : \mathrm{CH}_0(\widetilde{T}) \to \mathrm{CH}_{c-1}(X)$. Now, we know by assumption that the kernel of $\mathrm{cl} : \mathrm{CH}_{c-1}(X) \to H_B^{2n-2c+2}(X, \mathbb{Z})$ is torsion. Thus, the map $\widetilde{Z}'_*$ maps $\mathrm{CH}_0(\widetilde{T})_{\mathrm{hom}}$ to the torsion of $\mathrm{CH}_{c-1}(X)$. By a Baire countability argument, it follows that there exists an integer M such that $M\widetilde{Z}'_* = 0$ on $\mathrm{CH}_0(\widetilde{T})_{\mathrm{hom}}$.

For each component $\widetilde{T}_i$ of $\widetilde{T}$, choose a point $t_i \in \widetilde{T}_i$ as above, and let $W_i = \widetilde{Z}'_*(t_i)$. The cycle

$$Z'' = M\left(\widetilde{Z}' - \sum_i \widetilde{T}_i \times W_i\right) \subset \widetilde{T} \times X$$

then satisfies the property that for any $t \in \widetilde{T}$, $Z''_*(t) = 0$ in $\mathrm{CH}_{c-1}(X)$. We then apply Theorem 3.1 to conclude that there exists a cycle $\widetilde{Z}'' \subset \widetilde{T}' \times X$, where $\widetilde{T}' \subset \widetilde{T}$ is a proper closed algebraic subset, and an integer M', such that

$$M'M\left(\widetilde{Z}' - \sum_i \widetilde{T}_i \times W_i\right) = \widetilde{Z}'' \text{ in } \mathrm{CH}_n(\widetilde{T} \times X).$$

Setting $T' = \tau(\widetilde{T}')$ and $Z'' = \tau_*\widetilde{Z}''$, we obtain

$$M'M\left(Z' - \sum_i T_i \times W_i\right) = Z'' \text{ in } \mathrm{CH}_n(X \times X),$$

with Z'' supported on $T' \times X$ and $\operatorname{codim} T' \geq c$. Combining this equality with (3.15), we obtain the desired decomposition. □

Theorem 3.18 can be used to give a proof of the following result due to Lewis [67].

THEOREM 3.19 (Lewis 1995). *Let X be a smooth projective complex variety such that the cycle class map $\mathrm{cl} : \mathrm{CH}^i(X)_{\mathbb{Q}} \to H^{2i}(X, \mathbb{Q})$ is injective for all $i \geq 0$. Then $H_B^{2i+1}(X, \mathbb{Q}) = 0$ for all $i \geq 0$ and the cycle class map $\mathrm{cl} : \mathrm{CH}^i(X)_{\mathbb{Q}} \to H_B^{2i}(X, \mathbb{Q})$ is an isomorphism for all $i \geq 0$.*

PROOF. Indeed, by Theorem 3.18, we have under these assumptions a complete decomposition of the diagonal

$$\Delta_X = \sum_{i,j} n_{ij} W_i \times W_j \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}}.$$

The corresponding cohomological decomposition is written as

$$[\Delta_X] = \sum_{i,j} n_{ij}\, \mathrm{pr}_1^*[W_i] \cup \mathrm{pr}_2^*[W_j] \text{ in } H_B^{2n}(X \times X)_{\mathbb{Q}}.$$

For any $\alpha \in H_B^*(X, \mathbb{Q})$ we thus get

$$\alpha = [\Delta_X]_* \alpha = \sum_{i,j} n_{ij} \big\langle \alpha, [W_i] \big\rangle\, [W_j],$$

which shows that the rational cohomology of X is generated by classes of algebraic cycles. □

3.2.2 Generalized Bloch conjecture

The main application of the generalized decomposition of the diagonal is the following result.

THEOREM 3.20 (Laterveer 1996, Lewis 1995, Paranjape 1994, Schoen 1993). *Let X be a smooth projective variety of dimension m. Assume that the cycle class map*

$$\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2m-2i}(X, \mathbb{Q})$$

is injective for $i \leq c-1$. Then we have $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$).

More precisely, the Hodge structures on $H_B^k(X, \mathbb{Q})^{\perp \mathrm{alg}}$ are all of Hodge coniveau $\geq c$; in fact they even are of geometric coniveau $\geq c$, hence they satisfy the generalized Hodge conjecture (Conjecture 2.40) for coniveau c.

Here $H_B^k(X, \mathbb{Q})^{\perp \mathrm{alg}}$ denotes the transcendental part of the cohomology, defined as the subspace of $H_B^k(X, \mathbb{Q})$ consisting of classes orthogonal to all classes of algebraic cycles (of degree $2n - k$). Of course it is different from $H_B^k(X, \mathbb{Q})$ only if k is even.

PROOF OF THEOREM 3.20. Let us write the cohomological generalized decomposition of the diagonal of Theorem 3.18:

$$m[\Delta_X] = [Z_0] + \cdots + [Z_{c-1}] + [Z'] \in H^{2n}(X \times X, \mathbb{Q}).$$

By hypothesis, the Z_i are of the form

$$Z_i = \sum_j n_{i,j} W'_{i,j} \times W_{i,j}, \tag{3.16}$$

where the $W_{i,j}$ and $W'_{i,j}$ are irreducible components of W_i and W'_i, respectively. Moreover, Z' is supported in $T \times X$ with $\operatorname{codim} T \geq c$. Let $l : \widetilde{T} \to X$ be a desingularization of T, and let $\widetilde{Z}'$ in $\mathrm{CH}_n(\widetilde{T} \times X)$ be such that $(l, \mathrm{Id})_*(\widetilde{Z}') = Z'$. The above decomposition gives a decomposition of the corresponding morphisms of Hodge structures for every k:

$$m[\Delta_X]^* = m\,\mathrm{Id} = [Z_0]^* + \cdots + [Z_k]^* + [Z']^* : H^k_B(X, \mathbb{Q}) \to H^k_B(X, \mathbb{Q}).$$

Now, by (3.16), we clearly have

$$[Z_i]^*(\alpha) = \sum_j n_{i,j} \langle \alpha, [W_{i,j}] \rangle \, [W'_{i,j}], \tag{3.17}$$

where $\langle\, , \rangle$ is the intersection form on $H^*_B(X, \mathbb{Q})$. In particular, we have

$$[Z_i]^*(\alpha) = 0 \;\forall\, \alpha \in H^{p,q}(X), \quad p \neq q.$$

For α satisfying this hypothesis, we thus have

$$m\alpha = [Z']^*(\alpha) = l_*([\widetilde{Z}']^*(\alpha)) \text{ in } H^{p,q}(X). \tag{3.18}$$

Thus, if $\alpha \in H^{p,q}(X)$ with $p \neq q$, we have

$$m\alpha \in \operatorname{Im} l_* \cap H^{p,q}(X).$$

Now, as we may assume $\operatorname{codim} T = c$, the Gysin morphism

$$l_* : H^{p+q-2c}_B(\widetilde{T}, \mathbb{Z}) \to H^{p+q}_B(X, \mathbb{Z})$$

is a morphism of Hodge structures of bidegree (c, c), so that

$$\operatorname{Im} l_* \cap H^{p,q}(X) = 0, \quad q \leq c - 1.$$

Hence we have $m\alpha = 0$ for

$$\alpha \in H^{p,q}(X), \quad p \neq q, \quad q \leq c - 1.$$

This proves the first statement.

For the second statement, formula (3.17) shows more precisely that $[Z_i]^*(\alpha) = 0$ for $\alpha \in H^k_B(X, \mathbb{Q})^{\perp \mathrm{alg}}$. For such α, formula (3.18) holds and shows that $\alpha \in \operatorname{Im} l_*$, which completes the proof since $\operatorname{Im} l_*$ vanishes away from T and T has codimension $\geq c$. □

The major open problem in the theory of algebraic cycles, which by the above theorem would also solve many instances of the generalized Hodge conjecture, is the following conjecture which is explicitly stated in [58], relating the Hodge coniveau and Chow groups. This converse of Theorem 3.20 is a vast generalization of the Bloch conjecture for surfaces [13].

Conjecture 3.21. *Let X be a smooth projective variety of dimension m satisfying the condition $H^{p,q}(X)=0$ for $p\neq q$ and $p<c$ (or $q<c$). Then for any integer $i\leq c-1$, the cycle class map*

$$\mathrm{cl}:\mathrm{CH}_i(X)_{\mathbb{Q}}\to H_B^{2m-2i}(X,\mathbb{Q})$$

is injective.

Remark 3.22. This conjecture can be split into two parts. Indeed, as we proved in the previous section, if the conclusion of the conjecture holds true, then the cohomology of X (or rather its transcendental part) is supported on a closed algebraic subset of codimension $\geq c$. This is predicted by the generalized Hodge conjecture (Conjecture 2.40).

If we assume Conjecture 2.40, the generalized Bloch conjecture can be reformulated as follows, without any mention of Hodge structures.

Conjecture 3.23. *If the transcendental cohomology of X is supported on a closed algebraic subset of codimension $\geq c$, then for any integer $i\leq c-1$, the cycle class map*

$$\mathrm{cl}:\mathrm{CH}_i(X)_{\mathbb{Q}}\to H_B^{2m-2i}(X,\mathbb{Q})$$

is injective.

In the following section, we sketch the ideas of Kimura, which lead to a proof of Conjecture 3.23 for varieties dominated by products of curves. In a rather different geometric setting, in Section 4.3 we will prove Conjecture 3.23 for very general complete intersections in a variety with trivial Chow groups, assuming the Lefschetz standard conjecture.

3.2.3 Nilpotence conjecture and Kimura's theorem

Let X be smooth projective, and $\Gamma\subset X\times X$ a correspondence. Recall that correspondences between smooth projective varieties can be composed (Section 2.1.3), so that the self-correspondences of X form a ring.

The following represents the starting point of Kimura's work, and remains completely open.

Conjecture 3.24 (Nilpotence conjecture). *Suppose that $\Gamma\in\mathrm{CH}(X\times X)$ is homologous to 0. Then there exists a positive integer N such that $\Gamma^{\circ N}=0$ in* $\mathrm{CH}(X\times X)_{\mathbb{Q}}$.

Note that this conjecture is implied by the Bloch–Beilinson conjecture (Conjecture 2.19). Indeed, assuming the Bloch–Beilinson filtration F exists satisfying all the properties stated in Conjecture 2.19, if Γ is cohomologous to 0, then $\Gamma^{\circ N}$ belongs to $F^N\,\mathrm{CH}(X \times X)_{\mathbb{Q}}$. As this filtration is conjectured to satisfy $F^{k+1}\,\mathrm{CH}^k(X \times X) = 0$, we must have $\Gamma^{\circ N} = 0$ for $N > 2n$, $n := \dim X$.

For cycles algebraically equivalent to 0, the following result is proved independently in [96] and [99].

Theorem 3.25 (Voevodsky 1995, Voisin 1994). *The nilpotence conjecture holds for cycles in $X \times X$ that are algebraically equivalent to* 0.

Proof. Let $\Gamma \in \mathrm{CH}^d(X \times X)$, $d = \dim X$ be algebraically equivalent to 0. This means that there is a curve C that we may assume to be smooth, a 0-cycle $z \in \mathrm{CH}_0(C)$ homologous to 0, and a correspondence $Z \in \mathrm{CH}^d(C \times X \times X)$ such that

$$\Gamma = Z_*(z) \text{ in } \mathrm{CH}^d(X \times X).$$

For any integer k, we can construct a correspondence $Z_k \in \mathrm{CH}^d(C^k \times X \times X)$ obtained using the composition of the cycles Z_t, $t \in C$, namely we define Z_k by the formula

$$Z_k(t_1, \ldots, t_k) = Z(t_1) \circ \cdots \circ Z(t_k), \quad t_1, \ldots, t_k \in C.$$

By definition, we get

$$\Gamma^{\circ k} = Z_{k*}(z^k),$$

where the product $z^k \in \mathrm{CH}_0(C^k)$ is defined as $\mathrm{pr}_1^* z \cdot \cdots \cdot \mathrm{pr}_k^* z$.

The proof concludes with the following easy fact (see [101, II, Lemma 11.33]).

Lemma 3.26. *For a 0-cycle z homologous to 0 on a smooth curve C, the cycle z^k vanishes in $\mathrm{CH}_0(C^k)$ for k large enough.*

□

Coming back to the general situation, Kimura proved in [59] the remarkable result that Conjecture 3.24 is implied by the *finite-dimensionality property* for X, a notion that we will now describe. We refer to [3] for an expanded exposition of Kimura's ideas (which seem to have been developed independently by O'Sullivan) on the notion of finite-dimensionality and we just sketch the general idea: For any smooth projective variety X, we have $\bigwedge^N H^*(X, \mathbb{Q}) = 0$ for $N > \dim H^*(X, \mathbb{Q})$. Assume for simplicity that X has only even degree cohomology. Then $\bigwedge^N H^*(X, \mathbb{Q})$ identifies to the skew invariant part of $H^*(X^N, \mathbb{Q})$ under the action of the symmetric group $\mathfrak{S}_N$, which is also the image of the projector

$$\alpha \mapsto \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}^N} \epsilon(\sigma)\sigma^*\alpha$$

acting on $H^*(X^N, \mathbb{Q})$. Consider now the skew-symmetric motive $\bigwedge^N X := (X^N, \pi^{\rm skew})$, where

$$\pi^{\rm skew} := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}^N} \epsilon(\sigma)\Gamma_\sigma \in \mathrm{CH}^{nN}(X^N \times X^N), \quad n = \dim X.$$

(As usual we denote by Γ_σ the graph of σ acting on X^N.) This is a motive whose cohomology is exactly $\bigwedge^N H^*(X, \mathbb{Q}) = 0$. This motive is then expected to be 0, in the sense that $\pi^{\rm skew} = 0$ in $\mathrm{CH}^{nN}(X^N \times X^N)_{\mathbb{Q}}$. Indeed, this cycle is a projector whose class is 0, so we have

$$\pi^{\rm skew} = (\pi^{\rm skew})^{\circ k} \text{ in } \mathrm{CH}^{nN}(X^N \times X^N)_{\mathbb{Q}},$$

and furthermore if the Bloch–Beilinson filtration F exists, the right-hand side belongs to $F^k \mathrm{CH}^{nN}(X^N \times X^N)_{\mathbb{Q}}$ for any k, hence is 0 for large k, and thus $\pi^{\rm skew}$ itself is expected to be 0.

In general, X will also have odd degree cohomology. In this case, the hope is that the motive of X splits as $X^+ + X^-$, where X^- has only odd degree cohomology and X^+ has only even degree cohomology (in other words, we need projectors π^+, $\pi^- \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ satisfying the condition that the action π^+_* on cohomology is the projector on $H^{\rm even}(X)$ and the action of π^- on cohomology is the projector on $H^{\rm odd}(X)$). For the motive X^-, observe that the vanishing $\bigwedge^N H^{\rm odd}(X, \mathbb{Q})$, $N > \dim H^{\rm odd}(X, \mathbb{Q})$ says that the $\mathfrak{S}_N$-*invariant* cohomology of $(X^-)^N$ vanishes, or that the motive $S^N X^- := (X^N, \pi^{\rm inv} \circ (\pi^-)^N)$, where now $\pi^{\rm inv} := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}^N} \Gamma_\sigma \in \mathrm{CH}^{nN}(X^N \times X^N)$ has zero cohomology.

Kimura's conjecture (presented in a simplified form) is the following.

CONJECTURE 3.27. *For any smooth projective variety X, the motive of X is finite-dimensional, which means that it decomposes as $X^+ + X^-$, where X^- has only odd degree cohomology and X^+ has only even degree cohomology, and for large N, we have*

$$\bigwedge^N X^+ = 0, \qquad S^N X^- = 0.$$

A remarkable result due to Kimura [59] is the following theorem that proves a strengthening of the nilpotence conjecture under the assumption of finite-dimensionality.

THEOREM 3.28 (Kimura 2005). *Assume the motive of X is finite-dimensional. Then for any cycle $Z \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$, Z satisfies a polynomial equation in the ring $\mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ of self-correspondences of X of the form*

$$Z^{\circ N} = \alpha_{N-1} Z^{\circ N-1} + \cdots + \alpha_0 \Delta_X,$$

where the α_i's are rational numbers that vanish if Z is cohomologous to 0. In particular, if Z is cohomologous to 0, Z is nilpotent.

Sketch of proof. We will just show how the proof works in a fictive case, namely the motive of X satisfies $\bigwedge^2 X = 0$ and the cycle Z is the graph of a map $f : X \to X$ with finitely many fixed points. Hence we assume that

$$\Gamma := \Delta_{X\times X} - \Gamma_\tau = 0 \text{ in } \mathrm{CH}^{2n}(X \times X \times X \times X)_{\mathbb{Q}}, \tag{3.19}$$

where Γ_τ is the graph of the involution $\tau : (x, y) \mapsto (y, x)$ acting on $X \times X$. We have $\Delta_{X\times X} = \{(x, y, x, y),\, x \in X,\, y \in X\}$ and $\Gamma_\tau = \{(x, y, y, x),\, x \in X,\, y \in X\}$.

It follows that we also have $\Gamma \circ \Gamma_{(f,f)} = 0$ in $\mathrm{CH}^{2n}(X \times X \times X \times X)_{\mathbb{Q}}$. We thus have

$$0 = \Gamma \circ \Gamma_{(f,f)} = \Gamma_1 - \Gamma_2 \in \mathrm{CH}^{2n}(X \times X \times X \times X)_{\mathbb{Q}}, \tag{3.20}$$

where $\Gamma_1 = \{(x, y, f(x), f(y)),\, x \in X,\, y \in X\}$ and $\Gamma_2 = \{(x, y, f(y), f(x)),\, x \in X,\, y \in X\}$. Intersect Γ with $p_{13}^*\Delta_X$: from(3.20) we get

$$0 = (\Gamma \circ \Gamma_{(f,f)}) \cdot p_{13}^*(\Delta_X) = \Gamma_1' - \Gamma_2' \text{ in } \mathrm{CH}^{3n}(X \times X \times X \times X)_{\mathbb{Q}}, \tag{3.21}$$

where $\Gamma_1' = \{(x, y, x, f(y)),\, x \in X, f(x) = x,\, y \in X\}$ and $\Gamma_2' = \{(f(y), y, f(y), f \circ f(y)),\ y \in X\}$. Project the relation (3.21) to $X \times X$ via p_{24}. We get

$$\Gamma_1'' - \Gamma_2'' = 0 \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}}, \tag{3.22}$$

where

$$\Gamma_1'' = \alpha\Gamma_f, \quad \alpha := \deg(\Delta_X \cdot \Gamma_f), \qquad \Gamma_2'' = \Gamma_{f\circ f} = \Gamma_f^{\circ 2}.$$

Hence we proved that

$$\deg(\Delta_X \cdot \Gamma_f)\Gamma_f + \Gamma_f^{\circ 2} = 0 \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}},$$

where the number $\deg(\Delta_X \cdot \Gamma_f)$ depends only on the cohomology class of Γ_f, which is what we wanted. ☐

The main concrete result concerning Conjecture 3.27 was established by Kimura [59] who proved the following statement.

Theorem 3.29 (Kimura 2005). *If X is dominated by a product of curves then X satisfies the finite-dimensionality conjecture (Conjecture 3.27).*

The proof of this theorem is rather tricky, and would be out of place here. If X is a surface, it is in fact sufficient that X be *rationally* dominated by the product of two curves in order to conclude the validity of Conjecture 3.24.

Next we have the following beautiful application of the finite-dimensionality property (see [59]).

Theorem 3.30 (Kimura 2005). *Conjecture 3.24 implies the Bloch conjecture (Conjecture 1.9) for surfaces with $p_g = q = 0$. In particular, the Bloch conjecture is valid for surfaces with $p_g = q = 0$ that are rationally dominated by curves.*

PROOF. Let X be a surface satisfying these assumptions. By the Lefschetz theorem on $(1,1)$-classes, since $p_g(X)=0$, we know that

$$H_B^2(X,\mathbb{Z})=\langle[C_i]\rangle$$

is generated by classes of curves. Since $q=0$, the Künneth decomposition of the diagonal takes the following simple form:

$$[\Delta]\in H_B^0(X,\mathbb{Q})\otimes H_B^4(X,\mathbb{Q})\,\oplus\,H_B^2(X,\mathbb{Q})\otimes H_B^2(X,\mathbb{Q})\oplus H_B^4(X,\mathbb{Q})\otimes H_B^0(X,\mathbb{Q}),$$

and as $H_B^2(X,\mathbb{Q})$ is generated by the classes $[C_i]$, for any chosen point $x\in X$, we may write

$$[\Delta_X]\;=\;[X\times\{x\}]\;+\;\sum n_{ij}[C_i\times C_j]\;+[\{x\}\times X].\tag{3.23}$$

Consider the 2-cycle

$$\Gamma\;=\;\Delta_X-(X\times\{x\})-\sum n_{ij}C_i\times C_j-(\{x\}\times X)\tag{3.24}$$

in $X\times X$. Then by (3.23), $[\Gamma]=0$. The nilpotence conjecture implies that there exists a positive integer N such that $\Gamma^{\circ N}=0$ in $\mathrm{CH}_2(X\times X)_{\mathbb{Q}}$. Hence,

$$(\Gamma_*)^{\circ N}=(\Gamma^{\circ N})_*:\;\;\mathrm{CH}_0(X)_{\mathbb{Q}}\to\mathrm{CH}_0(X)_{\mathbb{Q}}$$

is 0. But Γ_* acts as the identity on the group $\mathrm{CH}_0(X)_{\mathrm{hom}}$ of 0-cycles of degree 0, since Δ_X acts as the identity, but the other terms in the right-hand side of (3.24) act trivially on $\mathrm{CH}_0(X)_{\mathrm{hom}}$. (For example, $\gamma=X\times\{x\}$ acts trivially on $\mathrm{CH}_0(X)_{\mathrm{hom}}$ because $\gamma_*(z)=(\deg z)x$.) Thus $\mathrm{CH}_0(X)_{\mathrm{hom}}=0$. □

REMARK 3.31. Looking more closely at the proof and introducing Murre's Chow–Künneth decomposition [74], one sees that the proof above would also show that Conjecture 3.24 implies the Bloch conjecture (Conjecture 1.11) for surfaces with $p_g=0$.

The next theorem is a direct generalization of Theorem 3.30. It proves the Bloch conjecture (Conjecture 1.11) and Conjecture 3.21 for $c=1$, under the generalized Hodge conjecture and the nilpotence conjecture.

THEOREM 3.32. *Let X be a smooth projective variety with $h^{k,0}(X)=0$ for $k>r$. Assume that the following conditions hold:*

(i) The generalized Hodge conjecture holds for X in coniveau 1.

(ii) The Hodge conjecture is true for $Y\times X$ with $\dim Y\leq\dim X$.

(iii) X satisfies the nilpotence conjecture (Conjecture 3.24).

Then $\mathrm{CH}_0(X)$ *is supported on a closed algebraic subset $X_r\subset X$ of dimension $\leq r$.*

Sketch of proof. Let us do it first for $r = 0$ (this is the situation of Conjecture 3.21 with $c = 1$). Again we work with the diagonal $\Delta_X \subset X \times X$ and its cohomology class

$$[\Delta_X] \in H^0_B(X, \mathbb{Q}) \otimes H^{2n}_B(X, \mathbb{Q}) \oplus \cdots .$$

We have

$$[\Delta_X] = [X \times \{x\}] \mod \bigoplus_{k>0} H^k_B(X, \mathbb{Q}) \otimes H^{2n-k}_B(X, \mathbb{Q}).$$

The generalized Hodge conjecture and the assumption that $H^{k,0}(X) = 0$ for $k > 0$ imply that there exist Y of codimension 1 and a resolution $\tilde{i}\colon \widetilde{Y} \to Y \to X$, so that

$$\tilde{i}_*\colon H^{k-2}_B(\widetilde{Y}, \mathbb{Q}) \to H^k_B(X, \mathbb{Q})$$

is surjective for any $k > 0$, as follows from Theorem 2.39. Hence

$$[\Delta_X - X \times \{x\}] \in \mathrm{Im}((\tilde{i}, \mathrm{id})_* : H^{2n-2}_B(\widetilde{Y} \times X, \mathbb{Q}) \to H^{2n}_B(X \times X, \mathbb{Q})).$$

The Gysin morphism $(\tilde{i}, \mathrm{id})_*$ is a morphism of Hodge structures. Corollary 2.24 implies that

$$[\Delta_X - X \times \{x\}] = (\tilde{i}, \mathrm{id})_*\beta, \tag{3.25}$$

for some Hodge class β on $\widetilde{Y} \times X$. We now finish the argument as before: The Hodge conjecture on $\widetilde{Y} \times X$ implies that β is the class $[Z]$ of a cycle Z on $\widetilde{Y} \times X$. Put

$$\Gamma = \Delta_X - X \times \{x\} - (\tilde{i}, \mathrm{id})_* Z,$$

so that $[\Gamma] = 0$ by (3.25). It follows that Γ_* is nilpotent, yet Γ_* acts as the identity on $\mathrm{CH}_0(X)_{\mathrm{hom}}$, since $((\tilde{i}, \mathrm{id})_* Z)_* = 0$ on $\mathrm{CH}_0(X)_\mathbb{Q}$ (because $(\tilde{i}, \mathrm{id})_* Z$ is supported on $Y \times X$ with $Y \subsetneq X$) and for $z \in \mathrm{CH}_0(X)_\mathbb{Q}$, $(X \times \{x\})_*(z) = (\deg z)\, x$ in $\mathrm{CH}_0(X)_\mathbb{Q}$. Therefore $\mathrm{CH}_0(X)_\mathbb{Q} = \mathbb{Q}$ and this implies that $\mathrm{CH}_0(X) = \mathbb{Z}$ because $\mathrm{CH}_0(X)$ has no torsion under our assumptions by Roitman's theorem (see [87]). Indeed, our assumptions imply that $\mathrm{Alb}\, X = 0$.

For the general case (r arbitrary), we just have to add the following argument.

First of all, since we assumed that the Hodge conjecture holds for $X \times X$, the Künneth components of the diagonal are algebraic, so we can write

$$[\Delta_X] = \sum_i \delta_i \text{ in } H^{2n}_B(X \times X, \mathbb{Q}),$$

with $\delta_i = [Z_i]$, $Z_i \in \mathrm{CH}^n(X \times X)_\mathbb{Q}$, and $\delta_i \in H^i_B(X, \mathbb{Q}) \otimes H^{2n-i}_B(X, \mathbb{Q})$.

For $i > r$, we know by assumption that the Hodge coniveau of $H^i_B(X, \mathbb{Q})$ is at least 1 and thus by the same arguments as above, we conclude using assumptions (i) and (ii) that we may assume the Z_i's are supported on $Y \times X$, where $Y \subsetneq X$

is a proper closed algebraic subset. Next assume $i \leq r$. Then $2n - i \geq 2n - r$, and thus by the Lefschetz theorem on hyperplane sections, if we choose for X_r a smooth complete intersection of $n - r$ ample hypersurfaces in X, and denote by $j_r : X_r \to X$ the inclusion, then the Gysin map $j_{r*} : H_B^{k-2n+2r}(X_r, \mathbb{Q}) \to H_B^k(X, \mathbb{Q})$ is surjective for $k \geq 2n - r$. Arguing as before, we conclude that under assumption (ii), we may assume the cycles Z_i for $i \geq r$ are supported on $X \times X_r$. In conclusion, assuming (i) and (ii), we find a decomposition of cycle classes,

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H_B^{2n}(X \times X, \mathbb{Q}),$$

where $Z_1 \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ is supported on $Y \times X$ for some proper closed algebraic subset $Y \subsetneqq X$ and $Z_2 \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ is supported on $X \times X_r$.

We then conclude using assumption (iii) that the cycle $\Delta_X - Z_1 - Z_2 \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$, being cohomologous to 0, is nilpotent, and as $Z_{1*} = 0$ on $\mathrm{CH}_0(X)_{\mathbb{Q}}$ and $\mathrm{Im}\, Z_{2*} : \mathrm{CH}_0(X)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}$ is supported on X_r, this implies that $j_{r*} : \mathrm{CH}_0(X_r)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}$ is surjective. $\square$

Chapter Four

Chow groups of large coniveau complete intersections

Our goal in this chapter, which is mainly based on [107] and [114], is to investigate the generalized Bloch conjecture for hypersurfaces or complete intersections in either projective space, or more generally, varieties with trivial Chow groups. We will first recall, following Griffiths [48], how the Hodge coniveau of such complete intersections is computed. We then turn to the study of coniveau 2 *complete intersections in projective space, for which neither the generalized Hodge conjecture nor the generalized Bloch conjecture are known to hold. We will sketch a strategy to attack the generalized Hodge conjecture for coniveau* 2*, which does not use Theorem 3.20, hence does not pass through the computations of the Chow groups* CH_0 *and* CH_1*. The main result proved in this chapter is Theorem 4.16, which says that, assuming Conjecture 2.29 (or the Lefschetz standard conjecture), for a very general complete intersection* Y *of very ample hypersurfaces in a variety with trivial Chow groups, if the cohomology of* Y *has geometric coniveau* $\geq c$*, then its Chow groups* $\mathrm{CH}_i(Y)_{\mathbb{Q}}$ *are trivial for* $i \leq c-1$*. In particular, proving the generalized Hodge conjecture for such a* Y *is equivalent to proving the generalized Bloch conjecture.*

4.1 HODGE CONIVEAU OF COMPLETE INTERSECTIONS

Consider a smooth complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degree $d_1 \leq \cdots \leq d_r$. By the Lefschetz hyperplane section theorem (see [101, II, 1.2.2]), the only interesting Hodge structure in the cohomology of X is the Hodge structure on $H_B^{n-r}(X,\mathbb{Q})$, and in fact on the primitive part of it (that is, the orthogonal of the restriction of $H_B^*(\mathbb{P}^n,\mathbb{Q})$ with respect to the intersection pairing). We will say that X has Hodge coniveau c if the Hodge structure on $H_B^{n-r}(X,\mathbb{Q})_{\mathrm{prim}}$ has coniveau c.

The Hodge coniveau of a complete intersection in projective space is computed as follows (see [48] for the case of hypersurfaces, [38] for the complete intersection case).

THEOREM 4.1. *X has coniveau $\geq c$ if and only if*

$$n \geq \sum_i^r d_i + (c-1)d_r. \tag{4.1}$$

Conjecture 3.21 thus predicts the following statement for complete intersections in projective space.

CONJECTURE 4.2. *Let X be a complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degree $d_1 \leq \cdots \leq d_r$. Then if $n \geq \sum_i d_i + (c-1)d_r$, the cycle class map* cl *is injective on* $\mathrm{CH}_i(X)_\mathbb{Q}$ *for $i \leq c-1$.*

The generalized Hodge conjecture (Conjecture 2.40) on the other hand predicts the following.

CONJECTURE 4.3. *Let X be a complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degree $d_1 \leq \cdots \leq d_r$. Then if $n \geq \sum_i d_i + (c-1)d_r$, the primitive cohomology $H_B^{n-r}(X, \mathbb{Q})_{\mathrm{prim}}$ is supported on a closed algebraic subset of codimension $\geq c$.*

In the next subsection, we will recall the proof of Theorem 4.1. The rest of the chapter will be devoted to the study of Conjectures 4.2 and 4.3. Note that according to Theorem 3.20, Conjecture 4.2 implies Conjecture 4.3.

Our main result in this chapter is taken from [114]. It says conversely that, assuming Conjecture 2.29, if a very general complete intersection X as above satisfies Conjecture 4.3, then it satisfies Conjecture 4.2.

Let us conclude this section by exhibiting for every d, following [100], smooth hypersurfaces X of degree d in $\mathbb{P}^n$ satisfying the conclusion of Conjecture 4.2. The examples constructed there are as follows: We write $n = cd + s$ and choose homogeneous degree d polynomials

$$f_1 \in \mathbb{C}[X_0, \ldots, X_d], \quad f_2, \ldots, f_{c-1} \in \mathbb{C}[Y_1, \ldots, Y_d], \quad f_c \in \mathbb{C}[Y_1, \ldots, Y_{d+s}].$$

We set

$$\begin{aligned} f =& f_1(X_0, \ldots, X_d) + f_2(X_{d+1}, \ldots, X_{2d}) + \cdots + f_{c-1}(X_{(c-2)d+1}, \ldots, X_{(c-1)d}) \\ &+ f_c(X_{(c-1)d+1}, \ldots, X_{cd+s}). \end{aligned} \tag{4.2}$$

The hypersurface X defined by f is smooth if and only if the hypersurfaces defined by the f_i's are smooth. The following is proved in [100].

THEOREM 4.4 (Voisin 1996). *Let X be as above and smooth. Then the cycle class map* cl *is injective on* $\mathrm{CH}_i(X)_\mathbb{Q}$ *for $i \leq c-1$.*

The proof that we will give below is closer to [37], [79]. It is in fact a consequence of the following result proved in [37].

THEOREM 4.5 (Esnault, Levine, and Viehweg 1997). *Let X be a smooth hypersurface of $\mathbb{P}^n$, covered by a family of projective spaces $\mathbb{P}^r$. Then the groups $\mathrm{CH}_l(X)_{\mathrm{hom}}$ are torsion for $l \leq r-1$, and $\mathrm{CH}_r(X)_\mathbb{Q}$ is generated as a group by the classes of the $\mathbb{P}^r$'s in the family.*

PROOF. The proof first uses the following observation.

LEMMA 4.6. *Let $h = c_1(\mathcal{O}_X(1))$. Then the map*

$$d\,h : \mathrm{CH}_l(X)_{\mathrm{hom}} \to \mathrm{CH}_{l-1}(X)_{\mathrm{hom}},$$

$$z \mapsto d\,z \cdot h$$

is equal to 0.

PROOF. Indeed, by the definition of the map j^*, when j is the inclusion of a Cartier divisor, and by the fact that the class of X in $\mathrm{Pic}(\mathbb{P}^n)$ is equal to dh, we find that

$$j^* \circ j_* = d\,h : \mathrm{CH}_l(X)_{\mathrm{hom}} \to \mathrm{CH}_{l-1}(X)_{\mathrm{hom}}.$$

But as $\mathrm{CH}_*(\mathbb{P}^n)_{\mathrm{hom}} = 0$, the map

$$j_* : \mathrm{CH}_l(X)_{\mathrm{hom}} \to \mathrm{CH}_l(\mathbb{P}^n)_{\mathrm{hom}}$$

is 0. $\square$

Now let F be the considered family of r-planes contained in X. Up to replacing F by a desingularization, we may assume that F is smooth.

Let $Q = \{(x, P) \in X \times F \mid x \in P\}$ be the incidence variety, and p, q the two projections:

$$\begin{array}{ccc} Q & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F. & & \end{array}$$

By hypothesis, q is surjective, and up to replacing F by a subvariety, we may assume that q is generically finite of degree $N > 0$. Then we know that the image of the map

$$q_* : \mathrm{CH}_l(Q)_{\mathrm{hom}} \to \mathrm{CH}_l(X)_{\mathrm{hom}}$$

contains $N\,\mathrm{CH}_l(X)_{\mathrm{hom}}$, since $q_* \circ q^* = N \cdot \mathrm{Id}$ on $\mathrm{CH}_l(X)_{\mathrm{hom}}$.

Moreover, the projection p makes Q into a $\mathbb{P}^r$-bundle on F, and $q^*(\mathcal{O}_X(1))$ is a divisor $\mathcal{O}_Q(1)$ on this projective bundle $Q \to F$. Writing $H = c_1(q^*(\mathcal{O}_X(1))) \in \mathrm{CH}^1(Q)$, we can thus apply the computation of Chow groups of a projective bundle (see [43, 3.3], [101, II, 9.3.2]), which yields

$$\mathrm{CH}_l(Q) = \oplus_{0 \le k \le r, l-r+k \ge 0} H^k p^* \,\mathrm{CH}_{l-r+k}(F).$$

By the fact that we also have the corresponding decomposition on the cohomology

$$H^m_B(Q, \mathbb{Z}) = \oplus_{0 \le k \le r, 2k \le m} [H]^k p^* H_B^{m-2k}(F, \mathbb{Z}), \quad [H] \in H^2_B(Q, \mathbb{Z}),$$

for every m (see [101, I, 7.3.3]), we immediately deduce the decomposition

$$\mathrm{CH}_l(Q)_{\mathrm{hom}} = \oplus_{0 \le k \le r, l-r+k \ge 0} H^k p^* \,\mathrm{CH}_{l-r+k}(F)_{\mathrm{hom}}.$$

In particular, for $l < r$, the map

$$H : \mathrm{CH}_{l+1}(Q)_{\mathrm{hom}} \to \mathrm{CH}_l(Q)_{\mathrm{hom}}$$

is surjective.

Thus, for $Z \in \mathrm{CH}_l(X)_{\mathrm{hom}}$, with $l < r$, there exist $Z' \in \mathrm{CH}_l(Q)_{\mathrm{hom}}$, $Z'' \in \mathrm{CH}_{l+1}(Q)_{\mathrm{hom}}$ such that

$$NZ = q_*Z' \in \mathrm{CH}_l(X), \quad Z' = HZ'' \in \mathrm{CH}_l(Q). \tag{4.3}$$

Now, if $Z'' \in \mathrm{CH}_{l+1}(Q)$, then by the projection formula, we have

$$q_*(H \cdot Z'') = q_*(q^*h \cdot Z'') = h \cdot q_*Z'' \text{ in } \mathrm{CH}_l(X), \tag{4.4}$$

where $h := c_1(\mathcal{O}_X(1)) \in \mathrm{CH}^1(X)$. Finally, we obtain from (4.3) and (4.4),

$$dN\,Z = d\,h \cdot q_*Z'',$$

which is equal to 0 by Lemma 4.6.

For $l = r$, the above reasoning shows that the map

$$\mathrm{CH}_0(F)_{\mathrm{hom},\mathbb{Q}} \oplus \mathrm{CH}_{r+1}(X)_{\mathrm{hom},\mathbb{Q}} \to \mathrm{CH}_r(X)_{\mathrm{hom},\mathbb{Q}},$$

$$(z, Z) \mapsto q_*p^*z + h \cdot Z$$

is surjective. Applying Lemma 4.6, we conclude that

$$\mathrm{CH}_0(F)_{\mathrm{hom},\mathbb{Q}} \to \mathrm{CH}_r(X)_{\mathrm{hom},\mathbb{Q}}$$

is surjective, as desired. □

Proof of Theorem 4.4. Using Theorem 4.5, the proof finishes by observing that a hypersurface X as above is covered by a family of linear spaces $L \cong \mathbb{P}^{c-1}$ which have the property that there is a linear space $L' \subset \mathbb{P}^n$ with

$$L' \cap X = dL \tag{4.5}$$

(or $L' \subset L \subset X$). Having this, we know on the one hand by Theorem 4.5 that the classes of these L's generate $\mathrm{CH}_{c-1}(X)_{\mathbb{Q}}$ and on the other hand, by (4.5), their classes are all equal to $\frac{h^{n-c}}{d}$ in $\mathrm{CH}_{c-1}(X)_{\mathbb{Q}}$. Thus $\mathrm{CH}_{c-1}(X)_{\mathbb{Q}} = \mathbb{Q}$ and $\mathrm{CH}_{c-1}(X)_{\mathrm{hom},\mathbb{Q}} = 0$.

Let us exhibit these spaces L'. We will do it for $c = 2$, so $n = 2d + s$, $s \geq 0$, and we have to exhibit lines L in X, sweeping out X, and having the property that for some plane $P \subset \mathbb{P}^n$, $P \cap X = dL$ or $L \subset P \subset X$. Let $P_1 \cong \mathbb{P}^d$ and $P_2 \cong \mathbb{P}^{d-1+s} \subset \mathbb{P}^n$, $n = 2d + s$ be the linear subspaces of $\mathbb{P}^n$ defined by $X_i = 0$, $i > d$ and $X_i = 0$, $i \leq d$, respectively. Let $x \in X$, which we write as (x_1, x_c) according to the decomposition of the set of coordinates appearing in (4.2). We may assume that both x_i's are nonzero vectors. The hypersurface $X_1 \subset P_1$ defined by f_1 is Fano, and it is a result due to Roitman [88] that for any $x_1 \in X_1$ there exists a line Δ in $\mathbb{P}^d$ passing through x_1 and such that $f_{1|\Delta} = U^d$, where U is a linear equation defining x_1 in Δ. The line Δ contains the point x_1. Let $P_2' := \langle x_1, P_2 \rangle \cong \mathbb{P}^{d+s} \subset \mathbb{P}^n$. The point x belongs to P_2' and again by Roitman's result, there is a line $\Delta' \subset P_2'$ passing through x such that

$f_{|P'_2} = V^d$. Let $P := \langle \Delta, \Delta' \rangle$. Then $P \cong \mathbb{P}^3$, $x \in P$ and one checks that the restriction of f to P is of the form $V^d + W^d - U^d$, for linear forms V, W, U on P vanishing at x. Restricting to the plane $P' \subset P$ defined by $W - U = 0$, we find that $f_{|P'} = V^d$ for some linear form on P' vanishing at x. The line $L \subset P'$ defined by $V = 0$ thus satisfies the desired conclusion. □

4.1.1 Proof of Theorem 4.1 in the case of hypersurfaces

We prove Theorem 4.1 using Griffiths' method comparing the pole order and Hodge filtration (see [48] or [101], II, 6.1.2). Let W be a projective variety, and $X \overset{l}{\hookrightarrow} W$ a smooth hypersurface. In the following, all varieties are endowed with the classical topology and analytic structural sheaves, so the notation Ω_W below is what we denoted $\Omega_{W_{\mathrm{an}}}$ in Section 2.2. Set

$$U = W - X \overset{j}{\hookrightarrow} W.$$

Recall (see [101, I, 8.2.2]) that the logarithmic de Rham complex $(\Omega^\bullet_W(\log X), d)$ is the complex of free $\mathcal{O}_X$-modules defined as

$$\Omega^k_W(\log X) = \bigwedge^k \Omega_X(\log X),$$

where $\Omega_W(\log X)$ is the sheaf of free $\mathcal{O}_W$-modules generated locally by Ω_W and by $\frac{df}{f}$, where f is a local holomorphic equation for X. The differential is the exterior differential. This complex can be viewed as a subcomplex of the complex $j_*\mathcal{A}^\bullet_U$, and we have the following result (see [101, I, 8.2.3]).

THEOREM 4.7. *The inclusion*

$$\Omega^\bullet_W(\log X) \hookrightarrow j_*\mathcal{A}^\bullet_U,$$

where $\mathcal{A}^k_U$ is the sheaf of $\mathcal{C}^\infty$ complex differential k-forms on U, is a quasi isomorphism.

We have two morphisms of complexes: the inclusion

$$\Omega^\bullet_W \hookrightarrow \Omega^\bullet_W(\log X),$$

which in cohomology induces the restriction

$$H^k_B(W, \mathbb{C}) \to H^k_B(U, \mathbb{C}), \tag{4.6}$$

and the residue

$$\mathrm{Res} : \Omega^\bullet_W(\log X) \to \Omega^{\bullet-1}_X,$$

which to $\alpha \wedge \frac{df}{f}$ associates $\mathrm{Res}(\alpha \wedge \frac{df}{f}) = 2i\pi\alpha_{|X}$. The map induced by Res in cohomology is the topological residue

$$H^k_B(U, \mathbb{C}) \to H^{k-1}_B(X, \mathbb{C}), \tag{4.7}$$

which we can also define as the composition

$$H_B^k(U,\mathbb{C}) \to H_B^{k+1}(W,U,\mathbb{C}) \cong H^{k+1}(T,\partial T) \cong H_B^{k-1}(X,\mathbb{C}),$$

where T is a tubular neighborhood of X in W, the first arrow is the connection morphism in the long exact sequence of relative cohomology, the second is the excision isomorphism, and the last is the Thom isomorphism.

The morphisms (4.6) and (4.7) are defined over $\mathbb{Z}$ and are compatible with the Hodge filtrations (see Section 2.2), because the Hodge filtration on $H_B^*(U,\mathbb{C})$ and $H_B^*(X,\mathbb{C})$ is induced by the naive filtration on the complexes $\Omega_W^\bullet(\log X)$ and $\Omega_X^\bullet$, respectively. (More precisely, the morphism Res sends $F^pH_B^k(U,\mathbb{C})$ to $F^{p-1}H_B^{k-1}(X,\mathbb{C})$.)

Let us now assume that $X \subset W$ is ample and that the pair satisfies the following condition:

$$\text{for every } k>0,\ i>0,\ j\geq 0, \text{ we have } H^i(W,\Omega_W^j(kX))=0. \tag{4.8}$$

These hypotheses are satisfied by Bott's vanishing theorem if W is the projective space $\mathbb{P}^n$. Under these hypotheses, we have the following result.

THEOREM 4.8 (Griffiths 1969). *For every integer p such that $1 \leq p \leq n = \dim W$, the image of the natural map*

$$H^0(W,K_W(pX)) \to H_B^n(U,\mathbb{C}), \tag{4.9}$$

which to a section α (viewed as a meromorphic form on W of degree n, and therefore closed, holomorphic on U, and having a pole of order p along X) associates its de Rham cohomology class, is equal to $F^{n-p+1}H_B^n(U,\mathbb{C})$.

We refer to [48], [101, II, 6.1.2] for the proof of this result.

Now assume that W is the projective space $\mathbb{P}^n$, and that X is a smooth hypersurface of degree d, with equation $f=0$. We know that $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, a generator of $H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(n+1))$ being given by

$$\begin{aligned}\Omega &= \sum_i (-1)^i X_i dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n \\ &= X_0 \cdots X_n \sum_i (-1)^i \frac{dX_0}{X_0} \wedge \cdots \wedge \widehat{\frac{dX_i}{X_i}} \wedge \cdots \wedge \frac{dX_n}{X_n},\end{aligned}$$

where the X_i's are homogeneous coordinates on $\mathbb{P}^n$. As $\mathcal{O}_{\mathbb{P}^n}(X) = \mathcal{O}_{\mathbb{P}^n}(d)$, Theorem 4.8 shows that for every p such that $1 \leq p \leq n$, we have a surjective map

$$\begin{aligned}\alpha_p : H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(pd-n-1\right)\right) \to & F^{n-p+1}H_B^n(U,\mathbb{C}) \\ \cong & F^{n-p}H_B^{n-1}(X,\mathbb{C})_{\text{prim}},\end{aligned}$$

which to a polynomial P associates the residue of the class of the meromorphic form $\frac{P\Omega}{f^p}$.

In particular, if $n \geq dc$, we find that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(pd - n - 1)) = 0$ for $p \leq c$, hence that $F^{n-c}H_B^{n-1}(X, \mathbb{C})_{\text{prim}} = 0$. This proves the "if" part in Theorem 4.1 for hypersurfaces, because the vanishing of $F^{n-c}H_B^{n-1}(X, \mathbb{C})_{\text{prim}}$ is equivalent to the fact that the coniveau of the weight $(n-1)$ Hodge structure on $H_B^{n-1}(X, \mathbb{C})_{\text{prim}}$ is $\geq c$. We refer to [101, II, 6.1.3] for the "only if" part.

4.1.2 Complete intersections

In order to get Theorem 4.1 for complete intersections, we reduce to the case of hypersurfaces by the following trick due to Terasoma (see [38], [94]).

Let $L_1, \ldots, L_r$ be ample hypersurfaces on W, with W smooth projective of dimension n. Let $\mathcal{E} := L_1 \oplus \cdots \oplus L_r$ and $W_E := \mathbb{P}(\mathcal{E})$, which is equipped with the line bundle

$$\mathcal{O}_{W_E}(1) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$$

satisfying the property that $\pi_* \mathcal{O}_{W_E}(1) = \mathcal{E}$, where $\pi : W_E \to W$ is the structural map (so we follow the Grothendieck convention here). Given hypersurfaces $X_i \in |L_i|$ defined by equations $f_i \in H^0(W, L_i)$, we get a section $\tau := (f_1, \ldots, f_r)$ of $\mathcal{E}$. Identifying $H^0(W, \mathcal{E})$ to $H^0(W_E, \mathcal{O}_{W_E}(1))$, we thus get a hypersurface

$$X_E \subset W_E, \quad X_E \in |\mathcal{O}_{W_E}(1)|,$$

defined by an equation $f \in H^0(W_E, \mathcal{O}_{W_E}(1))$. The hypersurface X_E is ample since the vector bundle $\mathcal{E}$ is ample on W. Furthermore X_E is smooth if and only if $X := \cap_i X_i$ is a smooth complete intersection.

The canonical bundle of W_E is computed by the formula (which is a consequence of the relative Euler exact sequence)

$$K_{W_E} = \mathcal{O}_{W_E}(-r) + \pi^* \det \mathcal{E} + \pi^* K_W.$$

We have $\det \mathcal{E} = \otimes_i L_i$.

When $W = \mathbb{P}^n$, which we assume from now on, the line bundles L_i are of the form $\mathcal{O}(d_i)$, $d_i > 0$ and the vanishing conditions (4.8) are satisfied by the line bundle $\mathcal{O}_{W_E}(1)$, as a consequence of Bott's vanishing theorem on $\mathbb{P}^n$.

Let Ω be a trivializing section of the line bundle $K_{W_E} \otimes \mathcal{O}_{W_E}(r) \otimes \pi^* \mathcal{O}(-\sum_i d_i + n + 1)$. Theorem 4.8 shows that for every p such that $1 \leq p \leq n + r - 1$, we have a surjective map

$$\begin{aligned}\alpha_p : \ & H^0\Big(W_E, \mathcal{O}_{W_E}(p - r) \otimes \pi^* \mathcal{O}\Big(\sum_i d_i - n - 1\Big)\Big) \\ & \to F^{n+r-p} H_B^{n+r-1}(U, \mathbb{C}) \cong F^{n+r-2} H_B^{n+r-1}(X_E, \mathbb{C})_{\text{van}},\end{aligned}$$

which to a section P associates the residue of the class of the meromorphic form $\frac{P\Omega}{f^p}$.

We thus deduce that for the hypersurface X_E we have

$$F^{n+r-p-1} H_B^{n+r-2}(X_E, \mathbb{C})_{\text{van}} = 0 \tag{4.10}$$

if $H^0(W_E, \mathcal{O}_{W_E}(p-r) \otimes \pi^*\mathcal{O}(\sum_i d_i - n - 1)) = 0$.

We now compute

$$\begin{aligned} &H^0\left(W_E, \mathcal{O}_{W_E}(p-r) \otimes \pi^*\mathcal{O}\left(\sum_i d_i - n - 1\right)\right) \\ &= H^0\left(\mathbb{P}^n, \left(S^{p-r}(\oplus_i \mathcal{O}_{\mathbb{P}^n}(d_i))\right) \otimes \mathcal{O}\left(\sum_i d_i - n - 1\right)\right), \end{aligned} \tag{4.11}$$

where S^i denotes the ith symmetric power of the considered vector bundle. Combining (4.10) and (4.11) we conclude that

$$F^{n+r-p-1}H_B^{n+r-2}(X_E, \mathbb{C})_{\text{van}} = 0 \text{ if } \sum_i d_i + (p-r)\operatorname{Sup}\{d_i\} - n - 1 < 0.$$

To conclude the proof of Theorem 4.1 (or rather of the "if" part) for complete intersections, it just remains to show the following.

Proposition 4.9 (Esnault, Nori, and Srinivas 1992; Terasoma 1990). *The hypersurface $X_E \subset W_E$ contains the $\mathbb{P}^{r-1}$-bundle $\mathbb{P}(\mathcal{E}_{|X})$. Let*

$$k : \mathbb{P}(\mathcal{E}_{|X}) \hookrightarrow X_E, \qquad \pi_0 : \mathbb{P}(\mathcal{E}_{|X}) \to X$$

be the natural maps. Then the morphism

$$k_* \circ \pi_0^* : H_B^{n-r}(X, \mathbb{Q})_{\text{prim}} \to H_B^{n+r-2}(X_E, \mathbb{Q})_{\text{van}}$$

is an isomorphism of Hodge structures of bidegree $(r-1, r-1)$.

Proof. The morphism above is obviously a morphism of Hodge structures. It suffices thus to prove that it induces an isomorphism between these groups. It is not hard to see that it sends the primitive part $H_B^{n-r}(X, \mathbb{Q})_{\text{prim}}$ to the vanishing part $H_B^{n+r-2}(X_E, \mathbb{Q})_{\text{van}}$. To see that it induces an isomorphism between both, we introduce the open set $V := W \setminus X$ and observe that $\pi : U = W_E \setminus X_E \to V = W \setminus X$ is an affine bundle (with fiber $\mathbb{P}^{r-1} \setminus \{f_w = 0\}$ over $w \in W \setminus X$). It follows that π^* induces an isomorphism

$$\pi^* : H_B^n(V, \mathbb{Q}) \cong H_B^n(U, \mathbb{Q}).$$

Let us denote by j the inclusion of X into W, and by j_E the inclusion of X_E into W_E. The result is then obtained by comparing the relative exact sequences of the pairs (W_E, U) and (W, V); we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} H_B^*(W, \mathbb{Z}) & \longrightarrow & H_B^*(V, \mathbb{Z}) & \longrightarrow & H_B^{*-2r+1}(X, \mathbb{Z}) & \xrightarrow{j_*} & H_B^{*+1}(W, \mathbb{Z}) \dots \\ \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow \\ H_B^*(W_E, \mathbb{Z}) & \longrightarrow & H_B^*(U, \mathbb{Z}) & \longrightarrow & H_B^{*-1}(X_E, \mathbb{Z}) & \xrightarrow{j_{E*}} & H_B^{*+1}(W_E, \mathbb{Z}) \dots . \end{array} \tag{4.12}$$

Here we identified by the Thom isomorphism the relative cohomology group $H_B^{*+1}(W_E, U, \mathbb{Z})$ to $H_B^{*-1}(X_E, \mathbb{Z})$ and the relative cohomology $H_B^{*+1}(W, V, \mathbb{Z})$ to $H_B^{*-2r+1}(X, \mathbb{Z})$. The vertical maps are pull-back maps π^*, except for the third one, which can be shown to be $k_* \circ \pi_0^*$.

As we already noted, the second pull-back map π^* is an isomorphism in all degrees. Furthermore, by definition of vanishing cohomology, the kernels of the maps j_* and j_{E*} are $H_B^{*-2r+1}(X, \mathbb{Z})_{\text{van}}$ and $H_B^{*-2r+1}(X_E, \mathbb{Z})_{\text{van}}$, respectively. Thus our diagram becomes the following exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_B^*(W,\mathbb{Z})/\operatorname{Im} j_* & \longrightarrow & H_B^*(V,\mathbb{Z}) & \longrightarrow & H_B^{*-2r+1}(X,\mathbb{Z})_{\text{van}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & H_B^*(W_E,\mathbb{Z})/\operatorname{Im} j_{E*} & \longrightarrow & H_B^*(U,\mathbb{Z}) & \longrightarrow & H_B^{*-1}(X_E,\mathbb{Z})_{\text{van}} & \longrightarrow & 0. \end{array} \tag{4.13}$$

We now consider what happens for $* = n + r - 1$. Then we claim that the two groups $H_B^{n+r-1}(W, \mathbb{Z})/\operatorname{Im} j_*$ and $H_B^{n+r-1}(W_E, \mathbb{Z})/\operatorname{Im} j_{E*}$ are naturally isomorphic via the map π^*. Indeed, we recall that X_E is a hypersurface in the ample linear system $|\mathcal{O}_{W_E}(1)|$. Using the Lefschetz theorem on hyperplane sections (see [101, II, 1.2.2]), it follows that

$$\operatorname{Im} j_{E*} : H_B^{n+r-3}(X_E, \mathbb{Z}) \to H_B^{n+r-1}(W_E, \mathbb{Z})$$

is equal to

$$\operatorname{Im} l\cup : H_B^{n+r-3}(W_E, \mathbb{Z}) \to H_B^{n+r-1}(W_E, \mathbb{Z}),$$

where $l := c_1(\mathcal{O}_{W_E}(1))$. We now use the fact that W_E is a $\mathbb{P}^{r-1}$-bundle over W, so that its cohomology is described as

$$H_B^*(W_E, \mathbb{Z}) = \oplus_{i \leq r-1} l^i \cup \pi^* H_B^{*-2i}(W, \mathbb{Z})$$

so that

$$H_B^*(W_E, \mathbb{Z})/\operatorname{Im} l\cup \cong H_B^*(W, \mathbb{Z})/\operatorname{Im} c_r\cup,$$

where c is the constant term of the Chern polynomial satisfied by l:

$$l^r = -\sum_{i \leq r-1} (-1)^{r-i} l^i \cup \pi^* c_{r-i}(E).$$

On the other hand, the class of X in W is equal to $c_r(E)$. It thus follows that

$$\operatorname{Im} c_r\cup : H_B^{n-r-1}(W, \mathbb{Z}) \to H_B^{n+r-1}(W, \mathbb{Z}) \tag{4.14}$$

is contained in

$$\operatorname{Im} j_* : H_B^{n-r-1}(X, \mathbb{Z}) \to H_B^{n+r-1}(W, \mathbb{Z}). \tag{4.15}$$

If we now recall that E is a direct sum of ample line bundles on W, we have that X is a complete intersection of ample hypersurfaces in W, hence the restriction map

$$j^* : H_B^{n-r-1}(W, \mathbb{Z}) \to H_B^{n-r-1}(X, \mathbb{Z})$$

is surjective by the Lefschetz theorem on hyperplane sections and (4.15) is in fact equal to (4.14), which concludes the proof of the claim.

We then conclude from diagram (4.13) and the last claim that

$$H_B^{n-r}(X, \mathbb{Z})_{\text{van}} \overset{k_* \circ \pi_0^*}{\cong} H_B^{n+r-2}(X_E, \mathbb{Z})_{\text{van}}.$$

□

4.1.3 Coniveau 1 and further examples

For $c = 1$, the estimate in Theorem 4.1 is obvious, as coniveau$(X) \geq 1$ is equivalent to $H^{n-r,0}(X) = H^0(X, K_X) = 0$, that is, X is a Fano complete intersection. In this case, the generalized Hodge–Grothendieck conjecture (Conjecture 2.40) for the coniveau 1 Hodge structure on $H^{n-r}(X, \mathbb{Q})_{\text{prim}}$ is known to be true. Let us give two proofs of this.

First of all, we can do it explicitly using the correspondence between X and its Fano variety of lines F. Denoting by

$$\begin{array}{ccc} P & \stackrel{q}{\to} & X \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array}$$

the incidence correspondence, where p is the tautological $\mathbb{P}^1$-bundle on F, one can show (see, for example, [92]) that taking an $(n-r-2)$-dimensional complete intersection $F_{n-r-2} \subset F$ and restricting P to it, the resulting morphism of Hodge structures

$$q'_* \circ p'^* : H_B^{n-r-2}(F_{n-r-2}, \mathbb{Q}) \to H_B^{n-r}(X, \mathbb{Q})$$

is surjective, where $P' = p^{-1}(F_{n-r-2})$ and p', q' are the restrictions of p, q to P'. It follows that $H_B^{n-r}(X, \mathbb{Q})$ vanishes on the complement of the (singular) hypersurface $q(P') \subset X$.

The second proof is by a direct application of Theorem 3.16. Indeed, Fano varieties are rationally connected (see [65]), hence have trivial CH_0 group. Theorem 3.16 then says that their cohomology of positive degree is supported on a proper closed algebraic subset.

4.2 CONIVEAU 2 COMPLETE INTERSECTIONS

4.2.1 A conjecture on effective cones

Let Y be smooth projective; for any integer k, denote by $H_B^{2k}(Y, \mathbb{R})_{\text{alg}}$ the subspace of $H_B^{2k}(Y, \mathbb{R})$ generated over $\mathbb{R}$ by cycle classes and by $E^{2k}(Y) \subset$

$H_B^{2k}(Y,\mathbb{R})_{\rm alg}$ the effective cone, that is, the convex cone generated by classes of subvarieties $Z \subset Y$ of codimension k.

Cones of effective cycles have been very much studied in codimension 1 or in dimension 1 (see [16]), but essentially nothing is known in intermediate (co)dimensions. Let us say that an algebraic cohomology class is big if it belongs to the interior of the effective cone. Note the following easy lemma (see [107]).

LEMMA 4.10. *Let $h \in H^2(Y,\mathbb{R})$ be the first Chern class of an ample line bundle on Y. A class $\alpha \in H_B^{2k}(Y,\mathbb{R})_{\rm alg}$ is big if and only if, for some $\epsilon > 0$, $\alpha - \epsilon h^k \in E^{2k}(Y)$.*

PROOF. Indeed, the "only if" is obvious, because if α is in the interior of $E^{2k}(Y)$, a small deformation $\alpha - \epsilon h^k$ also belongs to $E^{2k}(Y)$. In the other direction, it suffices to prove that h^k belongs to the interior of the effective cone. This is true because for any variety $Z \subset Y$ of codimension k, there is an integer N_Z such that $N_Z h^k - Z$ is effective. Thus both $h^k - \frac{1}{N_Z}[Z]$ and $h^k + \frac{1}{N_Z}[Z]$ belong to the cone $E^{2k}(Y)$. As the classes $[Z]$ generate the vector space $H_B^{2k}(Y,\mathbb{R})_{\rm alg}$, we can choose a basis $[Z_i]$ of $H_B^{2k}(Y,\mathbb{R})_{\rm alg}$, and letting $\epsilon = {\rm Inf}\{\frac{1}{N_{Z_i}}\}$, we get that the open neighborhood

$$\left\{h^k + \sum_i \epsilon_i[Z_i],\ |\epsilon_i| < \epsilon\right\}$$

of h^k in $H_B^{2k}(Y,\mathbb{R})_{\rm alg}$ is contained in $E^{2k}(Y)$. The proof is finished. □

In [81], it is shown that when $\dim W = 1$, and $W \subset V$ is moving and has ample normal bundle, its class $[W]$ is big. In [107] we give an example that works in any dimension ≥ 4 and in codimension 2, showing that in higher dimensions, a moving variety $W \subset V$ with ample normal bundle may not have a big class. Here, by *moving* we mean that a generic deformation of W in V may be imposed to pass through a generic point of V.

DEFINITION 4.11. A smooth k-dimensional subvariety $V \subset Y$, is *very moving* if it has the following property: through a general point $y \in Y$, and given a general vector subspace $W \subset T_{Y,y}$ of rank k, there is a deformation $V' \subset Y$ of V in Y that is smooth and passes through y with tangent space equal to W at y.

We make the following conjecture for very moving subvarieties.

CONJECTURE 4.12. *Let Y be smooth and projective and let $V \subset Y$ be a very moving subvariety. Then the class $[V]$ of V in Y is big.*

4.2.2 On the generalized Hodge conjecture for coniveau 2 complete intersections

Let $X \subset \mathbb{P}^n$ be a generic complete intersection of multidegree $d_1 \leq \cdots \leq d_r$. So X has dimension $(n-r)$ and the interesting cohomology of X is supported in degree $(n-r)$. Let F be the variety of lines contained in X. Then by genericity, F is smooth of dimension $(2n-2-\sum_i d_i - r)$. Note that if $n \geq \sum_i d_i$, the morphism

$$P^* : H_B^{n-r}(X,\mathbb{Q}) \to H_B^{n-r-2}(F,\mathbb{Q})$$

induced by the universal $\mathbb{P}^1$-bundle $P \subset F \times X$ is injective (see Section 4.1.3).

For a generic section $G \in H^0(X, \mathcal{O}_X(n - \sum_i d_i - 1))$ with zero set $X_G \subset X$, consider the subvariety $F_G \subset F$ of the variety of lines contained in X_G. Again by genericity, F_G is smooth of dimension $(n-r-2)$. The following result is obtained in [107] by studying the deformations of F_G in F induced by deformations of G.

Proposition 4.13. *When $n \geq \sum_i d_i + d_r$, that is, when X has Hodge coniveau ≥ 2, the subvariety $F_G \subset F$ is very moving in the sense of Definition 4.11.*

Following [107], let us now deduce from this proposition the following result.

Theorem 4.14 (Voisin 2010). *If the very moving varieties F_G satisfy Conjecture 4.12, that is, $[F_G]$ is big, then the generalized Hodge conjecture for coniveau 2 is satisfied by X.*

Proof. Assume that $[F_G]$ is big. Then it follows by Lemma 4.10 that for some positive large integer N and for some effective cycle E of codimension $(n - \sum_i d_i)$ on F, one has

$$N[F_G] = l^{n-\sum_i d_i} + [E], \tag{4.16}$$

where l is the restriction to F of the first Chern class of the Plücker line bundle on the Grassmannian $G(1,n)$. (Here we could work as well with real coefficients, but as we actually want to do geometry on E, it is better if E is a true cycle.) For $a \in H_B^{n-r}(X,\mathbb{Q})_{\text{prim}}$, let $\eta = p_* q^* a \in H_B^{n-r-2}(F,\mathbb{Q})$. We use the fact, essentially due to Shimada [92] (see also [107, Lemma 1.1]), that η is primitive with respect to l and furthermore vanishes on F_G, with $\dim F_G = n-r-2$.

Let us now assume that $a \in H^{p,q}(X)_{\text{prim}}$ and integrate $(-1)^{k(k-1)/2} i^{p-q} \eta \cup \overline{\eta}$, $k = p+q-2 = n-r-2$ over both sides in (4.16). We thus get

$$0 = \int_F (-1)^{k(k-1)/2} i^{p-q} l^{n-\sum_i d_i} \cup \eta \cup \overline{\eta} + \int_E (-1)^{k(k-1)/2} i^{p-q} \eta \cup \overline{\eta}.$$

As η is primitive with respect to l, and nonzero if a is nonzero, by the second Hodge–Riemann bilinear relations (see [101], I, 6.3.2), we have

$$\int_F (-1)^{k(k-1)/2} i^{p-q} l^{n-\sum_i d_i} \cup \eta \cup \overline{\eta} > 0.$$

It thus follows that

$$\int_E (-1)^{k(k-1)/2} i^{p-q} \eta \cup \overline{\eta} < 0. \tag{4.17}$$

Let $\widetilde{E} = \sqcup \widetilde{E_j}$ be a desingularization of the support of $E = \sum_j m_j E_j$, $m_j > 0$. Thus we have

$$\sum_j m_j \int_{\widetilde{E_j}} (-1)^{k(k-1)/2} i^{p-q} \eta \cup \overline{\eta} < 0.$$

It thus follows from (4.17) that there exists at least one E_j such that

$$\int_{\widetilde{E_j}} (-1)^{k(k-1)/2} i^{p-q} \eta \cup \overline{\eta} < 0. \tag{4.18}$$

Choose an ample divisor H_j on each $\widetilde{E_j}$. By the second Hodge–Riemann bilinear relations, inequality (4.18) implies that $\eta_{|\widetilde{E_j}}$ is not primitive with respect to the polarization given by H_j, that is, $\eta \cup [H_j] \neq 0$ and in particular,

$$\eta_{|H_j} \neq 0.$$

In conclusion, we have proved that the composed map

$$H_B^{n-r}(X, \mathbb{Q})_{\rm prim} \stackrel{p_* q^*}{\to} H_B^{n-r-2}(F, \mathbb{Q}) \to \bigoplus H_B^{n-r-2}(H_j, \mathbb{Q})$$

is injective, where the second map is given by pull-back to H_j via the composed map $H_j \hookrightarrow \widetilde{E_j} \to F$. If we dualize this, recalling that $\dim H_j = n - r - 3$, we conclude that

$$\bigoplus H_B^{n-r-4}(H_j, \mathbb{Q}) \to H_B^{n-r}(X, \mathbb{Q})_{\rm prim}$$

is surjective, where we consider the pull-backs of the incidence diagrams to H_j,

$$\begin{array}{ccc} P_j & \stackrel{q_j}{\to} & X \\ {\scriptstyle p_j}\downarrow & & \\ H_j & & \end{array}$$

and the map is the sum of the maps $q_{j*} p_j^*$, followed by orthogonal projection onto primitive cohomology. As for $n - r$ even and $n - r \geq 3$, the image of $\sum_j q_{j*} p_j^*$ also contains the class $h^{(n-r)/2}$ (up to adding to the H_j, if necessary, the class of a linear section of F), it follows immediately that the map

$$\sum_j q_{j*} p_j^* : \bigoplus H_B^{n-r-4}(H_j, \mathbb{Q}) \to H_B^{n-r}(X, \mathbb{Q})$$

is also surjective, if $n - r \geq 3$ (the case $n - r = 2$ is trivial). This implies that $H_B^{n-r}(X, \mathbb{Q})$ is supported on the $(n - r - 2)$-dimensional variety $\cup_j q_j(P_j)$, that is, it vanishes on $X \setminus \cup_j q_j(P_j)$. The result is proved. □

4.3 EQUIVALENCE OF GENERALIZED BLOCH AND HODGE CONJECTURES FOR GENERAL COMPLETE INTERSECTIONS

Our main result in this section concerns very general complete intersections X_b in a variety X with trivial Chow groups, with the following meaning.

Definition 4.15. A smooth complex algebraic variety (not necessarily projective) is said to have trivial Chow groups if it satisfies the property that the cycle class map

$$\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H_B^{2*}(X, \mathbb{Q})$$

is injective.

Smooth projective varieties with trivial Chow groups include all smooth toric varieties, for example, the projective space, and varieties admitting a stratification by affine spaces, like the Grassmannians.

For such X_b's we will prove the generalized Bloch conjecture as formulated in Remark 3.22. The statement is that if the transcendental cohomology of a smooth projective variety Y has *geometric* coniveau c, then the cycle class map is injective on cycles of Y of dimension $\leq c-1$. The generalized Bloch conjecture (Conjecture 3.21) states that if the cohomology of Y has *Hodge* coniveau c modulo classes of algebraic cycles, then the cycle class map is injective on cycles of Y of dimension $\leq c-1$. Thus the two statements differ by the generalized Hodge conjecture (Conjecture 2.40).

In Section 4.3.4 we will prove the following result (see [114]).

Theorem 4.16 (Voisin 2011). *Assume the "standard" conjecture (Conjecture 2.29) holds for degree $(2n-2r)$ cycle classes. Let X be a smooth complex projective variety with trivial Chow groups. Let $L_1, \ldots, L_r$, $r \leq n := \dim X$ be very ample line bundles on X. Assume that for a very general complete intersection $X_b = X_1 \cap \cdots \cap X_r$ of hypersurfaces $X_i \in |L_i|$, the Hodge structure on $H^{n-r}(X_b, \mathbb{Q})_{\mathrm{prim}}$ is supported on a closed algebraic subset $Y_b \subset X_b$ of codimension $\geq c$. Then for the general such X_b (hence in fact for all), the cycle map $\mathrm{cl} : \mathrm{CH}_i(X_b)_{\mathbb{Q}} \to H^{2n-2r-2i}(X_b, \mathbb{Q})$ is injective for any $i < c$.*

Remark 4.17. The conclusion would also apply in the more general situation where we have a very ample vector bundle E of rank r on X, and the X_b's are zero sets of sections of E. In fact, by Terasoma's trick described in Section 4.1.2, one can always reduce to the hypersurface case by working on the projective bundle $\mathbb{P}(E)$, so this generalization is almost immediate.

There are two cases where Conjecture 2.29 will be automatically satisfied, namely when the fibers X_b are either surfaces or threefolds. Furthermore, in the surface case, due to the Lefschetz theorem on $(1,1)$-classes, the geometric coniveau is equal to the Hodge coniveau. In the surface case, Theorem 4.16 proves the Bloch conjecture (Conjecture 3.21) for complete intersection surfaces inside a variety with trivial Chow groups and also a variant of it for complete intersection surfaces with group action (see Section 4.3.5.1).

4.3.1 Varieties with trivial Chow groups

We start by stating, mostly without proofs, a number of easy results concerning smooth quasi-projective complex varieties X with trivial Chow groups. The missing proofs can be found in [114].

LEMMA 4.18. *Let X be a smooth complex variety with trivial Chow groups. Then any projective bundle $p : \mathbb{P}(E) \to X$, where E is a locally free sheaf on X, has trivial Chow groups.*

LEMMA 4.19. *Let X be a smooth complex algebraic variety with trivial Chow groups and let $Y \subset X$ be a smooth closed subvariety with trivial Chow groups. Then the blow-up $\widetilde{X}_Y \to X$ of X along Y has trivial Chow groups.*

LEMMA 4.20. *Assume Conjecture 2.29. Let X be a smooth projective variety with trivial Chow groups. Then any Zariski open set $U \subset X$ has trivial Chow groups.*

PROOF. Write $U = X \setminus Y$. Let Z be a codimension k cycle on U with vanishing cohomology class. Then Z is the restriction to U of a cycle $\overline{Z}$ on X, which has the property that

$$[\overline{Z}]_{|U} = 0 \text{ in } H_B^*(U, \mathbb{Q}).$$

Conjecture 2.29 says that there is a cycle Z' supported on Y such that $[\overline{Z}] = [Z']$ in $H^{2k}(X, \mathbb{Q})$. The cycle $\overline{Z} - Z'$ is thus cohomologous to 0 on X. As X has trivial Chow groups, $\overline{Z} - Z'$ is rationally equivalent to 0 on X modulo torsion, and so is its restriction to U, which is equal to Z. □

Let us conclude with two more properties.

LEMMA 4.21. *Let $\phi : X \to X'$ be a projective surjective morphism, where X and X' are smooth complex algebraic varieties. If X has trivial Chow groups, so does X'.*

PROPOSITION 4.22. *Let X be a smooth projective variety with trivial Chow groups. Then $X \times X$ has trivial Chow groups.*

PROOF. This uses the fact (proved in Section 3.2.1) that a variety with trivial Chow groups has a complete decomposition of the diagonal as a combination of products of algebraic cycles (see [66], [80], [101, II, 10.3.1]):

$$\Delta_X = \sum_{i,j} n_{ij} Z_i \times Z_j \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}},$$

where $n_{ij} \in \mathbb{Q}$, and $\dim Z_i + \dim Z_j = n = \dim X$ when $n_{ij} \neq 0$. It follows that the variety $Z := X \times X$ also admits such a decomposition, since $\Delta_Z = p_{13}^* \Delta_X \cdot p_{24}^* \Delta_X$ in $\mathrm{CH}^{2n}(Z \times Z)$, where p_{ij} is the projection of $Z \times Z = X^4$ to the product $X \times X$ of the ith and jth summand.

But this in turn implies that the cycle class map $\mathrm{cl} : \mathrm{CH}^*(Z)_{\mathbb{Q}} \to H_B^{2*}(Z, \mathbb{Q})$ is injective. Indeed, write

$$\Delta_Z = \sum_{i,j} m_{ij} W_i \times W_j \text{ in } \mathrm{CH}^{2n}(Z \times Z).$$

Then any cycle $\gamma \in \mathrm{CH}(Z)_{\mathbb{Q}}$ satisfies

$$\gamma = \Delta_{Z*}\gamma = \sum_{i,j} m_{ij} \deg(\gamma \cdot W_i) W_j \text{ in } \mathrm{CH}(Z)_{\mathbb{Q}}.$$

It immediately follows that if γ is homologous to 0, it vanishes in $\mathrm{CH}(Z)_{\mathbb{Q}}$. $\square$

The same proof also shows the following result, as noted by Lie Fu (oral communication).

PROPOSITION 4.23. *Let X, Y be smooth projective varieties with trivial Chow groups. Then $X \times Y$ has trivial Chow groups.*

4.3.2 A consequence of Conjecture 2.29

We first observe in this section that assuming the Lefschetz standard conjecture, or more precisely Conjecture 2.29, the generalized Hodge conjecture for sub-Hodge structures on X can be rephrased by saying that a certain class on $X \times X$ is supported on a product $Y \times Y$. This will be quite important for the rest of the proof.

Let X be a smooth complex projective variety of dimension n, and let L be a sub-Hodge structure of $H^n_B(X, \mathbb{Q})_{\mathrm{prim}}$, where the subscript "prim" stands for "primitive with respect to a given polarization on X." We know then by the second Hodge–Riemann bilinear relations [101, I, 6.3.2] that the intersection form $\langle\, , \rangle$ restricted to L is nondegenerate. Let $\pi_L : H^n_B(X, \mathbb{Q}) \to L$ be the orthogonal projector onto L. We assume that π_L is algebraic, that is, there is an n-cycle $\Delta_L \subset X \times X$, such that

$$[\Delta_L]_* = \pi_L : H^n_B(X, \mathbb{Q}) \to L \subset H^n_B(X, \mathbb{Q}),$$

$$[\Delta_L]_* = 0 : H^i_B(X, \mathbb{Q}) \to H^i_B(X, \mathbb{Q}),\ i \neq n.$$

LEMMA 4.24. *Assume that there exists a closed algebraic subset $Y \subset X$ such that L vanishes under the restriction map $H^n_B(X, \mathbb{Q}) \to H^n_B(X \setminus Y, \mathbb{Q})$. Then if Conjecture 2.29 holds, there is a cycle Z'_L with $\mathbb{Q}$-coefficients supported on $Y \times Y$ such that*

$$[Z'_L] = [\Delta_L] \text{ in } H^{2n}_B(X \times X, \mathbb{Q}).$$

PROOF. Indeed, because π_L is the orthogonal projector on L, the class $[\Delta_L]$ belongs to $L \otimes L \subset H^{2n}_B(X \times X, \mathbb{Q})$. As L vanishes in $H^n_B(X \setminus Y, \mathbb{Q})$, the class $[\Delta_L] \in L \otimes L$ vanishes in $H^{2n}_B(X \times X \setminus (Y \times Y), \mathbb{Q})$. Conjecture 2.29 then guarantees the existence of a cycle Z'_L supported on $Y \times Y$ such that $[Z'_L] = [\Delta_L]$ in $H^{2n}_B(X \times X, \mathbb{Q})$. $\square$

4.3.3 A spreading result

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism and let $(\pi, \pi) : \mathcal{X} \times_B \mathcal{X} \to B$ be the fibered self-product of $\mathcal{X}$ over B. Let $\mathcal{Z} \subset \mathcal{X} \times_B \mathcal{X}$ be a codimension k algebraic cycle. We denote by $\mathcal{X}_b$ and $\mathcal{Z}_b$ the fibers, so $\mathcal{X}_b = \pi^{-1}(b)$, $\mathcal{Z}_b = \mathcal{Z}_{|\mathcal{X}_b \times \mathcal{X}_b}$.

Proposition 4.25. *Assume that for a very general point $b \in B$, there exist a closed algebraic subset $Y_b \subset \mathcal{X}_b \times \mathcal{X}_b$ of codimension c, and an algebraic cycle $Z'_b \subset Y_b \times Y_b$ with $\mathbb{Q}$-coefficients, such that*

$$[Z'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

Then there exist a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c, and a codimension k algebraic cycle $\mathcal{Z}'$ with $\mathbb{Q}$-coefficients on $\mathcal{X} \times_B \mathcal{X}$, which is supported on $\mathcal{Y} \times_B \mathcal{Y}$ and such that for any $b \in B$,

$$[\mathcal{Z}'_b] = [\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}). \tag{4.19}$$

Remark 4.26. This proposition is a crucial observation of the paper [114]. The key point is the fact that we do not need to make any base change for this specific problem. This will be crucial because the total space of the family $\mathcal{X} \times_B \mathcal{X}$ is very easy to describe, while it can become very complicated after an arbitrary base change. The idea of spreading out cycles has become very important in the theory of algebraic cycles since Nori's paper [76] (see [47], [89]). For most problems however, we usually need to work over a generically finite extension of the base, due to the fact that cycles existing at the general point will exist on the total space of the family only after a base change.

Proof of Proposition 4.25. There are countably many algebraic varieties $M_i \to B$ parametrizing data (b, Y_b, Z'_b) as above, and we can assume that each M_i parametrizes universal objects

$$\mathcal{Y}_i \to M_i, \quad \mathcal{Y}_i \subset \mathcal{X}_{M_i}, \qquad \mathcal{Z}'_i \subset \mathcal{Y}_i \times_{M_i} \mathcal{Y}_i, \tag{4.20}$$

satisfying the property that for $m \in M_i$, with $\mathrm{pr}_1(m) = b \in B$,

$$[\mathcal{Z}'_{i,b}] = [\mathcal{Z}_{i,b}] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

By assumption, a very general point of B belongs to the union of the images of the first projections $M_i \to B$. By a Baire category argument, we conclude that one of the morphisms $M_i \to B$ is dominating. Taking a subvariety of M_i if necessary, we may assume that $\phi_i : M_i \to B$ is generically finite. We may also assume that it is proper and carries the families $\mathcal{Y}_i \to M_i$, $\mathcal{Y}_i \subset \mathcal{X}_{M_i}$, $\mathcal{Z}'_i \subset \mathcal{Y}_i \times_{M_i} \mathcal{Y}_i$. Denote by $r_i : \mathcal{X}_{M_i} \to \mathcal{X}$ the proper generically finite morphism induced by ϕ_i. Let

$$\mathcal{Y} := r_i(\mathcal{Y}_i) \subset \mathcal{X}.$$

Note that because r_i is generically finite, $\operatorname{codim}\mathcal{Y} \geq c$. Let $r_i' : \mathcal{Y}_i \to \mathcal{Y}$ be the restriction of r_i to $\mathcal{Y}_i$ and let $\mathcal{Z}' := (r_i', r_i')_*(\mathcal{Z}_i')$, which is a codimension k cycle in $\mathcal{X} \times_B \mathcal{X}$ supported in

$$(r_i', r_i')(\mathcal{Y}_i \times_{M_i} \mathcal{Y}_i) \subset \mathcal{Y} \times_B \mathcal{Y}.$$

It is obvious that for any $b \in B$,

$$[\mathcal{Z}_b'] = N[\mathcal{Z}_b] \text{ in } H_B^{2k}(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}),$$

where N is the degree of r_i. Finally, since we work with cycles with $\mathbb{Q}$-coefficients, we can replace $\mathcal{Z}'$ by $\frac{1}{N}\mathcal{Z}'$ in order to achieve the desired equality (4.19). $\square$

4.3.4 Proof of Theorem 4.16

We will start with a few preparatory lemmas. Consider a smooth projective variety X of dimension n with trivial Chow groups. Let L_i, $i = 1, \ldots, r$ be very ample line bundles on X. Let $j : X_b \hookrightarrow X$ be a very general complete intersection of hypersurfaces in $|L_i|$, $i = 1, \ldots, r$. Then X_b is smooth of dimension $(n - r)$, and the Lefschetz theorem on hyperplane sections implies that

$$H_B^*(X_b, \mathbb{Q}) = H_B^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}} \oplus H_B^*(X, \mathbb{Q})_{|X_b}, \tag{4.21}$$

where the vanishing cohomology $L_b := H^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}}$ is defined as $\operatorname{Ker}(j_* : H_B^{n-r}(X_b, \mathbb{Q}) \to H_B^{n+r}(X, \mathbb{Q}))$ and the direct sum above is orthogonal with respect to Poincaré duality on X_b. Indeed, for $k < n - r$ the restriction map $H_B^k(X, \mathbb{Q}) \to H_B^k(X_b, \mathbb{Q})$ is surjective, and choosing an ample line bundle H on X with first Chern class h, the hard Lefschetz theorem on X_b says that $\cup h^s_{|X_b} : H_B^{n-r-s}(X_b, \mathbb{Q}) \to H_B^{n-r+s}(X_b, \mathbb{Q})$ is surjective for $s \geq 0$. We thus deduce that the restriction map $H_B^k(X, \mathbb{Q}) \to H_B^k(X_b, \mathbb{Q})$ is also surjective for $k > n - r$. Note that

$$H_B^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}} \subset H_B^{n-r}(X_b, \mathbb{Q})_{\mathrm{prim}},$$

where "prim" stands for "primitive with respect to a very ample line bundle coming from X," and thus, by the second Hodge–Riemann bilinear relations, the intersection form $\langle\, , \rangle$ on $H_B^{n-r}(X_b, \mathbb{Q})$ remains nondegenerate after restriction to L_b.

Since X has trivial Chow groups, we know that $H_B^*(X, \mathbb{Q})$ is generated by classes of algebraic cycles (see Theorem 3.19) and so is the restriction $H_B^*(X, \mathbb{Q})_{|X_b}$. This implies the following.

LEMMA 4.27. *The orthogonal projector π_{L_b} on L_b is algebraic.*

PROOF. In fact, we can construct an almost canonical algebraic cycle $\Delta_{b,\mathrm{van}}$ with $\mathbb{Q}$-coefficients on $X_b \times X_b$ whose class $[\Delta_{b,\mathrm{van}}]$ is equal to π_{L_b}. More precisely, the cycle is canonically determined by the choice of the ample line bundle

H on X. For this, we choose a basis of $\oplus_{i\leq n-r}H^{2i}_B(X,\mathbb{Q})$. As we know that X has trivial Chow groups, this basis consists of classes $[z_{i,j}]$ of algebraic cycles $z_{i,j}$ on X, with $\operatorname{codim} z_{i,j}=i\leq n-r$. We note that by the hard Lefschetz theorem applied to X_b, the classes $[h]^{n-r-i}\cup[z_{i,j}]_{|X_b}$, together with the classes $[z_{i,j}]_{|X_b}$, form a basis of $H^*_B(X,\mathbb{Q})_{|X_b}$. Here $[h]\in H^2_B(X_b,\mathbb{Q})$ is the topological first Chern class of $H_{|X_b}$. The intersection form on $H^*_B(X_b,\mathbb{Q})$ is nondegenerate when restricted to $H^*_B(X,\mathbb{Q})_{|X_b}$, and L_b is the orthogonal complement of $H^*_B(X,\mathbb{Q})_{|X_b}$ with respect to the intersection pairing on $H^*_B(X_b,\mathbb{Q})$. We thus have the decomposition of $\mathrm{Id}_{H^*_B(X_b,\mathbb{Q})}$ as the sum of two orthogonal projectors:

$$\pi_{L_b}+\pi_{H^*_B(X,\mathbb{Q})_{|X_b}}=\mathrm{Id}_{H^*_B(X_b,\mathbb{Q})}\,.$$

But the orthogonal projector $\pi_{H^*_B(X,\mathbb{Q})_{|X_b}}$ is given by the class of an algebraic cycle on $X_b\times X_b$, which is in fact almost canonical. Indeed, it suffices to choose an orthogonal basis e_i of $H^*_B(X,\mathbb{Q})_{|X_b}$ for the intersection form on $H^*_B(X_b,\mathbb{Q})$, satisfying $\langle e_i,e_i\rangle=\epsilon_i\in\mathbb{Q}^*$; then the class $\pi_{H^*_B(X,\mathbb{Q})_{|X_b}}$ is equal to

$$\sum_i \epsilon_i^{-1}e_i\otimes e_i\in H^*(X_b,\mathbb{Q})\otimes H^*(X_b,\mathbb{Q})\cong \mathrm{End}(H^*(X_b,\mathbb{Q})).$$

Each class e_i comes from a canonically defined class $\tilde{e}_i\in H^*(X,\mathbb{Q})$ using the basis $([z_{i,j|X_b}],\,h^{n-r-i}[z_{i,j|X_b}])$ constructed above. Furthermore, as X has trivial Chow groups, the cycles $z_{i,j},\,h^{n-r-i}z_{i,j}\in \mathrm{CH}(X)_{\mathbb{Q}}$ are determined by their cohomology classes, hence the classes $\tilde{e}_i\in H^*(X,\mathbb{Q})$ lift canonically to cycles $Z_i\in\mathrm{CH}(X)_{\mathbb{Q}}$. We thus conclude that $\pi_{H^*_B(X,\mathbb{Q})_{|X_b}}$ is the class of the cycle $\sum_i\epsilon_i^{-1}\,\mathrm{pr}_1^*\,Z_{i|X_b}\cdot\mathrm{pr}_2^*\,Z_{i|X_b}$ on $X_b\times X_b$, where pr_i, $i=1,2$ are the projectors from $X_b\times X_b$ to its summands.

As $\mathrm{Id}_{H^*_B(X_b,\mathbb{Q})}$ corresponds to the class of the diagonal of X_b, we find that

$$\pi_{L_b}=\left[\Delta_{X_b}-\sum_i\epsilon_i^{-1}\,\mathrm{pr}_1^*\,Z_{i|X_b}\cdot\mathrm{pr}_2^*\,Z_{i|X_b}\right]\in H^{2n-2r}(X_b\times X_b,\mathbb{Q})$$

which concludes the proof, with

$$\Delta_{b,\mathrm{van}}=\Delta_{X_b}-\sum_i\epsilon_i^{-1}\,\mathrm{pr}_1^*\,Z_{i|X_b}\cdot\mathrm{pr}_2^*\,Z_{i|X_b}\in\mathrm{CH}^{n-r}(X_b\times X_b)_{\mathbb{Q}}.$$

□

We now assume that there is a closed algebraic subset $Y_b\subset X_b$ of codimension c such that the sub-Hodge structure L_b vanishes on $X_b\setminus Y_b$. Then, under Conjecture 2.29, Lemma 4.24 tells us that there is an algebraic cycle Z_b supported on $Y_b\times Y_b$ such that $[Z_b]=[\Delta_{b,\mathrm{van}}]$. We now put this information in family.

NOTATION 4.28. *Let X be a smooth projective with trivial Chow groups. Let $\mathbb{P}_i := \mathbb{P}(H^0(X, L_i))$. Let $B \subset \prod_i \mathbb{P}_i$ be the open set parametrizing smooth complete intersections and let*

$$\mathcal{X} \subset B \times X, \qquad \pi : \mathcal{X} \to B$$

be the universal family. We will denote by $X_b \subset \mathcal{X}$ the fiber $\pi^{-1}(b)$ for $b \in B$.

We will apply Proposition 4.25 to

$$\mathcal{D}_{\mathrm{van}} \in \mathrm{CH}^{n-r}(\mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}, \tag{4.22}$$

the corrected relative diagonal with fiber over $b \in B$, the $\Delta_{b,\mathrm{van}}$ introduced at the end of the proof of Lemma 4.27. (Note that $\mathcal{D}_{\mathrm{van}}$ is not in fact canonically defined even if its restriction to $X_b \times X_b$ is canonically defined, because it may be modified by adding cycles which are restrictions to $\mathcal{X}$ of cycles in $\mathrm{CH}^{>0}(B) \otimes \mathrm{CH}(X) \subset \mathrm{CH}(B \times X)$.)

We then get the following lemma.

LEMMA 4.29. *Assume that for a general point $b \in B$, there is a codimension c closed algebraic subset $Y_b \subset X_b$ such that $L_b = H_B^{n-r}(X_b, \mathbb{Q})_{\mathrm{van}}$ vanishes on $X_b \setminus Y_b$. If furthermore Conjecture 2.29 holds, there exist a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c, and a codimension $(n-r)$ algebraic cycle $\mathcal{Z}'$ on $\mathcal{X} \times_B \mathcal{X}$ with $\mathbb{Q}$-coefficients, which is supported on $\mathcal{Y} \times_B \mathcal{Y}$ and such that for any $b \in B$,*

$$[\mathcal{Z}'_b] = [\Delta_{b,\mathrm{van}}] \text{ in } H_B^{2n-2r}(X_b \times X_b, \mathbb{Q}).$$

PROOF. This is a direct application of Proposition 4.25, because we know from Lemma 4.24 that under Conjecture 2.29, our assumption implies that there exists for a very general point $b \in B$ an algebraic cycle $Z'_b \subset Y_b \times Y_b$ such that $[Z'_b] = [\Delta_{b,\mathrm{van}}]$ in $H^{2n-2r}(X_b \times X_b, \mathbb{Q})$. ☐

Next, we have the following lemma.

LEMMA 4.30. *With Notation 4.28, let $\alpha \in H_B^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ be a cohomology class whose restriction to the fibers $X_b \times X_b$ is 0. Then we can write*

$$\alpha = \alpha_1 + \alpha_2,$$

where α_1 is the restriction to $\mathcal{X} \times_B \mathcal{X}$ of a class $\beta_1 \in H_B^{2n-2r}(X \times \mathcal{X}, \mathbb{Q})$, and α_2 is the restriction to $\mathcal{X} \times_B \mathcal{X}$ of a class $\beta_2 \in H_B^{2n-2r}(\mathcal{X} \times X, \mathbb{Q})$.

More precisely we can take $\beta_1 \in \oplus_{i<n-r} H^i(X, \mathbb{Q}) \otimes L^1 H^{2n-2r-i}(\mathcal{X}, \mathbb{Q})$, and $\beta_2 \in \oplus_{i<n-r} L^1 H^{2n-2r-i}(\mathcal{X}, \mathbb{Q}) \otimes H^i(X, \mathbb{Q})$, where L stands for the Leray filtration on $H^*(\mathcal{X}, \mathbb{Q})$ with respect to the morphism $\pi : \mathcal{X} \to B$.

PROOF. Consider the smooth proper morphism

$$(\pi, \pi) : \mathcal{X} \times_B \mathcal{X} \to B.$$

The relative Künneth decomposition gives

$$R^k(\pi,\pi)_*\mathbb{Q} = \bigoplus_{i+j=k} H^i_{\mathbb{Q}} \otimes H^j_{\mathbb{Q}},$$

where $H^i_{\mathbb{Q}} := R^i\pi_*\mathbb{Q}$. The Leray spectral sequence of (π,π), which degenerates at E_2 (see [30]), gives the Leray filtration L on $H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$ with graded pieces

$$\begin{aligned}\mathrm{Gr}^l_L\, H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q}) &= H^l_B(B, R^{2n-2r-l}(\pi,\pi)_*\mathbb{Q})\\ &= \bigoplus_{i+j=2n-2r-l} H^l_B(B, H^i_{\mathbb{Q}}\otimes H^j_{\mathbb{Q}}).\end{aligned}$$

Our assumption on α exactly says that it vanishes in the first quotient

$$H^0(B, R^{2n-2r}(\pi,\pi)_*\mathbb{Q})$$

for the Leray filtration, or equivalently, $\alpha \in L^1H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$. Consider now the other graded pieces

$$H^l_B(B, H^i_{\mathbb{Q}}\otimes H^j_{\mathbb{Q}}),\quad l>0,\quad i+j=2n-2r-l.$$

Since $l>0$, and $i+j=2n-2r-l$, we have either $i<n-r$ or $j<n-r$. Let us consider the case where $i<n-r$. Then, the Lefschetz hyperplane section theorem tells us that the sheaf $H^i_{\mathbb{Q}}$ is the constant sheaf with stalk $H^i_B(X,\mathbb{Q})$. Thus we find that $H^l_B(B,H^i_{\mathbb{Q}}\otimes H^j_{\mathbb{Q}}) = H^i_B(X,\mathbb{Q})\otimes H^l_B(B,H^j_{\mathbb{Q}})$, which is a Leray graded piece of $H^i_B(X,\mathbb{Q})\otimes H^{l+j}_B(\mathcal{X})$. Similarly analyzing the case where $j<n-r$, we conclude that the natural map

$$\bigoplus_{i<n-r} H^i_B(X,\mathbb{Q})\otimes L^1H^{2n-2r-i}_B(\mathcal{X},\mathbb{Q}) \oplus \bigoplus_{j<n-r} L^1H^{2n-2r-j}_B(\mathcal{X},\mathbb{Q})\otimes H^j_B(X,\mathbb{Q})$$

$$\to L^1H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$$

is surjective. This proves the existence of the classes β_1, β_2. ☐

In the case where X has trivial Chow groups, we get extra information.

LEMMA 4.31. *With the same notation as above, assume that X has trivial Chow groups and that α is the class of an algebraic cycle on $\mathcal{X}\times_B\mathcal{X}$. Then we can choose the β_i's to be the restriction of classes of algebraic cycles on $B\times X\times X$.*

PROOF. It suffices to show that we can choose the β_i's to be the classes of algebraic cycles on $X\times\mathcal{X}$. Indeed, these classes will lift to classes on $B\times X\times X$ for the following reason: $\mathcal{X}$ is a Zariski open set in the natural fibration

$$f:\mathbb{P}\to X,\quad \mathbb{P}\subset\prod_i \mathbb{P}_i\times X,$$

$$\mathbb{P} := \{(\sigma_1, \ldots, \sigma_r, x),\ \sigma_i(x) = 0\ \forall\, 1 \leq i \leq r\}.$$

This is a fibration into products of projective spaces, because we assumed the L_i's are globally generated. It follows that $X \times \mathcal{X}$ is also a Zariski open set in the corresponding fibration $X \times \mathbb{P} \to X \times X$ into products of projective spaces. The restriction map

$$\mathrm{CH}\left(X \times X \times \prod_i \mathbb{P}_i\right) \to \mathrm{CH}(X \times \mathbb{P})$$

is then surjective, by the computation of the Chow groups of a projective bundle fibration ([101, II, 9.3.2]) and the restriction map $\mathrm{CH}(X \times \mathbb{P}) \to \mathrm{CH}(X \times \mathcal{X})$ to the Zariski open set $X \times \mathcal{X}$ is also surjective by the localization exact sequence (2.2). Hence the composition $\mathrm{CH}(X \times X \times \prod_i \mathbb{P}_i) \to \mathrm{CH}(X \times \mathcal{X})$, and a fortiori the restriction map $\mathrm{CH}(X \times X \times B) \to \mathrm{CH}(X \times \mathcal{X})$, are surjective.

It remains to show that if α is algebraic, we can choose the β_i's to be the restrictions of classes of algebraic cycles on $X \times \mathcal{X}$.

By the proof of Lemma 4.30 we have

$$\alpha = \beta_{1|\mathcal{X} \times_B \mathcal{X}} + \beta_{2|\mathcal{X} \times_B \mathcal{X}}, \tag{4.23}$$

where $\beta_1 \in H_B^{*<n-r}(X, \mathbb{Q}) \otimes L^1 H_B^*(\mathcal{X}, \mathbb{Q})$ and $\beta_2 \in L^1 H_B^*(\mathcal{X}, \mathbb{Q}) \otimes H_B^{*<n-r}(X, \mathbb{Q})$. We know that the cohomology of X is generated by classes of algebraic cycles $[z_{i,j}] \in H_B^{2i}(X, \mathbb{Q})$. Let us choose a basis $[z_{i,j}]$, $2i < n-r$ of $H_B^{*<n-r}(X, \mathbb{Q})$. Then we can choose cycle classes $[z_{i,j}]^* \in H_B^{2n-2r-2i}(X, \mathbb{Q})$ in such a way that the restricted classes $[z_{i,j}]^*_{|X_b}$ form the dual basis of $H_B^{*>n-r}(X_b, \mathbb{Q})$ for the intersection pairing on X_b. Observe that for every i such that $2i < n-r$, the cycle classes

$$\sum_j p_{1,X}^*[z_{i,j}] \cup p_{2,X}^*[z_{i,j}]^* \in H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}),$$

seen as cohomological relative self-correspondences of $\mathcal{X}$ over B, provide (maybe up to shrinking B) projectors

$$\pi_{2i} : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q},$$

which act as the identity on the cohomology $R^{2i}\pi_*\mathbb{Q}$, for $2i < n-r$. Similarly the cycle classes

$$\sum_j p_{1,X}^*[z_{i,j}]^* \cup p_{2,X}^*[z_{i,j}] \in H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$$

give projectors $\pi_{2n-2r-2i}$ of $R\pi_*\mathbb{Q}$ acting as the identity on $R^{2n-2r-2i}\pi_*\mathbb{Q}$ for $2i < n-r$. Furthermore, these projectors satisfy the condition that $\pi_k \circ \pi_l = \pi_l \circ \pi_k = 0$ for $k \neq l$. It follows that denoting

$$\pi_{n-r} := \mathrm{Id} - \sum_{2i<n-r} \pi_{2i} - \sum_{2i<n-r} \pi_{2n-2r-2i},$$

we get a decomposition in the derived category of B,

$$R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i], \tag{4.24}$$

which in turn induces a similar decomposition by the relative Künneth decomposition

$$R(\pi,\pi)_*\mathbb{Q} \cong \oplus_i R^i(\pi,\pi)_*\mathbb{Q}[-i] = \oplus_{p+q=i}(R^p\pi_*\mathbb{Q}\otimes R^q\pi_*\mathbb{Q})[-i]. \tag{4.25}$$

Taking cohomology on both sides, we get a decomposition

$$H^k(\mathcal{X}\times_B\mathcal{X},\mathbb{Q}) \cong \oplus_{i+p+q=k}H^i(B, R^p\pi_*\mathbb{Q}\otimes R^q\pi_*\mathbb{Q}). \tag{4.26}$$

This is nothing but an explicit form of the Deligne decomposition already mentioned, except that it is clear now that the projector to each summand is induced by an algebraic relative self-correspondence of $\mathcal{X}\times_B\mathcal{X}$, hence sends a cycle class to a cycle class.

Applying (4.26) to our class α and recalling that α belongs to $L^1H^*(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$, we get that

$$\alpha = \sum_{i>0,i+p+q=2n-2r}\alpha_{i,p,q}, \tag{4.27}$$

with $\alpha_{i,p,q}\in H^i(B,R^p\pi_*\mathbb{Q}\otimes R^q\pi_*\mathbb{Q})$ being a cycle class on $\mathcal{X}\times_B\mathcal{X}$. It now suffices to show that each $\alpha_{i,p,q}$ lifts to a cycle class either on $X\times\mathcal{X}$ or on $\mathcal{X}\times X$.

We have $i+p+q=2n-2r$ with $i>0$ so either $p<n-r$ or $q<n-r$. Assume $p<n-r$; then p has to be even, $p=2m$. The sheaf $R^{2m}\pi_*\mathbb{Q}$ is trivial, with basis given by the pull-back to $\mathcal{X}$ of the classes $[z_{2m,j}]$. We can thus write

$$\alpha_{i,2m,2n-2r-2m-i} = \sum_{i>0,j}p_{1,X}^*[z_{2m,j}]\cup \mathrm{pr}_2^*\,\gamma_{i,2m,j}, \tag{4.28}$$

where $\gamma_{i,2m,j}\in H^i(B,R^{2n-2r-2m-i}\pi_*\mathbb{Q})$ is a cohomology class on $\mathcal{X}$. Here $\pi_2':\mathcal{X}\times_B\mathcal{X}\to\mathcal{X}$ is the second projection

The class $\alpha_{i,2m,2n-2r-2m-i}$ being algebraic, so is the class $\pi'_{2*}(p_{1,X}^*[z_{2m,j}]^*\cup\alpha_{i,2m,2n-2r-2m-i})$ for any j. However, we have the equality

$$\gamma_{i,2m,j} = \pi'_{2*}(p_{1,X}^*[z_{2m,j}]^*\cup\alpha_{i,2m,2n-2r-2m-i}), \tag{4.29}$$

which follows from (4.28), from the projection formula and from the fact that

$$\pi'_{2*}(p_{1,X}^*[z_{2m,j}]^*\cup p_{1,X}^*[z_{2m,k}]) = 0 \text{ in } H^0(\mathcal{X},\mathbb{Q}) \text{ for } j\neq k,$$

$$\pi'_{2*}(p_{1,X}^*[z_{2m,j}]^*\cup p_{1,X}^*[z_{2m,k}]) = 1 \text{ in } H^0(\mathcal{X},\mathbb{Q}) \text{ for } j=k.$$

Formula (4.29) obviously implies that the $\gamma_{i,2m,j}$'s are algebraic, hence that $\alpha_{i,2m,2n-2r-2m-i}$ is algebraic by (4.28). □

Proof of Theorem 4.16. We keep Notation 4.28 and assume now that the vanishing cohomology $H^{n-r}_B(X_b,\mathbb{Q})_{\rm van}$ is supported on a codimension c closed algebraic subset $Y_b\subset X_b$ for any $b\in B$. Consider the corrected diagonal $\mathcal{D}_{\rm van}$ introduced in (4.22), which is a codimension $(n-r)$ cycle of $\mathcal{X}\times_B\mathcal{X}$ with $\mathbb{Q}$-coefficients.

By Lemma 4.29, it follows that there exist a codimension c closed algebraic subset $\mathcal{Y}\subset\mathcal{X}$ and a codimension $(n-r)$ cycle $\mathcal{Z}$ on $\mathcal{X}\times_B\mathcal{X}$ with $\mathbb{Q}$-coefficients, which is supported on $\mathcal{Y}\times_B\mathcal{Y}$ and such that

$$[\mathcal{Z}_b]=[\mathcal{D}_{{\rm van},b}]=[\Delta_{b,{\rm van}}]\ \forall\, b\in B.$$

Thus the class $[\mathcal{Z}]-[\Delta_{b,{\rm van}}]\in H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$ vanishes on the fibers $X_b\times X_b$.

Using Lemmas 4.30 and 4.31, we conclude that there is a cycle $\Gamma\in{\rm CH}^{n-r}(B\times X\times X)_{\mathbb{Q}}$ such that

$$[\mathcal{Z}]=[\mathcal{D}_{\rm van}]+[\Gamma_{|\mathcal{X}\times_B\mathcal{X}}]\ \text{in}\ H^{2n-2r}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q}). \tag{4.30}$$

Lemma 4.32. *Assume Conjecture 2.29. If X has trivial Chow groups, the cycle class map*

$${\rm CH}^*(\mathcal{X}\times_B\mathcal{X})_{\mathbb{Q}}\to H^{2*}_B(\mathcal{X}\times_B\mathcal{X},\mathbb{Q})$$

is injective (in other words, $\mathcal{X}\times_B\mathcal{X}$ has trivial Chow groups).

Proof. Consider the blow-up $\widetilde{X\times X}$ of $X\times X$ along the diagonal. Applying Proposition 4.22 and Lemma 4.19, $\widetilde{X\times X}$ has trivial Chow groups. A point of $\widetilde{X\times X}$ parametrizes a couple (x,y) of points of X, together with a subscheme z of length 2 of X, with associated cycle $x+y$. We thus have the following natural variety of $\prod_i\mathbb{P}_i\times\widetilde{X\times X}$:

$$Q=\{(\sigma_1,\ldots,\sigma_r,x,y,z),\,\sigma_i\in\mathbb{P}_i,\,\sigma_{i|z}=0,\ \forall\, i=1,\ldots,r\}.$$

As the L_i's are assumed to be very ample, the map $Q\to\widetilde{X\times X}$ is a fibration with fiber over $(x,y,z)\in\widetilde{X\times X}$ a product of projective spaces $\mathbb{P}_{i,z}$ of codimension 2 in $\mathbb{P}_i$. By Lemma 4.18, Q also has trivial Chow groups. Let $Q_0\subset Q$ be the inverse image of B under the projection $Q\to\prod_i\mathbb{P}_i$. Then Q_0 is Zariski open in Q, so by Lemma 4.20 (for which we need Conjecture 2.29), the cycle class map is also injective on cycles of Q_0. Finally, Q_0 maps naturally onto $\mathcal{X}\times_B\mathcal{X}$ via the map

$$\prod_i\mathbb{P}_i\times\widetilde{X\times X}\to\prod_i\mathbb{P}_i\times X\times X.$$

The morphism $Q_0\to\mathcal{X}\times_B\mathcal{X}$ being projective and dominant, we conclude by Lemma 4.21 that the cycle class map is injective on cycles of $\mathcal{X}\times_B\mathcal{X}$. $\square$

The proof of Theorem 4.16 is then finished as follows. From the equality (4.30) of cohomology classes, we deduce by Lemma 4.32 the following equality of cycles:

$$\mathcal{Z} = \mathcal{D}_{\mathrm{van}} + \Gamma_{|\mathcal{X}\times_B\mathcal{X}} \text{ in } \mathrm{CH}^{n-r}(\mathcal{X}\times_B\mathcal{X})_{\mathbb{Q}}. \tag{4.31}$$

We now fix b and restrict this equality to $X_b \times X_b$. Then we find

$$\mathcal{Z}_b = \Delta_{b,\mathrm{van}} + \Gamma'_{|X_b\times X_b} \text{ in } \mathrm{CH}^{n-r}(X_b\times X_b)_{\mathbb{Q}},$$

where $\Gamma' \in \mathrm{CH}(X\times X)_{\mathbb{Q}}$ is the restriction of Γ to $b\times X\times X$.

Recalling that $\Delta_{b,\mathrm{van}} = \Delta_{X_b} + \Gamma''_{|X_b\times X_b}$ for some codimension $(n-r)$ cycle with $\mathbb{Q}$-coefficients Γ'' on $X\times X$, we conclude that

$$\Delta_{X_b} = \mathcal{Z}_b + \Gamma_{1|X_b\times X_b}, \tag{4.32}$$

where $\Gamma_1 \in \mathrm{CH}^{n-r}(X\times X,\mathbb{Q})$ and the cycle $\mathcal{Z}_b$ is by construction supported on $Y_b\times Y_b$, with $Y_b \subset X_b$ of codimension $\geq c$ for general b.

Let $z \in \mathrm{CH}_i(X_b)_{\mathbb{Q}}$, with $i < c$. Then $(\mathcal{Z}_b)_* z = 0$ since we may find a cycle rationally equivalent to z in X_b and disjoint from Y_b. Applying both sides of (4.32) to z thus gives

$$z = (\Gamma_{1|X_b\times X_b})_* z \text{ in } \mathrm{CH}_i(X_b)_{\mathbb{Q}}. \tag{4.33}$$

But it is obvious that

$$(\Gamma_{1|X_b\times X_b})_* : \mathrm{CH}(X_b)_{\mathbb{Q}} \to \mathrm{CH}(X_b)_{\mathbb{Q}}$$

factors through $j_{b*} : \mathrm{CH}(X_b)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$. Now, if z is homologous to 0 on X_b, $j_{b*}(z)$ is homologous to 0 on X, and thus it is rationally equivalent to 0 on X because X has trivial Chow groups. It follows from (4.33) that $z = 0$ in $\mathrm{CH}_i(X_b)_{\mathbb{Q}}$. Hence we proved that the cycle class map with $\mathbb{Q}$-coefficients is injective on $\mathrm{CH}_i(X_b)_{\mathbb{Q}}$ for $i < c$, which concludes the proof of the theorem. □

4.3.5 Further applications

4.3.5.1 Complete intersections with group action

Theorem 4.16 applies to general complete intersections in projective space. The relation (4.1) gives the Hodge coniveau for them, hence conjecturally the geometric coniveau c, according to the generalized Hodge conjecture (Conjecture 2.40). Hence solving the generalized Bloch conjecture for them is equivalent to solving a strong form of the generalized Hodge conjecture, implied by the generalized Hodge conjecture together with the standard Lefschetz conjecture.

There are interesting variants coming from the study of complete intersections X_b of r hypersurfaces in projective space $\mathbb{P}^n$, or in any ambient variety X with trivial Chow groups, invariant under a finite group action. Let G act on

X_b, and let $\pi \in \mathbb{Q}[G]$ be a projector, that is, $\pi^2 = \pi$. This projector gives (see Example 3.6) an algebraic cycle $\Gamma_\pi \in \mathrm{CH}^{n-r}(X_b \times X_b)_\mathbb{Q}$, which is a projector, that is,

$$\Gamma_\pi \circ \Gamma_\pi = \Gamma_\pi.$$

Then consider the sub-Hodge structure

$$L^\pi := \mathrm{Im}([\Gamma_\pi]^* : H^{n-r}(X_b, \mathbb{Q})_{\mathrm{prim}} \to H^{n-r}(X_b, \mathbb{Q})_{\mathrm{prim}}).$$

In general, it has a larger coniveau than X_b. For example, if X_b is a quintic surface in $\mathbb{P}^3$, defined by an invariant polynomial under the linearized group action of $G \cong \mathbb{Z}/5\mathbb{Z}$ with generator g on $\mathbb{P}^3$ given by

$$g^* X_i = \zeta^i X_i, \quad i = 0, \ldots, 3,$$

where ζ is a nontrivial fifth root of unity, then the G-invariant cohomology $H^2(S, \mathbb{Q})^{\mathrm{inv}}$ has no $(2, 0)$-part, hence is of coniveau 1, while $H^{2,0}(S) \neq 0$ so the coniveau of $H^2(S, \mathbb{Q})_{\mathrm{prim}}$ is 0. The quotient surface S/G is a quintic Godeaux surface (see [98]).

Coming back to the general situation, note that if $\pi = \sum_{g \in G} \alpha_g g$, the Hodge structure L^π is the image of the projector $[\Gamma_\pi]^* = \frac{1}{|G|} \sum_{g \in G} \alpha_g g^*$ acting on L. On the other hand, X_b equipped with the projector Γ_π is a motive and the generalized Bloch conjecture (Conjecture 3.21) extended to motives predicts the following.

CONJECTURE 4.33. *Assume L^π has coniveau $\geq c$. Then the cycle class map is injective on*

$$\mathrm{CH}_i(X_b)^\pi_\mathbb{Q} := \mathrm{Im}(\Gamma_{\pi *} : \mathrm{CH}_i(X_b)_\mathbb{Q} \to \mathrm{CH}_i(X_b)_\mathbb{Q})$$

for $i < c$.

If $\pi = \frac{1}{|G|} \sum_{g \in G} g$ is the projector onto the invariant part, this conjecture is essentially equivalent to the previous one by considering X_b/G or a desingularization of it. Even in this case, one needs to make assumptions on the linearized group action in order to apply the same strategy as in the proof of Theorem 4.16. The case of more general projectors cannot be reduced to the previous case.

In order to apply a strategy similar to the one applied for the proof of Theorem 4.16, we need some assumptions. Indeed, if the group G is too big, like the automorphisms group of the Fermat hypersurface, there are too few invariant complete intersections to play on the geometry of the universal family $\mathcal{X} \to B$ of G-invariant complete intersections.

In any case, what we get by mimicking the proof of Theorem 4.16 is the following (see [114]): X is, as before, a smooth projective variety of dimension n with trivial Chow groups and G is a finite group acting on X. We fix a projector $\pi \in \mathbb{Q}[G]$. We study complete intersections $X_b \subset X$ of r G-invariant ample hypersurfaces $X_i \in |L_i|^G$: Let G act via the character χ_i on the considered component of $|L_i|^G$. The basis B parametrizing such complete intersections is thus a Zariski open set in $\prod_i \mathbb{P}(H^0(X, L_i)^{\chi_i})$. As before we denote by $\mathcal{X} \to B$ the universal G-invariant complete intersection.

THEOREM 4.34 (Voisin 2011). *Assume the following conditions:*

(i) The variety $\mathcal{X} \times_B \mathcal{X}$ has trivial Chow groups.

(ii) The Hodge structure on $H_B^{n-r}(X_b, \mathbb{Q})^{\pi}_{\rm van}$ is supported on a closed algebraic subset $Y_b \subset X_b$ of codimension c. (According to Conjecture 2.40, this should be satisfied if the Hodge coniveau of $H_B^{n-r}(X_b, \mathbb{Q})^{\pi}_{\rm van}$ is $\geq c$.)

(iii) Conjecture 2.29 holds for codimension $(n-r)$ cycles.

Then the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X_b)^{\pi}_{\mathbb{Q}} \to H_B^{2n-2r-2i}(X_b, \mathbb{Q})^{\pi}$ *is injective for $i < c$ and for any $b \in B$.*

REMARK 4.35. In the case where the X_b are surfaces with $H^{2,0}(X_b)^{\pi} = 0$, by the Lefschetz theorem on $(1,1)$-classes, assumptions (ii) (for coniveau 1) and (iii) above are automatically satisfied. We thus get an alternative proof of the main theorem of [98], where the Bloch conjecture is proved for general Godeaux surfaces (quotients of quintic surfaces by a free action of $\mathbb{Z}/5\mathbb{Z}$, or quotients of complete intersections of four quadrics in $\mathbb{P}^6$ by a free action of $\mathbb{Z}/8\mathbb{Z}$).

In the case of threefolds X_b of Hodge coniveau 1, we can also conclude that $\mathrm{CH}_0(X_b)^{\pi}_0 = 0$ if assumption (i) above is satisfied and the generalized Hodge conjecture is satisfied by the coniveau 1 Hodge structure on $H^3(X_b, \mathbb{Q})^{\pi}$. Indeed, we used Conjecture 2.29 in the proof in two places: The first place is in the proof of Lemma 4.24, which says that if a certain Hodge structure $L \subset H_B^*(X_b, \mathbb{Q})$ is supported on a codimension c closed algebraic subset Y_b, the corresponding projector has a class that comes from the class of a cycle supported in $Y_b \times Y_b$. This will be satisfied if $\dim X_b = 3$, and $L \subset H_B^3(X, \mathbb{Q})^{\chi}$ is supported on Y_b because we know then that the degree 6 Hodge class of the projector π_L is supported on the codimension 2 closed algebraic subset $Y_b \times Y_b$ (or rather a desingularization of it), so that we can apply Lemma 2.31. The second place is in the proof of Lemma 4.32. However, in the threefold case, it is possible to prove it directly without using Conjecture 2.29 (see [114]).

In this way, the second result of [98] (quintic hypersurfaces with involutions) and the main application of [82] (three-dimensional complete intersections in weighted projective space) are re-proved; in both cases we are reduced to proving the generalized Hodge conjecture for the coniveau 1 Hodge structure on their cohomology of degree 3.

4.3.5.2 Application to self-products

Let Y be a smooth projective variety. There is a natural surjective map

$$\mathrm{CH}_0(Y) \otimes \mathrm{CH}_0(Y) \to \mathrm{CH}_0(Y \times Y),$$

sending (z, z') to $\mathrm{pr}_1^* z \cdot \mathrm{pr}_2^* z'$. More generally one can study the product map

$$\mathrm{CH}_0(Y)^{\otimes m} \to \mathrm{CH}_0(Y^m)$$

that is defined similarly, and is compatible with the action of the symmetric group $\mathfrak{S}_m$ on both sides.

This study was undertaken in [59], [99] and a variant of it (where the emphasis is on cycles of given codimension, instead of cycles of given dimension) is developed in [60].

Note that if we fix a point $o \in Y$, we always have an injection

$$\mathrm{CH}_0(Y) \to \mathrm{CH}_0(Y \times Y), \qquad z \mapsto z \times o = \mathrm{pr}_1^* z \cdot \mathrm{pr}_2^* o,$$

because the composition of this map with $\mathrm{pr}_{1*} : \mathrm{CH}_0(Y \times Y) \to \mathrm{CH}_0(Y)$ is the identity. We can argue similarly after exchanging factors. The conclusion of this is that the interesting map is the following:

$$\mathrm{CH}_0(Y)_{\mathrm{hom}} \otimes \mathrm{CH}_0(Y)_{\mathrm{hom}} \to \mathrm{CH}_0(Y \times Y)_{\mathrm{hom}}. \tag{4.34}$$

This map is rather mysterious. It is proved in [99] that if Y is a surface and this map, or only the symmetric part of it, is trivial, then the surface Y (which necessarily has $h^{2,0}(Y) = 0$) satisfies the Bloch conjecture.

Let us spell out what predicts the generalized Bloch conjecture (adapted to motives) for the map (4.34), or rather its antisymmetric or symmetric versions.

We start with the following lemma.

LEMMA 4.36. *Let H be a Hodge structure of weight m. Then for $k > h^{m,0} := \dim H^{m,0}$, the Hodge structure of weight km on $\bigwedge^k L$ has coniveau ≥ 1. In particular, if $h^{m,0} = 1$, the Hodge structure of weight $2m$ on $\bigwedge^2 H$ has coniveau ≥ 1.*

PROOF. Indeed, the $(km, 0)$-component of the Hodge structure on $\bigwedge^k H$ is equal to $\bigwedge^k H^{m,0}$, hence it is 0 for $k > h^{m,0}$. □

Let Y be a smooth projective variety. Assume that $H^{i,0}(Y) = 0$ for $i \neq 0$, m (this will be the case if Y is an m-dimensional complete intersection of ample hypersurfaces in a projective variety with trivial Chow groups). Conjecture 3.21 (or rather its generalization to motives), together with Lemma 4.36, predicts the following (see below for more detail).

CONJECTURE 4.37. *Assume Y satisfies the above assumption and has $h^{m,0}(Y) = 1$. Then, for any z, $z' \in \mathrm{CH}_0(Y)$ with $\deg z = \deg z' = 0$, one has $z \times z' - z' \times z = 0$ in $\mathrm{CH}_0(Y \times Y)$ for m even and $z \times z' + z' \times z = 0$ in $\mathrm{CH}_0(Y \times Y)$ for m odd.*

Indeed, the transcendental cohomology of the skew-symmetric motive (see Section 3.2.3) $\bigwedge^2 X$ if m is even and of the symmetric motive $S^2 X$ if m is odd has Hodge coniveau ≥ 1. The case $m = 2$ is particularly interesting, as noted in [99]. In this case, we have the following statement.

Lemma 4.38. *Let $(H, H^{p,q})$ be a weight 2 Hodge structure of $K3$ type, namely $h^{2,0} = 1$. Then the Hodge structures on $\bigwedge^{2k} H$ all have niveau ≤ 2 (that is, coniveau $\geq k-1$).*

Proof. Write $H = H^{2,0} \bigoplus H^{1,1} \bigoplus H^{0,2}$. Then

$$\bigwedge^{k} H = H^{2,0} \otimes \bigwedge^{k-1} H^{1,1} \bigoplus \left(\bigwedge^{k} H^{1,1} \oplus H^{2,0} \otimes H^{0,2} \otimes \bigwedge^{k-2} H^{1,1} \right) \bigoplus \bigwedge^{k-1} H^{1,1} \otimes H^{0,2}$$

is the Hodge decomposition of $\bigwedge^k H$, whose first nonzero term is of type $(k+1, k-1)$. ☐

When $k > \dim H$, we of course have that the Hodge structure on $\bigwedge^k H$ is trivial. Applying these observations to the case where $H = H^2_B(S, \mathbb{Q})$, where S is an algebraic $K3$ surface, we find that Conjecture 3.21 (or rather, its extension to motives) predicts the following (see [99]).

Conjecture 4.39.

(i) Let S be an algebraic $K3$ surface. Then for any $k \geq 2$, and $i \leq k-2$, the projector $\pi_{\mathrm{alt}} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\epsilon(\sigma)} \sigma \in \mathrm{CH}^{2k}(S^k \times S^k)$ composed with the Chow–Künneth projector $\pi_2^{\otimes k}$ (see [74]) acts as 0 on $\mathrm{CH}_i(S^k)_{\mathbb{Q}}$ for $i \leq k-2$.

(ii) For $k > b_2(S)$, this projector is identically 0.

Note that (ii) above is essentially Kimura's finite-dimensionality conjecture [59] and applies to any regular surface. One may wonder whether it could be attacked by the methods used for the proof of Theorem 4.16 for the case of quartic $K3$ surfaces. The question would be essentially to study whether the fibered product $\mathcal{X}^{2k/B}$ of the universal such $K3$ surface has trivial Chow groups. For small k this is easy, but we would need to know this in the range $k \geq 22$ in order to prove the Kimura conjecture. This seems to be very hard.

The fact that this is true for small k (see below) shows that Conjecture 4.39 is implied by the generalized Hodge conjecture for the self-products S^k and the coniveau $(k-1)$ Hodge structures $\bigwedge^k H^2_B(S, \mathbb{Q}) \subset H^{2k}_B(S^k, \mathbb{Q})$.

Example 4.40 (Abelian varieties). Conjecture 4.37 also works with varieties replaced by motives. Consider, for example, an abelian variety A of dimension g. The Künneth standard conjecture (Conjecture 2.27) holds for them, and a stronger version of it, namely, there is a decomposition of the diagonal of A as a sum of projectors π_i in $\mathrm{CH}(A \times A)_{\mathbb{Q}}$, where the class of p_i is the Künneth component δ_i (see [74]). Consider the motive $M_g(A) := (A, \pi_g)$. It has $h^{i,0} = 0$ for $i \neq g$ and $h^{g,0} = h^{g,0}(A) = 1$.

What is thus predicted in this case is the fact that the morphism

$$\mathrm{CH}_0(M_g(A)) \otimes \mathrm{CH}_0(M_g(A)) \to \mathrm{CH}_0(M_g(A) \otimes M_g(A))$$

is $(-1)^g$-symmetric. This is indeed true using the Beauville decomposition [8] which is also used in [35] to construct the projectors π_i. This decomposition is a splitting of the Chow groups of any abelian variety (in our case A and $A \times A$) into a sum of eigenspaces under the action of homotheties

$$\mu_i : A \to A, \qquad a \mapsto ia.$$

The Chow group $\mathrm{CH}_0(M_g(A))$ is identified (by the construction of π_g) to the subgroup of $\mathrm{CH}_0(A)_{\mathbb{Q}}$ where μ_i acts by multiplication by i^g. It is also generated by products $D_1 \cdot \cdots \cdot D_g$, where the D_i's are divisors homologous to 0 on A (see [8], [12]). We now follow [99]: It follows from the above that the image of the morphism

$$\mathrm{CH}_0(M_g(A)) \otimes \mathrm{CH}_0(M_g(A)) \to \mathrm{CH}_0(M_g(A) \otimes M_g(A))$$

consists of 0-cycles generated by products of divisors,

$$\mathrm{pr}_1^* D_1 \cdot \cdots \cdot \mathrm{pr}_1^* D_g \cdot \mathrm{pr}_2^* D_1' \cdot \cdots \cdot \mathrm{pr}_2^* D_g', \tag{4.35}$$

where the D_i's and D_i''s are divisors homologous to 0 on A. We want to show that the involution $\tau : A \times A \to A \times A$, $(a, b) \mapsto (b, a)$ acts by $(-1)^g$ on products (4.35). Consider the map

$$\sigma : A \times A \to A \times A, \qquad (a, b) \mapsto (a + b, a - b).$$

This is an isogeny of $A \times A$, since $\sigma \circ \sigma = 2\,\mathrm{Id}_{A\times A}$. It follows that it induces an isomorphism (in fact a homothety) at the level of Chow groups with $\mathbb{Q}$-coefficients. On the other hand, $\sigma \circ \tau \circ \sigma = (2\,\mathrm{Id}_A, -2\,\mathrm{Id}_A)$. It thus suffices to show that $(2\,\mathrm{Id}_A, -2\,\mathrm{Id}_A)$ acts by $(-1)^g$ on products (4.35), which is obvious because $-\mathrm{Id}_A$ acts by $-\mathrm{Id}$ on $\mathrm{Pic}^0(A)$.

Let us state explicitly what the arguments of the proof of Theorem 4.16 give in the case of general Calabi–Yau complete intersections and for $k = 2$. Let X_b be a smooth Calabi–Yau complete intersection of dimension m in projective space $\mathbb{P}^n$. Let $\Delta_{b,\mathrm{van}} \in \mathrm{CH}^m(X_b \times X_b)_{\mathbb{Q}}$ be the corrected diagonal, whose action on $H_B^*(X_b, \mathbb{Q})$ is the projection on $H_B^m(X_b, \mathbb{Q})_{\mathrm{van}}$. On $X_b \times X_b \times X_b \times X_b$, there is the induced $2m$-cycle

$$\Delta_{b,\mathrm{van},2} := p_{13}^* \Delta_{b,\mathrm{van}} \cdot p_{24}^* \Delta_{b,\mathrm{van}},$$

where p_{ij} is the projection from X_b^4 to the product X_b^2 of its ith and jth factor. The action on $\Delta_{b,\mathrm{van},2}$ seen as a self-correspondence of X_b^2 on $H^*(X_b^2, \mathbb{Q})$ is the orthogonal projector on

$$p_1^* H_B^m(X_b, \mathbb{Q})_{\mathrm{van}} \otimes p_2^* H_B^m(X_b, \mathbb{Q})_{\mathrm{van}} \subset H_B^{2m}(X_b \times X_b, \mathbb{Q}).$$

If instead of $\Delta_{b,\mathrm{van},2}$, we consider

$$\Delta_{b,\mathrm{van},2}^{\tau} := p_{14}^* \Delta_{b,\mathrm{van}} \cdot p_{23}^* \Delta_{b,\mathrm{van}},$$

then the action on $\Delta_{b,\mathrm{van},2}$ seen as a self-correspondence of X_b^2 on $H_B^*(X_b^2,\mathbb{Q})$ is the composition of the previous projector with the permutation

$$\tau_* : H_B^m(X_b,\mathbb{Q})_{\mathrm{van}} \otimes H_B^m(X_b,\mathbb{Q})_{\mathrm{van}} \to H_B^m(X_b,\mathbb{Q})_{\mathrm{van}} \otimes H_B^m(X_b,\mathbb{Q})_{\mathrm{van}}$$

exchanging summands. Note that the inclusion

$$H_B^m(X_b,\mathbb{Q})_{\mathrm{van}} \otimes H_B^m(X_b,\mathbb{Q})_{\mathrm{van}} \subset H_B^{2m}(X_b \times X_b,\mathbb{Q})$$

sends the anti-invariant part on the left to the anti-invariant part under τ on the right if m is even, and to the invariant part under τ on the right if m is odd. This is due to the fact that the cup-product on cohomology is graded commutative.

Hence we conclude that

$$\Delta^{\sharp}_{b,\mathrm{van},2} := \Delta_{b,\mathrm{van},2} - \Delta^{\tau}_{b,\mathrm{van},2}$$

acts on $H_B^*(X_b^2,\mathbb{Q})$ as twice the projector onto $\bigwedge^2 H_B^m(X_b \times X_b,\mathbb{Q})_{\mathrm{van}}$ if m is even, and that

$$\Delta^{\mathrm{inv}}_{b,\mathrm{van},2} := \Delta_{b,\mathrm{van},2} + \Delta^{\tau}_{b,\mathrm{van},2}$$

acts on $H_B^*(X_b^2,\mathbb{Q})$ as twice the projector onto $\bigwedge^2 H_B^m(X_b \times X_b,\mathbb{Q})_{\mathrm{van}}$ if m is odd.

In both cases, using Lemma 4.36, we get that this is twice the orthogonal projector associated to a sub-Hodge structure of coniveau ≥ 1.

Restricting to the case of Calabi–Yau hypersurfaces in $\mathbb{P}^n$ (so $m = n-1$), an easy adaptation of the proof of Theorem 4.16 now gives the following result.

THEOREM 4.41. *Assume Conjecture 2.29 is true and the generalized Hodge conjecture holds for the coniveau* 1 *Hodge structure on* $\bigwedge^2 H_B^{n-1}(X_b \times X_b,\mathbb{Q})_{\mathrm{van}} \subset H_B^{2n-2}(X_b \times X_b,\mathbb{Q})$*, where* X_b *is a very general Calabi–Yau hypersurface in projective space. Then the general such* X_b *has the following properties:*

(i) If $n-1$ *is even, for any two* 0*-cycle* z, z' *of degree* 0 *on* X_b*, we have* $z \times z' - z' \times z = 0$ *in* $\mathrm{CH}_0(X \times X)$.

(ii) If $n-1$ *is odd, for any two* 0*-cycle* z, z' *of degree* 0 *on* X_b*, we have* $z \times z' + z' \times z = 0$ *in* $\mathrm{CH}_0(X \times X)$.

Note that the proof has to be adapted, because we are interested in the self-correspondence $\Delta^{\sharp}_{b,\mathrm{van},2}$ or $\Delta^{\mathrm{inv}}_{b,\mathrm{van},2}$ of $X_b \times X_b$, which is a cycle in X_b^4. This means that we have to work with cycles on the fourth fibered self-product of the universal family $\mathcal{X} \to B$. We refer to [114] for the extra arguments needed.

4.4 FURTHER APPLICATIONS TO THE BLOCH CONJECTURE ON 0-CYCLES ON SURFACES

As we have already mentioned in the previous section, the results obtained by this method are unconditional in the surface case. Furthermore, they can be improved to get further cases of the Bloch conjecture for 0-cycles on surfaces, or of the nilpotence conjecture for self-correspondences of surfaces. These improvements have been worked out in [109] which we follow closely.

Let $\mathcal{S} \to B$ be a smooth projective morphism with two-dimensional connected fibers, where B is quasi-projective. Let $\Gamma \in \mathrm{CH}^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$ be a relative 0-self-correspondence. Let $\Gamma_t := \Gamma_{|\mathcal{S}_t \times \mathcal{S}_t}$ be the restricted cycle, with cohomology class $[\Gamma_t] \in H^4(\mathcal{S}_t \times \mathcal{S}_t, \mathbb{Q})$. We have the actions

$$\Gamma_{t*} : \mathrm{CH}_0(\mathcal{S}_t)_{\mathbb{Q}} \to \mathrm{CH}_0(\mathcal{S}_t)_{\mathbb{Q}}, \qquad [\Gamma_t]^* : H^{i,0}(\mathcal{S}_t) \to H^{i,0}(\mathcal{S}_t).$$

Theorem 4.42 (Voisin 2012). *Assume the following:*

(1) The fibers $\mathcal{S}_t$ satisfy $h^{1,0}(\mathcal{S}_t) = 0$ and $[\Gamma_t]^* : H^{2,0}(\mathcal{S}_t) \to H^{2,0}(\mathcal{S}_t)$ is equal to 0.

(2) A smooth projective (equivalently any smooth projective) completion $\overline{\mathcal{S} \times_B \mathcal{S}}$ of the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ is rationally connected.

Then $\Gamma_{t*} : \mathrm{CH}_0(\mathcal{S}_t)_{\mathrm{hom}} \to \mathrm{CH}_0(\mathcal{S}_t)_{\mathrm{hom}}$ is nilpotent for any $t \in B$.

We refer to [109] for geometric applications of this statement. They include a proof of the Bloch conjecture for Catanese surfaces (and, as a by-product, for determinantal Barlow surfaces).

Sketch of proof of Theorem 4.42. We first construct a cycle

$$\Gamma' \in \mathrm{CH}^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$$

with the properties that Γ'_t is cohomologous to 0 in $\mathcal{S}_t \times \mathcal{S}_t$ and that Γ'_t acts as Γ_t on $\mathrm{CH}_0(\mathcal{S}_t)_{\mathrm{hom}}$. The existence of Γ' follows from the assumption that $H^{1,0}(\mathcal{S}_t) = 0$ and $[\Gamma_t]^* = 0$ on $H^{2,0}(\mathcal{S}_t)$, which says equivalently that the cohomology class of Γ_t belongs to $\mathrm{pr}_1^* H^4(\mathcal{S}_t) \oplus \mathrm{pr}_2^* H^4(\mathcal{S}_t) \oplus \mathrm{pr}_1^* \mathrm{NS}(\mathcal{S}_t)_{\mathbb{Q}} \otimes \mathrm{NS}(\mathcal{S}_t)_{\mathbb{Q}}$.

As $\Gamma'_{t*} = \Gamma_t t*$ on $\mathrm{CH}_0(\mathcal{S}_t)_{\mathrm{hom}}$, it suffices to prove the conclusion for Γ'. The same arguments as in the proof of Theorem 4.16 then show that there exist codimension 2 algebraic cycles $\mathcal{Z}'_1$, $\mathcal{Z}'_2$ with $\mathbb{Q}$-coefficients on $\mathcal{S}$ such that

$$[\Gamma' - \mathcal{Z} - p_1^*\mathcal{Z}'_1 - p_2^*\mathcal{Z}'_2] = 0 \text{ in } H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q}).$$

We now claim that under assumption (2) the following hold:

(i) The cycle $\Gamma - \mathcal{Z} - p_1^*\mathcal{Z}'_1 - p_2^*\mathcal{Z}'_2$ is algebraically equivalent to 0 on $\mathcal{S} \times_B \mathcal{S}$.

(ii) The restriction to the fibers $\mathcal{S}_t \times \mathcal{S}_t$ of the codimension 2 cycle $\mathcal{Z}' = \Gamma' - \mathcal{Z} - p_1^*\mathcal{Z}'_1 - p_2^*\mathcal{Z}'_2$ is a nilpotent element (with respect to the composition of self-correspondences) of $\mathrm{CH}^2(\mathcal{S}_t \times \mathcal{S}_t)_{\mathbb{Q}}$.

This is clearly sufficient to conclude the proof of Theorem 4.42 since the cycle $\mathcal{Z}'_t$ acts as Γ'_t on $\mathrm{CH}_0(\mathcal{S}_t)$.

To prove the claim, we work with a smooth projective completion $\overline{\mathcal{S}\times_B\mathcal{S}}$. Let $D := \overline{\mathcal{S}\times_B\mathcal{S}}\setminus\mathcal{S}\times_B\mathcal{S}$ be the divisor at infinity. Let $\widetilde{D}\stackrel{j}{\to}\overline{\mathcal{S}\times_B\mathcal{S}}$ be a desingularization of D. The codimension 2 cycle $\mathcal{Z}'$ extends to a cycle $\overline{\mathcal{Z}'}$ over $\overline{\mathcal{S}\times_B\mathcal{S}}$. We know that

$$[\overline{\mathcal{Z}'}]_{|\mathcal{S}\times_B\mathcal{S}} = 0 \text{ in } H^4(\mathcal{S}\times_B\mathcal{S},\mathbb{Q})$$

and this implies, by Corollary 2.24, that there is a degree 2 Hodge class α on $\widetilde{D}$ such that

$$j_*\alpha = [\overline{\mathcal{Z}'}] \text{ in } H^4(\overline{\mathcal{S}\times_B\mathcal{S}},\mathbb{Q}).$$

By the Lefschetz theorem on $(1,1)$-classes, α is the class of a codimension 1 cycle $\mathcal{Z}''$ of $\widetilde{D}$ and we conclude that

$$[\overline{\mathcal{Z}'} - j_*\mathcal{Z}''] = 0 \text{ in } H^4(\overline{\mathcal{S}\times_B\mathcal{S}},\mathbb{Q}).$$

Replacing $\overline{\mathcal{Z}'}$ by $\overline{\mathcal{Z}'}-j_*\mathcal{Z}''$, we have thus proved that the codimension 2 cycle $\mathcal{Z}'$ which is cohomologous to 0 on $\mathcal{S}\times_B\mathcal{S}$ extends to a cycle $\overline{\mathcal{Z}'}$ on $\overline{\mathcal{S}\times_B\mathcal{S}}$ *which is also cohomologous to* 0.

We use now assumption (2) which says that the variety $\overline{\mathcal{S}\times_B\mathcal{S}}$ is rationally connected. It then has trivial CH_0, and so any codimension 2 cycle homologous to 0 on $\overline{\mathcal{S}\times_B\mathcal{S}}$ is algebraically equivalent to 0 by Theorem 3.14. We thus conclude that $\overline{\mathcal{Z}'}$ is algebraically equivalent to 0 on $\overline{\mathcal{S}\times_B\mathcal{S}}$, hence that $\mathcal{Z}' = (\overline{\mathcal{Z}'})_{|\mathcal{S}\times_B\mathcal{S}}$ is algebraically equivalent to 0 on $\mathcal{S}\times_B\mathcal{S}$.

Statement (ii) is a direct consequence of (i), using the nilpotence theorem (Theorem 3.25). □

Chapter Five

On the Chow ring of $K3$ surfaces and hyper-Kähler manifolds

This chapter is devoted to a completely different application of Theorem 3.1. We will consider varieties whose Chow ring has rather special properties. This includes abelian varieties, $K3$ surfaces, and Calabi–Yau hypersurfaces in projective space. For $K3$ surfaces S, it was discovered in [11] that they have a canonical 0-cycle o of degree 1 with the property that the product of two divisors of S is a multiple of o in $\mathrm{CH}_0(S)$. In [110], we extended this result to Calabi–Yau hypersurfaces in projective space. Another feature is a decomposition in $\mathrm{CH}(X \times X \times X)_{\mathbb{Q}}$ of the small diagonal $\Delta \subset X \times X \times X$ that was established for $K3$ surfaces in [11], and is partially extended in [110] to Calabi–Yau hypersurfaces. Finally, we use this decomposition and the spreading principle (Theorem 3.1) to show, following [110], that for families $\pi : \mathcal{X} \to B$ of smooth projective $K3$ surfaces, there is a decomposition isomorphism

$$R\pi_*\mathbb{Q} \cong \oplus R^i\pi_*\mathbb{Q}[-i]$$

that is multiplicative (that is, compatible with the cup-product on both sides) over a nonempty Zariski dense open set of B. Numerous examples show that this statement, which is also true for families of abelian varieties, is rarely satisfied.

5.1 TAUTOLOGICAL RING OF A $K3$ SURFACE

The following theorem is proved in [11].

THEOREM 5.1 (Beauville and Voisin 2004). *Let S be a $K3$ surface, $D_i \in \mathrm{CH}^1(S)$ be divisors on S and n_{ij} be integers. Then if the 0-cycle $\sum_{i,j} n_{ij} D_i D_j \in \mathrm{CH}_0(S)$ is cohomologous to 0 on S, it is equal to 0 in $\mathrm{CH}_0(S)$.*

PROOF. It suffices to prove that there is a 0-cycle o of degree 1, with the property that for any two line bundles L, L' on S,

$$L \cdot L' = \deg(L \cdot L')\, o \text{ in } \mathrm{CH}_0(S). \tag{5.1}$$

The cycle o is defined to be the class of any point of S contained in a (singular) rational curve $C \subset S$. We claim that this cycle does not depend on the choice of rational curve. This follows from the fact that there are rational curves C_0 in any *ample* linear system on S. If C, C' are two rational curves on

S, then any point in the intersection of C and C_0 is rationally equivalent to any point supported on C or on C_0, and also to any point supported on C' if $C' \cap C_0$ is nonempty. As C_0 is ample, the intersections $C_0 \cap C$, $C_0 \cap C'$ are nonempty, and this concludes the proof of the claim.

The proof of (5.1) is obtained by reducing to the case where L and L' are ample line bundles. Then there are rational curves C, C' in $|L|$ and $|L'|$. The intersection $L \cdot L' = C \cdot C'$ is then by definition proportional to o. □

The cycle o has the following quite remarkable property (which makes it intrinsically defined).

Proposition 5.2 (Beauville and Voisin 2004). *We have the following equality:*

$$c_2(T_S) = 24\, o \text{ in } \mathrm{CH}_0(S).$$

Proposition 5.2 is a consequence of the following result from [11], which will be proved in Section 5.2. Let $\Delta = \{(x, x, x),\, x \in S\}$ be the small diagonal in $S \times S \times S$.

Theorem 5.3 (Beauville and Voisin 2004). *We have the following equality in* $\mathrm{CH}^4(S \times S \times S)_{\mathbb{Q}}$*:*

$$\Delta = \Delta_{12} \cdot o_3 + (\text{perm.}) - (o_1 \cdot o_2 + (\text{perm.})). \tag{5.2}$$

Here $\Delta \subset S \times S \times S$ is the small diagonal $\{(x, x, x),\, x \in S\}$. The class $o \in \mathrm{CH}_0(S)$ is the class of any point as above and the o_i's are its pull-back in $\mathrm{CH}^2(S \times S \times S)_{\mathbb{Q}}$ via the various projections. The cycle $\Delta_{12} \cdot o_3$ is then the class of the algebraic subset $\{(x, x, o),\, x \in S\}$ of $S \times S \times S$. The terms "+(perm.)" mean that we symmetrize the given expression in the indices $1, 2, 3$. As our cycles $\Delta_{12} \cdot o_3$ and $o_1 \cdot o_2 \times S$ are invariant under the transposition exchanging 1 and 2, there are only three terms of each sort in these sums.

Proof of Proposition 5.2. Let us restrict equality (5.2) to

$$(j_S, \mathrm{Id})(S \times S) \subset S \times S \times S,$$

where $j_S : S \to S \times S$ is the inclusion of the diagonal. We get the following equality in $\mathrm{CH}^4(S \times S)_{\mathbb{Q}}$:

$$\begin{aligned}(j_S, \mathrm{Id})^* \Delta = {} & (\Delta_{12} \cdot o_3)_{|(j_S,\mathrm{Id})(S\times S)} + (\Delta_{13} \cdot o_2)_{|(j_S,\mathrm{Id})(S\times S)} \\ & + (o_1 \cdot \Delta_{23})_{|(j_S,\mathrm{Id})(S\times S)} - (o_1 \cdot o_2)_{|(j_S,\mathrm{Id})(S\times S)} \\ & - (o_1 \cdot o_3)_{|(j_S,\mathrm{Id})(S\times S)} - (o_2 \cdot o_3)_{|(j_S,\mathrm{Id})(S\times S)}.\end{aligned} \tag{5.3}$$

The left-hand side is clearly equal to the self-intersection $\Delta_S^2 \in \mathrm{CH}^4(S \times S)_{\mathbb{Q}}$, that is, to $j_{S*} c_2(T_S)$. The right-hand side is equal to

$$c_2 \times o + (o, o) + (o, o) - (o, o) - (o, o) = c_2 \times o.$$

Hence we obtain the equality

$$j_{S*}c_2(T_S) = c_2 \times o \text{ in } \mathrm{CH}_0(S \times S)_{\mathbb{Q}}.$$

Applying the second projection to this equality, we conclude that

$$c_2(T_S) = \deg(c_2(T_S))\, o = 24\, o \text{ in } \mathrm{CH}_0(S)_{\mathbb{Q}},$$

and the equality is in fact also true in $\mathrm{CH}_0(S)$ since $\mathrm{CH}_0(S)$ has no torsion by Roitman's theorem [87]. □

REMARK 5.4. Theorem 5.3 also implies Theorem 5.1 because we can see the small diagonal Δ as a correspondence between $S \times S$ and S. Then we have

$$\Delta_* \left(\sum_{i,j} n_{ij} \operatorname{pr}_1^* D_i \cdot \operatorname{pr}_2^* D_j \right) = \sum_{i,j} n_{ij} D_i \cdot D_j \text{ in } \mathrm{CH}_0(S)_{\mathbb{Q}},$$

and the left-hand side is computed using formula (5.2). The conclusion is that the cycle $\sum_{i,j} n_{ij} D_i D_j$ is a multiple of o in $\mathrm{CH}_0(S)_{\mathbb{Q}}$ and thus it is trivial if it is homologous to 0. However, one should be warned that Theorem 5.1 is in fact used in the proof of Theorem 5.3 (see below) so that it cannot be seen as a consequence of it.

We conclude with another remarkable property of the cycle o.

LEMMA 5.5. *Let $j_S : S \hookrightarrow S \times S$ be the diagonal inclusion. Then for any line bundle $L \in \operatorname{Pic} S$, we have*

$$j_{S*}L = L \times o + o \times L \text{ in } \mathrm{CH}_1(S \times S). \tag{5.4}$$

PROOF. Both sides are $\mathbb{Z}$-linear in L. We use the fact that $\operatorname{Pic} S$ is generated by $\mathcal{O}_S(C)$, where C is a (singular) rational curve in S. For the normalization $\widetilde{C} \cong \mathbb{P}^1$ of C, we have

$$\Delta_{\widetilde{C}} = \widetilde{C} \times o_1 + o_1 \times \widetilde{C} \text{ in } \mathrm{CH}_1(\widetilde{C} \times \widetilde{C}) \tag{5.5}$$

for any point o_1 of $\widetilde{C}$. Let $n : \widetilde{C} \to S$ be the natural map. By definition of o, we have $n_* o_1 = o$ in $\mathrm{CH}_0(S)$, and applying $(n, n)_*$ to (5.5) thus gives the desired result. □

5.1.1 Other hyper-Kähler manifolds

Recall that (irreducible) hyper-Kähler manifolds are simply connected projective complex varieties with $H^{2,0}(X) = \mathbb{C}\eta$, where η is an everywhere-nondegenerate holomorphic 2-form on X. One famous series of examples is constructed in [7]: one considers the Hilbert scheme $S^{[n]}$ of length n subschemes of an algebraic $K3$ surface S. Such hyper-Kähler varieties admit projective deformations that are not obtained by the same construction, but they are not well understood except in a few cases, namely the four different families of hyper-Kähler fourfolds constructed in the papers [10], [29], [55], [77]. The varieties constructed by Beauville and Donagi are obtained as Fano varieties of lines of smooth cubic fourfolds in $\mathbb{P}^5$.

In [9], Beauville conjectured that a result similar to Theorem 5.1 holds for algebraic hyper-Kähler varieties.

Conjecture 5.6 (Beauville 2007). *Let Y be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_i)]) = 0 \text{ in } H^*(Y, \mathbb{Q}), \quad L_i \in \operatorname{Pic} Y$$

already holds at the level of Chow groups:

$$P(c_1(L_i)) = 0 \text{ in } \operatorname{CH}(Y)_{\mathbb{Q}}.$$

Beauville proved this conjecture in [9] in the case of the second and third punctual Hilbert scheme of an algebraic $K3$ surface.

In the paper [106], we stated the following more general conjecture concerning the Chow ring of an (irreducible algebraic) hyper-Kähler variety. Namely, a synthesis of Theorem 5.1 and Proposition 5.2 is the statement that any polynomial relation between the cohomology classes $[c_2(T_S)], [c_1(L_i)]$ in $H^*(S, \mathbb{Q})$ already holds between the cycles $c_2(T_S)$, $c_1(L_i)$ in $\operatorname{CH}(S)$.

Conjecture 5.7 (Voisin 2008). *Let Y be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_j)], [c_i(T_Y)]) = 0 \text{ in } H^{2k}(Y, \mathbb{Q}), \quad L_j \in \operatorname{Pic} Y$$

already holds at the level of Chow groups:

$$P(c_1(L_j), c_i(T_Y)) = 0 \text{ in } \operatorname{CH}^k(Y)_{\mathbb{Q}}.$$

The following results are proved in [106].

Theorem 5.8 (Voisin 2008).

(1) Conjecture 5.7 holds for $Y = S^{[n]}$, where $S^{[n]}$ is the Hilbert scheme of length n subschemes of an algebraic $K3$ surface S, in the range $n \leq 2b_2(S)_{\mathrm{tr}} + 4$.

(2) Conjecture 5.7 is true for any k when Y is the Fano variety of lines of a cubic fourfold.

We will not comment on the proof of (2). Let us just say that the result in (2) was partially extended by Ferretti in [40] to the case of O'Grady fourfolds (see [77]).

In item (1) above, the number $b_2(S)_{\mathrm{tr}}$ is equal to $b_2(S) - \rho(S) = 22 - \rho(S)$, where $\rho(S)$ is the Picard number of S. In particular, we have $b_2(S)_{\mathrm{tr}} \geq$

$2h^{2,0}(S) = 2$ and the conjecture is true for $Y = S^{[n]}$, where S is any $K3$ surface and $n \leq 8$. More importantly, the method of proof for (1) shows Theorem 5.10 below.

The next theorem is conditional on the following conjecture, involving only the usual products S^m of S.

CONJECTURE 5.9 (Voisin 2008). *Let S be an algebraic $K3$ surface. For any integer m, let $P \in \mathrm{CH}(S^m)_{\mathbb{Q}}$ be a polynomial expression in*

$$\mathrm{pr}_i^* c_1(L_s), \quad L_s \in \mathrm{Pic}\, S, \qquad \mathrm{pr}_j^* o, \qquad \mathrm{pr}_{kl}^* \Delta_S,$$

where the pr_i and pr_{kl} are the projections from S^m to S and $S \times S$, respectively.

Then if $[P] = 0$ in $H^*(S^m, \mathbb{Q})$, we have $P = 0$ in $\mathrm{CH}(S^m)_{\mathbb{Q}}$.

Note that by Theorems 5.1 and 5.3, we may assume in the above Conjecture 5.9 that the polynomial P involves only monomials

$$\Pi_{(i,j,k,l)\in\{1,\ldots,m\}^4}\, \mathrm{pr}_i^* c_1(L_s) \cdot \mathrm{pr}_j^* o \cdot \mathrm{pr}_{kl}^* \Delta_S$$

with four different indices i, j, k, l. Indeed, relation (5.2) can be written as a relation of the form

$$p_{12}^*\Delta_S \cdot p_{23}^*\Delta_S = P(p_{ij}^*\Delta_S, p_k^* o) \text{ in } \mathrm{CH}^4(S \times S \times S)_{\mathbb{Q}},$$

where the polynomial P involves only monomials where the three indices i, j, k are distinct. Similarly, relation (5.4) can be written as

$$\Delta_S \cdot p_2^* L = P'(p_i^* L, p_j^* o) \text{ in } \mathrm{CH}^3(S \times S)_{\mathbb{Q}},$$

where the polynomial P' involves only monomials where the two indices i, j are distinct. Note also that Conjecture 5.9 is very strong because for $K3$ surfaces it implies Kimura's finite-dimensionality conjecture (Conjecture 3.27). Indeed, for a $K3$ surface S, we have $\dim H^*(S, \mathbb{Q}) = 24$, so $\bigwedge^{25} H^*(S, \mathbb{Q}) = 0$. This also says that the cycle

$$\Gamma^{\mathrm{alt}} \in \mathrm{CH}^{50}(S^{50})_{\mathbb{Q}},$$

which is the skew-symmetric projector defining the motive $\bigwedge^{25} S$ (see Section 3.2.3), that is,

$$\Gamma^{\mathrm{alt}} = \frac{1}{25!} \sum_{\sigma \in \mathfrak{S}_{25}} \epsilon(\sigma)\Gamma_\sigma,$$

is cohomologous to 0. On the other hand, each Γ_σ, being the graph of a permutation, can be expressed as a product of diagonals $\mathrm{pr}_{ij}^* \Delta_S$. So Conjecture 5.9 would imply that Γ^{alt} is rationally equivalent to 0 in S^{50}, hence that $\bigwedge^{25} S = 0$.

THEOREM 5.10. *Conjecture 5.7 is implied by Conjecture 5.9.*

It could be the case that the converse is also true, and this might be proved by looking more closely at the proof of Theorem 5.10 (and more precisely Proposition 5.14) below.

Let us give an idea of the proof of these theorems. It mainly uses the inductive method of [36] and the result of [27] computing (additively) the Chow groups of $\mathrm{CH}(S^{[m]})_{\mathbb{Q}}$ in terms of Chow groups of strata of $S^{(m)}$ (see below).

This inductive method necessitates proving a more general statement (Theorem 5.11), as follows.

There are two natural vector bundles on $S^{[n]}$, namely $\mathcal{O}_{[n]}$ on the one hand, which is defined as $R^0p_*\mathcal{O}_{\Sigma_n}$, where

$$\Sigma_n \subset S^{[n]} \times S, \qquad p = \mathrm{pr}_1 : \Sigma_n \to S^{[n]}$$

is the incidence subscheme, and on the other hand the tangent bundle T_n of $S^{[n]}$. It is not clear that the Chern classes of $\mathcal{O}_{[n]}$ can be expressed as polynomials in $c_1(\mathcal{O}_{[n]})$ and the Chern classes of T_n.

Theorem 5.11 (Voisin 2008). *Let $n \leq 2b_2(S)_{\mathrm{tr}} + 4$, and let $P \in \mathrm{CH}(S^{[n]})_{\mathbb{Q}}$ be any polynomial expression in the variables*

$$c_1(L), \quad L \in \operatorname{Pic} S \subset \operatorname{Pic} S^{[n]}, \qquad c_i(\mathcal{O}_{[n]}), \qquad c_j(T_n) \in \mathrm{CH}(S^{[n]})_{\mathbb{Q}}.$$

Then if P is cohomologous to 0, we have $P = 0$ in $\mathrm{CH}(S^{[n]})_{\mathbb{Q}}$.

This implies Theorem 5.8 for the nth Hilbert scheme of $K3$ surface S with $n \leq 2b_2(S)_{\mathrm{tr}} + 4$, because we have $c_1(\mathcal{O}_{[n]}) = -\delta$, where $2\delta \equiv E$ is the class of the exceptional divisor of the resolution $S^{[n]} \to S^{(n)}$, and it is well known that $\operatorname{Pic} S^{[n]}$ is generated by $\operatorname{Pic} S$ and δ.

The proof of this theorem uses the following proposition.

Proposition 5.12. *Let $P \in \mathrm{CH}(S^m)_{\mathbb{Q}}$ be a polynomial expression in the variables*

$$\mathrm{pr}_i^*\left(\frac{1}{24}c_2(T)\right) = \mathrm{pr}_i^*\, o, \qquad \mathrm{pr}_j^*\, c_1(L_s), \quad L_s \in \operatorname{Pic} S, \qquad \mathrm{pr}_{kl}^*\, \Delta_S, \quad k \neq l,$$

where $\Delta_S \subset S \times S$ is the diagonal. Assume that one of the following assumptions is satisfied:

(1) $m \leq 2b_2(S)_{\mathrm{tr}} + 1$.

(2) P is invariant under the action of the symmetric group $\mathfrak{S}_{m-2}$ acting on the first $m-2$ indices.

Then if P is cohomologous to 0, it is equal to 0 in $\mathrm{CH}(S^m)_{\mathbb{Q}}$.

Using the results of [11] described in the previous sections (Theorem 5.1 and Proposition 5.2), this proposition is a consequence of the following lemma (see [106] for the proof).

LEMMA 5.13. *The polynomial relations* $[P] = 0$ *in the cohomology ring* $H^*(S^m)$*, satisfying one of the above assumptions on* m, P*, are all generated (as elements of the ring of all polynomial expressions in the variables above) by the following polynomial relations, the list of which will be denoted by* $(*)$*:*

(1) $[\mathrm{pr}_i^*(c_1(L)) \cdot \mathrm{pr}_i^* o] = 0$, $L \in \mathrm{Pic}\, S$, $[\mathrm{pr}_i^*(o) \cdot \mathrm{pr}_i^*(o)] = 0$.

(2) $[\mathrm{pr}_i^*(c_1(L)^2 - [c_1(L)]^2 o)] = 0$, $L \in \mathrm{Pic}\, S$.

(3) $[\mathrm{pr}_{ij}^*(\Delta_S \cdot p_1^* o - (o, o))] = 0$*, where* p_1 *here is the first projection of* $S \times S$ *to* S*, and* $(o, o) = p_1^* o \cdot p_2^* o$.

(4) $[\mathrm{pr}_{ij}^*(\Delta_S . p_1^* c_1(L) - c_1(L) \times o - o \times c_1(L))] = 0$, $L \in \mathrm{Pic}\, S$*, where* p_1 *here is the first projection of* $S \times S$ *to* S*, and* $c_1(L) \times o = p_1^* c_1(L) \cdot p_2^* o$.

(5) $[\mathrm{pr}_{ijk}^*(\Delta - p_{12}^*\Delta_S \cdot p_3^* o - p_1^* o \cdot p_{23}^* \Delta_S - p_{13}^* \Delta_S \cdot p_2^* o + p_{12}^*(o, o) + p_{23}^*(o, o) + p_{13}^*(o, o))] = 0$.

(6) $[\mathrm{pr}_{ij}^* \Delta_S]^2 = 24\, \mathrm{pr}_{ij}^*(o, o) = 24\, \mathrm{pr}_i^* o \cdot \mathrm{pr}_j^* o$.

In (5) above, Δ is, as before, the small diagonal of S^3, and the p_i and p_{ij} are the various projections from S^3 to S and $S \times S$, respectively. Note that Δ can be expressed as $p_{12}^*\Delta_S \cdot p_{23}^*\Delta_S$. Furthermore we have

$$\mathrm{pr}_{ij}^* \circ p_1^* = \mathrm{pr}_i^*, \qquad \mathrm{pr}_{ijk}^* \circ p_{12}^* = \mathrm{pr}_{ij}^*, \qquad \mathrm{pr}_{ijk}^* \circ p_i^* = \mathrm{pr}_i^* .$$

Thus all the relations in $(*)$ are actually polynomial expressions in the variables

$$[\mathrm{pr}_i^* o], \qquad [\mathrm{pr}_j^* c_1(L)], \quad L \in \mathrm{Pic}\, S, \qquad [\mathrm{pr}_{kl}^* \Delta_S], \quad k \neq l.$$

PROOF OF PROPOSITION 5.12. Using Lemma 5.13, we conclude that under one of the assumptions of Proposition 5.12 on m, P, all polynomial relations $[P] = 0$ in the variables $\mathrm{pr}_i^* o$, $\mathrm{pr}_j^* c_1(L)$, $L \in \mathrm{Pic}\, S$, $\mathrm{pr}_{kl}^* \Delta_S$, $k \neq l$ that hold in $H^*(S^m, \mathbb{Q})$ also hold in $\mathrm{CH}(S^m)_\mathbb{Q}$, because we know by Theorem 5.1, Proposition 5.2, and Theorem 5.3 that the cohomological relations listed in $(*)$ hold in $\mathrm{CH}(S^m)_\mathbb{Q}$. In fact (apart from the relations (1) and (3) above, which obviously hold in $\mathrm{CH}(S^m)_\mathbb{Q}$), these relations are pulled back, via the maps pr_i, pr_{ij}, and pr_{ijk}, from relations in $\mathrm{CH}(S)_\mathbb{Q}$, $\mathrm{CH}(S^2)_\mathbb{Q}$, and $\mathrm{CH}(S^3)_\mathbb{Q}$, respectively, which are established there.

Similarly, for any m, the same conclusion holds for polynomial relations invariant under $\mathfrak{S}_{m-2}$.

This concludes the proof of Proposition 5.12. □

In order to sketch the proof of Theorem 5.11, let us introduce the following notation: Let

$$\mu = \{\mu_1, \ldots, \mu_m\}, \quad m = m(\mu), \quad \sum_i |\mu_i| = n$$

be a partition of $\{1,\dots,n\}$. Such a partition determines a partial diagonal

$$S_\mu \cong S^m \subset S^n,$$

defined by the conditions

$$x = (x_1,\dots,x_n) \in S_\mu \Leftrightarrow x_i = x_j \text{ if } i,\, j \in \mu_l \text{ for some } l.$$

Consider the quotient map

$$q_\mu : S^m \cong S_\mu \to S^{(n)},$$

and denote by E_μ the following fibered product:

$$E_\mu := S_\mu \times_{S^{(n)}} S^{[n]} \subset S^m \times S^{[n]}.$$

We view E_μ as a correspondence between S^m and $S^{[n]}$ and as usual we will denote by $E_\mu^* : \mathrm{CH}(S^{[n]})_{\mathbb{Q}} \to \mathrm{CH}(S^m)_{\mathbb{Q}}$ the map

$$\alpha \mapsto \mathrm{pr}_{1*}(\mathrm{pr}_2^*(\alpha)\cdot E_\mu).$$

Let us denote by $\mathfrak{S}_\mu$ the subgroup of $\mathfrak{S}_m$ permuting only the indices i, j for which the cardinalities of μ_i, μ_j are equal. The group $\mathfrak{S}_\mu$ can be seen as the quotient of the global stabilizer of S_μ in S^n by its pointwise stabilizer. In this way the action of $\mathfrak{S}_\mu$ on $S_\mu \cong S^m$ is induced by the action of $\mathfrak{S}_n$ on S^n.

We have the following result.

Proposition 5.14. *Let* $P \in \mathrm{CH}(S^{[n]})_{\mathbb{Q}}$ *be a polynomial expression in the variables* $c_i(\mathcal{O}_{[n]})$, $c_j(T_n)$. *Then for any* μ *as above,* $E_\mu^*(P) \in \mathrm{CH}(S^m)$ *is a polynomial expression in* $\mathrm{pr}_s^* o$, $\mathrm{pr}_{lk}^* \Delta_S$. *Furthermore,* $E_\mu^*(P)$ *is invariant under the group* $\mathfrak{S}_\mu$.

Note that the last statement is obvious, since $\mathfrak{S}_\mu$ leaves invariant the correspondence $E_\mu \subset S_\mu \times S^{[n]}$.

The proof of this proposition is rather painful and we refer to [106] for the detail. It is here that we use the Ellingsrud–Göttsche–Lehn method.

Admitting this proposition, we give now the proofs of the theorems, following [106].

Proof of Theorem 5.11. From the work of de Cataldo and Migliorini [27], we know that the map

$$(E_\mu^*)_{\mu\in\mathrm{Part}(\{1,\dots,n\})} : \mathrm{CH}(S^{[n]})_{\mathbb{Q}} \to \bigoplus_\mu \mathrm{CH}(S^{m(\mu)})_{\mathbb{Q}}$$

is injective. Now let $P \in \mathrm{CH}(S^{[n]})_{\mathbb{Q}}$ be a polynomial expression in $c_1(L)$, $L \in \mathrm{Pic}\, S \subset \mathrm{Pic}\, S^{[n]}$, $c_i(\mathcal{O}_{[n]})$, $c_j(T_n) \in \mathrm{CH}(S^{[n]})_{\mathbb{Q}}$. Note first that for $L \in \mathrm{Pic}\, S$, and for each μ, the restriction of $\mathrm{pr}_2^* L$ to $E_\mu \subset S_\mu \times S^{[n]}$ is a pull-back $\mathrm{pr}_1^* L_{\mu|E_\mu}$, where $L_\mu \in \operatorname{Pic} S_\mu = \operatorname{Pic} S^m$ is equal to $L^{\otimes|\mu_1|} \boxtimes \cdots \boxtimes L^{\otimes|\mu_m|}$. This follows from the fact that L is the pull-back of a line bundle on $S^{(n)}$. Note that L_μ is invariant under $\mathfrak{S}_\mu$.

Thus it follows from Proposition 5.14 and the projection formula that for each partition μ, $E_\mu^*(P)$ is a polynomial expression in $\mathrm{pr}_i^* c_1(L)$, $\mathrm{pr}_k^* o$, $\mathrm{pr}_{lm}^* \Delta$ which is invariant under the group $\mathfrak{S}_\mu$.

Now, if P is cohomologous to 0, each $E_\mu^*(P)$ is cohomologous to 0. Let us now verify that the assumptions of Proposition 5.12 are satisfied. Recall that we assume $n \leq 2b_2(S)_{\mathrm{tr}} + 4$. If $m(\mu) \leq 2b_2(S)_{\mathrm{tr}} + 1$, Proposition 5.12 applies. Otherwise, $m(\mu) \geq 2b_2(S)_{\mathrm{tr}} + 2$ and, as $n \leq 2b_2(S)_{\mathrm{tr}} + 4$, it follows that the partition μ contains at most two sets of cardinality ≥ 2. Thus the group $\mathfrak{S}_\mu$ contains in this case a group conjugate to $\mathfrak{S}_{m(\mu)-2}$. Proposition 5.12 thus applies, and gives $E_\mu^*(P) = 0$ in $\mathrm{CH}(S_\mu)_\mathbb{Q}$ for all μ.

It follows that $P = 0$ by the result of de Cataldo and Migliorini. This concludes the proof of Theorem 5.11. □

To conclude, let us note that Proposition 5.14 and the end of the proof of Theorem 5.11 also prove Theorem 5.10.

5.2 A DECOMPOSITION OF THE SMALL DIAGONAL

In this section we give the proof of Theorem 5.3. We first recall its statement.

THEOREM 5.15 (Beauville and Voisin 2004). *We have equality of cycles in* $\mathrm{CH}^4(S \times S \times S)_\mathbb{Q}$*:*

$$\Delta = \Delta_{12} \cdot o_3 + (\text{perm.}) - (o_1 \cdot o_2 + (\text{perm.})). \tag{5.6}$$

REMARK 5.16. This decomposition is very particular. Even for curves (say general curves of genus $g \geq 3$ and for cycles modulo algebraic equivalence), a similar decomposition of the small diagonal does not exist. This follows from Ceresa's result [18] as explained in [11]. Note that for curves, and for cycles modulo algebraic equivalence, there are relations established by Colombo and van Geemen [25] in $\mathrm{CH}_1(C^k)_\mathbb{Q}$ involving the various diagonals, that is, 1-cycles of the form (up to permutation of factors) $\{(x, \ldots, x, o, \ldots, o),\, x \in C\}$ with r terms x and $k - r$ terms o, where o is a fixed point, but the integer k depends on the gonality of the curve.

The proof of Theorem 5.15 will use the analogous result for elliptic curves.

PROPOSITION 5.17. *Let E be an elliptic curve and let l be a divisor of degree d on E:*

(i) For any $x \in E$, we have the following equality in $\mathrm{CH}_0(E \times E)$*:*

$$d^2(x, x) = d(x \times l + l \times x) - l \times l. \tag{5.7}$$

(ii) We have the following equality of cycles in $\mathrm{CH}^2(E\times E\times E)_{\mathbb{Q}}$*:*

$$d^2\Delta = d[\Delta_{12}\times l_3 + (\text{perm.})] - (l_1\times l_2\times E + (\text{perm.})), \tag{5.8}$$

where again $\Delta\subset E\times E\times E$ *is the small diagonal, and* $l_1\times l_2\times E := \mathrm{pr}_1^* l\cdot \mathrm{pr}_2^* l$ *etc.*

Proof. (i) Indeed, both cycles in (5.7) are symmetric with respect to the involution ι exchanging factors of $E\times E$. Hence (up to torsion) they come from cycles in $\mathrm{CH}_0(E\times E/\iota)$. The group $\mathrm{CH}_0(E\times E/\iota)$ is representable, because the quotient $E\times E/\iota = E^{(2)}$ is a $\mathbb{P}^1$-bundle over E. In other words, the Chow group $\mathrm{CH}_0(E^{(2)})$ is an extension of $\mathbb{Z}$ by the Albanese variety of $S^{(2)}E$. In order to check (5.7) (at least up to torsion), it thus suffices to verify that both sides have the same degree and that their difference has a trivial Albanese invariant, which is elementary. This proves a priori the result only up to torsion, but as Roitman's theorem [87] says that the Albanese map is injective on the torsion of CH_0, the result is actually true in $\mathrm{CH}_0(E\times E)$.

(ii) We use (i) and apply Corollary 3.8 to the small diagonal of E seen as a correspondence between E and $E\times E$. We thus deduce that there are points p_i of E and cycles $Z_i\in \mathrm{CH}_1(E\times E)_{\mathbb{Q}}$ such that the following equality holds in $\mathrm{CH}^2(E\times E\times E)_{\mathbb{Q}}$:

$$d^2\Delta = d[\Delta_{12}\times l_3 + (\text{perm.})] - (l_1\times l_2\times E + (\text{perm.})) + \sum_i p_i\times Z_i. \tag{5.9}$$

Using the fact that the 1-cycle $d^2\Delta - d[\Delta_{12}\times l_3 + (\text{perm.})] + (l_1\times l_2\times E + (\text{perm.}))$ is invariant under the symmetric group $\mathfrak{S}_3$, we conclude that the 1-cycle $\sum_i p_i\times Z_i$ in $\mathrm{CH}^2(E\times E\times E)_{\mathbb{Q}}$ is also invariant under the group $\mathfrak{S}_3$. It follows that it is cohomologous to 0 and Abel–Jacobi equivalent to 0. But the Deligne cycle class (with value in Deligne cohomology $H^4_{\mathcal{D}}(E\times E\times E,\mathbb{Q}(2))$) is injective on the invariant part $\mathrm{CH}^2(E\times E\times E)_{\mathbb{Q}}^{\mathfrak{S}_3}$, because $E^{(3)}$ is a $\mathbb{P}^2$-bundle over E. □

Note that from Proposition 5.17(i) we can deduce the following property satisfied by the cycle o (which is a weak version of Theorem 5.15).

Corollary 5.18. *For any 0-cycle* $z\in \mathrm{CH}_0(S)$*, one has*

$$i_{S*}(z) = z\times o + o\times z - \deg z\,(o,o) \ \text{ in }\ \mathrm{CH}_0(S\times S),$$

where $i_S : S\to S\times S$ *is the inclusion of the diagonal of* $S\times S$*.*

Proof. It suffices to prove the result when z is a point of S. We know that S is swept out by (singular) elliptic curves, that is, curves C whose normalization E is a smooth elliptic curve. A general point x of S belongs to a smooth point of such a curve C, and we will denote by $\tilde{x}$ the corresponding point of E. Denoting by $\tilde{j} : E\to S$ the natural map, so that $x = \tilde{j}(\tilde{x})$, we now choose for the line bundle l on E any line bundle of the form $\tilde{j}^*L$, where $L \in \operatorname{Pic} S$ has nonzero degree d on C. We apply (5.7) on E and then apply $\tilde{j}_*$. We then get

$$d^2(x,x) = d(x \times L \cdot C + L \cdot C \times x) - (\tilde{j},\tilde{j})_*(\tilde{j}^*L \times \tilde{j}^*L) \text{ in } \mathrm{CH}_0(S \times S).$$

We know by Theorem 5.1 that $L \cdot C = d\,o$ in $\mathrm{CH}_0(S)$. It follows also that

$$\tilde{j}_*(\tilde{j}^*L) = d\,o \text{ in } \mathrm{CH}_0(S).$$

Hence we conclude that

$$d^2(x,x) = d^2(x \times o + o \times x) - d^2(o,o) \text{ in } \mathrm{CH}_0(S \times S),$$

which proves the result since $\mathrm{CH}_0(S \times S)$ has no torsion by Roitman's theorem. □

Proof of Theorem 5.15. We know that S is swept out by a one-parameter family of elliptic curves; thus there is a smooth surface Σ that admits an elliptic fibration

$$p : \Sigma \to B$$

and a generically finite dominating morphism

$$q : \Sigma \to S.$$

In fact, we can even assume that all fibers Σ_b are reduced irreducible, if S has the property that $\operatorname{Pic} S$ is generated by the class of an ample line bundle L on S. In that case, there is a one-parameter family of elliptic curves in $|L|$ and all of them are irreducible and reduced. Once we have the result for S satisfying this property, the result for any S follows by specialization.

We set $d := \deg q^*L_{|\Sigma_b}$. We apply Proposition 5.17(ii) to the elliptic curve Σ_b endowed with the line bundle L_b. This gives us for the general point $b \in B$ a formula for the small diagonal of Σ_b in $\mathrm{CH}_1(\Sigma_b \times \Sigma_b \times \Sigma_b)_{\mathbb{Q}}$. We view $\Sigma_b \times \Sigma_b \times \Sigma_b$ as the fiber of the map

$$p_3 : \Sigma \times_B \Sigma \times_B \Sigma \to B,$$

and observe that the small diagonal Δ^Σ of Σ is contained in $\Sigma \times_B \Sigma \times_B \Sigma$ and that its restriction to the general fiber $\Sigma_b \times \Sigma_b \times \Sigma_b$ is equal to the small diagonal of Σ_b. We thus conclude from Corollary 3.8 that there are finitely many points $b_i \in B$ and 2-cycles $Z_i \subset \Sigma_{b_i} \times \Sigma_{b_i} \times \Sigma_{b_i}$ with $\mathbb{Q}$-coefficients such that the following equation holds in $\mathrm{CH}_2(\Sigma \times_B \Sigma \times_B \Sigma)_{\mathbb{Q}}$:

$$d^2\Delta^\Sigma = d[\Delta_{12}^{\mathrm{rel}} \cdot \mathrm{pr}_3^*(q^*L) + (\mathrm{perm.})] - (\mathrm{pr}_1^*\, q^*L \cdot \mathrm{pr}_2^*\, q^*L + (\mathrm{perm.})) + \sum_i Z_i, \tag{5.10}$$

where $\Delta_{12}^{\mathrm{rel}}$ is defined as the inverse image $\mathrm{pr}_{12}^{-1}(\Delta_{\Sigma/B}) \subset \Sigma \times_B \Sigma \times_B \Sigma$ of the relative diagonal of Σ over B.

Furthermore, we observe that up to changing the base B, we may assume there is an involution $\iota : \Sigma \to \Sigma$, with the properties that

$$\iota^*(q^*L) = q^*L, \qquad p \circ \iota = p$$

acting on each smooth fiber Σ_b in such a way that $\Sigma_b/\iota \cong \mathbb{P}^1$. Then the cycle

$$d^2\Delta^\Sigma - d[\Delta_{12}^{\mathrm{rel}} \cdot \mathrm{pr}_3^*(q^*L) + (\mathrm{perm.})] + (\mathrm{pr}_1^* q^*L \cdot \mathrm{pr}_2^* q^*L + (\mathrm{perm.})) \\ \in \mathrm{CH}_2(\Sigma \times_B \Sigma \times_B \Sigma)_{\mathbb{Q}}$$

being invariant under ι, we can assume by averaging that each cycle Z_i is invariant under ι.

We now push-forward this equality to $S \times S \times S$ via the composition of the inclusion

$$k : \Sigma \times_B \Sigma \times_B \Sigma \hookrightarrow \Sigma \times \Sigma \times \Sigma$$

and the map

$$(q, q, q) : \Sigma \times \Sigma \times \Sigma \to S \times S \times S.$$

Let $N := \deg q$. Then we clearly have

$$(q, q, q)_*\Delta^\Sigma = N\Delta. \tag{5.11}$$

Next we have the following lemma.

Lemma 5.19. *The following equality holds in* $\mathrm{CH}_2(S \times S \times S)_{\mathbb{Q}}$*:*

$$((q, q, q) \circ k)_*(\mathrm{pr}_i^* q^*L \cdot \mathrm{pr}_j^* q^*L) = Nd^2 \mathrm{pr}_{ij}^* o \times o + Nd \,\mathrm{pr}_i^* o \times \mathrm{pr}_j^* L \times \mathrm{pr}_l^* L \\ + Nd \,\mathrm{pr}_j^* o \times \mathrm{pr}_i^* L \times \mathrm{pr}_l^* L, \tag{5.12}$$

where $\{i, j, l\} = \{1, 2, 3\}$*. Furthermore,*

$$((q, q, q) \circ k)_*(\Delta_{12}^{\mathrm{rel}} \cdot \mathrm{pr}_3^*(q^*L)) = Nd\Delta_{12} \cdot \mathrm{pr}_3^* o + N \mathrm{pr}_1^* L \times o_2 \times \mathrm{pr}_3^* L \\ + N \mathrm{pr}_1^* o \cdot \mathrm{pr}_2^* L \cdot \mathrm{pr}_3^* L \\ \in \mathrm{CH}_2(S \times S \times S)_{\mathbb{Q}}. \tag{5.13}$$

Proof. We may assume that $i = 1$, $j = 2$. Let $P = \mathbb{P}(H^0(S, L)) = \mathbb{P}^r$. The curve B is a curve of degree N in P. The cycle $((q, q, q) \circ k)_*(\mathrm{pr}_1^* q^*L \cdot \mathrm{pr}_2^* q^*L) \in \mathrm{CH}_2(S \times S \times S)$ is N times the cycle

$$((q', q', q') \circ k')_*(\mathrm{pr}_1^* q^*L \cdot \mathrm{pr}_2^* q^*L) \in \mathrm{CH}_2(S \times S \times S),$$

where B is replaced by a pencil $\mathbb{P}^1 \subset P$,

$$q' : \Sigma' \to S, \qquad p' : \Sigma' \to \mathbb{P}^1$$

are obtained by blowing up S at the base points of the pencil, and k' is the inclusion of $\Sigma' \times_{\mathbb{P}^1} \Sigma' \times_{\mathbb{P}^1} \Sigma'$ in $\Sigma' \times \Sigma' \times \Sigma'$. But the small diagonal of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has its class equal to

$$\sum_{i \neq j} \pi_i^* H \cdot \pi_j^* H, \qquad H := \mathcal{O}_{\mathbb{P}^1}(1),$$

where $\pi_j : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the jth projection. Hence the class of $\Sigma' \times_{\mathbb{P}^1} \Sigma' \times_{\mathbb{P}^1} \Sigma'$ in $\Sigma' \times \Sigma' \times \Sigma'$ is equal to $\sum_{i \neq j} p'^*_i H \cdot p'^*_j H$, where $p'_l = \pi_l \circ (p', p', p')$. Our cycle in (5.12) is thus equal to

$$N(q', q', q')_* \left[\left(\sum_{i \neq j} {p'_i}^* H \cdot {p'_j}^* H \right) \cdot \mathrm{pr}_1^* q^* L \cdot \mathrm{pr}_2^* q^* L \right],$$

which by the projection formula is also equal to

$$N \,\mathrm{pr}_1^* L \cdot \mathrm{pr}_2^* L \cdot \left((q', q', q')_* \left(\sum_{i \neq j} {p'_i}^* H \cdot {p'_j}^* H \right) \right).$$

As $q'_* H = L$, this is clearly equal to

$$N \,\mathrm{pr}_1^* L \cdot \mathrm{pr}_2^* L \cdot \left(\sum_{i \neq j} \mathrm{pr}_i^* L \cdot \mathrm{pr}_j^* L \right).$$

For $\{i, j\} = \{1, 2\}$ we get $Nd^2 o_1 \times o_2 \times S$ using (5.1). For the same reason the two other terms give $Ndo_1 \times L \times L$ and $NdL \times o_2 \times L$. This proves (5.12).

Formula (5.13) is proved in a similar way, using Lemma 5.5. □

We now deduce from (5.10), (5.11), and (5.12), by applying $(q, q, q)_* \circ k_*$, the following equality in $\mathrm{CH}_2(S \times S \times S)_{\mathbb{Q}}$:

$$\begin{aligned} Nd^2 \Delta = {} & d[Nd\Delta_{12} \cdot \mathrm{pr}_3^* o + N \,\mathrm{pr}_1^* L \cdot \mathrm{pr}_2^* o \cdot \mathrm{pr}_3^* L + N \,\mathrm{pr}_1^* o \cdot \mathrm{pr}_2^* L \cdot \mathrm{pr}_3^* L] \\ & - [Nd^2 o \times o \times S - Ndo \times L \times L - NdL \times o \times L + (\text{perm.})] \\ & + \sum_i Z_i, \end{aligned} \tag{5.14}$$

where the Z_i are 2-cycles supported on products of curves $C_i \times C_i \times C_i$, where the $C_i \in |L|$ are elliptic, and the Z_i are invariant under the symmetric group $\mathfrak{S}_3$ and the action of the involution ι.

We have now, denoting by $n : \widetilde{C_i} \to C_i$ the normalization of C_i, the following lemma.

Lemma 5.20. *2-cycles in $\widetilde{C_i} \times \widetilde{C_i} \times \widetilde{C_i}$, which are invariant under the symmetric group $\mathfrak{S}_3$ and the action of the involution ι, are generated over $\mathbb{Q}$ by $\sum_j (n \circ \mathrm{pr}_j)^* L_{|C_i}$ and by the big diagonal $\sum_{k,l} \mathrm{pr}_{kl}^* \Delta_{\widetilde{C_i}}$.*

This is elementary as $\widetilde{C_i}$ is either elliptic or rational, and $n^*(L_{|C_i})$ generates over $\mathbb{Q}$ the invariant part under ι of $\mathrm{Pic}\, \widetilde{C_i}$.

Using (5.1) and Lemmas 5.5 and 5.20, we conclude that the cycles Z_i are rationally equivalent in $S \times S \times S$ to $\mathrm{pr}_1^* L \cdot \mathrm{pr}_2^* o \cdot \mathrm{pr}_3^* L + (\text{perm.})$.

We thus deduce from (5.14) an equality

$$\Delta-(\Delta_{12}\cdot \mathrm{pr}_3^*o+(\text{perm.}))+o\times o\times S+(\text{perm.})=\mu(\mathrm{pr}_1^*L\cdot \mathrm{pr}_2^*o\cdot \mathrm{pr}_3^*L+(\text{perm.}))$$

in $\mathrm{CH}_2(S\times S\times S)_{\mathbb{Q}}$, where $\mu\in\mathbb{Q}$.

Comparing cohomology classes, we find that $\mu=0$ which concludes the proof of Theorem 5.15. □

5.2.1 Calabi–Yau hypersurfaces

In the case of smooth Calabi–Yau hypersurfaces X in projective space $\mathbb{P}^n$, that is, hypersurfaces of degree $(n+1)$ in $\mathbb{P}^n$, we have the following result proved in [110], which partially generalizes Theorem 5.15 and provides some information on the Chow ring of X. Denote by $o\in \mathrm{CH}_0(X)_{\mathbb{Q}}$ the class of the degree 1 0-cycle $\frac{h^{n-1}}{n+1}$, where $h:=c_1(\mathcal{O}_X(1))\in \mathrm{CH}^1(X)$. Again we denote by Δ the small diagonal of X in X^3.

Theorem 5.21 (Voisin 2011). *The following relation is satisfied in the group* $\mathrm{CH}^{2n-2}(X\times X\times X)_{\mathbb{Q}}$*:*

$$\Delta=\Delta_{12}\cdot o_3+(\text{perm.})+Z+\Gamma', \tag{5.15}$$

where Z is the restriction to $X\times X\times X$ of a cycle on $\mathbb{P}^n\times\mathbb{P}^n\times\mathbb{P}^n$, and Γ' is a multiple of the following effective cycle of dimension $(n-1)$:

$$\Gamma:=\cup_{l\in F(X)}\mathbb{P}^1_l\times\mathbb{P}^1_l\times\mathbb{P}^1_l.$$

Here $F(X)$ is the variety of lines contained in X. It is of dimension $n-4$ for general X. For a point $l\in F(X)$, $\mathbb{P}^1_l\subset X$ denotes the corresponding line.

This result has been generalized by Lie Fu [42] to Calabi–Yau varieties obtained as zero sets of transverse sections of very ample vector bundles on projective space, under the assumption that a certain number computed from the Chern classes of this vector bundle does not vanish (this is satisfied by complete intersections, that is, in the case where the vector bundle is split). In particular, the main consequence below (Theorem 5.25) also holds for them. It would be very interesting to understand the class of Calabi–Yau varieties satisfying conclusions analogous to Theorem 5.21 and Theorem 5.25.

Remark 5.22. Note that Theorem 5.21 gives an alternative proof of Theorem 5.3 for quartic $K3$ surfaces S in $\mathbb{P}^3$ (in dimension 2, the cycle Γ above is empty for general S, and the result for general S implies the result for any S). Similarly, the results of Lie Fu re-prove Theorem 5.3 for complete intersection $K3$ surfaces.

Proof of Theorem 5.21. Observe first of all that it suffices to prove the following equality of $(n-1)$-cycles on X_0^3, where $X_0^3:=X^3\setminus\Delta$:

$$\Gamma_{|X_0^3}=(n+1)![\Delta_{12|X_0^3}\cdot o_3+(\text{perm.})]+Z \text{ in } \mathrm{CH}^{2n-2}(X_0^3)_{\mathbb{Q}}, \tag{5.16}$$

where Z is the restriction to X_0^3 of a cycle on $(\mathbb{P}^n)^3$. Indeed, by the localization exact sequence (2.2), (5.16) implies the equality

$$N\Delta = \Delta_{12} \cdot o_3 + (\text{perm.}) + Z + \Gamma' \text{ in } \mathrm{CH}^{2n-2}(X \times X \times X)_{\mathbb{Q}}, \tag{5.17}$$

for some integer N. Projecting to X^2 and taking cohomology classes, then we easily conclude that $N = 1$. (We use here the fact that X has some transcendental cohomology, so that the cohomology class of the diagonal of X does not vanish on products $U \times U$, where $U \subset X$ is Zariski open.)

In order to prove (5.16), we do the following: First of all we compute the class in $\mathrm{CH}^{n-1}(X_0^3)$ of the $(2n-2)$-dimensional subvariety

$$X^3_{0,\mathrm{col,sch}} \subset X_0^3,$$

parametrizing 3-uples of collinear points satisfying the following property.

Let $\mathbb{P}^1_{x_1x_2x_3} := \langle x_1, x_2, x_3 \rangle$ be the line generated by the x_i's. Then the subscheme $x_1 + x_2 + x_3$ of $\mathbb{P}^1_{x_1x_2x_3} \subset \mathbb{P}^n$ is contained in X.

We will denote by

$$X^3_{0,\mathrm{col}} \subset X_0^3,$$

the $(2n-2)$-dimensional subvariety parametrizing 3-uples of collinear points. Obviously $X^3_{0,\mathrm{col,sch}} \subset X^3_{0,\mathrm{col}}$. We will see that it is in fact one irreducible component of it.

Next we observe that there is a natural morphism $\phi : X^3_{0,\mathrm{col}} \to G(2, n+1)$ to the Grassmannian of lines in $\mathbb{P}^n$, which to (x_1, x_2, x_3) associates the line $\mathbb{P}^1_{x_1x_2x_3}$. This morphism is well defined on $X^3_{0,\mathrm{col}}$ because at least two of the points x_i are distinct, so that this line is unique. The morphism ϕ corresponds to a tautological rank 2 vector bundle $\mathcal{E}$ on $X^3_{0,\mathrm{col}}$, with fiber $H^0(\mathcal{O}_{\mathbb{P}^1_{x_1x_2x_3}}(1))$ over the point (x_1, x_2, x_3).

We then observe that $\Gamma \subset X^3_{0,\mathrm{col,sch}}$ is defined by the condition that the line $\mathbb{P}^1_{x_1x_2x_3}$ is contained in X. In other words, the equation f defining X has to vanish on the line $\mathbb{P}^1_{x_1x_2x_3}$. This equation can be seen globally as giving a section σ,

$$\sigma((x_1, x_2, x_3)) = f_{|\mathbb{P}^1_{x_1x_2x_3}},$$

of the vector bundle $S^{n+1}\mathcal{E}$.

This section σ is not transverse (in fact the rank of $S^{n+1}\mathcal{E}$ is $n+2$, while the codimension of Γ is $n-1$), but the reason for this is very simple: indeed, at a point (x_1, x_2, x_3) of $X^3_{0,\mathrm{col,sch}}$, the equation f vanishes by definition on the length 3 subscheme $x_1 + x_2 + x_3$ of $\mathbb{P}^1_{x_1x_2x_3}$. Another way to express this is to say that σ is in fact a section of the rank $(n-1)$ bundle

$$\mathcal{F} \subset S^{n+1}\mathcal{E}, \tag{5.18}$$

where $\mathcal{F}_{(x_1,x_2,x_3)}$ consists of degree $(n+1)$ polynomials vanishing on the subscheme $x_1 + x_2 + x_3$ of $\mathbb{P}^1_{x_1x_2x_3}$.

The section σ of $\mathcal{F}$ is transverse and thus we conclude that we have the equality

$$\Gamma_{|X_0^3} = j_*(c_{n-1}(\mathcal{F})) \text{ in } \mathrm{CH}^{2n-2}(X_0^3)_{\mathbb{Q}}, \tag{5.19}$$

where j is the inclusion of $X^3_{0,\mathrm{col,sch}}$ in X_0^3.

We now observe that the vector bundles $\mathcal{E}$ and $\mathcal{F}$ come from vector bundles on the variety $(\mathbb{P}^n)^3_{0,\mathrm{col}}$ parametrizing 3-uples of collinear points in $\mathbb{P}^n$, at least two of them being distinct.

The variety $(\mathbb{P}^n)^3_{0,\mathrm{col}}$ is smooth irreducible of dimension $2n+1$ (hence of codimension $(n-1)$ in $(\mathbb{P}^n)^3$), being Zariski open in a $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$-bundle over the Grassmannian $G(2, n+1)$. We have now the following lemma.

Lemma 5.23. *The intersection $(\mathbb{P}^n)^3_{0,\mathrm{col}} \cap X_0^3$ is reduced, of pure dimension $(2n-2)$. It decomposes as*

$$(\mathbb{P}^n)^3_{0,\mathrm{col}} \cap X_0^3 = X^3_{0,\mathrm{col,sch}} \cup \Delta_{0,12} \cup \Delta_{0,13} \cup \Delta_{0,23}, \tag{5.20}$$

where $\Delta_{0,ij} \subset X_0^3$ is defined as $\Delta_{ij} \setminus \Delta$ with Δ_{ij} the big diagonal $x_i = x_j$.

Proof. The set-theoretic equality in (5.20) is obvious. The fact that each component on the right has dimension $2n-2$ and thus is a component of the right dimension of this intersection is also obvious. Hence the only point to check is the fact that these intersections are transverse at the generic point of each component in the right-hand side. The generic point of the irreducible variety $X^3_{0,\mathrm{col,sch}}$ parametrizes a triple of distinct collinear points that are on a line Δ not tangent to X. At such a triple, the intersection $(\mathbb{P}^n)^3_{0,\mathrm{col}} \cap X_0^3$ is smooth of dimension $2n-2$ because $(\mathbb{P}^n)^3_{0,\mathrm{col}}$ is the triple self-product $P \times_{G(2,n+1)} P \times_{G(2,n+1)} P$ of the tautological $\mathbb{P}^1$-bundle P over the Grassmannian $G(2, n+1)$, and the intersection with X_0^3 is defined by the three equations

$$p \circ \mathrm{pr}_1^* f, \qquad p \circ \mathrm{pr}_2^* f, \qquad p \circ \mathrm{pr}_3^* f,$$

where the pr_i's are the projections $P^{3/G(2,n+1)} \to P$ and $p : P \to \mathbb{P}^n$ is the natural map. These three equations are independent since they are independent after restriction to $\mathbb{P}^1_{x_1x_2x_3} \times \mathbb{P}^1_{x_1x_2x_3} \times \mathbb{P}^1_{x_1x_2x_3} \subset (\mathbb{P}^n)^3_{0,\mathrm{col}}$ at the point (x_1, x_2, x_3) because $\mathbb{P}^1_{x_1x_2x_3}$ is not tangent to X.

Similarly, the generic point of the irreducible variety $\Delta_{0,12} \subset X^3_{0,\mathrm{col}}$ parametrizes a triple (x, x, y) with the property that $x \neq y$ and the line $\mathbb{P}^1_{xy} := \langle x, y \rangle$ is not tangent to X. Again, the intersection $(\mathbb{P}^n)^3_{0,\mathrm{col}} \cap X_0^3$ is smooth of dimension $2n-2$ near (x, x, y) because the restrictions to $\mathbb{P}^1_{xy} \times \mathbb{P}^1_{xy} \times \mathbb{P}^1_{xy}$ of the equations

$$p \circ \mathrm{pr}_1^* f, \qquad p \circ \mathrm{pr}_2^* f, \qquad p \circ \mathrm{pr}_3^* f,$$

defining X^3 are independent. $\square$

Combining (5.20), (5.19), and the fact that the vector bundle $\mathcal{F}$ already exists on $(\mathbb{P}^n)^3_{0,\mathrm{col}}$, we find that

$$\Gamma_{|X_0^3} = J_*(c_{n-1}(\mathcal{F}_{|(\mathbb{P}^n)^3_{0,\mathrm{col}}\cap X_0^3})) - \sum_{i\neq j} J_{ij*}c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}}) \text{ in } \mathrm{CH}^{2n-2}(X_0^3)_{\mathbb{Q}},$$

where $J : (\mathbb{P}^n)^3_{0,\mathrm{col}} \cap X_0^3 \hookrightarrow X_0^3$ is the inclusion and similarly for $J_{0ij} : \Delta_{0,ij} \hookrightarrow X_0^3$. This provides us with the formula

$$\Gamma_{|X_0^3} = (K_*c_{n-1}(\mathcal{F}))_{|X_0^3} - \sum_{i\neq j} J_{ij*}c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}}) \text{ in } \mathrm{CH}^{2n-2}(X_0^3)_{\mathbb{Q}}, \quad (5.21)$$

where $K : (\mathbb{P}^n)^3_{0,\mathrm{col}} \hookrightarrow (\mathbb{P}^n)^3_0$ is the inclusion map.

The first term comes from $\mathrm{CH}((\mathbb{P}^n)^3_0)$, and this only contributes to the term Z in Theorem 5.21, so to conclude, we only have to compute the terms $J_{ij*}c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}})$. This is however very easy, because the vector bundles $\mathcal{E}$ and $\mathcal{F}$ are very simple on $\Delta_{0,ij}$: Assume for simplicity $i = 1, j = 2$. Points of $\Delta_{0,ij}$ are points (x, x, y), $x \neq y \in X$. The line $\phi((x, x, y))$ is the line $\mathbb{P}^1_{xy} = \langle x, y\rangle$, and it follows that

$$\mathcal{E}_{|\Delta_{0,12}} = \mathrm{pr}_2^* \mathcal{O}_X(1) \oplus \mathrm{pr}_3^* \mathcal{O}_X(1). \quad (5.22)$$

The projective bundle $\mathbb{P}(\mathcal{E}_{|\Delta_{0,12}})$ has two sections on Δ_{12} which give two divisors

$$D_2 \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathrm{pr}_3^* \mathcal{O}_X(-1)|, \qquad D_3 \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_X(-1)|.$$

The length 3 subscheme $2D_2 + D_3 \subset \mathbb{P}(\mathcal{E}_{|\Delta_{0,1,2}})$ with fiber $2x + y$ over the point (x, x, y) is thus the zero set of a section α of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3) \otimes \mathrm{pr}_3^* \mathcal{O}_X(-2) \otimes \mathrm{pr}_2^* \mathcal{O}_X(-1)$. We thus conclude that the vector bundle $\mathcal{F}_{|\Delta_{0,12}}$ is isomorphic to

$$\mathrm{pr}_3^* \mathcal{O}_X(2) \otimes \mathrm{pr}_2^* \mathcal{O}_X(1) \otimes S^{n-2}\mathcal{E}_{|\Delta_{0,12}}.$$

Combining with (5.22), we conclude that $c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}})$ can be expressed as a polynomial of degree $(n-1)$ in $h_2 = c_1(\mathrm{pr}_2^* \mathcal{O}_X(1))$ and $h_3 = c_1(\mathrm{pr}_3^* \mathcal{O}_X(1))$ on $\Delta_{0,12}$. The proof of (5.16) is completed by the following lemma.

Lemma 5.24. *Let $\Delta_X \subset X \times X$ be the diagonal. Then the codimension n cycles*

$$\mathrm{pr}_1^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X, \qquad \mathrm{pr}_2^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X$$

of $X \times X$ are restrictions to $X \times X$ of cycles in $\mathrm{CH}^n(\mathbb{P}^n \times \mathbb{P}^n)_{\mathbb{Q}}$.

Proof. Indeed, let $j_X : X \hookrightarrow \mathbb{P}^n$ be the inclusion of X in $\mathbb{P}^n$, and $j_{X,1}$, and $j_{X,2}$ the corresponding inclusions of $X \times X$ in $\mathbb{P}^n \times X$ and $X \times \mathbb{P}^n$, respectively. Then as X is a degree $(n + 1)$ hypersurface, we have

$$(n + 1)\, \mathrm{pr}_1^* c_1(\mathcal{O}_X(1)) = j^*_{X,1} \circ j_{X,1*} : \mathrm{CH}(X \times X) \to \mathrm{CH}(X \times X),$$

and similarly for the second inclusion. On the other hand, $j_{X,1*}(\Delta_X) \subset \mathbb{P}^n \times X$ is obviously the (transpose of the) graph of the inclusion of X in $\mathbb{P}^n$, hence its class is the restriction to $\mathbb{P}^n \times X$ of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. We argue similarly for the second inclusion. □

It follows from this lemma that a monomial of degree $n-1$ in

$$h_2 = c_1(\mathrm{pr}_2^* \mathcal{O}_X(1)) \text{ and } h_3 = c_1(\mathrm{pr}_3^* \mathcal{O}_X(1))$$

on $\Delta_{0,12}$, seen as a cycle in $X \times X \times X$, will be the restriction to $X \times X \times X$ of a cycle with $\mathbb{Q}$-coefficients, unless it is proportional to h_3^{n-1}. Recalling that $c_1(\mathcal{O}_X(1))^{n-1} = (n+1)o \in \mathrm{CH}_0(X)$, we finally proved that modulo restrictions of cycles coming from $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, the term $J_{12*}c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$ is a multiple of $\Delta_{12} \times o_3$ in $\mathrm{CH}^{2n-2}(X \times X \times X)_{\mathbb{Q}}$. The precise coefficient is in fact given by the argument above. Indeed, we just saw that modulo restrictions of cycles coming from $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, the term $J_{12*}c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$ is equal to

$$\mu\Delta_{12} \cdot \mathrm{pr}_3^*(c_1(\mathcal{O}_X(1))^{n-1}) = \mu(n+1)\Delta_X \times o_3, \tag{5.23}$$

with $c_1(\mathcal{O}_X(1))^{n-1} = (n+1)o$ in $\mathrm{CH}_0(X)$, and where the coefficient μ is the coefficient of h_3^{n-1} in the polynomial in h_2, h_3 computing $c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$.

Now we use the isomorphism

$$\mathcal{F}_{|\Delta_{0,12}} \cong \mathrm{pr}_3^* \mathcal{O}_X(2) \otimes \mathrm{pr}_2^* \mathcal{O}_X(1) \otimes S^{n-2}\mathcal{E}_{|\Delta_{0,12}},$$

where $\mathcal{E}_{|\Delta_{0,12}} \cong \mathrm{pr}_2^* \mathcal{O}_X(1) \oplus \mathrm{pr}_3^* \mathcal{O}_X(1)$ according to (5.22). Hence we conclude that the coefficient μ is equal to $n!$, and this concludes the proof of (5.16), using (5.23) and (5.21). □

We have the following consequence of Theorem 5.21, which is a generalization of Theorem 5.1 to Calabi–Yau hypersurfaces.

Theorem 5.25 (Voisin 2011). *Let $X \subset \mathbb{P}^n$ be a smooth Calabi–Yau hypersurface. Let z, z' be cycles on X such that*

$$\operatorname{codim} z > 0, \qquad \operatorname{codim} z' > 0, \qquad \operatorname{codim} z + \operatorname{codim} z' = n-1.$$

Then $z \cdot z'$ is proportional to o in $\mathrm{CH}_0(X)$. Equivalently, let z_i, z'_i, $i = 1, \ldots, N$ be cycles on X such that $\operatorname{codim} z_i > 0$, $\operatorname{codim} z'_i > 0$, for all i, $\operatorname{codim} z_i + \operatorname{codim} z'_i = n-1$. Then if we have a cohomological relation

$$\sum_i n_i[z_i] \cup [z'_i] = 0 \text{ in } H^{2n-2}(X, \mathbb{Q}),$$

this relation already holds at the level of Chow groups:

$$\sum_i n_i z_i \cdot z'_i = 0 \text{ in } \mathrm{CH}^{n-1}(X)_{\mathbb{Q}}.$$

The two statements are equivalent since $o = \frac{h^{n-1}}{n+1}$. As already mentioned, this result holds more generally for Calabi–Yau complete intersections in projective space, as proved by Lie Fu [42].

PROOF OF THEOREM 5.25. Indeed, let us view formula (5.15) as an equality of correspondences between $X \times X$ and X. The left-hand side applied to $z \times z'$ is

$$\Delta_*(z \times z') = z \cdot z' \text{ in } \mathrm{CH}(X)_{\mathbb{Q}}.$$

The right-hand side is a sum $Z_*(z \times z') + \Gamma_*(z \times z')$. But we observe that as Z is the restriction of a cycle $Z' \in \mathrm{CH}^{2n-2}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)_{\mathbb{Q}}$, the 0-cycle $Z_*(z \times z')$ is equal to

$$j^*(Z'^*((j,j)_*(z \times z'))) \in \mathrm{CH}^{n-1}(X)_{\mathbb{Q}}.$$

Thus the 0-cycle $Z_*(z \times z')$ belongs to

$$\mathrm{Im}(j^* : \mathrm{CH}^{n-1}(\mathbb{P}^n)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}) = \mathbb{Q}o.$$

Consider now the term $\Gamma_*(z \times z')$: Let $\Gamma_0 \subset X$ be the locus swept out by lines. We observe that for any line $\Delta \subset X$, any point on Δ is rationally equivalent to the 0-cycle $h \cdot \Delta$, which is in fact proportional to o, since

$$(n+1)h \cdot \Delta = j^* \circ j_*(\Delta) \text{ in } \mathrm{CH}(X)$$

and $j_*(\Delta) = c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-1}$ in $\mathrm{CH}^{n-1}(\mathbb{P}^n)$. Hence all points of Γ_0 are rationally equivalent to o in X. It follows that the 0-cycle $\Gamma_*(z \cdot z')$, which is supported on Γ_0, is proportional to o. □

5.3 DELIGNE'S DECOMPOSITION THEOREM FOR FAMILIES OF $K3$ SURFACES

5.3.1 Deligne's decomposition theorem

Let $\phi : X \to Y$ be a submersive and proper morphism of complex varieties. Recall that the morphism ϕ is projective if there exists a holomorphic embedding

$$i : X \hookrightarrow Y \times \mathbb{P}^n$$

such that $\phi = \mathrm{pr}_1 \circ i$. Giving such an embedding provides a class $\omega \in H^2(X, \mathbb{Z})$ defined by

$$\omega = (\mathrm{pr}_2 \circ i)^* c_1(\mathcal{O}_{\mathbb{P}^n}(1)).$$

As $\mathrm{pr}_2 \circ i_{|X_t}$ is a holomorphic immersion on each fiber X_t of ϕ, the restriction

$$\omega_t := \omega_{|X_t} \in H^2(X_t, \mathbb{Z})$$

is a Kähler class, and the morphism

$$\omega_t \cup : H^k(X_t, \mathbb{Q}) \to H^{k+2}(X_t, \mathbb{Q})$$

is a Lefschetz operator on $H^*(X_t, \mathbb{Q})$. Moreover, as ω is closed, it induces as above a morphism of local systems

$$L := \omega \cup : R^*\phi_*\mathbb{Q} \to R^{*+2}\phi_*\mathbb{Q},$$

which is equal to $L_t = \omega_t \cup$ on the stalk at the point t. The operator L is called the relative Lefschetz operator. If $n = \dim X_t$ is the relative dimension of ϕ, we know that L_t satisfies the hard Lefschetz theorem, that is,

$$L_t^{n-k} : H^k(X_t, \mathbb{Q}) \to H^{2n-k}(X_t, \mathbb{Q}) \tag{5.24}$$

is an isomorphism for $k \leq n$.

One deduces formally from the Lefschetz isomorphisms the Lefschetz decomposition

$$H^k(X_t, \mathbb{Q}) = \oplus_{2r \leq k} L^r H^{k-2r}(X_t, \mathbb{Q})_{\text{prim}} \text{ for } k \leq n, \tag{5.25}$$

where

$$H^{k-2r}(X_t,\mathbb{Q})_{\text{prim}} := \operatorname{Ker}\left(L^{n-k+2r+1} : H^{k-2r}(X_t, \mathbb{Q})_{\text{prim}} \to H^{2n-k+2r+2}(X_t, \mathbb{Q})\right).$$

The corresponding decomposition for $k \geq n$ is obtained by applying the isomorphism (5.24).

The relative Lefschetz operator thus gives relative Lefschetz isomorphisms

$$L^{n-k} : R^k\phi_*\mathbb{Q} \cong R^{2n-k}\phi_*\mathbb{Q}$$

and a relative Lefschetz decomposition

$$R^k\phi_*\mathbb{Q} = \oplus_{2r \leq k} L^r R^{k-2r}\phi_*\mathbb{Q}_{\text{prim}}, \quad k \leq n.$$

The relative Lefschetz operator L as well as its powers L^k induce endomorphisms (of degree $2k$) of the Leray spectral sequence of ϕ, that is, morphisms L_r^k of the complexes $(E_r^{p,q}, d_r)$ (of degree $2k$ on the second index), and the morphism $H^*(X, \mathbb{Q}) \to H^{*+2k}(X, \mathbb{Q})$ induced by L^k is compatible with the Leray filtration on $H^*(X, \mathbb{Q})$ and with each L_∞^k on $\operatorname{Gr}_L H^*(X, \mathbb{Q})$. Finally, the Lefschetz isomorphism (5.24) shows that

$$L_2^{n-k} : H^l(Y, R^k\phi_*\mathbb{Q}) \to H^l(Y, R^{2n-k}\phi_*\mathbb{Q}) \tag{5.26}$$

is an isomorphism for $k \leq n$.

Let us recall the proof of the following theorem, proved in [30].

THEOREM 5.26 (Deligne 1968). *If $\phi : X \to Y$ is a submersive projective morphism, then the Leray spectral sequence of ϕ degenerates at E_2.*

Proof. Let us first show that $d_2 = 0$. For this, note that if $q \geq n$, we have the following commutative diagram:

$$\begin{array}{ccc} E_2^{p,2n-q} = H^p(Y, R^{2n-q}\phi_*\mathbb{Q}) & \xrightarrow{L_2^{q-n}} & E_2^{p,q} = H^p(Y, R^q\phi_*\mathbb{Q}) \\ \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2} \\ E_2^{p+2,2n-q-1} = H^{p+2}(Y, R^{2n-q+1}\phi_*\mathbb{Q}) & \longrightarrow & E_2^{p+2,q-1} = H^{p+2}(Y, R^{q-1}\phi_*\mathbb{Q}), \end{array}$$

where the upper horizontal arrow is an isomorphism. Thus it suffices to show that $d_2 = 0$ on $E_2^{p,q}$ with $q \leq n$. We have the decomposition

$$E_2^{p,q} = \oplus_{2r \leq q} L_2^r H^p(Y, R^{q-2r}\phi_*\mathbb{Q}_{\text{prim}})$$

induced by the relative Lefschetz decomposition, and it suffices to show that $d_2 = 0$ on $L_2^r H^p(Y, R^{q-2r}\phi_*\mathbb{Q}_{\text{prim}})$. As L_2^r commutes with d_2, it suffices to show that $d_2 = 0$ on $H^p(Y, R^{q-2r}\phi_*\mathbb{Q}_{\text{prim}}) \subset E_2^{p,q-2r}$.

Setting $k = q - 2r$, we have the following commutative diagram:

$$\begin{array}{lccc} L_2^{n-k+1}: & H^p(Y, R^k\phi_*\mathbb{Q}_{\text{prim}}) & \longrightarrow & H^p(Y, R^{2n-k+2}\phi_*\mathbb{Q}) \\ & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2} \\ L_2^{n-k+1}: & H^{p+2}(Y, R^{k-1}\phi_*\mathbb{Q}_{\text{prim}}) & \longrightarrow & H^{p+2}(Y, R^{2n-k+1}\phi_*\mathbb{Q}). \end{array}$$

The upper arrow is 0 by definition of the primitive cohomology, while the lower arrow is the isomorphism (5.26). Thus, the first arrow d_2 is 0.

To show that the arrows d_r, $r > 2$ are also zero, we proceed in exactly the same way, using the morphisms of spectral sequences L_r^k and noting that if $d_s = 0$ for $2 \leq s < r$, then $E_r^{p,q} = E_2^{p,q}$, so that we can use the Lefschetz decomposition as above on $E_r^{p,q}$. □

As explained in [30], the proof above has the following much stronger consequence.

Theorem 5.27 (Deligne 1968). *In the derived category of sheaves of $\mathbb{Q}$-vector spaces on B, there is a decomposition*

$$R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]. \tag{5.27}$$

Proof. We simply observe that the arguments given for the degeneracy at E_2 of the Leray spectral sequence of π relative to the constant sheaf $\mathbb{Q}$ or $\mathbb{R}$ also prove the degeneracy at E_2 of the Leray spectral sequence of π relative to the locally constant sheaves $\pi^{-1}((R^k\pi_*\mathbb{Q})^*)$.

We deduce from this that the natural map

$$H^k(X, \pi^{-1}((R^k\pi_*\mathbb{Q})^*)) \to H^0(B, R^k\pi_*(\pi^{-1}((R^k\pi_*\mathbb{Q})^*)))$$

is surjective. The right-hand side is equal to $H^0(B, R^k\pi_*\mathbb{Q} \otimes (R^k\pi_*\mathbb{Q})^*)$ and thus contains the identity β_k of $R^k\pi_*\mathbb{Q}$. There is thus a class

$$\alpha_k \in H^k(X, \pi^{-1}((R^k\pi_*\mathbb{Q})^*)) = H^k(B, (R^k\pi_*\mathbb{Q})^* \otimes R\pi_*\mathbb{Q})$$

that induces β_k.

We view α_k as giving a morphism

$$R^k\pi_*\mathbb{Q}[-k] \to R\pi_*\mathbb{Q}.$$

This morphism by definition induces the identity on cohomology of degree k and 0 otherwise. The direct sum of the morphisms α_k thus provides the desired quasi isomorphism. ☐

5.3.2 Multiplicative decomposition isomorphisms

Note that both sides of (5.27) carry a cup-product. On the right, we put the direct sum of the relative cup-product maps $\mu_{i,j} : R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q} \to R^{i+j}\pi_*\mathbb{Q}$. On the left, one needs to choose an explicit representation of $R\pi_*\mathbb{Q}$ by a complex C^*, together with an explicit morphism of complexes $\mu : C^* \otimes C^* \to C^*$ which induces the cup-product in cohomology. When passing to coefficients $\mathbb{R}$ or $\mathbb{C}$, one can take $C^* = \pi_*\mathcal{A}^*_{\mathcal{X}}$, where $\mathcal{A}^*_{\mathcal{X}}$ is the sheaf of $\mathcal{C}^\infty$ real or complex differential forms on $\mathcal{X}$ and for μ the wedge product of forms. For rational coefficients, the explicit construction of the cup-product at the level of complexes (for example, Čech complexes) is more painful (see [45, 6.3]). The resulting cup-product morphism μ will be canonical only in the derived category. The rest of this chapter is devoted to the study of the following question.

QUESTION 5.28. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a decomposition as in (5.27) that is multiplicative, that is, compatible with the morphism*

$$\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

given by cup-product?

Note that a multiplicative decomposition isomorphism for the morphism $\pi : \mathcal{X} \to B$ induces a bigrading of the cohomology algebra $H^*(\mathcal{X}, \mathbb{Q})$. Indeed, the induced decompositions

$$H^k(\mathcal{X}, \mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(B, R^q\pi_*\mathbb{Q})$$

are then compatible with the cup-product on $H^*(\mathcal{X}, \mathbb{Q})$.

In fact, we will rather consider the following variant: For which class of varieties X does there exist a multiplicative decomposition isomorphism as above for any family of deformations of X?

The simplest example is that of projective bundles $\pi : \mathbb{P}(\mathcal{E}) \to B$, where $\mathcal{E}$ is a locally free sheaf on B.

LEMMA 5.29 (See Voisin 2012 [110]). *Assume that $c_1^{\mathrm{top}}(\mathcal{E}) = 0$ in $H^2(B, \mathbb{Q})$. Then, if there exists a multiplicative decomposition isomorphism for $\pi : \mathbb{P}(\mathcal{E}) \to B$, one has $c_i^{\mathrm{top}}(\mathcal{E}) = 0$ in $H^{2i}(B, \mathbb{Q})$ for all $i > 0$.*

PROOF. Let $h = c_1^{\mathrm{top}}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. It is standard that

$$H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q}) = \pi^* H^2(B, \mathbb{Q}) \oplus \mathbb{Q}h,$$

where $\pi^* H^2(B, \mathbb{Q})$ identifies canonically with the deepest term $H^2(B, R^0\pi_*\mathbb{Q})$ in the Leray spectral sequence. A multiplicative decomposition isomorphism as in (5.27) induces, by taking cohomology, another decomposition of $H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q})$ as $\pi^* H^2(B, \mathbb{Q}) \oplus \mathbb{Q}h'$, where $h' = h + \pi^*\alpha$, for some $\alpha \in H^2(B, \mathbb{Q})$. In this multiplicative decomposition, h' will generate a summand isomorphic to $H^0(B, R^2\pi_*\mathbb{Q})$. Let $r = \operatorname{rank} \mathcal{E}$. As $c_1^{\mathrm{top}}(\mathcal{E}) = 0$, one has $\pi_* h^r = 0$ in $H^2(B, \mathbb{Q})$. As $(h')^r = 0$ in $H^0(B, R^{2r}\pi_*\mathbb{Q})$, and $(h')^r$ belongs, by multiplicativity, to a direct summand naturally isomorphic (by restriction to fibers) to $H^0(B, R^{2r}\pi_*\mathbb{Q}) = 0$, one must also have $(h')^r = 0$ in $H^{2r}(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. On the other hand, $(h')^r = h^r + rh^{r-1}\pi^*\alpha + \cdots + \pi^*\alpha^r$, and it follows that

$$\pi_*(h')^r = 0 = \pi_* h^r + r\alpha \text{ in } H^2(B, \mathbb{Q}).$$

Thus $\alpha = 0$, $h' = h$, and $h^r = 0$ in $H^{2r}(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. The definition of Chern classes and the fact that $h^r = 0$ shows then that $c_i^{\mathrm{top}}(\mathcal{E}) = 0$ for all $i > 0$. ☐

In this example, the obstructions to the existence of a multiplicative decomposition isomorphism are given by cycle classes $c_i^{\mathrm{top}}(\mathcal{E})$ on the base B. They vanish on dense Zariski open sets of B, and this suggests studying the following variant of Question 5.28.

QUESTION 5.30. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a Zariski dense open set B^0 of B, and a multiplicative decomposition isomorphism as in (5.27) for the restricted family $\mathcal{X}^0 \to B^0$?*

We will give a simple example where Question 5.30 has a negative answer. It is based on the following criterion (Proposition 5.31): Let $\pi : \mathcal{X} \to B$ be a projective family of smooth complex varieties without irregularity, parametrized by a complex quasi-projective variety B. Let $\mathcal{L}_i$, $i = 1, \ldots, m$ be line bundles on $\mathcal{X}$ and $l_i := c_1^{\mathrm{top}}(\mathcal{L}_i) \in H^2(\mathcal{X}, \mathbb{Q})$.

PROPOSITION 5.31. *Assume that there is a multiplicative decomposition isomorphism*

$$R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i].$$

Then for any fiberwise cohomological relation

$$P(l_{i,b}) = 0 \text{ in } H^{2r}(\mathcal{X}_b, \mathbb{Q}) \ \forall\, b \in B,$$

where P is a homogeneous polynomial of degree r in m variables with rational coefficients, the class $P(l_i) \in H^{2r}(\mathcal{X}, \mathbb{Q})$ vanishes locally over B in the Zariski topology, that is, B is covered by Zariski open sets $B^0 \subset B$, such that

$$P(l_i)_{|\mathcal{X}^0} = 0 \text{ in } H^{2r}(\mathcal{X}^0, \mathbb{Q}),$$

where $\mathcal{X}^0 = \pi^{-1}(B^0)$.

Proof. We will assume for simplicity that B is smooth although a closer look at the proof shows that this assumption is not necessary. The multiplicative decomposition isomorphism induces, by taking cohomology and using the fact that the fibers have no degree 1 rational cohomology, a decomposition

$$H^2(\mathcal{X}, \mathbb{Q}) = H^0(B, R^2\pi_*\mathbb{Q}) \oplus \pi^* H^2(B, \mathbb{Q}), \tag{5.28}$$

which is compatible with cup-product, so that the cup-product map on the first term factors through the map induced by cup-product:

$$\mu_r : H^0(B, R^2\pi_*\mathbb{Q})^{\otimes r} \to H^0(B, R^{2r}\pi_*\mathbb{Q}).$$

We write in this decomposition $l_i = l_i' + \pi^* k_i$, where

$$k_i \in H^2(B, R^0\pi_*\mathbb{Q}) = H^2(B, \mathbb{Q}) \overset{\pi^*}{\cong} \pi^* H^2(B, \mathbb{Q}).$$

We claim that the k_i are divisor classes on B. Indeed, take any line bundle $\mathcal{L}$ on $\mathcal{X}$. Let $l = c_1^{\text{top}}(\mathcal{L}) \in H^2(\mathcal{X}, \mathbb{Q})$ and decompose as above $l = l' + \pi^* k$, where l' has the same image as l in $H^0(B, R^2\pi_*\mathbb{Q})$ and k belongs to $H^2(B, \mathbb{Q})$. Denoting by n the dimension of the fibers, we get

$$\begin{aligned} l^n l_i &= \left(\sum_p \binom{n}{p} l'^p \pi^* k^{n-p} \right) (l_i' + \pi^* k_i) \\ &= \sum_p \binom{n}{p} l'^p \pi^* k^{n-p} l_i' + \sum_p \binom{n}{p} l'^p \pi^* k^{n-p} \pi^* k_i. \end{aligned} \tag{5.29}$$

Recall now that the decomposition is multiplicative. The class $l'^n l_i'$ thus belongs to a direct summand of $H^{2n+2}(\mathcal{X}, \mathbb{Q})$ isomorphic to $H^0(B, R^{2n+2}\pi_*\mathbb{Q}) = 0$. Hence it follows that it is identically 0. Applying π_* to (5.29), we then get

$$\begin{aligned} \pi_*(l^n l_i) &= \deg_{\mathcal{X}_b}(l'^n) k_i + n \deg_{\mathcal{X}_b}(l'^{n-1} l_i') k \\ &= \deg_{\mathcal{X}_b}(l^n) k_i + n \deg_{\mathcal{X}_b}(l^{n-1} l_i) k. \end{aligned} \tag{5.30}$$

Observe that the term on the left is a divisor class on B. If the fiberwise self-intersection $\deg_{\mathcal{X}_b}({l_i}^n)$ is nonzero, we can take $\mathcal{L} = \mathcal{L}_i$ and (5.30) shows that k_i is a divisor class on B as claimed. If it is 0, choose a line bundle $\mathcal{L}$ on $\mathcal{S}$ such that the fiberwise intersection numbers $\deg_{\mathcal{X}_b}(l^{n-1} l_i)$ and $\deg_{\mathcal{X}_b}(l^n)$ do not vanish (such an $\mathcal{L}$ exists because the morphism π is projective). Then, applying (5.30) to both pairs $(\mathcal{L}, \mathcal{L})$ and $(\mathcal{L}, \mathcal{L}_i)$ shows that both k and $\deg_{\mathcal{X}_b}(l^n)k_i + n\deg_{\mathcal{X}_b}(l^{n-1}l_i)k$ are divisor classes on B, so that k_i is also a divisor class on B.

As divisor classes are locally trivial on B for the Zariski topology, we thus have that, locally on B for the Zariski topology, $k_i = 0$ and thus l_i belongs to the first summand $H^0(B, R^2\pi_*\mathbb{Q})$ in (5.28). Let B^0 be a Zariski open set where this is the case and let $\mathcal{X}^0 = \pi^{-1}(B^0)$. It then follows by multiplicativity that any polynomial expression $P(l_i)_{|\mathcal{X}^0}$ belongs to a direct summand of $H^{2r}(\mathcal{X}^0, \mathbb{Q})$ isomorphic by the natural projection to $H^0(B^0, R^{2r}\pi_*\mathbb{Q})$.

Consider now our fiberwise cohomological polynomial relation $P(l_{i,b}) = 0$ in $H^{2r}(\mathcal{X}_b, \mathbb{Q})$, for $b \in B$. It says equivalently that $P(l_i)$ vanishes in $H^0(B^0, R^{2r}\pi_*\mathbb{Q})$. It follows then from the previous statement that it vanishes in $H^{2r}(\mathcal{X}^0, \mathbb{Q})$. $\square$

We consider now a smooth projective surface S, and set

$$X = \widetilde{S \times S}_\Delta, \qquad B = S, \qquad \pi = \mathrm{pr}_2 \circ \tau,$$

where $\tau : \widetilde{S \times S}_\Delta \to S$ is the blow-up of the diagonal.

Proposition 5.32. *Assume that $h^{1,0}(S) = 0$, $h^{2,0}(S) \neq 0$. Then for the morphism $\pi : X \to B$ above, there is no multiplicative decomposition isomorphism over any Zariski dense open set of $B = S$.*

Proof. Let H be an ample line bundle on S, and $d := \deg c_1(H)^2$. On X, we have then two line bundles, namely $L := \tau^*(\mathrm{pr}_1^* H)$ and $L' = \mathcal{O}_X(E)$, where E is the exceptional divisor of τ. On the fibers of π, we have the relation

$$\deg c_1(L)^2 = -d \deg c_1(E)^2.$$

If there existed a multiplicative decomposition isomorphism over a Zariski dense open set of $B = S$, we would have by Proposition 5.31, using the fact that the fibers of π are regular surfaces, a Zariski dense open set $U \subset S$ and the relation

$$c_1^{\mathrm{top}}(L)^2 = -d c_1^{\mathrm{top}}(E)^2 \tag{5.31}$$

in $H^4(X_U, \mathbb{Q})$. If we apply τ_* to this relation, we now get

$$\mathrm{pr}_1^* c_1^{\mathrm{top}}(H)^2 = -d[\Delta] \tag{5.32}$$

in $H^4(S \times U, \mathbb{Q})$.

This relation implies that the class $\mathrm{pr}_1^* c_1^{\mathrm{top}}(H)^2 + d[\Delta] \in H^4(S \times S, \mathbb{Q})$ comes from a class $\gamma \in H^2(S \times \widetilde{D}, \mathbb{Q})$, where $D := S \setminus U$ and $\widetilde{D}$ is a desingularization of D. Denoting by $\tilde{j} : \widetilde{D} \to S$ the natural map, we then conclude that for any class $\alpha \in H^2(S, \mathbb{Q})$,

$$d\alpha \in H^2(S, \mathbb{Q}) = \tilde{j}_*(\gamma_*\alpha)$$

is supported on D. This contradicts the assumption $h^{2,0}(S) \neq 0$. $\square$

5.3.3 Families of abelian varieties

In the abelian case, the existence of a multiplicative decomposition isomorphism is essentially due to Deninger and Murre [35]. The proof below will be based on the following lemma, applied to the category of sheaves of $\mathbb{Q}$-vector spaces on B.

Let A be an abelian category in which morphisms are $\mathbb{Q}$-vector spaces, and let $\mathcal{D}(A)$ be the corresponding derived category of left bounded complexes. Let $M \in \mathcal{D}(A)$ be an object with bounded cohomology such that $\operatorname{End} M$ is finite-dimensional. Assume M admits a morphism $\phi : M \to M$ such that

$$H^i(\phi) : H^i(M) \to H^i(M)$$

is equal to $\lambda_i \operatorname{Id}_{H^i(M)}$, where all the $\lambda_i \in \mathbb{Q}$ are distinct.

Lemma 5.33. *The morphism ϕ induces a canonical decomposition*

$$M \cong \oplus_i H^i(M)[-i], \tag{5.33}$$

characterized by the following properties:

(1) The induced map on cohomology is the identity map.

(2) One has

$$\phi \circ \pi_i = \lambda_i \pi_i : M \to M, \tag{5.34}$$

where π_i corresponds via the isomorphism (5.33) to the ith projector pr_i.

Proof. Using the arguments of [30], we first prove that M is decomposed, namely there is an isomorphism

$$f : M \cong \oplus_i H^i(M)[-i].$$

For this, given an object $K \in \operatorname{Ob} A$, we consider the left exact functor T from A to the category of $\mathbb{Q}$-vector spaces defined by $T(N) = \operatorname{Hom}_A(K, N)$, and for any integer i the induced functor, denoted by T_i, $N \mapsto \operatorname{Hom}_{\mathcal{D}(A)}(K[-i], N)$ on $\mathcal{D}^b(A)$. For any $N \in \mathcal{D}^b(A)$, there is the hypercohomology spectral sequence with E_2-term,

$$E_2^{p,q} = R^p T_i(H^q(N)) = \operatorname{Ext}_A^{p+i}(K, H^q(N)) \Rightarrow \mathbb{R}^{p+q} T_i(N).$$

Under our assumptions, this spectral sequence for $N = M$ degenerates at E_2. Indeed, the morphism ϕ acts then on the above spectral sequence starting from E_2. The differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$,

$$\operatorname{Ext}_A^{p+i}(K, H^q(M)) \Rightarrow \operatorname{Ext}_A^{p+2+i}(K, H^{q-1}(M)) \tag{5.35}$$

commutes with the action of ϕ. On the other hand, ϕ acts as $\lambda_q \operatorname{Id}$ on the left-hand side and as $\lambda_{q-1} \operatorname{Id}$ on the right-hand side of (5.35). Thus we conclude that $d_2 = 0$ and similarly that all d_r, $r \geq 2$ are 0.

Now we take $K = H^i(M)$. We conclude, from the degeneracy at E_2 of the above spectral sequence, that the map

$$\mathrm{Hom}_{\mathcal{D}(A)}\left(H^i(M)[-i], M\right) \to \mathrm{Hom}_A(H^i(M), H^i(M)) = E_2^{-i,i}$$

is surjective, so that there is a morphism

$$f_i : H^i(M)[-i] \to M$$

inducing the identity on degree i cohomology. The direct sum $f = \sum f_i$ is a quasi isomorphism that gives the desired splitting.

The morphism ϕ can thus be seen as a morphism of the split object $\oplus_i H^i(M)[-i]$. Such a morphism is given by a block upper-triangular matrix

$$\phi_{j,i} \in \mathrm{Ext}_A^{i-j}(H^i(M), H^j(M)),\ i \geq j,$$

with $\lambda_i \,\mathrm{Id}$ on the ith diagonal block. Let ψ be the endomorphism of $\mathrm{End}\, M$ given by left multiplication by ϕ. By the above description of ϕ we have

$$\prod_{i, H^i(M) \neq 0} (\psi - \lambda_i \,\mathrm{Id}_{\mathrm{End}\, M}) = 0, \tag{5.36}$$

which shows that the endomorphism ψ is diagonalizable. More precisely, as ψ is block upper triangular in an adequately ordered decomposition

$$\mathrm{End}\, M = \oplus_{i \geq j} \mathrm{Ext}_A^{i-j}(H^i(M), H^j(M)),$$

with term $\lambda_j \,\mathrm{Id}$ on the block diagonals $\mathrm{Ext}_A^{i-j}(H^i(M), H^j(M))$, hence in particular on $\mathrm{End}_A\, H^j(M)$, we conclude that there exists $\pi'_i \in \mathrm{End}\, M$ such that π'_i acts as the identity on $H^i(M)$, and $\phi \circ \pi'_i = \lambda_i \pi'_i$.

Let $\rho_i := \pi'_i \circ f_i : H^i(M)[-i] \to M$. Then $\rho := \sum \rho_i$ gives another decomposition $\oplus_i H^i(M)[-i] \cong M$ and we have $\phi \circ \rho_i = \lambda_i \rho_i$, which gives $\phi \circ \pi_i = \lambda_i \pi_i$, where $\pi_i = \rho \circ \mathrm{pr}_i \circ \rho^{-1}$.

The uniqueness of the π_i satisfying properties (1) and (2) is obvious, since these properties force the equality $\pi_i = \frac{\prod_{j \neq i}(\phi - \lambda_j \,\mathrm{Id}_M)}{\prod_{j \neq i} \lambda_i - \lambda_j}$. □

Corollary 5.34. *For any family $\pi : \mathcal{A} \to B$ of abelian varieties (or complex tori), there is a multiplicative decomposition isomorphism $R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]$.*

Proof. Choose an integer $n \neq \pm 1$ and consider the multiplication map

$$\mu_n : \mathcal{A} \to \mathcal{A}, \qquad a \mapsto na.$$

We then get morphisms $\mu_n^* : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ with the property that the induced morphism on each $R^i\pi_*\mathbb{Q} = H^i(R\pi_*\mathbb{Q})$ is multiplication by n^i. Now we use Lemma 5.33 to deduce from such a morphism a canonical splitting

$$R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i], \tag{5.37}$$

characterized by the properties that the induced map on cohomology is the identity map, and

$$\mu_n^* \circ \pi_i = n^i \pi_i : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}, \tag{5.38}$$

where π_i is the endomorphism of $R\pi_*\mathbb{Q}$ that identifies to the ith projector pr_i via the isomorphism (5.37). On the other hand, the morphism $\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ given by cup-product is compatible with μ_n^* in the sense that

$$\mu \circ (\mu_n^* \otimes \mu_n^*) = \mu_n^* \circ \mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}.$$

Combining this last equation with (5.38), we find that

$$\begin{aligned} \mu \circ (\mu_n^* \otimes \mu_n^*) \circ (\pi_i \otimes \pi_j) &= n^{i+j} \mu \circ (\pi_i \otimes \pi_j) \\ &= \mu_n^* \circ \mu \circ (\pi_i \otimes \pi_j) : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}, \end{aligned}$$

from which it follows, again applying (5.38), that $\mu \circ (\pi_i \otimes \pi_j)$ factors through $R^{i+j}\pi_*[-i-j]$, or equivalently that in the splitting (5.37), the cup-product morphism μ maps $R^i\pi_*\mathbb{Q}[-i] \otimes R^j\pi_*\mathbb{Q}[-j]$ to the summand $R^{i+j}\pi_*[-i-j]$. $\square$

5.3.4 A multiplicative decomposition theorem for families of $K3$ surfaces

The following is one of the main results of [110]. It provides an unexpected application of Theorem 5.3.

Theorem 5.35 (Voisin 2011).

(i) For any smooth projective family $\pi : \mathcal{X} \to B$ of $K3$ surfaces, there exist a nonempty Zariski open subset B^0 of B, and a multiplicative decomposition isomorphism as in (5.27) for the restricted family $\pi : \mathcal{X}^0 \to B^0$.

(ii) Furthermore, the class of the diagonal $[\Delta_{X^0/B^0}] \in H^4(X \times_B X, \mathbb{Q})$ belongs to the direct summand $H^0(B^0, R^4(\pi,\pi)_*\mathbb{Q})$ of $H^4(X^0 \times_{B^0} X^0, \mathbb{Q})$, for the induced decomposition of $R(\pi,\pi)_*\mathbb{Q}$.

(iii) For any algebraic line bundle $\mathcal{L}$ on $\mathcal{X}$, there is a dense Zariski open set B^0 of B such that the topological Chern class $c_1^{\mathrm{top}}(\mathcal{L}) \in H^2(\mathcal{X}, \mathbb{Q})$ restricted to $\mathcal{X}^0$ belongs to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$ of $H^2(\mathcal{X}^0, \mathbb{Q})$ induced by this decomposition.

In the second statement, $(\pi,\pi) : X^0 \times_{B^0} X^0 \to B^0$ denotes the natural map. A decomposition $R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i]$ induces a decomposition

$$R(\pi,\pi)_*\mathbb{Q} = \oplus_i R^i(\pi,\pi)_*\mathbb{Q}[-i]$$

by the relative Künneth isomorphism

$$R(\pi,\pi)_*\mathbb{Q} \cong R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q}.$$

Theorem 5.35(i) is definitely wrong if we do not restrict to a Zariski open set (see [110] for an example).

REMARK 5.36. It follows from Proposition 5.31 that in Theorem 5.35, statement (iii) is in fact a consequence of statement (i).

PROOF OF THEOREM 5.35. We use the existence of the canonical 0-cycle $o_t \in \mathrm{CH}_0(X_t)$ (see Section 5.1). This cycle may not be spread-up on the total space $X \to B$, but this can be done on a generically finite proper cover $r : B' \to B$, thus providing a cycle $o_{X'} \in \mathrm{CH}^2(X')$, where $X' = X \times_B B'$, with $o_{X'|X'_t} = o_t$ in $\mathrm{CH}_0(X'_t)$. The cycle

$$\frac{1}{\deg r} r'_* o_{X'} =: o_X \in \mathrm{CH}^2(X)_{\mathbb{Q}},$$

where $r' : X' = X \times_B B' \to X$ is the first projection, then has the property that $o_{X|X_t} = o_t$ in $\mathrm{CH}_0(X_t)_{\mathbb{Q}}$. In particular it has degree 1.

The cohomology classes

$$\mathrm{pr}_1^*[o_X] =: [Z_1], \qquad \mathrm{pr}_2^*[o_X] =: [Z_2] \in H^4(X \times_B X, \mathbb{Q})$$

of the two codimension 2 cycles $\mathrm{pr}_1^* o_X$ and $\mathrm{pr}_2^* o_X$, where

$$\mathrm{pr}_i : X \times_B X \to B$$

are the two projections, provide morphisms in the derived category:

$$\begin{aligned} &P_1 : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}, && P_2 : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}, \\ &P_1 := \mathrm{pr}_{2*} \circ (\mathrm{pr}_1^*[o_X] \cup) \circ \mathrm{pr}_1^*, && P_2 := \mathrm{pr}_{2*} \circ (\mathrm{pr}_2^*[o_X] \cup) \circ \mathrm{pr}_1^*. \end{aligned} \tag{5.39}$$

LEMMA 5.37.

(i) The morphisms P_1, P_2 are projectors of $R\pi_*\mathbb{Q}$.

(ii) $P_1 \circ P_2 = P_2 \circ P_1 = 0$ over a Zariski dense open set of B.

PROOF. (i) We compute $P_1 \circ P_1$. From (5.39), (2.5), and the projection formula (2.4), we get that $P_1 \circ P_1$ is the morphism $R\pi_* \to R\pi_*$ induced by the cycle class

$$p_{13*}(p_{12}^*[o_X^1] \cup p_{23}^*[o_X^1]) \in H^4(X \times_B X, \mathbb{Q}), \tag{5.40}$$

where the p_{ij} are the various projections from $X \times_B X \times_B X$ to $X \times_B X$. We now use the facts that $p_{12}^*[o_X^1] = p_1^*[o_X]$ and $p_{23}^*[o_X^1] = p_2^*[o_X]$, where the p_i's are the various projections from $X \times_B X \times_B X$ to X, so that (5.40) is equal to

$$p_{13*}(p_1^*[o_X] \cup p_2^*[o_X]). \tag{5.41}$$

Using the projection formula, this class is equal to

$$\mathrm{pr}_1^*[o_X] \cup \mathrm{pr}_2^*(\pi_*[o_X]) = \mathrm{pr}_1^*[o_X] \cup \mathrm{pr}_2^*(1_B) = \mathrm{pr}_1^*[o_X].$$

This completes the proof for P_1 and the proof for P_2 is exactly similar.

(ii) We compute $P_1 \circ P_2$: From (5.39) and (2.5), we get that $P_1 \circ P_2$ is the morphism $R\pi_* \to R\pi_*$ induced by the cycle class

$$p_{13*}(p_{12}^*[o_X^2] \cup p_{23}^*[o_X^1]) \in H^4(X \times_B X, \mathbb{Q}), \tag{5.42}$$

where the p_{ij} are the various projections from $X \times_B X \times_B X$ to $X \times_B X$. We now use the facts that $p_{12}^*[o_X^2] = p_2^*[o_X]$ and $p_{23}^*[o_X^1] = p_2^*[o_X]$, where the p_i's are the various projections from $X \times_B X \times_B X$ to X, so that (5.42) is equal to

$$p_{13*}(p_2^*[o_X] \cup p_2^*[o_X]). \tag{5.43}$$

But the class $p_2^*[o_X] \cup p_2^*[o_X] = p_2^*([o_X \cdot o_X])$ vanishes over a Zariski dense open set of B since the cycle $o_X \cdot o_X$ has codimension 4 in X. This shows that $P_1 \circ P_2 = 0$ over a Zariski dense open set of B and the proof for $P_2 \circ P_1$ works in the same way. □

Using Lemma 5.37, we get (up to passing to a Zariski dense open set of B) a third projector

$$P := \mathrm{Id} - P_1 - P_2$$

acting on $R\pi_*\mathbb{Q}$ and commuting with the other two.

It is well known (see [74]) that the actions of these three projectors are

$$\begin{aligned}
P_{1*} &= 0 \text{ on } R^2\pi_*\mathbb{Q},\ R^4\pi_*\mathbb{Q}, & P_{1*} &= \mathrm{Id} \text{ on } R^0\pi_*\mathbb{Q};\\
P_{2*} &= 0 \text{ on } R^2\pi_*\mathbb{Q},\ R^0\pi_*\mathbb{Q}, & P_{2*} &= \mathrm{Id} \text{ on } R^4\pi_*\mathbb{Q};\\
P_* &= \mathrm{Id} \text{ on } R^2\pi_*\mathbb{Q}, & P_* &= 0 \text{ on } R^0\pi_*\mathbb{Q},\ R^4\pi_*\mathbb{Q}.
\end{aligned}$$

As a consequence, we get (for example, using Lemma 5.33) a decomposition

$$R\pi_*\mathbb{Q} \cong \oplus R^i\pi_*\mathbb{Q}[-i], \tag{5.44}$$

where the corresponding projectors π_0, π_2, and π_4 of $R\pi_*\mathbb{Q}$ identify to P_1, P, and P_2, respectively.

We now prove the following result.

Proposition 5.38. *Assume that the cohomology class of the relative small diagonal $\Delta \subset X \times_B X \times_B X$ satisfies the equality*

$$[\Delta] = p_1^*[o_X] \cup p_{23}^*[\Delta_X] + (\text{perm.}) - (p_1^*[o_X] \cup p_2^*[o_X] + (\text{perm.})), \tag{5.45}$$

where the p_{ij}'s and p_i's are as above and Δ_X is the relative diagonal $X \subset X \times_B X$; then, over some Zariski dense open set $B^0 \subset B$, we have the following:

(i) The decomposition (5.44) is multiplicative.

(ii) The class of the diagonal $[\Delta_X] \in H^4(X \times_B X, \mathbb{Q})$ belongs to the direct summand $H^0(B, R^4(\pi,\pi)_*\mathbb{Q})$ induced by the decomposition (5.44).

Admitting Proposition 5.38, the end of the proof of Theorem 5.35 is as follows: By Theorem 5.3, we know that the relation

$$\Delta_t = p_1^* o_{X_t} \cdot p_{23}^* \Delta_{X_t} + (\text{perm.}) - (p_1^* o_{X_t} \cdot p_2^* o_{X_t} + (\text{perm.}))$$

holds in $\mathrm{CH}_2(X_t, \mathbb{Q})$ for any $t \in B$. By Corollary 1.3, we conclude that there exists a Zariski dense open set B^0 of B such that (5.45) holds in $H^8(X \times_B X \times_B X, \mathbb{Q})$. Proposition 5.38 thus implies Theorem 5.35. ☐

Proof of Proposition 5.38. (i) We want to show that

$$\pi_k \circ \cup \circ (\pi_i \otimes \pi_j) : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

vanishes for $k \neq i + j$. We note that

$$\cup : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

is induced, via the relative Künneth decomposition

$$R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \cong R(\pi, \pi)_*\mathbb{Q},$$

by the class $[\Delta]$ of the small relative diagonal in $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$, seen as a relative correspondence between $\mathcal{X} \times_B \mathcal{X}$ and $\mathcal{X}$, while $P_1 = \pi_0$, $P_2 = \pi_4$, $P = \pi_2$ are induced by the cycle classes $[Z_1]$, $[Z_2]$, $[Z] \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$, where $Z := \Delta_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X}$. It thus suffices to show that the cycle classes

$$\begin{aligned}
&[Z_2 \circ \Delta \circ (Z_1 \times_B Z_1)], \quad [Z \circ \Delta \circ (Z_1 \times_B Z_1)], \\
&[Z_1 \circ \Delta \circ (Z \times_B Z)], \quad [Z \circ \Delta \circ (Z \times_B Z)], \\
&[Z_1 \circ \Delta \circ (Z_2 \times_B Z_2)], \quad [Z \circ \Delta \circ (Z_2 \times_B Z_2)], \quad [Z_2 \circ \Delta \circ (Z_2 \times_B Z_2)], \\
&[Z_1 \circ \Delta \circ (Z_1 \times_B Z_2)], \quad [Z \circ \Delta \circ (Z_1 \times_B Z_2)], \\
&[Z_1 \circ \Delta \circ (Z \times_B Z_2)], \quad [Z \circ \Delta \circ (Z \times_B Z_2)], \quad [Z_2 \circ \Delta \circ (Z \times_B Z_2)], \\
&[Z_1 \circ \Delta \circ (Z_1 \times_B Z)], \quad [Z_2 \circ \Delta \circ (Z_1 \times_B Z)]
\end{aligned}$$

vanish in $H^8(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ over a dense Zariski open set of B. Here, all the compositions of correspondences are over B.

Equivalently, it suffices to prove the following equality of cycle classes in $H^8(\mathcal{X}^0 \times_B \mathcal{X}^0 \times_B \mathcal{X}^0, \mathbb{Q})$, $\mathcal{X}^0 = \pi^{-1}(B^0)$ for a Zariski dense open set B^0 of B:

$$\begin{aligned}
[\Delta] = {}& [Z_1 \circ \Delta \circ (Z_1 \times_B Z_1)] + [Z_2 \circ \Delta \circ (Z \times_B Z)] \\
&+ [Z \circ \Delta \circ (Z_1 \times_B Z)] + [Z \circ \Delta \circ (Z \times_B Z_1)] \\
&+ [Z_2 \circ \Delta \circ (Z_1 \times_B Z_2)] + [Z_2 \circ \Delta \circ (Z_2 \times_B Z_1)]. \qquad (5.46)
\end{aligned}$$

Replacing Z by $\Delta_{\mathcal{X}} - Z_1 - Z_2$, we get

$$\begin{aligned}
Z \times_B Z = {}& \Delta_{\mathcal{X}} \times_B \Delta_{\mathcal{X}} - \Delta_{\mathcal{X}} \times_B Z_1 - \Delta_{\mathcal{X}} \times_B Z_2 - Z_1 \times_B \Delta_{\mathcal{X}} \\
&- Z_2 \times_B \Delta_{\mathcal{X}} + Z_1 \times_B Z_1 + Z_2 \times_B Z_2 + Z_1 \times_B Z_2 + Z_2 \times_B Z_1,
\end{aligned}$$

and thus (5.46) becomes

$$\begin{aligned}[\Delta] = {} & [Z_1 \circ \Delta \circ (Z_1 \times_B Z_1)] + [Z_2 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B \Delta_X)] - [Z_2 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_1)] \\ & - [Z_2 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_2)] - [Z_2 \circ \Delta \circ (Z_1 \times_B \Delta_{\mathcal{X}})] - [Z_2 \circ \Delta \circ (Z_2 \times_B \Delta_{\mathcal{X}})] \\ & + [Z_2 \circ \Delta \circ (Z_1 \times_B Z_1)] + [Z_2 \circ \Delta \circ (Z_2 \times_B Z_2)] + [Z_2 \circ \Delta \circ (Z_1 \times_B Z_2)] \\ & + [Z_2 \circ \Delta \circ (Z_2 \times_B Z_1)] + [Z \circ \Delta \circ (Z_1 \times_B \Delta_{\mathcal{X}})] - [Z \circ \Delta \circ (Z_1 \times_B Z_1)] \\ & - [Z \circ \Delta \circ (Z_1 \times_B Z_2)] + [Z \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_1)] - [Z \circ \Delta \circ (Z_1 \times_B Z_1)] \\ & - [Z \circ \Delta \circ (Z_2 \times_B Z_1)] + [Z_2 \circ \Delta \circ (Z_1 \times_B Z_2)] + [Z_2 \circ \Delta \circ (Z_2 \times_B Z_1)]. \end{aligned} \tag{5.47}$$

We now have the following lemma.

LEMMA 5.39. *We have the following equalities of cycles in* $\mathrm{CH}^4(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$ *(or relative correspondences between* $\mathcal{X} \times_B \mathcal{X}$ *and* $\mathcal{X}$*):*

$$\Delta \circ (Z_1 \times_B Z_1) = p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}, \tag{5.48}$$
$$\Delta \circ (\Delta_{\mathcal{X}} \times_B \Delta_{\mathcal{X}}) = \Delta, \tag{5.49}$$
$$\Delta \circ (\Delta_{\mathcal{X}} \times_B Z_1) = p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}, \tag{5.50}$$
$$\Delta \circ (\Delta_{\mathcal{X}} \times_B Z_2) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{5.51}$$
$$\Delta \circ (Z_1 \times_B \Delta_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}, \tag{5.52}$$
$$\Delta \circ (Z_2 \times_B \Delta_{\mathcal{X}}) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{5.53}$$
$$\Delta \circ (Z_2 \times_B Z_2) = p_3^* (o_{\mathcal{X}} \cdot o_{\mathcal{X}}), \tag{5.54}$$
$$\Delta \circ (Z_1 \times_B Z_2) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{5.55}$$
$$\Delta \circ (Z_2 \times_B Z_1) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{5.56}$$

where the p_i*'s, for* $i = 1, 2, 3$*, are the projections from* $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ *to* $\mathcal{X}$ *and the* p_{ij}*'s are the projections from* $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ *to* $\mathcal{X} \times_B \mathcal{X}$*.*

PROOF. Equation (5.49) is obvious. Equations (5.48), (5.54), (5.55), (5.56) are all similar. We will only prove (5.55). The cycle Z_2 is $\mathcal{X} \times_B o_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X}$, and similarly $Z_1 = o_{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X} \times_B \mathcal{X}$, hence $Z_1 \times_B Z_2$ is the cycle

$$\{(o_{\mathcal{X}_b}, x, y, o_{\mathcal{X}_b}),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}. \tag{5.57}$$

(It turns out that in this case, we do not have to take care with the ordering we take for the last inclusion.) Composing over B with $\Delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ is done by taking the pull-back of (5.57) under $p_{1234} : \mathcal{X}^{5/B} \to \mathcal{X}^{4/B}$, intersecting with $p_{345}^* \Delta$, and projecting the resulting cycle to $\mathcal{X}^{3/B}$ via p_{125}. The resulting cycle is obviously

$$\{(o_{\mathcal{X}_b}, x, o_{\mathcal{X}_b}),\ x \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X},$$

which proves (5.55).

For the remaining formulas, which are all of the same shape, let us just prove (5.50). Recall that $Z_1 = o_{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X} \times_B \mathcal{X}$. Thus $\Delta_{\mathcal{X}} \times_B Z_1$ is the cycle

$$\{(x, x, o_{\mathcal{X}_b}, y),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}.$$

But we have to see this cycle as a relative self-correspondence of $\mathcal{X} \times_B \mathcal{X}$, for which the right ordering is

$$\{(x, o_{\mathcal{X}_b}, x, y),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}. \qquad (5.58)$$

Composing over B with $\Delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ is again done by taking the pull-back of (5.58) by $p_{1234} : \mathcal{X}^{5/B} \to \mathcal{X}^{4/B}$, intersecting with $p_{345}^*\Delta$, and projecting the resulting cycle to $\mathcal{X}^{3/B}$ via p_{125}. Since $\Delta = \{(z, z, z),\ z \in \mathcal{X}\}$, the considered intersection is $\{(x, o_{\mathcal{X}_b}, x, x, x),\ x \in \mathcal{X}_b,\ b \in B\}$, and thus the projection via p_{125} is $\{(x, o_{\mathcal{X}_b}, x),\ x \in \mathcal{X}_b,\ b \in B\}$, thus proving (5.50). □

Using Lemma 5.39 and the fact that the cycle $p_3^*(o_{\mathcal{X}} \cdot o_{\mathcal{X}})$ vanishes, for reasons of dimension, over a dense Zariski open set of B, then after passing to a Zariski open set of B if necessary, (5.47) becomes

$$\begin{aligned}[\Delta] = {} & [Z_1 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_2 \circ \Delta] - [Z_2 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] \\ & - [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] - [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] - [Z_2 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] - [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})], \end{aligned} \qquad (5.59)$$

which can be rewritten as

$$\begin{aligned}[\Delta] = {} & [Z_1 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_2 \circ \Delta] - [Z_2 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] \\ & - [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] - [Z \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\ & + [Z \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - 2[Z \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})]. \end{aligned} \qquad (5.60)$$

To conclude, we use the following lemma.

LEMMA 5.40. *Up to passing to a dense Zariski open set of B, we have the* *following equalities in* $\mathrm{CH}^4(\mathcal{X}\times_B\mathcal{X}\times_B\mathcal{X})_{\mathbb{Q}}$*:*

$$Z_1\circ(p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})=p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}},\tag{5.61}$$
$$Z_2\circ\Delta=p_{12}^*\Delta_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.62}$$
$$Z_2\circ(p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})=p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.63}$$
$$Z_2\circ(p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}})=p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.64}$$
$$Z_2\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}})=p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.65}$$
$$Z_2\circ(p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})=0,\tag{5.66}$$
$$Z_2\circ(p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}})=p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.67}$$
$$Z\circ(p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}})=0,\tag{5.68}$$
$$Z\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}})=p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.69}$$
$$Z\circ(p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})=p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}},\tag{5.70}$$
$$Z\circ(p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})=0.\tag{5.71}$$

Proof. The proof of (5.63) is explicit, recalling that $Z_2=\{(x,o_{\mathcal{X}_b}),\,x\in\mathcal{X}_b,\,b\in B\}$ and that $p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}=\{(y,o_{\mathcal{X}_b},y),\,y\in\mathcal{X}_b,\,b\in B\}$. We then find that $Z_2\circ(p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}})$ is the cycle

$$\begin{aligned}p_{124}(p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}\cdot p_{34}^*(Z_2))&=p_{124}(\{(y,o_{\mathcal{X}_b},y,o_{\mathcal{X}_b}),\,y\in\mathcal{X}_b,\,b\in B\})\\&=\{(y,o_{\mathcal{X}_b},o_{\mathcal{X}_b}),\,y\in\mathcal{X}_b,\,b\in B\},\end{aligned}$$

which proves (5.63). The proofs of the remaining equations from (5.61) to (5.67) work similarly.

For the other proofs, we recall that

$$Z=\Delta_{\mathcal{X}}-Z_1-Z_2\subset\mathcal{X}\times_B\mathcal{X}.$$

Thus, since $\Delta_{\mathcal{X}}$ acts as the identity, we get

$$Z\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}})=p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}}-Z_1\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}})-Z_2\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}}).$$

We then compute the terms $Z_1\circ(p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}})$ explicitly as before, which gives (5.69).

The other proofs are similar. □

Using the cohomological version of Lemma 5.40, (5.60) becomes

$$\begin{aligned}[\Delta]=&[p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}]+[p_{12}^*\Delta_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]-[p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]\\&-[p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]+[p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]+[p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]\\&+[p_1^*o_{\mathcal{X}}\cdot p_{23}^*\Delta_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}]\\&+[p_{13}^*\Delta_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_1^*o_{\mathcal{X}}\cdot p_2^*o_{\mathcal{X}}-p_2^*o_{\mathcal{X}}\cdot p_3^*o_{\mathcal{X}}].\end{aligned}\tag{5.72}$$

This last equality is now satisfied by assumption (compare with (5.45)) and this concludes the proof of formula (5.46). Thus Proposition 5.38(i) is proved.

(ii) We just have to prove that

$$P_1 \otimes P_1([\Delta_{\mathcal{X}}]) = P_2 \otimes P_2([\Delta_{\mathcal{X}}]) = 0,$$
$$P_1 \otimes P([\Delta_{\mathcal{X}}]) = P_2 \otimes P([\Delta_{\mathcal{X}}]) = 0 \text{ in } H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}). \tag{5.73}$$

Indeed, the relative Künneth decomposition gives

$$R(\pi,\pi)_*\mathbb{Q} = R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q}$$

and the decomposition (5.44) induces a decomposition of the above tensor product on the right:

$$R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} = \oplus_{k,l} R^k\pi_*\mathbb{Q} \otimes R^l\pi_*\mathbb{Q}[-k-l], \tag{5.74}$$

where the decomposition is induced by the various tensor products of P_1, P_2, P. Taking cohomology in (5.74) gives

$$H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}) = \oplus_{s+k+l=4} H^s(B, R^k\pi_*\mathbb{Q} \otimes R^l\pi_*\mathbb{Q}).$$

The term $H^0(R^4(\pi,\pi)_*\mathbb{Q})$ is then exactly the term in the above decomposition of $H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ that is annihilated by the four projectors $P_1 \otimes P_1$, $P_1 \otimes P$, $P_2 \otimes P$, $P_2 \otimes P_2$ and those obtained by changing the order of factors, whose vanishing will be deduced from the others by symmetry.

The proof of (5.73) is elementary. Indeed, consider for example the term $P_1 \otimes P_1$, which is given by the cohomology class of the cycle

$$Z := \mathrm{pr}_1^* o_{\mathcal{X}} \cdot \mathrm{pr}_2^* o_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X},$$

which we see as a relative self-correspondence of $\mathcal{X} \times_B \mathcal{X}$ We have

$$Z_*(\Delta_{\mathcal{X}}) = p_{34*}(p_{12}^*\Delta_{\mathcal{X}} \cdot Z).$$

But the cycle on the right is trivially rationally equivalent to 0 on fibers $\mathcal{X}_t \times \mathcal{X}_t$. It thus follows from Corollary 1.3 that for some dense Zariski open set B^0 of B,

$$[Z]_*([\Delta_{\mathcal{X}}]) = 0 \text{ in } H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q}).$$

The other vanishing statements are proved similarly. □

Chapter Six

Integral coefficients

Up to now, we have been working with rational coefficients, and indeed most conjectures and statements made previously become wrong when passing to Chow groups or Betti cohomology with integral coefficients. There are two examples of this fact: First of all, it has been well known since the work of Atiyah and Hirzebruch [5] that the Hodge conjecture is not true for integral Hodge classes. Second, Theorem 3.1 and all its more precise forms concerning the decomposition of the diagonal are in fact decomposition results for a multiple of the considered cycle, or in other words, they provide a decomposition of the cycle itself only with $\mathbb{Q}$-coefficients. What we discuss in this chapter is a number of birational invariants that can be defined using $\mathbb{Z}$-coefficients instead. Of course, the important question is whether these birational invariants can be nonzero for rationally connected or even unirational varieties.

6.1 INTEGRAL HODGE CLASSES AND BIRATIONAL INVARIANTS

6.1.1 Atiyah–Hirzebruch–Totaro topological obstruction.

Atiyah and Hirzebruch [5] found counterexamples to the Hodge conjecture (Conjecture 2.25) stated for degree $2k$ *integral* Hodge classes (as opposed to *rational* Hodge classes) when $k \geq 2$. In degree 2, the most optimistic statements are true, due to the Lefschetz theorem on $(1,1)$-classes.

In [95], Totaro revisited the examples of Atiyah and Hirzebruch and reformulated more directly the obstructions they had found, using the complex cobordism graded ring $\mathrm{MU}^*(X)$ of X. Let us first describe this ring that is defined for all differentiable compact manifolds: Given such an X, we consider first of all the objects that are triples (V, f, ϵ), where V is a differentiable compact manifold, $f : V \to X$ is a differentiable map, and ϵ is a class of a *stable* complex structure on the virtual normal bundle $f^*T_X - T_V$. Here $f^*T_X - T_V$ is an element of the K_0 group of real vector bundles on V, and choosing a stable complex structure on it means we choose an element of the K_0 group of *complex* vector bundles on V that sends (via the natural forgetful map) to $f^*T_X - T_V + T$, where T is the trivial real vector bundle of rank 1 or 0.

One now makes the following construction, which is a slight variant of the Thom construction of the absolute complex cobordism ring $\mathrm{MU}^* = \mathrm{MU}^*(\text{point})$. Let us consider the free abelian group generated by such triples, and take its quotient by the complex cobordism relations; namely, for each differentiable compact manifold with boundary M, differentiable map $\phi : M \to X$, and stable complex structure ϵ on the virtual normal bundle $f^*T_X - T_M$, we observe that the restriction of T_M to the boundary of M is naturally isomorphic to $T_{\partial M} \oplus T$, where T is trivial of rank 1. Thus the stable complex structure ϵ on the virtual normal bundle $f^*T_X - T_M$ induces a stable complex structure ϵ_0 on the virtual normal bundle $f_0^*T_X - T_{\partial M}$, where f_0 is the restriction of f to ∂M.

We take the quotient of the free abelian group by the relations generated by

$$(\partial M, f_0, \epsilon_0) = 0,$$

$$(V_1 \sqcup V_2, f_1 \sqcup f_2, \epsilon_1 \sqcup \epsilon_2) = (V_1, f_1, \epsilon_1) + (V_2, f_2, \epsilon_2).$$

The result will be denoted by $\mathrm{MU}^*(X)$. Here the grading is given by $* = \dim X - \dim V$. The product structure is given by the fibered product over X.

Note that $\mathrm{MU}^*(X)$ is naturally an MU^*-module, since elements of MU^* are generated by data (W, ϵ_0), where W is differentiable compact and ϵ_0 is a stable complex structure on T_W (here the map to a point is necessarily constant). Then for (V, f, ϵ) and (W, ϵ_0) as above, we can consider the product $(V \times W, f \circ \mathrm{pr}_1, \epsilon + \epsilon_0)$.

Denote by $\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z}$ its tensor product with $\mathbb{Z}$ over MU^* (which maps by the degree to $\mathbb{Z} = H^0(\mathrm{point}, \mathbb{Z})$). Thus, in $\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z}$, one kills all the products $(V \times W, f \circ \mathrm{pr}_1, \epsilon_1 + \epsilon_2)$, where ϵ_2 is a stable complex structure on W, with $\dim W > 0$. Since, for such products, we have

$$(f \circ \mathrm{pr}_1)_*(1_{V \times W}) = 0,$$

there is a natural map,

$$\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z} \to H^*(X, \mathbb{Z}),$$

$$(V, f, \epsilon) \mapsto f_* 1_V.$$

(Here we note that, as we have a stable complex structure on the virtual normal bundle of f, the Gysin image $f_* 1_V$ is well defined. If X is oriented, V is also naturally oriented because the virtual normal bundle of f has a stable complex structure, and then $f_*(1_V)$ is the Poincaré dual cohomology class of the homology class $f_*([V]_{\mathrm{fund}})$.)

Coming back to the case where X is a complex projective (or more generally compact complex) manifold, Totaro [95] observed that the cycle class map

$$\mathcal{Z}^k(X) \to H^{2k}(X, \mathbb{Z}), \qquad Z \mapsto [Z],$$

where the left-hand side is the free abelian group generated by subvarieties (or irreducible closed analytic subsets) of codimension k of X, is the composite of two maps

$$\mathcal{Z}^k(X) \to \left(\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z}\right)^{2k} \to H^{2k}(X, \mathbb{Z}). \tag{6.1}$$

Here the second map was introduced above, and the first map is defined using the construction of the complex cobordism ring; indeed, for any inclusion of a (maybe singular) codimension k closed algebraic subset $Z \subset X$, there is a desingularization

$$\tau : \widetilde{Z} \to X,$$

and we have

$$[Z] = \tau_* 1_{\widetilde{Z}}.$$

On the other hand, the virtual normal bundle of τ has an obvious stable complex structure.

By the factorization (6.1), a torsion class in $H^{2k}(X, \mathbb{Z})$ that is not in the image of

$$\left(\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z}\right)^{2k} \to H^{2k}(X, \mathbb{Z})$$

cannot be algebraic. On the other hand, as it sends to 0 in $H^{2k}(X, \mathbb{C})$, it is obviously an integral Hodge class. This is a supplementary topological obstruction for an integral Hodge class to be algebraic. These obstructions are of torsion, as the map $(\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z})^{2k} \to H^{2k}(X, \mathbb{Z})$ becomes an isomorphism when tensored by $\mathbb{Q}$. In fact they essentially concern torsion classes, as explained by Totaro, as the map

$$\left(\mathrm{MU}^*(X) \underset{\mathrm{MU}^*}{\otimes} \mathbb{Z}\right)^{2k} \to H^{2k}(X, \mathbb{Z})$$

is an isomorphism if $H^*(X, \mathbb{Z})$ has no torsion. The Atiyah–Hirzebruch example [5] shows that these obstructions are effective.

6.1.2 Kollár's example

We start this section by describing a method due to Kollár [63], which produces examples of smooth projective complex varieties X, together with an even degree integral cohomology class α, which is not algebraic, that is, which is not the cohomology class of an algebraic cycle of X, while a nonzero multiple of α is algebraic. This is another sort of counterexample to the Hodge conjecture over the integers, since the class α is of course a Hodge class.

The examples are as follows: Consider a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d. For $l < n$, the Lefschetz theorem on hyperplane sections says that the restriction map

$$H^l(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^l(X, \mathbb{Z})$$

is an isomorphism. Since the left-hand side is isomorphic to $\mathbb{Z}H^k$ for $l = 2k < n$, where H is the cohomology class of a hyperplane, and is 0 for l odd or $l > n$, we conclude by Poincaré duality on X that for $2k > n$, we have $H^{2k}(X, \mathbb{Z}) = \mathbb{Z}\alpha$, where α is determined by the condition $\langle \alpha, h^{n-k} \rangle = 1$, with the notation $h =$

$H_{|X} = c_1(\mathcal{O}_X(1))$. Note that the class $d\cdot\alpha$ is equal to h^k (both have intersection number d with h^{n-k}), hence is algebraic.

In the sequel, we consider for simplicity the case where $n = 3$, $k = 2$. Then $d \cdot \alpha$ is the class of a plane section of X.

THEOREM 6.1 (Kollár 1990). *Assume that for some integer p coprime to 6, p^3 divides d. Then for very general X, any curve $C \subset X$ has degree divisible by p. Hence the class α is not algebraic.*

Recall that "very general" means that the defining equation for X has to be chosen away from a specified union of countably many Zariski closed proper subsets of the parameter space.

PROOF OF THEOREM 6.1. Let $d = p^3 s$, and let $Y \subset \mathbb{P}^4$ be a degree s smooth hypersurface. Let $\phi_0, \ldots, \phi_4$ be sections of $\mathcal{O}_{\mathbb{P}^4}(p)$ without common zeros. They provide a map

$$\phi : Y \to \mathbb{P}^4,$$

which for a generic choice of the ϕ_i's satisfies the following properties:

(1) ϕ is generically of degree 1 onto its image, which is a hypersurface $X_0 \subset \mathbb{P}^4$ of degree $p^3 s = d$.

(2) ϕ is two-to-one generically over a surface in X_0, three-to-one generically over a curve in X_0, at most finitely many points of X_0 have more than 3 preimages, and no point has more than 4 preimages.

Now let $X \subset \mathbb{P}^4$ be a smooth hypersurface that is very general in moduli. Let $C \subset X$ be a reduced curve. The idea is to degenerate the pair (X, C), $C \subset X$ to a pair (X_0, C_0), $C_0 \subset X_0$, where X_0 was defined above. This is possible because the point parametrizing X is very general, and because there are only countably many relative Hilbert schemes over the moduli space of X, parametrizing curves in the fibers X_t. Thus a curve $C \subset X$, with X very general in moduli, must correspond to a point of a relative Hilbert scheme that dominates the moduli space of X.

By flatness, the curve C_0 has the same degree as C. Recall the normalization map

$$\phi : Y \to X_0.$$

By property (2) above, there exists a 1-cycle $\tilde{z}_0$ in Y such that $\phi_*(\tilde{z}_0) = 6z_0$, where z_0 is the cycle associated to C_0. It follows that

$$6 \deg z_0 = \deg \phi_*(\tilde{z}_0).$$

On the other hand, the right-hand side is equal to the degree of the line bundle $\phi^*\mathcal{O}_{X_0}(1)$ computed on the cycle $\tilde{z}_0$. Since $\phi^*\mathcal{O}_{X_0}(1)$ is equal to $\mathcal{O}_Y(p)$, it follows that this degree is divisible by p. Hence we find that $6 \deg C = 6 \deg C_0 = 6 \deg z_0$ is divisible by p, and since p is coprime to 6, it follows that $\deg C$ is also divisible by p. □

REMARK 6.2. The argument above works only for very general X, which could a priori exclude all varieties X defined over a number field K. However Hassett and Tschinkel gave an alternative argument (replacing the degeneration above by specialization to an adequate closed fiber of a projective model of X defined over $\operatorname{Spec} \mathcal{O}_K$), showing that there exist varieties X satisfying the conclusions of Theorem 6.1 and defined over a number field.

As remarked in [93], Kollár's example, which is not topological, exhibits the following phenomenon that illustrates the complexity of the Hodge conjecture. We can have a family $\mathcal{X} \to B$ of smooth projective complex manifolds, and a locally constant integral Hodge class $\alpha_t \in H^4(X_t, \mathbb{Z})$, with the property that on a dense (for the Euclidean topology) subset $B_{\text{alg}} \subset B$, which is a countable union of closed proper algebraic subsets of B, the class α_t is algebraic, that is, is an integral combination $[Z_t] = \sum_i n_i [Z_{i,t}]$ of classes of codimension 2 subvarieties, but on its complementary set, which is a countable intersection of dense open subsets, the class α_t is not algebraic.

Indeed, it is shown in [93] that there exists a countable union of proper algebraic subsets, which is *dense for the usual topology* in the parameter space of all smooth hypersurfaces of degree d in $\mathbb{P}^4$, consisting of points parametrizing hypersurfaces X for which the class α is algebraic. For this, it suffices to prove that the set of smooth surfaces of degree d in $\mathbb{P}^3$ carrying an algebraic class $\lambda \in H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$, satisfying the property that $\langle \lambda, c_1(\mathcal{O}_S(1)) \rangle_S$ is coprime to d, is dense in the space of all surfaces of degree d in $\mathbb{P}^3$. Indeed, for any X containing such a surface, the class α is algebraic on X.

Now this fact follows from the density criterion for the Noether–Lefschetz locus explained in [101, II, 5.3.4], and from the fact that rational classes $\nu \in H^2(S, \mathbb{Q})$, such that a multiple $b\nu$ is integral and satisfies $\langle b\nu, c_1(\mathcal{O}_S(1)) \rangle_S = a$ with a coprime to d, are dense in $H^2(S, \mathbb{Q})$.

6.2 RATIONALLY CONNECTED VARIETIES AND THE RATIONALITY PROBLEM

A long-standing problem in algebraic geometry is the characterization of *rational varieties*, namely those smooth projective X that are birationally equivalent to $\mathbb{P}^n$, $n = \dim X$.

Beautiful obstructions to rationality, very efficient in dimension 3, even for unirational varieties (for which there exists a surjective rational map $\phi : \mathbb{P}^n \dashrightarrow X$), have been demonstrated in the papers [4], [21], [57].

In higher dimensions, the criteria above, and especially those of [4] and [21] are less useful. In [93] and [105], we observed that if X is a smooth projective variety that is birational to $\mathbb{P}^n$, then the Hodge conjecture holds for integral Hodge classes on X of degree $(2n-2)$ and 4. This is optimal, because in other degrees $2i$ (different from 0, 2, and $2n$, where the groups are always 0), we can blow up a copy of Kollár's example embedded in some projective space to get rational varieties X with nonalgebraic integral Hodge classes of degree $2i$. In fact, we have the following lemma (see [105]).

LEMMA 6.3. *The groups*

$$Z^4(X) := \mathrm{Hdg}^4(X,\mathbb{Z})/\,\mathrm{Hdg}^4(X,\mathbb{Z})_{\mathrm{alg}},$$
$$Z^{2n-2}(X) := \mathrm{Hdg}^{2n-2}(X,\mathbb{Z})/\,\mathrm{Hdg}^{2n-2}(X,\mathbb{Z})_{\mathrm{alg}}, \tag{6.2}$$

where the lower index "alg" means that we consider the group of integral Hodge classes which are algebraic, are both birational invariants of the complex projective variety X of dimension n.

PROOF. This follows from the resolution of singularities and the invariance under blow-ups, which is a consequence of the computation of the cohomology and the Chow groups of a blown-up variety (see [68] or [101, I, 7.3.3; II, 9.3.3]). For the degree 4 case, the new degree 4 integral Hodge classes appearing under blow-up come from degree 2 (or degree 0) integral Hodge classes on the center of the blow-up. Hence they are algebraic by the Lefschetz theorem on $(1,1)$-classes. For the other case, the new degree $(2n-2)$ integral Hodge classes appearing under blow-up of a connected smooth subvariety are multiples of the class of a line in a fiber of the blowing-down map, hence they are also algebraic. □

Note the following two facts concerning rational Hodge classes of degree 4 and $2n-2$:

(1) The Hodge conjecture is true for rational Hodge classes of degree $2n-2$ on smooth projective varieties of dimension n (see Section 2.2.2).

(2) The Hodge conjecture is true for rational Hodge classes of degree 4 on smooth projective varieties that have their CH_0 group supported on a closed algebraic subset of dimension ≤ 3 (see Section 3.1.2).

Thus it seems natural to consider in these situations the problem for integral Hodge classes. The Kollár examples lead, by taking products with $\mathbb{P}^r$, to other examples of nonalgebraic integral Hodge classes of degree 4 or $2n-2$, showing that we have to restrict strongly the considered class of varieties.

Rationally connected varieties, for which there passes a rational curve through any two points, have been the subject of intensive work since the seminal paper by Kollár, Miyaoka, and Mori [65]. Still they remain very mysterious from several points of view. They are as close as possible to unirational varieties (no example is known to be not unirational), and this is a birationally and deformation invariant class. In view of the birational invariance explained above, it is thus natural to consider the problem of integral Hodge classes for them.

QUESTION 6.4. *Let X be a smooth rationally connected variety of dimension n. Is the Hodge conjecture true for integral Hodge classes of degree 4 or $2n-2$ on X?*

It is tempting to believe that the answer should be yes for degree $(2n-2)$ classes and we will give one theoretical argument in favor of this in Section 6.2.1. On the other hand we will show in Section 6.2.2, following [24], that the answer is negative in degree 4 for X of dimension ≥ 6.

6.2.1 The group $Z^{2n-2}(X)$

For $n = 3$, we will have the equality $4 = 2n - 2$, hence there is only one degree to consider. In [103], we solved Question 6.4 in dimension 3, proving more generally the following result.

THEOREM 6.5 (Voisin 2006). *Let X be a uniruled threefold, or a Calabi–Yau threefold, namely a threefold with trivial canonical bundle and vanishing irregularity. Then the Hodge conjecture is satisfied by integral Hodge classes of degree 4 on X, that is, integral Hodge classes of degree 4 are generated by classes of curves in X.*

SKETCH OF PROOF. The Hodge conjecture is satisfied by integral Hodge classes of degree 2. This is the Lefschetz theorem on $(1,1)$-classes. Let $\alpha \in H^4(X,\mathbb{Z})$ and let $j : \Sigma \hookrightarrow X$ be the inclusion of a smooth ample surface into X. The Lefschetz theorem on hyperplane restriction says that the Gysin map

$$j_* : H^2(\Sigma,\mathbb{Z}) \to H^4(X,\mathbb{Z})$$

is surjective. Assume for simplicity that $H^2(X,\mathcal{O}_X) = 0$ and that X is uniruled. Then there is no $H^{3,1}$-part in the Hodge decomposition of $H^4(X,\mathbb{C})$ and thus we want to show that any integral degree 4 cohomology class is algebraic, that is, is a combination with integral coefficients of classes of curves in X. The key point (in the case where X is uniruled with $H^2(X,\mathcal{O}_X) = 0$) is then the following.

PROPOSITION 6.6. *If Σ is chosen ample enough (that is, Σ belongs to the linear system associated to a sufficiently high power of an ample line bundle on X) and satisfies the condition that*

$$\Sigma^2\, c_1(K_X) < 0,$$

then $H^2(\Sigma,\mathbb{Z})$ is generated over $\mathbb{Z}$ by classes that become algebraic on Σ_t, where Σ_t is a small deformation of Σ in X.

Note that if X is uniruled, after performing a birational transformation of X, we can construct a smooth projective variety X' with an ample line bundle L such that $c_1(L)^2 \cdot c_1(K_{X'}) < 0$. Thus, up to replacing X by X', we may assume surfaces Σ as above exist.

If now $\alpha \in H^2(\Sigma,\mathbb{Z})$ becomes algebraic under a small deformation of Σ in X, this means that for a deformation $j_t : \Sigma_t \hookrightarrow X$, the class α_t deduced from α

by flat transport satisfies $\alpha_t = [Z_t] \in H^2(\Sigma_t, \mathbb{Z})$, for some divisor $Z_t \in \mathrm{CH}_1(\Sigma_t)$, and thus

$$j_*\alpha = j_{t*}\alpha_t = j_{t*}[Z_t] = [j_{t*}(Z_t)] \in H^4(X, \mathbb{Z}).$$

Thus it follows from the surjectivity of j_* and from Proposition 6.6 that $H^4(X, \mathbb{Z})$ is generated by classes of 1-cycles on X.

The proof of Proposition 6.6 uses the study of infinitesimal variations of Hodge structures. Using the Lefschetz theorem on $(1,1)$-classes, one has equivalently to prove that $H^2(\Sigma, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes that become of type $(1,1)$ under a small deformation of Σ in X. Thus, this is mainly a question of showing that the spaces

$$H^{1,1}(\Sigma)_{\mathbb{R}} := H^{1,1}(\Sigma) \cap H^2(\Sigma, \mathbb{R})$$

"move enough" with $\Sigma \subset X$ inside $H^2(\Sigma, \mathbb{R})$ so as to fill in an open subset $V = \cup_{t \in U} H^{1,1}(\Sigma_t) \cap H^2(\Sigma_t, \mathbb{R}) \subset H^2(\Sigma, \mathbb{R})$. Here U is a simply connected open set of the space of smooth deformations of Σ in X, and we freely use the canonical identification $H^2(\Sigma_t, \mathbb{R}) \cong H^2(\Sigma, \mathbb{R})$ given by parallel transport.

As this open subset V is a cone, it will then be clear that integral points in this cone will generate over $\mathbb{Z}$ the whole lattice $H^2(\Sigma, \mathbb{Z})$.

The study of the deformations of the subspace $H^{1,1}(\Sigma)_{\mathbb{R}} \subset H^2(\Sigma, \mathbb{R})$ is done using Griffiths machinery of infinitesimal variations of Hodge structures for hypersurfaces (see [48], [101, II, 6.2]). □

Remark 6.7. It would be tempting to weaken the assumptions in Theorem 6.5 and to ask whether a smooth projective threefold X with trivial CH_0 group (or CH_0 group supported on a surface) satisfies the condition $Z^4(X) = 0$. This is essentially disproved in [24] (except that we do not know that the example we have indeed satisfies the conclusion that $\mathrm{CH}_0(X)$ is trivial). In fact, following Kollár, we produce an example of a smooth projective threefold X satisfying $H^i(X, \mathcal{O}_X) = 0$, $i > 0$ and with nontrivial $Z^4(X)$. By the Bloch conjecture (Conjecture 3.21) (and Roitman's theorem [87]), this X should satisfy $\mathrm{CH}_0(X) = \mathbb{Z}$.

Remark 6.8. Another question is whether in dimension 3, the assumptions on X are optimal in Theorem 6.5. In [24], examples of threefolds X with Kodaira dimension 1 and $Z^4(X) \neq 0$ are exhibited. A remaining question would be whether in Theorem 6.5 the assumption that X has trivial canonical bundle could be replaced by the condition that X has Kodaira dimension 0. In other words, is it true that a smooth projective threefold X with $\kappa(X) = 0$ has $Z^4(X) = 0$?

In dimension ≥ 4, Question 6.4 has been studied in [53]. We prove the following result.

Theorem 6.9 (Höring and Voisin 2010). *Let X be a Fano fourfold or a Fano fivefold of index 2. Then X satisfies $Z^{2n-2}(X) = 0$.*

For the proof, we first extend the Calabi–Yau case in Theorem 6.5 allowing X to have isolated canonical singularities. We then prove that a Fano fourfold contains an anticanonical divisor with isolated canonical singularities. The case of a Fano fivefold of index 2 works similarly: if $-K_X = 2H$, where $H \in \operatorname{Pic} X$, is an ample line bundle, we show that there is a complete intersection of two members of $|H|$ which is a threefold with isolated canonical singularities, and of course with trivial canonical bundle.

To conclude this section, let us prove the following result.

THEOREM 6.10 (See Voisin 2013 [112]). *The group $Z^{2n-2}(X)$ is locally deformation invariant for rationally connected n-folds. In particular its order is a deformation invariant.*

Let us first explain the meaning of the statement. Consider a smooth projective morphism $\pi : \mathcal{X} \to B$ between connected quasi-projective complex varieties, with n-dimensional fibers. Recall from [65] that if one fiber $X_b := \pi^{-1}(b)$ is rationally connected, so is every fiber, so that in particular, $H^{2n-2}(\mathcal{X}_b, \mathbb{Z}) = \operatorname{Hdg}^{2n-2}(\mathcal{X}_b, \mathbb{Z})$ for any $b \in B$. Let us endow everything with the usual topology. Then the sheaf $R^{2n-2}\pi_*\mathbb{Z}$ is locally constant on B. On any Euclidean open set $U \subset B$ where it is trivial, the group $Z^{2n-2}(X_b)$, $b \in U$ is the finite quotient of the *constant* group $H^{2n-2}(X_b, \mathbb{Z})$ by its subgroup $H^{2n-2}(X_b, \mathbb{Z})_{\mathrm{alg}}$. To say that $Z^{2n-2}(X_b)$ is locally constant means that on open sets U as above, the subgroup $H^{2n-2}(X_b, \mathbb{Z})_{\mathrm{alg}}$ of the constant group $H^{2n-2}(X_b, \mathbb{Z})$ does not depend on b.

PROOF OF THEOREM 6.10. We first observe that, due to the fact that relative Hilbert schemes parametrizing curves in the fibers of B are a countable union of varieties which are projective over B, given a simply connected open set $U \subset B$ (in the classical topology of B) and a class $\alpha \in \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ such that α_t is algebraic for $t \in V$, where V is a smaller nonempty open set $V \subset U$, then α_t is algebraic for any $t \in U$.

To prove the deformation invariance, we only need to use the above observation to prove that for U as above, $t \in U \subset B$, and for a curve $C \subset X_t$ with cohomology class $[C] \in H^{2n-2}(X_t, \mathbb{Z})$. Then the class $[C]_s$ is algebraic for s in U.

By the results of [65], there are rational curves $R_i \subset X_t$ with ample normal bundle which meet C transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number D of such curves with generically chosen tangent directions at the attachment points. We then know by [65] that the curve $C' = C \cup_{i \leq D} R_i$ is smoothable and that the result is a smooth unobstructed curve $C'' \subset X_t$, that is, $H^1(C'', N_{C'/X_t}) = 0$. This curve C'' then deforms with X_t (see [62], [64, II.1]) in the sense that the morphism from the deformation of the pair (C'', X_t) to B is smooth. So for $s \in U$, there is a curve $C''_s \subset X_s$ which is a deformation of $C'' \subset X_t$. The class

$[C''_s] = [C'']_s$ is thus algebraic on X_s. On the other hand, we have

$$[C''] = [C'] = [C] + \sum_i [R_i].$$

As the R_i's are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_i]_s$ also are algebraic on X_s. Thus $[C]_s = [C'']_s - \sum_i [R_i]_s$ is algebraic on X_s. The lemma, hence also the theorem, is proved. □

The same proof applies to prove invariance of the group $Z^{2n-2}(X)$ for X a rationally connected variety defined over a number field K, under specialization (in an adequate sense) to a point p of $\operatorname{Spec} \mathcal{O}_K$ such that $\mathcal{X}_p$ is smooth, where $\mathcal{X}$ is a projective model of X over $\operatorname{Spec} \mathcal{O}_K$. We refer to [112] for this result and the following Theorem 6.11 (which strongly suggests, in fact, the vanishing of $Z^{2n-2}(X)$ for X a rationally connected variety over $\mathbb{C}$).

THEOREM 6.11. *Assume the Tate conjecture is true for divisor classes on varieties over a finite field (see [69]). Then the group $Z^{2n-2}(X)$ vanishes for any rationally connected n-fold over $\mathbb{C}$.*

6.2.2 The group $Z^4(X)$ and unramified cohomology

Let X be a smooth projective complex variety. We will denote by X_{cl} the set $X(\mathbb{C})$ endowed with its classical (or Euclidean) topology, and by X_{Zar} the set $X(\mathbb{C})$ endowed with its Zariski topology.

Let

$$\pi : X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$$

be the identity of $X(\mathbb{C})$. This is a continuous map since a Zariski open set is open in the classical topology, and Bloch–Ogus theory [14] is the study of the Leray spectral sequence associated to this map and any constant sheaf with stalk A on X_{cl}. In applications, the abelian group A will be one of the groups $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Q}/\mathbb{Z}$.

We are thus led to introduce, on X_{Zar}, the sheaves

$$\mathcal{H}^i(A) := R^i\pi_*A.$$

By definition, $\mathcal{H}^i(A)$ is thus the sheaf associated to the presheaf $U \mapsto H^i_B(U, A)$ on X_{Zar}.

The Leray spectral sequence for π and A has terms

$$E_2^{p,q} = H^p(X_{\mathrm{Zar}}, \mathcal{H}^q(A)).$$

Unramified cohomology of X with value in A is defined by the formula (see [23])

$$H^i_{\mathrm{nr}}(X, A) = H^0(X_{\mathrm{Zar}}, \mathcal{H}^i_X(A)).$$

The main result of the paper by Bloch and Ogus [14] is the following Gersten–Quillen resolution for the sheaves $\mathcal{H}^i_X(A)$. For any closed subvariety $D \subset X$, let $i_D : D \to X$ be the inclusion map and $H^i(\mathbb{C}(D), A)$ the constant sheaf on D with stalk

$$\lim_{\substack{\to \\ U \subset D \\ \text{nonempty Zariski open}}} H^i(U(\mathbb{C}), A)$$

at any point of D. When $D' \subset D$ has codimension 1, there is a map induced by the topological residue (on the normalization of D) (see [101, II, 6.1.1]):

$$\mathrm{Res}_{D,D'} : H^i(\mathbb{C}(D), A) \to H^{i-1}(\mathbb{C}(D'), A).$$

For $r \geq 0$, let $X^{(r)}$ be the set of irreducible closed algebraic subsets of codimension r in X.

Theorem 6.12 (Bloch and Ogus 1974, Theorem 4.2). *For any A, and any integer $i \geq 1$, there is an exact sequence of sheaves on X_{Zar}*

$$0 \to \mathcal{H}^i_X(A) \to i_{X*}H^i(\mathbb{C}(X), A) \xrightarrow{\partial} \bigoplus_{D \in X^{(1)}} i_{D*}H^{i-1}(\mathbb{C}(D), A) \xrightarrow{\partial}$$
$$\cdots \xrightarrow{\partial} \bigoplus_{D \in X^{(i)}} i_{D*}A_D \to 0.$$

Here the components of the maps ∂ are induced by the maps $\mathrm{Res}_{D,D'}$ when $D' \subset D$ (and are 0 otherwise). The sheaf A_D on D_{Zar} identifies, of course, to the constant sheaf with stalk $H^0(\mathbb{C}(D), A)$.

Let us state a few consequences proved in [14]. First of all, denoting by $\mathrm{CH}^k(X)/\mathrm{alg}$ the group of codimension k cycles of X modulo algebraic equivalence, we get the Bloch–Ogus formula.

Corollary 6.13 (Bloch and Ogus 1974, Corollary 7.4). *If X is a smooth complex projective variety, there is a canonical isomorphism*

$$\mathrm{CH}^k(X)/\mathrm{alg} = H^k(X_{\mathrm{Zar}}, \mathcal{H}^k(\mathbb{Z})). \tag{6.3}$$

Proof. Indeed, the Bloch–Ogus resolution is acyclic, because a constant sheaf on an irreducible variety (equipped with the Zariski topology) is flasque hence acyclic (see [52, II, Exercise 1.16]), and this applies to the constant sheaves $H^i(\mathbb{C}(D), A)$. It is thus possible to compute $H^k(X_{\mathrm{Zar}}, \mathcal{H}^k(\mathbb{Z}))$ by taking global sections in the above resolution, which gives

$$H^k(X_{\mathrm{Zar}}, \mathcal{H}^k(\mathbb{Z})) = \mathrm{Coker}\left(\partial : \oplus_{D \in X^{(k-1)}} H^1(\mathbb{C}(W), \mathbb{Z}) \longrightarrow \oplus_{D \in X^{(k)}} \mathbb{Z}\right).$$

The group $\oplus_{D \in X^{(k)}} \mathbb{Z}$ is the group of codimension k cycles on X, and to conclude, one has to check that the image of the map ∂ above is the group of cycles algebraically equivalent to 0. This follows from the fact that on a smooth projective variety $\widetilde{W}$, a divisor D is cohomologous to 0 (hence algebraically equivalent to 0 by Example 3.7) if and only if there exists a degree 1 cohomology class $\alpha \in H^1(\widetilde{W} \setminus \mathrm{Supp}\, D, \mathbb{Z})$ such that $\mathrm{Res}\, \alpha = D$. ∎

By Theorem 6.12, the sheaf $\mathcal{H}^i(A)$ has an acyclic resolution of length $\leq i$. We thus get the following vanishing result.

Corollary 6.14. *For X smooth, A an abelian group, and $r > i$, one has*

$$H^r(X_{\mathrm{Zar}}, \mathcal{H}^i_X(A)) = 0. \tag{6.4}$$

Concerning the structure of the sheaves $\mathcal{H}^i(\mathbb{Z})$, we have the following result, which is a consequence of the Bloch–Kato conjecture recently proved by Rost and Voevodsky (we refer to [6], [15], [24] for more explanations concerning the way the very important result below is deduced from the Bloch–Kato conjecture).

Theorem 6.15. *The sheaves $\mathcal{H}^i(\mathbb{Z})$ of $\mathbb{Z}$-modules on X_{Zar} have no torsion.*

The following corollary gives an equivalent formulation of this theorem, by considering the long exact sequence associated to the short exact sequence of sheaves on X_{cl},

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

on X_{cl}, and the associated long exact sequence of sheaves on X_{Zar},

$$\cdots \to \mathcal{H}^i(\mathbb{Q}) \to \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}) \to \mathcal{H}^{i+1}(\mathbb{Z}) \to \mathcal{H}^{i+1}(\mathbb{Q}) \to \cdots.$$

Corollary 6.16. *For any integer i, there is a short exact sequence of sheaves on Z_{Zar},*

$$0 \to \mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Q}) \to \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}) \to 0.$$

Unramified cohomology with torsion coefficients (that is, $A = \mathbb{Z}/n\mathbb{Z}$ or $A = \mathbb{Q}/\mathbb{Z}$) plays an important role in the study of the Lüroth problem, that is, the study of unirational varieties that are not rational (see, for example, the papers [4], [23], and [84]). To start with, we have the following result concerning the invariant used by Artin and Mumford, which is the torsion in the group $H^3_B(X, \mathbb{Z})$.

Proposition 6.17. *Let X be a rationally connected variety. Then the torsion in the group $H^3_B(X, \mathbb{Z})$ is naturally isomorphic to the unramified cohomology group $H^2_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z})$.*

Proof. We consider the Bloch–Ogus spectral sequence for X and for integral coefficients, and compute its terms in degree 3. By Corollary 6.14, there are only two possibly nonzero E_2-terms, namely $H^1(X_{\mathrm{Zar}}, \mathcal{H}^2(\mathbb{Z}))$ and $H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z}))$. Because X is rationally connected, it has trivial CH_0 group and it follows from the Bloch–Srinivas decomposition of the diagonal (Theorem 3.10) and from functoriality of unramified cohomology under correspondences that $H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z}))$ is of torsion. By Theorem 6.15, the unramified cohomology group $H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z}))$ vanishes identically, and we conclude that the only nonzero E_2-term is $H^1(X_{\mathrm{Zar}}, \mathcal{H}^2(\mathbb{Z}))$. Because no nonzero differential d_r

can start from $H^1(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Z}))$ by Corollary 6.14, or arrives in $H^1(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Z}))$ for degree reasons, we conclude that

$$H^3(X, \mathbb{Z}) \cong H^1(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Z})).$$

It thus follows that $\mathrm{Tors}(H^3(X, \mathbb{Z})) \cong \mathrm{Tors}(H^1(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Z})))$, and the second term is computed writing the exact sequence

$$0 \to \mathcal{H}^2(\mathbb{Z}) \to \mathcal{H}^2(\mathbb{Q}) \to \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}) \to 0$$

of Corollary 6.16. Taking the associated long exact sequence, we get

$$\mathrm{Tors}(H^1(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Z}))) \cong H^0(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}))/\,\mathrm{Im}(H^0(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Q}))).$$

On the other hand, we have $H^0(X_{\rm Zar}, \mathcal{H}^2(\mathbb{Q})) = 0$ by the same argument as before, involving Bloch–Srinivas decomposition of the diagonal. □

Going further, in the paper [23], the authors exhibit unirational sixfolds with vanishing group $H^2_{\rm nr}(X, \mathbb{Q}/\mathbb{Z})$ but nonvanishing group $H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z})$. Their example is reinterpreted in the recent paper [24] using the groups $Z^4(X)$ introduced in (6.2). More precisely, in the paper [24] we give the following comparison result between $Z^4(X)$ and $H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z})$ (a similar result was in fact also established in [6]).

Theorem 6.18 (Colliot-Thélène and Voisin 2010).

(1) For any smooth projective X, there is an exact sequence

$$0 \to H^3_{\rm nr}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(Z^4(X)) \to 0.$$

(2) If $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset of dimension ≤ 3, then $\mathrm{Tors}(Z^4(X)) = Z^4(X)$. If $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset of dimension ≤ 2, then $H^3_{\rm nr}(X, \mathbb{Z}) = 0$.

(3) In particular, if X is rationally connected (so that $\mathrm{CH}_0(X) = \mathbb{Z}$), we have

$$H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z}) \cong Z^4(X).$$

As a consequence, we get a negative answer to Question 6.4 for degree 4 integral Hodge classes on certain rationally connected varieties (and even unirational varieties) of dimension at least 6.

Theorem 6.19. *The Colliot-Thélène–Ojanguren varieties X constructed in [23], which are unirational sixfolds, hence rationally connected, have $Z^4(X) \neq 0$.*

Indeed, as proved in [23], these varieties have a nontrivial unramified cohomology group $H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z})$.

weyllecturesformat September 3, 2013 6x9

Sketch of proof of Theorem 6.18. Assuming that the Hodge conjecture is satisfied for degree $2i$ rational Hodge classes on X, the group $Z^{2i}(X)$ is of torsion. The fact that the Hodge conjecture is satisfied for degree 4 rational Hodge classes on X if X has the property that $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset of dimension ≤ 3, is from Theorem 3.15 (see [15]). This proves the first part of statement (2). The second part of statement (2) is proved by applying the Bloch–Srinivas decomposition of the diagonal (Theorem 3.10) and letting both sides act on unramified cohomology (see [24, Appendix]). It thus follows that $H^3_{\mathrm{nr}}(X,\mathbb{Z})$ is of torsion. As it has no torsion by Theorem 6.15, it is in fact 0.

We now prove statement (1). We observe first of all that the torsion of the group $Z^{2i}(X) = \mathrm{Hdg}^{2i}(X,\mathbb{Z})/H^{2i}(X,\mathbb{Z})_{\mathrm{alg}}$ is always isomorphic to the torsion of the group $H^{2i}(X,\mathbb{Z})/H^{2i}(X,\mathbb{Z})_{\mathrm{alg}}$. We examine now the Bloch–Ogus spectral sequence of X with coefficients in $\mathbb{Z}$. The E_2-terms in degree 4 are

$$H^2(X_{\mathrm{Zar}},\mathcal{H}^2(\mathbb{Z})) = E_2^{2,2},\ \ H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z})) = E_2^{1,3},\ \ H^0(X_{\mathrm{Zar}},\,\mathcal{H}^4(\mathbb{Z})) = E_2^{0,4}.$$

No nonzero differential d_r, $r \geq 2$ starts from one of the groups $H^2(X_{\mathrm{Zar}},\mathcal{H}^2(\mathbb{Z}))$ or $H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z}))$ due to Corollary 6.14. No nonzero differential d_r, $r \geq 2$ arrives at $H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z}))$ or $H^0(X_{\mathrm{Zar}},\,\mathcal{H}^4(\mathbb{Z}))$ for degree reasons. Thus we have

$$E_2^{2,2} \twoheadrightarrow E_\infty^{2,2}, \qquad E_2^{1,3} = E_\infty^{1,3}, \qquad E_2^{0,4} \subset E_\infty^{0,4}.$$

It follows that there is a natural composed map

$$H^2(X_{\mathrm{Zar}},\mathcal{H}^2(\mathbb{Z})) \to E_\infty^{2,2} \subset H^4(X,\mathbb{Z}),$$

whose image is the deepest level $N^2H^4(X,\mathbb{Z})$ of the Leray–Bloch–Ogus filtration (which is in fact the coniveau filtration). This map identifies to the cycle class map (see [14]), so that its cokernel $H^4(X,\mathbb{Z})/N^2H^4(X,\mathbb{Z})$ is the group $H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{\mathrm{alg}}$. This group has a filtration induced by the Leray spectral sequence, and the graded pieces are

$$E_\infty^{1,3} = H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z})), \qquad E_\infty^{0,4} \subset H^0(X_{\mathrm{Zar}},\,\mathcal{H}^4(\mathbb{Z})).$$

By Theorem 6.15, the group $H^0(X_{\mathrm{Zar}},\,\mathcal{H}^4(\mathbb{Z}))$ has no torsion, hence it follows finally that we have an isomorphism,

$$\mathrm{Tors}(H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{\mathrm{alg}}) = \mathrm{Tors}(H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z}))).$$

The group on the left identifies to the group $\mathrm{Tors}(Z^4(X))$ by the observation above, while the group on the right is analyzed by the exact sequence of sheaves on X_{Zar} given in Corollary 6.16:

$$0 \to \mathcal{H}^3(\mathbb{Z}) \to \mathcal{H}^3(\mathbb{Q}) \to \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}) \to 0.$$

The induced long exact sequence immediately gives

$$\mathrm{Tors}(H^1(X_{\mathrm{Zar}},\mathcal{H}^3(\mathbb{Z}))) \cong H^0(X,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}))/\,\mathrm{Im}(H^0(X,\mathcal{H}^3(\mathbb{Q}))),$$

which finishes the proof. □

In the paper [111], we give a similar cycle-theoretic interpretation of the group $H^4_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$ (Theorem 6.22 below). For this we need to introduce some notation.

Let X be a smooth complex projective variety and k, l be integers. We have the subgroup $N^lH^k_B(X,\mathbb{Z}) \subset H^k_B(X,\mathbb{Z})$ of "coniveau l cohomology," defined as

$$N^lH^k_B(X,\mathbb{Z}) = \mathrm{Ker}\left(H^k_B(X,\mathbb{Z}) \to \varinjlim_{\mathrm{codim}\,W=l} H^k_B(X\setminus W,\mathbb{Z})\right),$$

where the $W \subset X$ considered here are the closed algebraic subsets of X of codimension l. Introducing a resolution of singularities $\widetilde{W}$ of W and its natural morphism $\tau_W : \widetilde{W} \to X$ to X, we have

$$N^lH^k_B(X,\mathbb{Q}) = \sum_{\mathrm{codim}\,W=l} \mathrm{Im}\left(\tau_{W*} : H^{k-2l}_B(\widetilde{W},\mathbb{Q}) \to H^k(X,\mathbb{Q})\right),$$

as explained in the proof of Theorem 2.39.

From now on, we restrict to the case $k = 2l+1$. For any $W \subset X$, $\tau_W : \widetilde{W} \to X$ as above, the Gysin morphism $\tau_{W*} : H^1_B(\widetilde{W},\mathbb{Z}) \to H^{2l+1}_B(X,\mathbb{Z})$ is a morphism of Hodge structures (of bidegree (l,l)), which induces a morphism between the intermediate Jacobians (see [101, I, 12.2]),

$$\begin{aligned}\tau_{W*} : \mathrm{Pic}^0(\widetilde{W}) = J^1(\widetilde{W}) &\to J^{2l+1}(X)\\ &:= \frac{H^{2l+1}_B(X,\mathbb{C})}{F^lH^{2l+1}_B(X,\mathbb{C}) \oplus H^{2l+1}_B(X,\mathbb{Z})/\mathrm{torsion}}.\end{aligned}$$

This map τ_{W*} is compatible in an obvious way with the Abel–Jacobi maps ϕ_W and ϕ_X, defined, respectively, on codimension 1 and codimension $(l+1)$ cycles of $\widetilde{W}$ and X which are homologous to 0.

The Deligne cycle class map

$$\mathrm{cl}^{l+1}_{\mathcal{D}} : \mathrm{CH}^{l+1}(X) \to H^{2l+2}_{\mathcal{D}}(X,\mathbb{Z}(l+1))$$

restricts to the Abel–Jacobi map ϕ^l_X on the subgroup of cycles homologous to 0 (see [101, I, 12.3.3]), and in particular on the subgroup of cycles algebraically equivalent to 0.

If $Z \in \mathrm{CH}^l(X)$ is algebraically equivalent to 0, there exist subvarieties $W_i \subset X$ of codimension l and cycles $Z_i \subset \widetilde{W}_i$ homologous to 0 such that $Z = \sum_i \tau_{W_i*}Z_i$ in $\mathrm{CH}^{l+1}(X)$. It follows from the previous considerations that $\mathrm{cl}^{l+1}_{\mathcal{D}}$ induces a morphism

$$\mathrm{cl}^{l+1}_{\mathcal{D},\mathrm{tr}} : \mathrm{CH}^l(X)/\mathrm{alg} \to H^{2l+2}_{\mathcal{D}}(X,\mathbb{Z}(l+1))_{\mathrm{tr}} := \frac{H^{2l+2}_{\mathcal{D}}(X,\mathbb{Z}(l+1))}{\left\langle \tau_{W*}J^1(\widetilde{W}),\ \mathrm{codim}\,W = l\right\rangle}.$$

Let $\mathcal{T}^{l+1}(X) := \mathrm{Tors}(\mathrm{Ker}\,\mathrm{cl}^{l+1}_{\mathcal{D},\mathrm{tr}})$.

Lemma 6.20. *This group identifies to the image of the subgroup* $\mathrm{Tors}(\mathrm{Ker}\,\mathrm{cl}_{\mathcal{D}}^{l+1})$ *in* $\mathrm{CH}^{l+1}(X)/\mathrm{alg}$.

Proof. This follows from the fact that the groups of cycles algebraically equivalent to 0 modulo rational equivalence are divisible. This implies that the natural map $\mathrm{Tors}(\mathrm{CH}^i(X)) \to \mathrm{Tors}(\mathrm{CH}^i(X)/\mathrm{alg})$ is surjective for any i. We then use the fact that for the $\widetilde{W}$'s introduced above, the map

$$\mathrm{CH}^1(\widetilde{W})_{\mathrm{alg}} \cong \mathrm{Pic}^0(\widetilde{W}) \to J^1(\widetilde{W})$$

is an isomorphism, where $\mathrm{CH}^1(\widetilde{W})_{\mathrm{alg}} \subset \mathrm{CH}^1(\widetilde{W})$ is the subgroup of cycles algebraically equivalent to 0. □

We have the following lemma.

Lemma 6.21. *The group* $\mathcal{T}^3(X)$ *is a birational invariant of* X.

Proof. It suffices to check invariance under blow-up. The Manin formulas (see [68], [101, II, 9.3.3]) for groups of cycles modulo rational or algebraic equivalence and for Deligne cohomology of a blow-up imply that it suffices to prove that the groups $\mathcal{T}^i(Y)$ are trivial for $i \leq 2$ and Y smooth projective. However this is an immediate consequence of the definition, of Lemma 6.20, and of the fact that the Deligne cycle class map $\mathrm{cl}_{\mathcal{D}} : \mathrm{CH}^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is injective on torsion cycles of codimension $i \leq 2$ (see [73]). □

The following interpretation of degree 4 unramified cohomology with finite coefficients is proved in [111].

Theorem 6.22 (Voisin 2011). *Assume that the group* $H^5_B(X,\mathbb{Z})/N^2H^5_B(X,\mathbb{Z})$ *has no torsion. Then the quotient of* $H^4_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$ *by* $H^4_{\mathrm{nr}}(X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z}$ *identifies to the group* $\mathcal{T}^3(X)$.

Equivalently, there is an exact sequence

$$\begin{aligned} 0 \to H^4_{\mathrm{nr}}(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} &\to H^4_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \\ &\to \mathrm{Tors}(\mathrm{CH}^3(X)/\mathrm{alg}) \stackrel{\mathrm{cl}^3_{\mathcal{D},\mathrm{tr}}}{\to} H^6_{\mathcal{D}}(X,\mathbb{Z}(3))_{\mathrm{tr}}. \end{aligned}$$

One can of course add that for a variety X with $\mathrm{CH}_0(X)$ supported on a three-dimensional closed algebraic subset $X' \subset X$ (for example, a rationally connected variety), the group $H^4_{\mathrm{nr}}(X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z}$ on the left is 0, so that the exact sequence above then gives rise to an isomorphism

$$H^4_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ker}\left(\mathrm{Tors}(\mathrm{CH}^3(X)/\mathrm{alg}) \stackrel{\mathrm{cl}^3_{\mathcal{D},\mathrm{tr}}}{\to} H^6_{\mathcal{D}}(X,\mathbb{Z}(3))_{\mathrm{tr}}\right).$$

Indeed, the vanishing of $H^4_{\mathrm{nr}}(X,\mathbb{Z})$ follows as before from the fact that this group has no torsion by Theorem 6.15, while under the assumptions made on X, and by the Bloch–Srinivas decomposition, it is annihilated by an integer N.

6.3 INTEGRAL DECOMPOSITION OF THE DIAGONAL AND THE STRUCTURE OF THE ABEL–JACOBI MAP

6.3.1 Structure of the Abel–Jacobi map and birational invariants

Recall the diagonal decomposition principle (Theorem 3.10) which says that if Y is a smooth variety such that $\mathrm{CH}_0(Y) = \mathbb{Z}$, there is an equality in $\mathrm{CH}^d(Y \times Y)$, $d = \dim Y$:

$$N\Delta_Y = Z_1 + Z_2, \tag{6.5}$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$\operatorname{Supp} Z_1 \subset D \times Y, \quad D \subsetneq Y, \qquad Z_2 = N(Y \times \mathrm{pt}).$$

Note that the integer N appearing above cannot in general be set equal to 1, and one purpose of this concluding section is to investigate the significance of this invariant, at least if we work on the level of cycles modulo homological equivalence. We will say that Y admits a *cohomological decomposition of the diagonal* as in (6.5) if there is an equality of cycle classes,

$$N[\Delta_Y] = [Z_1] + [Z_2], \tag{6.6}$$

in $H^{2d}(Y \times Y, \mathbb{Z})$, with $\operatorname{Supp} Z_1 \subset D \times Y$ and $Z_2 = N(Y \times \mathrm{pt})$. We will say that Y admits an *integral cohomological decomposition of the diagonal* if there is such a decomposition with $N = 1$.

Remark 6.23. The minimal positive integers N such that a decomposition as above exists in either the Chow-theoretic or the cohomological setting are birational invariants of Y. Indeed, if we have a birational morphism

$$\phi : X \to Y,$$

which is an isomorphism onto its image away from a divisor $E \subset X$, then we have in $\mathrm{CH}^d(X \times X)$,

$$\Delta_X = \phi^*(\Delta_Y) + Z,$$

where Z is a cycle supported on $E \times E$. As the term Z is absorbed in the term Z_1 of a diagonal decomposition for X, the same integers N work for X and Y.

Recall that the existence of a cohomological decomposition of the diagonal in the form (6.6) has strong consequences (see [15] and Sections 3.1.1 and 3.1.2):

(1) This implies the generalized Mumford theorem which says in this case that $H^i(Y, \mathcal{O}_Y) = 0$ for $i \geq 1$. In particular the Hodge structures on $H^2(Y, \mathbb{Q})$, hence on its Poincaré dual $H^{2d-2}(Y, \mathbb{Q})$, are trivial.

(2) The group $Z^4(Y) := \mathrm{Hdg}^4(Y)/\mathrm{Hdg}^4(Y)_{\mathrm{alg}}$ is of torsion (annihilated by the integer N above).

(3) The intermediate Jacobian $J^3(Y)$ defined by

$$J(Y) := H^3(Y,\mathbb{C})/(F^2H^3(Y) \oplus H^3(Y,\mathbb{Z})) \tag{6.7}$$

is an abelian variety because $H^3(Y,\mathcal{O}_Y) = 0$ (see [48], [101, I, 12.2.2]) and the Abel–Jacobi map $\mathrm{CH}^2(Y)_{\mathrm{hom}} \to J^3(Y)$ is surjective with finite kernel (annihilated by the integer N above).

Regarding the last point, much more is true assuming the existence of a Chow-theoretic decomposition of the diagonal; see [15].

Theorem 6.24 (Bloch and Srinivas 1983; see also Murre 1983/1985). *If Y is a smooth complex projective variety such that $\mathrm{CH}_0(Y) = \mathbb{Z}$, then the Abel–Jacobi map induces an isomorphism,*

$$\mathrm{AJ}_Y : \mathrm{CH}^2(Y)_{\mathrm{hom}} = \mathrm{CH}^2(Y)_{\mathrm{alg}} \cong J(Y). \tag{6.8}$$

Remark 6.25. In fact, the conclusion holds if we only assume that $\mathrm{CH}_0(Y)$ is supported on a curve.

This theorem is proved by delicate arguments from algebraic K-theory involving the Merkurjev–Suslin theorem (the degree 2 case of the Bloch–Kato conjecture), Gersten–Quillen resolution in K-theory, and Bloch–Ogus theory [14]. The fact that under the above assumptions $\mathrm{CH}^2(Y)_{\mathrm{hom}} = \mathrm{CH}^2(Y)_{\mathrm{alg}}$ is Theorem 3.14, and the fact that the Abel–Jacobi map is surjective with a torsion kernel is an immediate application of the Bloch–Srinivas decomposition of the diagonal. The fact that the Abel–Jacobi map is injective on torsion codimension 2 cycles in $\mathrm{CH}^2(Y)$ is true without any assumptions on Y and this is the hardest part, for which we refer to [73].

The group on the left in (6.8) a priori does not have the structure of an algebraic variety, unlike the group on the right which is an abelian variety. However it makes sense to say that AJ_Y is algebraic, with the meaning that for any smooth algebraic variety B, and any codimension 2 cycle $Z \in \mathrm{CH}^2(B \times Y)$, with $Z_b \in \mathrm{CH}^2(Y)_{\mathrm{hom}}$ for any $b \in B$, the induced map

$$\phi_Z : B \to J(Y), \qquad b \mapsto \mathrm{AJ}_Y(Z_b)$$

is a morphism of algebraic varieties. In fact, it is possible to construct abstractly the abelian variety $J(Y)$ (when the Abel–Jacobi map is surjective) as the universal target of morphisms $\phi_Z : B \to A$ with value in an abelian variety, induced by an algebraic cycle $Z \in \mathrm{CH}^2(B \times Y)$, and factoring set-theoretically via the induced morphism $B \to \mathrm{CH}^2(Y)$. We refer to [73] for this construction which generalizes Serre's construction of the Albanese variety of a smooth projective variety X as the target of the universal morphism to an abelian variety (see [91]).

Consider the case of a uniruled threefold Y with $\mathrm{CH}_0(Y) = \mathbb{Z}$. Then we also have Theorem 6.5 saying that the integral degree 4 cohomology $H^4(Y,\mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves, and thus the birational invariant

$$Z^4(Y) := \frac{\mathrm{Hdg}^4(Y,\mathbb{Z})}{\langle [Z],\, Z \subset Y,\, \mathrm{codim}\, Z = 2\rangle}$$

introduced in (6.2) is trivial in this case.

The conclusion of the above-mentioned Theorems 6.5 and 6.24 is that for a uniruled threefold with $\mathrm{CH}_0 = \mathbb{Z}$, all the interesting (and birationally invariant) phenomena concerning codimension 2 cycles, namely the kernel of the Abel–Jacobi map (Mumford [71]), the Griffiths group (Griffiths [48]) and the group $Z^4(X)$ versus degree 3 unramified cohomology with torsion coefficients (Soulé and Voisin [93], Colliot-Thélène and Voisin [24]) are trivial. In the rationally connected case, the only interesting cohomological or Chow-theoretic invariant could be the Artin–Mumford invariant (or degree 2 unramified cohomology with torsion coefficients; see [23]), which is also equal to the Brauer group since $H^2(Y,\mathcal{O}_Y) = 0$.

Still the geometric structure of the Abel–Jacobi map on families of 1-cycles on such threefolds is mysterious, in contrast to what happens in the curve case, where Abel's theorem shows that the Abel–Jacobi map on the family of effective 0-cycles of large degree has fibers isomorphic to projective spaces.

There are for example two natural questions (Questions 6.26 and 6.29) left open by Theorem 6.24.

QUESTION 6.26. *Let Y be a smooth projective threefold, such that $\mathrm{AJ}_Y : \mathrm{CH}_1(Y)_{\mathrm{alg}} \to J(Y)$ is surjective. Is there a codimension 2 cycle $Z \in \mathrm{CH}^2(J(Y) \times Y)$ with $Z_b \in \mathrm{CH}^2(Y)_{\mathrm{hom}}$ for $b \in J(Y)$, such that the induced morphism*

$$\phi_Z : J(Y) \to J(Y), \qquad \phi_Z(b) := \mathrm{AJ}_Y(Z_b)$$

is the identity?

Note that the surjectivity assumption is conjecturally implied by the vanishing $H^3(Y,\mathcal{O}_Y) = 0$, via the generalized Hodge conjecture (Conjecture 2.40) or by the Hodge conjecture for degree 4 rational Hodge classes on $Y \times J(Y)$ (see the proof of Theorem 2.42).

Stated in words, this question asks for the existence of a universal codimension 2 cycle on $J(Y) \times Y$. The analogous question for codimension 1 cycles is well known to have an affirmative answer; this is the universal divisor $\mathcal{L} \in \mathrm{Pic}(X) \times \mathrm{Pic}^0(X)$, which is itself induced by pull-back via the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ of the Poincaré divisor $\mathcal{P} \in \mathrm{Pic}(\mathrm{Alb}(X) \times \mathrm{Pic}^0(X))$ (see [72, pp. 74–82]). (Here we use the fact that the abelian varieties $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$ are dual abelian varieties.)

REMARK 6.27. One can more precisely introduce a birational invariant of Y defined as the gcd of the nonzero integers N for which there exist a variety B

and a cycle $Z \in \mathrm{CH}^2(B \times Y)$ as above, with $\deg \phi_Z = N$. Question 6.26 can then be reformulated by asking whether this invariant is equal to 1.

REMARK 6.28. Question 6.26 has a positive answer if the Hodge conjecture is satisfied by degree 4 *integral* Hodge classes on $Y \times J(Y)$. Indeed, the isomorphism $H_1(J(C), \mathbb{Z}) \cong H^3(Y, \mathbb{Z})$ is an isomorphism of Hodge structures which provides a degree 4 integral Hodge class α on $J(Y) \times Y$ (see the proof of Theorem 2.42 or [101, I, Lemma 11.41]). A codimension 2 algebraic cycle Z on $J(Y) \times Y$ with $[Z] = \alpha$ would provide a solution to Question 6.26.

The following question is an important variant of the previous one, which appears to be much more natural from a geometric point of view.

QUESTION 6.29. *Is the following property* $(*)$ *satisfied by* Y*?*

$(*)$ *There exist a smooth projective variety* B *and a codimension* 2 *cycle* $Z \in \mathrm{CH}^2(B \times Y)$*, with* $Z_b \in \mathrm{CH}^2(Y)_{\mathrm{hom}}$ *for any* $b \in B$*, such that the induced morphism* $\phi_Z : B \to J(Y)$ *is surjective with rationally connected general fiber.*

This question has been solved by Iliev and Markushevich and by Markushevich and Tikhomirov ([54], [70]; see also [51] for similar results obtained independently) in the case where Y is a smooth cubic threefold in $\mathbb{P}^4$. Their work solves Question 6.29 since they construct a family $M_{1,5}$ of curves in Y (a completion of the family of elliptic curves of degree 5), which they show to be zero sets of sections of associated rank 2 vector bundles on Y and they prove that the moduli space M_E of these vector bundles is birationally equivalent via the Abel–Jacobi map to $J(Y)$. The dominant rational map $M_{1,5} \dashrightarrow M_E$ has rationally connected fibers and it follows that the universal family of curves in Y parametrized by $M_{1,5}$ provides the desired codimension 2 cycle.

Note that this does not answer Question 6.26 since the rationally connected fibration $M_{1,5} \dashrightarrow M_E$ could have no section. This difficulty is related to the non-existence of an universal vector bundle on the product $M_E \times Y$. Question 6.26 is to our knowledge still open even for the cubic threefold.

The answer to Question 6.29 is also affirmative for the intersection X of two quadrics in $\mathbb{P}^5$ (see [85]): in this case the family of lines in X is a surface isomorphic via a choice of base point to the intermediate Jacobian $J(X)$.

Obviously a positive answer to Question 6.26 implies a positive answer to Question 6.29, as we can then just take $B = J(X)$. We will provide a more precise relation between these two questions in Section 6.3.2.1. However, it seems that Question 6.29 is more natural, especially if we go to the following stronger version (Question 6.30).

Here we choose an integral cohomology class $\alpha \in H^4(Y, \mathbb{Z})$. Assuming $\mathrm{CH}_0(Y)$ is supported on a curve, the Hodge structure on $H^4(Y, \mathbb{Q})$ is trivial and thus α is a Hodge class. Introduce the torsor $J(Y)_\alpha$ defined as follows: The Deligne cohomology group $H^4_D(Y, \mathbb{Z}(2))$ is defined as the cohomology $H^4(X, \mathbb{Z}_D(2))$ of the Deligne complex

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \Omega_X \to 0,$$

where $\mathbb{Z}$ is put in degree 0. The group $H^4_D(Y,\mathbb{Z}(2))$ is an extension,

$$0 \to J(Y) \to H^4_D(Y,\mathbb{Z}(2)) \stackrel{o}{\to} \mathrm{Hdg}^4(Y,\mathbb{Z}) \to 0, \tag{6.9}$$

where o is the natural map from Deligne to Betti cohomology induced by the morphism of complexes $\mathbb{Z}_D(2) \to \mathbb{Z}$ (see [101, I, Corollary 12.27]). Define

$$J(Y)_\alpha := o^{-1}(\alpha). \tag{6.10}$$

By definition, the Deligne cycle class map (see [39], [101, I, 12.3.3]), restricted to codimension 2 cycles of class α, takes values in $J(Y)_\alpha$. Furthermore, for any family of 1-cycles $Z \in \mathrm{CH}^2(B \times Y)$ of class $[Z_b] = \alpha$, $b \in B$, parametrized by an algebraic variety B, the map ϕ_Z induced by the Abel–Jacobi map (or rather the Deligne cycle class map) of Y, that is,

$$\phi_Z : B \to J(Y)_\alpha, \qquad \phi_Z(b) = \mathrm{AJ}_Y(Z_b),$$

is a morphism of complex algebraic varieties (note again that the twisted complex torus $J(Y)_\alpha$ is algebraic because $H^{3,0}(Y) = 0$; see [101, I, 12.2.2]). The following question makes sense for any smooth projective threefold Y satisfying the conditions $H^2(Y,\mathcal{O}_Y) = H^3(Y,\mathcal{O}_Y) = 0$.

Question 6.30. *Is the following property* $(**)$ *satisfied by* Y*?*

$(**)$ *For any degree 4 integral cohomology class* α *on* Y*, there is a "naturally defined" (up to birational transformations) smooth projective variety* B_α*, together with a codimension 2 cycle* $Z_\alpha \in \mathrm{CH}^2(B_\alpha \times Y)$*, with* $[Z_{\alpha,b}] = \alpha$ *in* $H^4(Y,\mathbb{Z})$ *for any* $b \in B$*, such that the morphism* $\phi_{Z_\alpha} : B_\alpha \to J(Y)_\alpha$ *is surjective with rationally connected general fiber.*

By "naturally defined," we have in mind that B_α should be determined by α by some natural geometric construction (for example, if α is sufficiently positive, a distinguished component of the Hilbert scheme of curves of class α and given genus, or a moduli space of vector bundles with $c_2 = \alpha$), which would imply that B_α is defined over the same definition field as Y.

This question is solved affirmatively by Castravet in [17] when Y is the complete intersection of two quadrics. It is also solved affirmatively in [113] for cubic threefolds.

We now explain the importance of Question 6.30 in relation to the Hodge conjecture with integral coefficients for degree 4 Hodge classes (that is, the study of the group Z^4 introduced in (6.2)). The important point here is that we want to consider fourfolds fibered over curves, or families of threefolds Y_t parametrized by a curve Γ. The generic fiber of this fibration is a threefold Y defined over $\mathbb{C}(\Gamma)$. Property $(**)$ essentially says that property $(*)$, being satisfied over the definition field, which is in this case $\mathbb{C}(\Gamma)$, holds in family. When we work in families, the necessity to look at all torsors $J(Y)_\alpha$, and not only at $J(Y)$, becomes obvious. For fixed Y the twisted Jacobians are all isomorphic (maybe not canonically) and if we can choose a cycle z_α in each given class α (for example if Y is uniruled so that $Z^4(Y) = 0$), we can use translations by the z_α to reduce the problem to the case where $\alpha = 0$; this is a priori not true in families, for example because there might not be any codimension 2 cycle of relative class α on the total space of the family.

We have the following application (Theorem 6.31) of property ($**$) to the Hodge conjecture for degree 4 integral Hodge classes on fourfolds fibered over curves. Let X be a smooth projective fourfold, and let $f : X \to \Gamma$ be a surjective morphism to a smooth curve Γ, whose general fiber X_t satisfies $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$. As already mentioned, for X_t, $t \in \Gamma$, the intermediate Jacobian $J(X_t)$ is an abelian variety, as a consequence of the vanishing $H^3(X_t, \mathcal{O}_{X_t}) = 0$. For any class

$$\alpha \in H^4(X_t, \mathbb{Z}) = \mathrm{Hdg}^4(X_t, \mathbb{Z}),$$

we introduced above a torsor $J(X_t)_\alpha$ under $J(X_t)$, which is an algebraic variety noncanonically isomorphic to $J(X_t)$.

Using the obvious extension of the formulas (6.7), (6.10) in the relative setting, the construction of $J(X_t)$, $J(X_t)_\alpha$ can be done in family on the Zariski open set $\Gamma_0 \subset \Gamma$, over which f is smooth. There is thus a family of abelian varieties $\mathcal{J} \to \Gamma_0$, and for any global section α of the locally constant system $R^4 f_* \mathbb{Z}$ on Γ_0, we get the twisted family $\mathcal{J}_\alpha \to \Gamma_0$. The construction of these families in the analytic setting (that is, as (twisted) families of complex tori) follows from Hodge theory (see [101, II, 7.1.1]) and from their explicit set-theoretic description given by formulas (6.7), (6.10). The fact that the resulting families are algebraic can be proved using the results of [73], when one knows that the Abel–Jacobi map is surjective. Indeed, we already mentioned that under this assumption the intermediate Jacobian is the universal abelian quotient of CH^2, and thus can be constructed algebraically in the same way as the Albanese variety.

Given a smooth algebraic variety B, a morphism $g : B \to \Gamma$ and a codimension 2 cycle $\mathcal{Z} \subset B \times_\Gamma X$ of relative class $[\mathcal{Z}_b] = \alpha_{g(b)} \in H^4(X_t, \mathbb{Z})$, the relative Abel–Jacobi map (or rather Deligne cycle class map) gives a morphism

$$\phi_{\mathcal{Z}} : B_0 \to \mathcal{J}_\alpha, \qquad b \mapsto \mathrm{AJ}_Y(\mathcal{Z}_b)$$

over Γ_0, where $B_0 := g^{-1}(\Gamma_0)$. Again, the proof that $\phi_{\mathcal{Z}}$ is holomorphic is quite easy (see [101, II, 7.2.1]), while the algebraicity is more delicate.

The following result, which illustrates the importance of condition ($**$) as opposed to condition ($*$), appears in [24]. As before, we assume that X is a smooth projective fourfold, and that $f : X \to \Gamma$ is a surjective morphism to a smooth curve whose general fiber X_t satisfies $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$.

THEOREM 6.31 (Colliot-Thélène and Voisin 2010). *Assume $f : X \to \Gamma$ satisfies the following assumptions:*

(1) The smooth fibers X_t have no torsion in $H^3_B(X_t, \mathbb{Z})$.

(2) The singular fibers of f are reduced with at worst ordinary quadratic singularities.

(3) For any section α of $R^4 f_*\mathbb{Z}$ on Γ_0, there exist a variety $g_\alpha : B_\alpha \to \Gamma$ and a codimension 2 cycle $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of relative class $g_\alpha^*\alpha$, such that the morphism $\phi_{\mathcal{Z}_\alpha} : B \to \mathcal{J}_\alpha$ is surjective with rationally connected general fiber.

Then the Hodge conjecture is true for integral Hodge classes of degree 4 on X.

Proof. An integral Hodge class $\tilde{\alpha} \in \mathrm{Hdg}^4(X,\mathbb{Z}) \subset H^4(X,\mathbb{Z})$ induces a section α of the constant system $R^4 f_*\mathbb{Z}$ which admits a lift to a section of the family of twisted Jacobians $\mathcal{J}_\alpha$ over Γ^0. This lift is obtained as follows: The class $\tilde{\alpha}$ being a Hodge class on X admits a lift β in the Deligne cohomology group $H^4_D(X,\mathbb{Z}(2))$ by the exact sequence (6.9) for X. Then our section σ is obtained by restricting β to the smooth fibers of f: $\sigma(t) := \beta_{|X_t}$. A crucial point is the fact that this lift is an *algebraic* section $\Gamma \to \mathcal{J}_\alpha$ of the structural map $\mathcal{J}_\alpha \to \Gamma^0$.

Recall that we have by hypothesis the morphism

$$\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha,$$

which is algebraic, surjective, with rationally connected general fiber. We can now replace $\sigma(\Gamma)$ by a 1-cycle $\Sigma = \sum_i n_i \Sigma_i$ rationally equivalent to it in $\mathcal{J}_\alpha$, in such a way that the fibers of $\phi_{\mathcal{Z}_\alpha}$ are rationally connected over the general points of each component Σ_i of $\mathrm{Supp}\,\Sigma$.

According to [46], the morphism $\phi_{\mathcal{Z}_\alpha}$ admits a lifting over each Σ_i, which provides curves $\Sigma_i' \subset B_\alpha$.

Recall next that there is a codimension 2 cycle $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of relative class α parametrized by a smooth projective variety B_α. We can restrict this cycle to each Σ_i', getting codimension 2 cycles $\mathcal{Z}_{\alpha,i} \in \mathrm{CH}^2(\Sigma_i' \times_\Gamma X)$. Consider the 1-cycle

$$Z := \sum_i n_i p_{i*} \mathcal{Z}_{\alpha,i} \in \mathrm{CH}^2(\Gamma \times_\Gamma X) = \mathrm{CH}^2(X),$$

where p_i is the restriction to Σ_i' of $p : B_\alpha \to \Gamma$. Recalling that Σ is rationally equivalent to $\sigma(\Gamma)$ in $\mathcal{J}_\alpha$, we find that the "normal function ν_Z associated to Z" (see [101, II, 7.2.1]), defined by

$$\nu_Z(t) = \mathrm{AJ}_{X_t}(Z_{|X_t}),$$

is equal to σ. We then deduce from [48] (see also [101, II, 8.2.2]), using the Leray spectral sequence of $f_U : X_U \to U$ and assumption (1), that the cohomology classes $[Z] \in H^4(X,\mathbb{Z})$ of Z and $\tilde{\alpha}$ coincide on any open set of the form X_U, where $U \subset \Gamma_0$ is an affine open subset of Γ over which f is smooth.

On the other hand, the kernel of the restriction map $H^4(X,\mathbb{Z}) \to H^4(X_U,\mathbb{Z})$ is generated by the groups $i_{t*}H_4(X_t,\mathbb{Z})$, where $t \in \Gamma \setminus U$, and $i_t : X_t \to X$ is the inclusion map. We conclude using assumption (2) and the fact that the general fiber of f has $H^2(X_t, \mathcal{O}_{X_t}) = 0$, which imply that all fibers X_t (singular or not) have their degree 4 integral homology generated by homology classes of algebraic cycles; indeed, it follows from this and the previous conclusion that $[Z] - \tilde{\alpha}$ is algebraic, so that $\tilde{\alpha}$ is also algebraic. □

In [113] we extend the results of [54] and answer Question 6.30 affirmatively for cubic threefolds. More precisely, Iliev and Markushevich provide a naturally defined parametrization with rationally connected fibers of the twisted intermediate Jacobian $J(Y)_\alpha$, where Y is a cubic threefold and α has degree -1 or 1 modulo 3, using the family of smooth elliptic quintic curves. We provide a similar parametrization of $J(Y)_\alpha$, where α has degree 0 modulo 3, using the universal family of degree 6 elliptic curves. Since there is a canonical 1-cycle of degree 3 on Y, these results are sufficient to answer Question 6.30 for all α.

Combining this result with Theorem 6.31, we get the following corollary.

Corollary 6.32 (Voisin 2010; Voisin 2013 [113]). *Let $f : X \to \Gamma$ be a fibration over a curve with general fiber either a smooth cubic threefold or a complete intersection of two quadrics in $\mathbb{P}^5$. If the fibers of f have at worst ordinary quadratic singularities, then the Hodge conjecture holds for degree 4 integral Hodge classes on X. In other words, the group $Z^4(X)$ is trivial.*

Remark 6.33. The difficulty here is to prove the result for integral Hodge classes. Indeed, the fact that degree 4 rational Hodge classes are algebraic for X as above can be proved by using either the results of [26] or Bloch and Srinivas [15] (see Section 3.1.2), since such an X is swept out by rational curves, hence has its CH_0 group supported on a three-dimensional closed algebraic subset, or by using the method of Zucker [115], who uses the theory of normal functions, which we have essentially followed here.

Corollary 6.32 in the case of a fourfold X fibered by complete intersections of two quadrics in $\mathbb{P}^5$ has been re-proved by Colliot-Thélène [22] without any assumptions on singular fibers. Note however that many such fourfolds X are rational over the base (that is, birational to $\Gamma \times \mathbb{P}^3$); this is the case for example if there is a section of the family of lines in the fibers of f, for which it suffices to have a Hodge class of degree 4 on X whose restriction to the fibers of f has degree 1. When X is rational over the base, the vanishing of $Z^4(X)$ is immediate because the group $Z^4(X)$ of (6.2) is a birational invariant of X (see Lemma 6.3).

6.3.2 Decomposition of the diagonal and structure of the Abel–Jacobi map

6.3.2.1 Relation between Questions 6.26 and 6.29

Following [113], we establish here the following relation between Questions 6.26 and 6.29.

Theorem 6.34 (Voisin 2010). *Assume that Question 6.29 concerning the existence of parametrizations of $J(Y)$ with rationally connected fibers has an affirmative answer for Y and that the intermediate Jacobian of Y admits a 1-cycle Γ such that $\Gamma^{*g} = g!\, J(Y)$, $g = \dim J(Y)$. Then Question 6.26 concerning the existence of a universal codimension 2 cycle in $J(Y) \times Y$ also has an affirmative answer for Y.*

Here we use the Pontryagin product $*$ on cycles of $J(Y)$ defined by

$$z_1 * z_2 = \mu_*(z_1 \times z_2),$$

where $\mu : J(Y) \times J(Y) \to J(Y)$ is the sum map (see [101, II, 11.3.1]). The condition $\Gamma^{*g} = g!\, J(Y)$ is satisfied if the class of Γ is equal to $\frac{[\Theta]^{g-1}}{(g-1)!}$, for some principal polarization Θ. This is the case if $J(Y)$ is a Jacobian. It is however a weaker assumption (see Remark 6.37 below).

Proof of Theorem 6.34. There exist by assumption a variety B, and a codimension 2 cycle $Z \in \mathrm{CH}^2(B \times Y)$ which is cohomologous to 0 on fibers $b \times Y$, such that the morphism

$$\phi_Z : B \to J(Y)$$

induced by the Abel–Jacobi map of Y is surjective with rationally connected general fibers. Consider the 1-cycle Γ of $J(Y)$. We may assume by a moving lemma, up to changing the representative of Γ modulo homological equivalence, that $\Gamma = \sum_i n_i \Gamma_i$ where, for each component Γ_i of the support of Γ, the general fiber of ϕ_Z over Γ_i is rationally connected. We may furthermore assume that the Γ_i's are smooth. According to [46], the inclusion $j_i : \Gamma_i \hookrightarrow J(Y)$ then has a lift $\sigma_i : \Gamma_i \to B$. Denote by $Z_i \subset \Gamma_i \times Y$ the codimension 2 cycle $(\sigma_i, \mathrm{Id}_Y)^* Z$. Then the morphism $\phi_i : \Gamma_i \to J(Y)$ induced by the Abel–Jacobi map is equal to j_i.

For each g-uple of components $(\Gamma_{i_1}, \ldots, \Gamma_{i_g})$ of $\mathrm{Supp}\,\Gamma$, consider $\Gamma_{i_1} \times \cdots \times \Gamma_{i_g}$, and the codimension 2 cycle

$$Z_{i_1,\ldots,i_g} := (\mathrm{pr}_1, \mathrm{Id}_Y)^* Z_{i_1} + \cdots + (\mathrm{pr}_g, \mathrm{Id}_Y)^* Z_{i_g} \in \mathrm{CH}^2(\Gamma_1 \times \cdots \times \Gamma_g \times Y).$$

The codimension 2 cycle

$$\mathcal{Z} := \sum_{i_1,\ldots,i_g} n_{i_1} \cdots n_{i_g} Z_{i_1,\ldots,i_g} \in \mathrm{CH}^2((\sqcup \Gamma_i)^g \times Y), \tag{6.11}$$

where $\sqcup \Gamma_i$ is the disjoint union of the Γ_i's (hence, in particular, is smooth), is invariant under the symmetric group $\mathfrak{S}_g$ acting on the factor $(\sqcup \Gamma_i)^g$ in the product $(\sqcup \Gamma_i)^g \times Y$. The part of $\mathcal{Z}$ dominating at least one component of $(\sqcup \Gamma_i)^g$ (which is the only part of $\mathcal{Z}$ we are interested in) is then the pull-back of a codimension 2 cycle $\mathcal{Z}_{\mathrm{sym}}$ on $(\sqcup \Gamma_i)^{(g)} \times Y$. Consider now the sum map

$$\sigma : (\sqcup \Gamma_i)^{(g)} \to J(Y).$$

Let $\mathcal{Z}_J := (\sigma, \mathrm{Id})_*(\mathcal{Z}_{\mathrm{sym}}) \in \mathrm{CH}^2(J(Y) \times Y)$. The proof concludes with the following lemma.

Lemma 6.35. *The Abel–Jacobi map,*

$$\phi_{\mathcal{Z}_J} : J(Y) \to J(Y),$$

is equal to $\mathrm{Id}_{J(Y)}$.

Proof. Instead of the symmetric product $(\sqcup \Gamma_i)^{(g)}$ and the descended cycle $\mathcal{Z}_{\mathrm{sym}}$, consider the product $(\sqcup \Gamma_i)^g$, the cycle $\mathcal{Z}$, and the sum map

$$\sigma' : (\sqcup \Gamma_i)^g \to J(Y).$$

Then we have $(\sigma', \mathrm{Id})_*(\mathcal{Z}) = g!(\sigma, \mathrm{Id})_*(\mathcal{Z}_{\mathrm{sym}})$ in $\mathrm{CH}^2(J(Y) \times Y)$, so that writing $\mathcal{Z}'_J := (\sigma', \mathrm{Id})_* \mathcal{Z}$, it suffices to prove that $\phi_{\mathcal{Z}'_J} : J(Y) \to J(Y)$ is equal to $g!\, \mathrm{Id}_{J(Y)}$.

This is done as follows: Let $j \in J(Y)$ be a general point, and let $\{x_1, \dots, x_N\}$ be the fiber of σ' over j. Thus each x_l parametrizes a g-uple $(i_1^l, \dots, i_g^l)$ of components of $\operatorname{Supp} \Gamma$, and points $\gamma_{i_1}^l, \dots, \gamma_{i_g}^l$ of $\Gamma_{i_1}, \dots, \Gamma_{i_g}$, respectively, such that

$$\sum_{1 \le k \le g} \gamma_{i_k}^l = j. \tag{6.12}$$

On the other hand, recall that

$$\gamma_{i_k}^l = \mathrm{AJ}_Y(Z_{i_k^l, \gamma_{i_k}^l}). \tag{6.13}$$

It follows from (6.12) and (6.13) that for each $l \in \{1, \dots, N\}$, we have

$$\mathrm{AJ}_Y\left(\sum_{1 \le k \le g} Z_{i_k, \gamma_{i_k}^l}^l\right) = \mathrm{AJ}_Y(Z_{i_1^l, \dots, i_g^l, (\gamma_{i_1}^l, \dots, \gamma_{i_g}^l)}) = j. \tag{6.14}$$

Recall now that $\Gamma = \sum_i n_i \Gamma_i$ and that $(\Gamma)^{*g} = g! J(Y)$, which is equivalent to the equality

$$\sigma'_*\left(\sum_{i_1, \dots, i_g} n_{i_1} \cdots n_{i_g} \Gamma_{i_1} \times \cdots \times \Gamma_{i_g}\right) = g! J(Y).$$

This exactly says that $\sum_{1 \le l \le N} \sum_{i_1^l, \dots, i_g^l} n_{i_1^l} \cdots n_{i_g^l} = g!$, which together with (6.11) and (6.14) proves the desired equality $\phi_{\mathcal{Z}'_J} = g!\, \mathrm{Id}_{J(Y)}$. ☐

The proof of Theorem 6.34 is now complete. ☐

REMARK 6.36. When $\mathrm{NS}(J(Y)) = \mathbb{Z}\Theta$, the existence of a 1-cycle Γ in $J(Y)$ such that $\Gamma^{*g} = g!\,J(Y)$, $g = \dim J(Y)$ is equivalent to the existence of a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$. The question of whether the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$ is unknown even for the cubic threefold. However it has a positive answer for $g \leq 3$ because any principally polarized abelian variety (ppav) of dimension ≤ 3 is the Jacobian of a curve.

REMARK 6.37. Totaro asked for examples of principally polarized abelian varieties (ppav's) (A, Θ) of dimension g, such that the minimal class $\frac{[\Theta]^{g-1}}{(g-1)!}$ is algebraic, but which are not Jacobians. Such examples exist and can be constructed as follows: For $g = 4$ or 5, it is known that any ppav (A, Θ) of dimension g is a Prym variety. This implies that the class $2\frac{[\Theta]^{g-1}}{(g-1)!}$ is algebraic for them. We now start from a general Jacobian (J, Θ_J) of dimension $g = 4$ or 5 (so $\mathrm{NS}(J) = \mathbb{Z}$), and consider ppav's (A, Θ_A) which are isogenous to J, the degree of the isogeny $A \to J$ being odd. For such an abelian variety, an odd multiple of $\frac{[\Theta_A]^{g-1}}{(g-1)!}$ is algebraic, since $\frac{[\Theta_J]^{g-1}}{(g-1)!}$ is algebraic. On the other hand, $2\frac{[\Theta_A]^{g-1}}{(g-1)!}$ is algebraic, as already noted. It follows that $\frac{[\Theta_A]^{g-1}}{(g-1)!}$ is algebraic. But the general such ppav is not a Jacobian. Indeed, they form a dense set in the moduli space of g-dimensional ppav's, while for $g \geq 4$, the Schottky locus parametrizing Jacobians is a proper closed algebraic subset of $\mathcal{A}_g$.

6.3.2.2 Decomposition of the diagonal modulo homological equivalence

This section is devoted to the study of Question 6.26 or condition $(*)$ of Question 6.29.

Assume Y is a smooth projective threefold such that $\mathrm{CH}_0(Y) = \mathbb{Z}$. The cohomological version of the Bloch–Srinivas decomposition of the diagonal (3.2) says that there exists a nonzero integer N such that, denoting by $\Delta_Y \subset Y \times Y$ the diagonal,

$$N[\Delta_Y] = [Z] + [Z'] \text{ in } H^6_B(Y \times Y, \mathbb{Z}), \tag{6.15}$$

where $Z' = N(X \times x)$, and the support of Z is contained in $D \times Y$, $D \subsetneqq Y$.

We wish to study the invariant of Y defined as the gcd of the nonzero integers N appearing above. This is a birational invariant of Y by Remark 6.23. The results below relate the triviality of this invariant, that is, the existence of an integral cohomological decomposition of the diagonal, to condition $(*)$ (among other things). They can be found in [113].

THEOREM 6.38 (Voisin 2010). *Let Y be a smooth projective threefold. Assume Y admits a cohomological decomposition of the diagonal as in (6.15). Then we have the following conditions:*

(i) The integer N annihilates the torsion of $H^p(Y, \mathbb{Z})$ for any p.

(ii) *The integer* N *annihilates* $Z^4(Y)$.

(iii) $H^i(Y, \mathcal{O}_Y) = 0$ *for all* $i > 0$ *and there exists a codimension* 2 *cycle* $Z \in \mathrm{CH}^2(J(Y) \times Y)$ *such that* ϕ_Z *is equal to* $N\ \mathrm{Id}_{J(Y)}$.

Corollary 6.39. *If* Y *admits an integral cohomological decomposition of the diagonal, then we have the following conditions:*

(i) $H^p(Y, \mathbb{Z})$ *is without torsion for any* p.

(ii) $Z^4(Y) = 0$.

(iii) *There exists a universal codimension* 2 *cycle in* $J(Y) \times Y$.

Remark 6.40. That the integral decomposition of the diagonal as in (6.15), with $N = 1$, and in the Chow group $\mathrm{CH}(Y \times Y)$ implies that $H^3(Y, \mathbb{Z})$ has no torsion was observed by Colliot-Thélène. Note that when $H^2(Y, \mathcal{O}_Y) = 0$, the torsion of $H^3(Y, \mathbb{Z})$ is the Brauer group of Y.

Proof of Theorem 6.38. There exist by assumption a proper algebraic subset $D \subsetneqq Y$, which one may assume to be of pure dimension 2, and a cycle $Z \in \mathrm{CH}^3(Y \times Y)$ with support contained in $D \times Y$ such that

$$N[\Delta_Y] = [Z] + [Z'] \text{ in } H^6(Y \times Y, \mathbb{Z}), \tag{6.16}$$

where $Z' = NY \times \mathrm{pt}$.

Codimension 3 cycles z of $Y \times Y$ act on $H^p(Y, \mathbb{Z})$ for any p and on the intermediate Jacobian of Y, and this action, which we will denote

$$z^* : H^p(Y, \mathbb{Z}) \to H^p(Y, \mathbb{Z}), \qquad z^* : J(Y) \to J(Y),$$

depends only on the cohomology class of z. As the diagonal of Y acts by the identity map on $H^p(Y, \mathbb{Z})$ for $p > 0$ and on $J(Y)$, one concludes that

$$N\ \mathrm{Id}_{H^p(Y,\mathbb{Z})} = Z'^* + Z^* : H^p(Y, \mathbb{Z}) \to H^p(Y, \mathbb{Z}) \text{ for } p > 0, \tag{6.17}$$

$$N\ \mathrm{Id}_{J(Y)} = Z'^* + Z^* : J(Y) \to J(Y). \tag{6.18}$$

It is clear that Z'^* acts trivially on $H^{*>0}(Y, \mathbb{Z})$ and on $J(Y)$ since Z' is supported over a point in Y. We thus conclude that

$$N\ \mathrm{Id}_{J(Y)} = Z^* : J(Y) \to J(Y),$$

$$N\ \mathrm{Id}_{H^{*>0}(Y,\mathbb{Z})} = Z^* : H^{*>0}(Y, \mathbb{Z}) \to H^{*>0}(Y, \mathbb{Z}). \tag{6.19}$$

Let $\tau : \widetilde{D} \to Y$ be a desingularization of D and $i_{\widetilde{D}} = i_D \circ \tau : \widetilde{D} \to Y$. The part of the cycle Z that dominates D can be lifted to a cycle $\widetilde{Z}$ in $\widetilde{D} \times Y$, and the remaining part acts trivially on $H^p(Y, \mathbb{Z})$ for $p \leq 3$ for reasons of codimension. Thus the map Z^* acting on $H^p(Y, \mathbb{Z})$ for $p \leq 3$ can be written as

$$Z^* = i_{\widetilde{D}*} \circ \widetilde{Z}^*. \tag{6.20}$$

We note now that the action of $\widetilde{Z}^*$ on cohomology sends $H^p(Y,\mathbb{Z})$, $p \leq 3$ to $H^{p-2}(\widetilde{D},\mathbb{Z})$, $p \leq 3$. The last groups have no torsion. It follows that $\widetilde{Z}^*$ annihilates the torsion of $H^p(Y,\mathbb{Z})$, $p \leq 3$. Formula (6.19) then implies that the torsion of $H^p(Y,\mathbb{Z})$ is annihilated by N Id for $1 \leq p \leq 3$. As there is no torsion in $H^0(Y,\mathbb{Z})$, this concludes the case $p \leq 3$.

To deal with the torsion of $H^p(Y,\mathbb{Z})$ with $p \geq 4$, we rather use the actions Z_*, Z'_* of Z, Z' on $H^p(Y,\mathbb{Z})$, $p = 4, 5$. This action again factors through $\widetilde{Z}_*$, $\widetilde{Z'}_*$. Now, $\widetilde{Z}_*$ factors through the restriction map

$$H^p(Y,\mathbb{Z}) \to H^p(\widetilde{D},\mathbb{Z}),$$

while $\widetilde{Z'}_*$ obviously annihilates $H^{*<6}(Y,\mathbb{Z})$. We can restrict to the range $3 < p < 6$ since $H^6(Y,\mathbb{Z})$ has no torsion. On the other hand, since $\dim \widetilde{D} \leq 2$, $H^p(\widetilde{D},\mathbb{Z})$ has no torsion for $p = 4, 5$. It follows that we also have $Z_*(H^p(Y,\mathbb{Z})_{\rm tors}) = 0$ for $p = 4, 5$, and since Z_* acts as N Id $= Z_*$ on these groups, we conclude that $N\, H^p(Y,\mathbb{Z})_{\rm tors} = 0$ for $p = 4, 5$. This proves (i).

(ii) Let us consider again the action $Z_* = N$ Id on the cohomology $H^4(Y,\mathbb{Z})$. Observe again that the part of Z not dominating D has a trivial action on $H^4(Y,\mathbb{Z})$, while the dominating part lifts as above to a cycle $\widetilde{Z}$ in $\widetilde{D} \times Y$. Then we find that $N\,{\rm Id}_{H^4(Y,\mathbb{Z})}$ factors as $\widetilde{Z}_* \circ i^*_{\widetilde{D}}$, hence through the restriction map $i^*_{\widetilde{D}} : H^4(Y,\mathbb{Z}) \to H^4(\widetilde{D},\mathbb{Z})$. As $\dim \widetilde{D} = 2$, the group on the right is generated by classes of algebraic cycles, and thus $(N\,(H^4(Y,\mathbb{Z})))$ is generated by classes of algebraic cycles on Y. Hence we have $N\, Z^4(Y) = 0$.

(iii) The vanishing $H^i(Y,\mathcal{O}_Y) = 0$ for all $i > 0$ is a consequence due to Bloch and Srinivas of the decomposition (6.15) of the diagonal (see Theorem 3.16).

It is well known, and this is a consequence of the Lefschetz theorem on $(1,1)$-classes applied to $\mathrm{Pic}^0(\widetilde{D}) \times \widetilde{D}$ (see Remark 6.28), that there exists a universal divisor $\mathcal{D} \in \mathrm{Pic}(\mathrm{Pic}^0(\widetilde{D}) \times \widetilde{D})$ such that the induced morphism $\phi_{\mathcal{D}} : \mathrm{Pic}^0(\widetilde{D}) \to \mathrm{Pic}^0(\widetilde{D})$ is the identity. On the other hand, we have the morphism

$$\widetilde{Z}^* : J(Y) \to J^1(\widetilde{D}) = \mathrm{Pic}^0(\widetilde{D}),$$

which is a morphism of abelian varieties.

Let us consider the cycle

$$\mathcal{Z} := (\mathrm{Id}_{J(Y)}, i_{\widetilde{D}})_* \circ (\widetilde{Z}^*, \mathrm{Id}_{\widetilde{D}})^*(\mathcal{D}) \in \mathrm{CH}^2(J(Y) \times Y).$$

Then $\phi_{\mathcal{Z}} : J(Y) \to J(Y)$ is equal to

$$i_{\widetilde{D}*} \circ \phi_{\mathcal{D}} \circ \widetilde{Z}^* : J(Y) \to J^1(\widetilde{D}) \to J^1(\widetilde{D}) \to J(Y).$$

As $\phi_{\mathcal{D}}$ is the identity map acting on $\mathrm{Pic}^0(\widetilde{D})$ and $i_{\widetilde{D}*} \circ \widetilde{Z}^*$ is equal to N Id acting on $J(Y)$ according to (6.19) and (6.20), one concludes that the endomorphism $\phi_{\mathcal{Z}}$ of $J(X)$ is equal to $N\,\mathrm{Id}_{J(Y)}$. □

A partial converse to Theorem 6.38 is as follows (see [113]).

THEOREM 6.41 (Voisin 2010). *Assume the smooth projective threefold Y satisfies the following conditions:*

(i) $H^i(Y, \mathcal{O}_Y) = 0$ *for* $i > 0$.

(ii) $Z^4(Y) = 0$.

(iii) $H^p(Y, \mathbb{Z})$ *has no torsion for any integer* p.

(iv) The intermediate Jacobian of Y *admits a* 1*-cycle* Γ *of class* $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = \dim J(Y)$.

Then if there exists a universal codimension 2 *cycle on* $J(Y) \times Y$, Y *admits an integral cohomological decomposition of the diagonal as in (6.15).*

By Theorem 6.5 which guarantees condition (ii), we conclude that for rationally connected threefolds Y with no torsion in $H^*(Y, \mathbb{Z})$ (or equivalently with no torsion in $H^3(Y, \mathbb{Z})$ since a rationally connected variety X has trivial fundamental group, hence no torsion in $H^{2n-1}(X, \mathbb{Z})$, $n = \dim X$) and satisfying condition (iv), the existence of a universal codimension 2 cycle in $J(Y) \times Y$ is equivalent to the fact that Y admits an integral cohomological decomposition of the diagonal as in (6.15).

REMARK 6.42. Under condition (iv) above, we have seen that the existence of a universal codimension 2 cycle is equivalent to the existence of a parametrization with rationally connected fibers of the intermediate Jacobian by a family of algebraic cycles (Theorem 6.34).

REMARK 6.43. Nothing is known on condition (iv) above, but it is satisfied if $\dim J(Y) \leq 3$. It is also satisfied by threefolds satisfying the Clemens–Griffiths criterion for rationality (see [21]), that is, the intermediate Jacobian $J(Y)$ is isomorphic as a principally polarized abelian variety to a direct sum of Jacobians of curves.

PROOF OF THEOREM 6.41. When the integral cohomology of Y has no torsion, the class of the diagonal $[\Delta_Y] \in H^6(Y \times Y, \mathbb{Z})$ has an integral Künneth decomposition,

$$[\Delta_Y] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{1,5} + \delta_{0,6},$$

where $\delta_{i,j} \in H^i(Y, \mathbb{Z}) \otimes H^j(Y, \mathbb{Z})$. The class $\delta_{0,6}$ is the class of $Y \times y$ for any point y of Y. By assumption we have the vanishing

$$H^1(Y, \mathcal{O}_Y) = 0, \qquad H^2(Y, \mathcal{O}_Y) = 0. \tag{6.21}$$

The first condition implies that the groups $H^1(Y, \mathbb{Q})$ and $H^5(Y, \mathbb{Q})$ are trivial, hence the groups $H^1(Y, \mathbb{Z})$ and $H^5(Y, \mathbb{Z})$ must be trivial since they have no torsion by assumption. It follows that $\delta_{5,1} = \delta_{1,5} = 0$.

Next, the second condition implies that the Hodge structure on $H^2(Y,\mathbb{Q})$, hence also on $H^4(Y,\mathbb{Q})$ by duality, is trivial. Hence $H^4(Y,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are generated by Hodge classes, and because we assumed $Z^4(Y)=0$, it follows that $H^4(Y,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are generated by cycle classes. From this, one concludes that $\delta_{4,2}$ and $\delta_{2,4}$ are represented by algebraic cycles whose support does not dominate Y by the first projection. The same is true for $\delta_{6,0}$, which is the class of $y\times Y$. The existence of a decomposition as in (6.16) with $N=1$ is thus equivalent to the fact that there exists a cycle $Z\subset Y\times Y$ such that the support of Z is contained in $D\times Y$, with $D\subsetneqq Y$, and $Z^*: H^3(Y,\mathbb{Z})\to H^3(Y,\mathbb{Z})$ is the identity map. This last condition is indeed equivalent to the fact that the component of type $(3,3)$ of $[Z]$ is equal to $\delta_{3,3}$.

Now let $\Gamma=\sum_i n_i\Gamma_i$ be a 1-cycle of $J(Y)$ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, where $\sigma_i:\Gamma_i\to J(Y)$ are smooth curves. By assumption (iv), there exists a codimension 2 cycle $\mathcal{Z}\in \mathrm{CH}^2(J(Y)\times Y)$ which is homologous to 0 on the fibers $b\times Y$, such that $\phi_{\mathcal{Z}}: J(Y)\to J(Y)$ is equal to the identity. Then for each i, $(\sigma_i,\mathrm{Id})^*\mathcal{Z}$ provides a codimension 2 cycle $Z_i\in\mathrm{CH}^2(\Gamma_i\times Y)$ of 1-cycles homologous to 0 in Y, parametrized by Γ_i, such that $\phi_{Z_i}:\Gamma_i\to J(Y)$ identifies to the inclusion σ_i of Γ_i in $J(Y)$.

Let us consider the cycle $Z\in\mathrm{CH}^3(Y\times Y)$ defined by

$$Z=\sum_i n_i Z_i\circ {}^tZ_i.$$

The proof that the cycle Z satisfies the desired property is then given in the following lemma. □

Lemma 6.44. *The map $Z^*: H^3(Y,\mathbb{Z})\to H^3(Y,\mathbb{Z})$ is the identity map.*

Proof. We have

$$Z^*=\sum_i n_i {}^tZ_i{}^*\circ Z_i^*.$$

Let us study the composite map

$${}^tZ_i^*\circ Z_i^*: H^3(Y,\mathbb{Z})\to H^1(\Gamma_i,\mathbb{Z})\to H^3(Y,\mathbb{Z}).$$

Recalling that $Z_i\in\mathrm{CH}^2(\Gamma_i\times Y)$ is the restriction to $\sigma_i(\Gamma_i)$ of $\mathcal{Z}\in\mathrm{CH}^2(J(Y)\times Y)$, one finds that this composite map can also be written as

$${}^tZ_i^*\circ Z_i^*={}^t\mathcal{Z}^*\circ([\sigma(\Gamma_i)]\cup)\circ\mathcal{Z}^*,$$

where $[\sigma(\Gamma_i)]\cup$ is the morphism of cup-product with the class $[\sigma(\Gamma_i)]$. One uses for this the fact that the composition

$$\sigma_{i*}\circ\sigma_i^*: H^1(J(Y),\mathbb{Z})\to H^{2g-1}(J(B),\mathbb{Z}),\quad g=\dim J(B)$$

is equal to $[\sigma(\Gamma_i)]\cup$.

We thus obtain

$$Z^* = {}^t\mathcal{Z}^* \circ \left(\sum_i n_i[\sigma(\Gamma_i)]\cup\right) \circ \mathcal{Z}^*.$$

But we know that the map $\phi_{\mathcal{Z}} : J(Y) \to J(Y)$ is the identity, which is equivalent to saying that $\mathcal{Z}^*$ is equal to the canonical isomorphism

$$H^3(Y,\mathbb{Z}) \cong H^1(J(Y),\mathbb{Z})$$

(which uses the fact that $H^3(Y,\mathbb{Z})$ is torsion free) and that ${}^t\mathcal{Z}^*$ is the dual canonical isomorphism

$$H^{2g-1}(J(Y),\mathbb{Z}) \cong H^3(Y,\mathbb{Z}).$$

Finally, the map $\sum_i n_i[\sigma(\Gamma_i)]\cup : H^1(J(Y),\mathbb{Z}) \to H^{2g-1}(J(Y),\mathbb{Z})$ is equal to the cup-product map with the class $\frac{[\Theta]^{g-1}}{(g-1)!}$. We have thus identified $Z^* : H^3(Y,\mathbb{Z}) \to H^3(Y,\mathbb{Z})$ to the composite map

$$H^3(Y,\mathbb{Z}) \cong H^1(J(Y),\mathbb{Z}) \stackrel{\frac{[\Theta]^{g-1}}{(g-1)!}\cup}{\rightarrow} H^{2g-1}(J(Y),\mathbb{Z}) \cong H^3(Y,\mathbb{Z}),$$

where the last isomorphism is the Poincaré dual of the first, and using the definition of the polarization Θ on $J(Y)$ (as given by Poincaré duality on Y: $H^3(Y,\mathbb{Z}) \cong H^3(Y,\mathbb{Z})^*$) we find that this composite map is the identity. $\square$

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Index