Hodge loci

Claire Voisin

Abstract. The goal of this expository article is first of all to show that Hodge theory provides naturally defined subvarieties of any moduli space parameterizing smooth varieties, the "Hodge loci", although only the Hodge conjecture would guarantee that these subvarieties are defined on a finite extension of the base field. We will show how these subsets can be studied locally in the Euclidean topology and introduce a number of related Hodge-theoretic notions. The article will culminate with two results by Deligne, Cattani-Deligne-Kaplan respectively. The first one says that Hodge classes are absolute Hodge on abelian varieties. This is a statement which we will rephrase in general in terms of Hodge loci and is enough to guarantee that Hodge loci are closed algebraic, defined on a finite extension of the base field. The second tells us that Hodge loci are in general closed algebraic, as predicted by the Hodge conjecture.

1. Introduction

These notes are devoted to the study of Hodge loci associated to a family of smooth complex projective varieties $\pi : \mathcal{X} \to B$. The Hodge loci are quite easy to define set theoretically and also, locally on B for the classical topology, as a countable union of analytic subschemes; the local components are indeed endowed with a natural analytic schematic structure, which we will describe and can be studied using Griffiths theory of variations of Hodge structures [16]. On the other hand, the right approach to give a global definition of the Hodge loci is the notion of "locus of Hodge classes" introduced in [8]. This locus of Hodge classes is not a subset of B, but a subset of some Hodge bundle on B, associated to π.

Griffiths' theory has several interesting local consequences on Hodge loci, eg their expected codimension and density properties. However, it only gives a local and transcendental approach to the subject, which makes it hard to understand the global structure of the Hodge loci. As we will explain in section 1.1 below, the Hodge conjecture predicts in fact that, assuming $\mathcal{X}$, B are quasi-projective, Hodge loci are closed algebraic subsets of B, which makes them relevant to the topic of this book. One of our main goals in this paper is to motivate and explain the main

Received by the editors .

1991 *Mathematics Subject Classification.* 14D05, 14D07, 14F25, 14F40, 14C30.

Key words and phrases. Variation of Hodge structure, Hodge classes, Monodromy, Mumford-Tate groups, Hodge loci.

theorem of [8] proving that Hodge loci are indeed closed algebraic, as predicted by the Hodge conjecture. As a consequence of this result, Hodge loci provide naturally defined subvarieties of any quasiprojective variety B parameterizing smooth projective varieties.

Finally, there are several arithmetic aspects of the question that we also want to describe. First of all, one question which is not addressed by Cattani, Deligne and Kaplan is whether, assuming everything is defined over a number field, the Hodge loci are also defined over a number field. Following [38], we will explain a partial result in this direction, together with the relation between this question and the question whether Hodge classes are "absolute Hodge".

A last aspect of the question, also related to arithmetic, is the study of the set of points of $B(\overline{\mathbb{Q}})$ not belonging to the Hodge locus, assuming as before the family and the base are defined over a number field. This study cannot be done within the framework of Betti cohomology and variations of Hodge structures. Complex geometers would usually just say that the Hodge locus is a countable union of proper closed algebraic subsets, hence that there are complex points in $B(\mathbb{C})$ outside the Hodge locus. But this countability argument does not say anything on the existence of points in $B(\overline{\mathbb{Q}})$ outside this locus. We will explain a partial approach to this, which uses results of Terasoma comparing Galois and monodromy groups, but works under an assumption which would be also implied by the Hodge conjecture, namely that "Hodge classes are Tate classes" (see subsection 1.2).

The rest of this introduction will make precise a number of notions used above.

1.1. Hodge structure and Hodge classes

Let X be a smooth projective complex variety, and k be a nonnegative integer. The Hodge structure on $H^{2k}(X,\mathbb{Q})$ is given by the Hodge decomposition of the complexified vector space $H^{2k}(X,\mathbb{C}) = H^{2k}(X,\mathbb{Q}) \otimes \mathbb{C}$ into components of type (p,q):

$$H^{2k}(X,\mathbb{C}) = \oplus_{p+q=2k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the set of complex cohomology classes which can be represented by a closed form of type (p,q). The Hodge symmetry property says that $\overline{H^{p,q}(X)} = H^{q,p}(X)$, where complex conjugation acts naturally on $H^{2k}(X,\mathbb{C}) = H^{2k}(X,\mathbb{R}) \otimes \mathbb{C}$.

Definition 1.1. *A Hodge class of degree $2k$ on X is a rational cohomology class $\alpha \in H^{2k}(X,\mathbb{Q})$ which is also in $H^{k,k}(X)$. The set of Hodge classes of degree $2k$ will be denoted by $Hdg^{2k}(X)$.*

Remark 1.2. One could of course also speak of integral Hodge class, and their study can lead to interesting invariants of varieties (cf. [10], [39]). Furthermore, even if we will focus here on rational Hodge classes, we will see that the integral structure on cohomology should be kept in mind, in order to establish finiteness results for the monodromy action (cf. subsection 4.1).

Remark 1.3. Notice that by Hodge symmetry, for a class $\alpha \in H^{2k}(X, \mathbb{Q})$, being in $H^{k,k}(X)$ is equivalent to being in

$$F^k H^{2k}(X, \mathbb{C}) := H^{2k,0}(X) \oplus H^{2k-1,1}(X) \oplus \ldots \oplus H^{k,k}(X).$$

The Hodge conjecture states the following:

Conjecture 1.4. *(Hodge 1950) Hodge classes on X are combinations with rational coefficients of classes $[Z] \in H^{2k}(X, \mathbb{Q})$ of algebraic subvarieties Z of X of codimension k.*

1.2. Hodge loci and moduli spaces

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism between irreducible complex analytic spaces. Fix a nonnegative integer k. We sketch several possible definitions of the Hodge locus for this family.

Definition 1.5. *The Hodge locus for degree $2k$ Hodge classes is the sublocus of B consisting of points t where the number $\rho^{2k}(\mathcal{X}_t) := \dim_{\mathbb{Q}} Hdg^{2k}(\mathcal{X}_t)$ does not take the minimal value.*

This locus is also sometimes called the jumping locus, because the number $\rho^{2k}(\mathcal{X}_t)$ is uppersemicontinuous. In fact, we have the following statement which will follow immediately from the local description given below (see also section 3.1):

Lemma 1.6. *Locally for the classical topology on B, for t taken away from a countable union of closed proper analytic subsets, the following holds: any Hodge class $\alpha \in Hdg^{2k}(\mathcal{X}_t)$ remains Hodge (that is of type (k, k)) on $\mathcal{X}_{t'}$, for t' in a neighborhood of t.*

Here we use the parallel transport of cohomology to see $\alpha \in Hdg^{2k}(\mathcal{X}_t) \subset H^{2k}(\mathcal{X}_t, \mathbb{Q})$ as a cohomology class on $\mathcal{X}_{t'}$ for t' close to t. Note that, by analytic continuation, the conclusion will then hold everywhere on B, as B is irreducible.

This lemma is a consequence of the following more precise result (Lemma 1.7): Here we choose a small contractible open set $U \subset B$, so that the family $\mathcal{X} \to B$ becomes topologically trivial over U. Let $t \in U$. The topological triviality shows that any degree $2k$ rational cohomology class α on $\mathcal{X}_t$ can be extended to a constant section $\tilde{\alpha}$ of the sheaf $R^{2k}\pi_*\mathbb{Q}$ on U. We will show in section 3.1 the following:

Lemma 1.7. *Given a class $\alpha \in H^{2k}(\mathcal{X}_t, \mathbb{Q})$, the set $U_\alpha \subset U$ consisting of points $t' \in U$ such that $\tilde{\alpha}_{t'}$ is a Hodge class is a closed analytic subset of U.*

It is then clear that the Hodge locus for our family restricted to U is the countable union over those α for which $U_\alpha \neq U$ of the corresponding U_α, as the $\mathbb{Q}$-vector space of Hodge classes on the fiber $\mathcal{X}_t$ is locally constant, hence of constant dimension, for t away from this locus. This dimension is then minimal as the Hodge classes on $\mathcal{X}_t$ for t very general as above, remain Hodge everywhere.

If we now assume the Hodge conjecture 1.4, we see that the expected structure of the Hodge locus is much more precise: indeed, the presence of a degree $2k$ Hodge class on $\mathcal{X}_t$ is then equivalent to the presence of an algebraic cycle of codimension k in $\mathcal{X}_t$. Such algebraic cycles are parameterized over B by a countable union of fiber-products of relative Hilbert schemes for the family $\mathcal{X} \to B$. A standard countability argument then shows that, assuming the Hodge conjecture, for each local component of the Hodge locus as described below, we can find a (local piece of) a relative Hilbert scheme which dominates it, and we then finally get by analytic continuation the following conclusion:

(*) *Assuming the Hodge conjecture, the Hodge locus is a countable union of proper closed algebraic subsets of B.*

This prediction has been made into a theorem by Cattani, Deligne and Kaplan (cf. [8]) as we will explain with some detail in section 7.

1.3. Arithmetic aspects: Absolute Hodge classes and Tate classes

The rough prediction above can be made more precise taking into account the definition field K of the family, which usually will be a number field, as we are interested in moduli spaces. We simply observe that the relative Hilbert schemes of the family $\pi : \mathcal{X} \to B$ are defined over K, so that its irreducible components are defined over a finite extension of K, and furthermore $\mathrm{Gal}\,(\overline{K}/K)$ acts on the set of irreducible components. We thus have the following refinement of (*) :

(**) *Assuming the Hodge conjecture, the Hodge locus is a countable union of proper closed algebraic subsets of B, defined over a finite extension of the definition field K. This union is stable under* $\mathrm{Gal}\,(\overline{K}/K)$.

There is no general result available on this question, but in sections 5 and 7.3, we will explain first of all that (**) is just a very weak version of the following conjecture, which is implied by the Hodge conjecture:

Conjecture 1.8. *Hodge classes are absolute Hodge.*

Here we are alluding only to the de Rham aspect of the notion of absolute Hodge class introduced by Deligne in [12]. We refer to [25] for the full definition involving also comparison with étale cohomology. Note that conjecture 1.8 has been proved by Deligne in [12] for abelian varieties. We will explain the strategy of the proof in section 6.

The general problem seems to be very hard, but we will give in subsection 7.3 a criterion for (**) to hold. This criterion however concerns only the "geometric aspect" of conjecture 1.8, and says something only when the components of the Hodge locus do not consist of isolated points.

We finish this introduction by giving some more detail on another arithmetic aspect of the study of the Hodge locus. As already mentioned, and assuming that everything is defined over a number field, it is not clear at all from the descriptions above that there are any points in $B(\overline{\mathbb{Q}})$ outside the Hodge locus.

The first result in this direction was due to Shioda. Recall that the Noether-Lefschetz theorem says that if $S \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$, the Picard number $\rho(S)$, defined as the rank of the Néron-Severi group of S, is equal to 1. Shioda proved in [32] the following:

Theorem 1.9. *For $d \geq 5$, d prime, there exist smooth surfaces $S \subset \mathbb{P}^3$ of degree d defined over $\mathbb{Q}$ and satisfying the conclusion of Noether-Lefschetz theorem.*

The proof by Shioda involved an explicit construction of surfaces with large automorphism group. Later on, Terasoma proved in [35] a result comparing Galois groups acting on étale cohomology of "most" closed geometric fibers (even defined over a number field) and the l-adic geometric monodromy group acting on the Betti cohomology of the same fiber with $\mathbb{Q}_l$-coefficients (cf. section 4.4).

The main consequence of Terasoma's result is the fact that, if the geometric monodromy group is big, which is one of the standard proofs towards Noether-Lefschetz type theorems (cf. [40, II, 3.3.2]), then so is the image of the Galois group. This allows to prove for many families that there are no Tate classes in the varying part of the cohomology of many fibers defined over a number field of a given family. Here Tate classes are defined as étale $\mathbb{Q}_l$-cohomology classes with finite Galois orbit.

In order to apply this to the original question concerning "jumping" Hodge classes, which must lie in the varying part of the cohomology of a family, on fibers defined over a number field, we see that what is needed is the answer to the following question:

Conjecture 1.10. *Hodge classes are Tate classes.*

This conjecture is of course implied by the Hodge conjecture, so it has a positive answer in the degree 2 case, where the Hodge conjecture is known as the Lefschetz $(1,1)$-theorem. It is also essentially related to the "étale aspect" of the question whether Hodge classes are absolute Hodge classes (cf. [12]).

The conclusion is then the following (cf. [2], [35] for various versions of this result).

Theorem 1.11. *Assuming conjecture 1.10, and the family $\mathcal{X} \to B$ is defined over a number field, there are many points in $B(\overline{\mathbb{Q}})$ outside the Hodge locus.*

Acknowledgements. I thank Gavril Farkas and Ian Morrison for inviting me to contribute to this beautiful volume. I also thank Pierre Deligne for allowing me to present the contents of his letter [13] here (cf. Theorem 4.14) and Michel Brion for his help. Finally, I thank Marc-Hubert Nicole and Ben Moonen for their comments on a preliminary version.

2. Variations of Hodge structures

2.1. Hodge bundles and Gauss-Manin connection

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism of complex varieties, which we first endow with the classical topology. Locally over B for the classical topology, the family $\mathcal{X}$ is topologically trivial, which provides local systems

$$H^i_{\mathbb{Z}} := R^i\pi_*\mathbb{Z},\ H^i_{\mathbb{C}} := R^i\pi_*\mathbb{C}.$$

Consider the associated holomorphic vector bundles on B

$$\mathcal{H}^i := H^i_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_B.$$

They are endowed with the Gauss-Manin connection

$$\nabla : \mathcal{H}^i \to \mathcal{H}^i \otimes \Omega_B,$$

which is the flat connection characterized by the fact that the sections of $H^i_{\mathbb{C}}$ are the sections of $\mathcal{H}^i$ annihilated by ∇ (i.e. the flat sections). The holomorphic bundles $\mathcal{H}^i$ can be described in the following way: let $\Omega_{\mathcal{X}/B,hol} : \Omega_{\mathcal{X}}/\pi^*\Omega_B$ where $\Omega_{\mathcal{X}}$ is the sheaf of holomorphic 1-forms, and more generally $\Omega^\bullet_{\mathcal{X}/B,hol} := \bigwedge^\bullet \Omega_{\mathcal{X}/B,hol}$, with differential $d_{\mathcal{X}/B}$ induced by the exterior differential. The relative holomorphic Poincaré lemma tells that $\Omega^\bullet_{\mathcal{X}/B,hol}$ is a resolution of the sheaf $\pi^{-1}\mathcal{O}_B$, so that

$$\mathcal{H}^i = \mathbb{R}^i\pi_*\Omega^\bullet_{\mathcal{X}/B,hol}.$$

The complex $\Omega^\bullet_{\mathcal{X}/B,hol}$ has the naïve (or Hodge) filtration

$$F^k\Omega^\bullet_{\mathcal{X}/B,hol} := \Omega^{\bullet \geq k}_{\mathcal{X}/B,hol}$$

whose graded pieces are the sheaves $Gr^k_F\Omega^\bullet_{\mathcal{X}/B,hol} = \Omega^k_{\mathcal{X}/B,hol}$.

The filtration above induces on each fiber $\mathcal{X}_t$ of π the Frölicher spectral sequence. Its degeneracy at E_1 for every t (cf. [40, I, 8.3.3]) implies that the coherent sheaves $R^q\pi_*\Omega^k_{\mathcal{X}/B}$ are locally free (and satisfy base change), and also that the coherent subsheaves

$$F^k\mathcal{H}^i := \mathrm{Im}\,(\mathbb{R}^i\pi_*\Omega^{\bullet\geq k}_{\mathcal{X}/B,hol} \to \mathbb{R}^i\pi_*\Omega^\bullet_{\mathcal{X}/B,hol})$$

are locally free subsheaves, in fact isomorphic to $\mathbb{R}^i\pi_*\Omega^{\bullet\geq k}_{\mathcal{X}/B,hol}$, and satisfying base-change. These bundles are called the Hodge bundles.

By Hodge theory, and degeneracy at E_1 of the Frölicher spectral sequence, the Hodge filtration $F^kH^i(\mathcal{X}_t,\mathbb{C}) = F^k\mathcal{H}^i_t$ can be described fiberwise as follows (see [40, I, 8.3.3]): Let $H^i(\mathcal{X}_t,\mathbb{C}) = \oplus_{p+q=i}H^{p,q}(\mathcal{X}_t)$ be the Hodge decomposition. then

$$F^kH^i(\mathcal{X}_t,\mathbb{C}) = \oplus_{p\geq k,p+q=i}H^{p,q}(\mathcal{X}_t). \tag{2.1}$$

2.2. Transversality

An extremely useful tool in the theory of infinitesimal variations of Hodge structure is the following *transversality property*, discovered by Griffiths ([16]), see also [40, I, 10.1.2].

Theorem 2.2. *The Hodge filtration is shifted infinitesimally by -1 under the Gauss-Manin connection:*

$$\nabla(F^k\mathcal{H}^i) \subset F^{k-1}\mathcal{H}^i \otimes \Omega_B.$$

Griffiths' proof of this fact involves the explicit description of the Gauss-Manin connection with respect to a vector field v on B as induced by the Lie derivative of forms along an horizontal vector field in $\mathcal{X}$ lifting v. In the next subsection, we will give another proof, based on the Katz-Oda description [20] of the Gauss-Manin connection. (Of course, there remains to prove that this description is equivalent to the differential geometric one. This can be found in [40, II, 5.1.1]).

We conclude with the following by-product of Griffiths computation: The transversality property (Theorem 2.2) allows to construct $\mathcal{O}_B$-linear maps

$$\overline{\nabla} : Gr^k_F\mathcal{H}^i =: \mathcal{H}^{k,i-k} \to Gr^{k-1}_F\mathcal{H}^i \otimes \Omega_B = \mathcal{H}^{k-1,i-k+1} \otimes \Omega_B.$$

The fiber of the bundle $\mathcal{H}^{p,q}$ at the point $t \in B$ identifies to $H^q(\mathcal{X}_t,\Omega^p_{\mathcal{X}_t})$. Let us specialize the map $\overline{\nabla}$ at the point t and see it by adjunction as a map

$${}^t\overline{\nabla}_t : T_{B,t} \to \mathrm{Hom}\,(H^{i-k}(\mathcal{X}_t,\Omega^k_{\mathcal{X}_t}), H^{i-k+1}(\mathcal{X}_t,\Omega^{k-1}_{\mathcal{X}_t})).$$

We then have:

Theorem 2.3. *(Griffiths) The map ${}^t\overline{\nabla}_t$ is the composition of the Kodaira-Spencer map $\rho : T_{B,t} \to H^1(\mathcal{X}_t,T_{\mathcal{X}_t})$ and of the contraction-cup-product map:*

$$H^1(\mathcal{X}_t,T_{\mathcal{X}_t}) \to \mathrm{Hom}\,(H^{i-k}(\mathcal{X}_t,\Omega^k_{\mathcal{X}_t}), H^{i-k+1}(\mathcal{X}_t,\Omega^{k-1}_{\mathcal{X}_t})).$$

2.3. Algebraic de Rham cohomology

Let us now see π as a morphism of algebraic varieties. We will then denote by π^{alg} the map π seen as a morphism of algebraic varieties endowed with the Zariski topology. We can consider the sheaf $\Omega_{\mathcal{X}/B,alg}$ of Kähler differentials, which is a locally free coherent sheaf on $\mathcal{X}$, and more generally the *relative algebraic de Rham complex* $\Omega^\bullet_{\mathcal{X}/B,alg}$, endowed with the natural differential $d_{\mathcal{X}/B}$. Here and as before

$\Omega^i_{\mathcal{X}/B,alg} = \bigwedge^i \Omega_{\mathcal{X}/B,alg}$. Using local parameters, it is obvious that the natural map

$$\Omega^{an}_{\mathcal{X}/B,alg} \to \Omega_{\mathcal{X}/B,hol}$$

is an isomorphism, where in the left hand side, the superscript "an" stands for the corresponding analytic coherent sheaf.

An immediate application of Serre's GAGA principle in the relative setting shows the following (see Grothendieck [19]):

Theorem 2.4. *One has a canonical isomorphism of analytic coherent sheaves on B:*

$$F^k\mathcal{H}^i = \mathbb{R}^i\pi_*\Omega^{\bullet\geq k}_{\mathcal{X}/B,hol} = (\mathbb{R}^i\pi^{alg}_*\Omega^{\bullet\geq k}_{\mathcal{X}/B,alg})^{an}. \tag{2.5}$$

The importance of this remark in our setting is first of all the fact that the total space of the locally free sheaf $F^k\mathcal{H}^i$ (a holomorphic vector bundle that we will denote by F^kH^i) is in fact an algebraic vector bundle on B, hence in particular an algebraic variety. If B was projective, this would follow from the GAGA principle [31] on B, but it is rarely the case that we have a smooth projective fibration defined over a projective basis B. In most cases, B is only quasi-projective. From a more arithmetic point of view, observe that if everything, namely $\mathcal{X}$, B and π^{alg}, is defined over a field K, so is the algebraic relative de Rham complex $\Omega^\bullet_{\mathcal{X}/B,alg}$ and its naïve filtration. It thus follows that the algebraic vector bundle F^kH^i is in fact an algebraic variety defined over K. If $K = \mathbb{Q}$, the field automorphisms of $\mathbb{C}$ act on the set of complex points of this algebraic variety. On the other hand, (2.5) together with the description given in (2.1) of the fibers of $F^k\mathcal{H}^i$ (or equivalently F^kH^i) shows that complex points of this algebraic variety are exactly pairs (t, α) where $t \in B(\mathbb{C})$, and $\alpha \in F^kH^i(\mathcal{X}_t, \mathbb{C}) = \oplus_{p\geq k}H^{p,i-p}(\mathcal{X}_t)$.

The action of $\mathrm{Aut}\,\mathbb{C}$ on the set of such pairs will allow us to define absolute Hodge classes (in the de Rham sense) in section 5.

To conclude, let us explain the Katz-Oda construction of the Gauss-Manin connection, which will make obvious that it is in fact algebraic and defined over the definition field K. Consider the exact sequence of holomorphic differentials on $\mathcal{X}$:

$$0 \to \pi^*\Omega_{B,hol} \to \Omega_{\mathcal{X},hol} \to \Omega_{\mathcal{X}/B,hol} \to 0.$$

This exact sequence induces a filtration $L^\bullet$ on the holomorphic de Rham complex of $\mathcal{X}$ (where L stands for "Leray", as this filtration is directly related to the Leray filtration, see [40, II, 4.1.3]):

$$L^r\Omega^i_{\mathcal{X},hol} := \pi^*\Omega^r_{B,hol} \wedge \Omega^{i-r}_{\mathcal{X},hol}.$$

We have then an exact sequence of complexes on $\mathcal{X}$:

$$0 \to \pi^*\Omega_{B,hol} \otimes \Omega^{\bullet-1}_{\mathcal{X}/B,hol} \to \Omega^\bullet_{\mathcal{X},hol}/L^2\Omega^\bullet_{\mathcal{X},hol} \to \Omega^\bullet_{\mathcal{X}/B,hol} \to 0.$$

Applying $\mathbb{R}\pi_*$ to this exact sequence, and using the fact that

$$\mathbb{R}^i\pi_*\Omega^\bullet_{\mathcal{X}/B,hol} = \mathcal{H}^i,\ \mathbb{R}^{i+1}\pi_*\Omega^{\bullet-1}_{\mathcal{X}/B,hol} = \mathcal{H}^i$$

gives a connecting map (which is only $\mathbb{C}$-linear, due to the fact that the exact sequence above is not an exact sequence of complexes of $\mathcal{O}_B$-modules)

$$\delta : \mathcal{H}^i \to \Omega_{B,hol} \otimes \mathcal{H}^i.$$

The result is then:

Theorem 2.6. *(Katz-Oda, see also* [40, II, 5.1.1]*) The map δ is the connection ∇.*

This formal construction shows that the Gauss-Manin connection is algebraic, as the exact sequence of relative differentials exists as well at the level of algebraic differentials. This exact sequence being defined over the definition field K, we also get that ∇ is defined over K.

3. Hodge loci

3.1. Local structure

In the abstract setting of a variation of Hodge structure of even weight $2k$ on B, described by the data

$$H^{2k}_{\mathbb{Q}},\ \mathcal{H}^{2k} = H^{2k}_{\mathbb{Q}} \otimes \mathcal{O}_B,\ F^i\mathcal{H}^{2k} \subset \mathcal{H}^{2k},$$

satisfying Griffiths transversality

$$\nabla F^i\mathcal{H}^{2k} \subset F^{i-1}\mathcal{H}^{2k} \otimes \Omega_B,$$

let us describe locally the analytic components of the Hodge locus.

We will give two equivalent descriptions: in both cases, we restrict to an Euclidean open set U of B which is a connected and simply connected neighborhood of 0. Then we have a trivialization of $H^{2k}_{\mathbb{Q}}$ on U which allows to identify $\alpha \in H^{2k}(\mathcal{X}_0, \mathbb{Q})$ to a section $\tilde{\alpha} \in H^{2k}_{\mathbb{Q}}$.

Using Remark 1.3, the Hodge locus in U is then the countable union of the U_α's, taken over the set of $\alpha \in H^{2k}(\mathcal{X}_0, \mathbb{Q})$ such that $U_\alpha \neq U$, where

$$U_\alpha = \{t \in U,\ \tilde{\alpha}_t \in F^k\mathcal{H}^{2k}_t\}.$$

Lemma 3.1. *(cf.* [41]*,* [40, II, 5.3.1]*) Each $U_\alpha \subset U$ is a closed analytic subset of U, which can be defined locally schematically by at most $h^{k-1,k+1}$ holomorphic equations, where $h^{k-1,k+1} := \operatorname{rank} F^{k-1}\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$.*

Proof. U_α is defined by the annulation of the projection in the quotient vector bundle $\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$ of the flat, hence holomorphic, section $\tilde{\alpha} \in \mathcal{H}^{2k}$. This proves the first statement and also provides a natural analytic schematic structure for U_α. The second statement is a consequence of transversality. Choose a holomorphic splitting of the quotient $\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$ above as

$$\mathcal{H}^{2k}/F^k\mathcal{H}^{2k} = \mathcal{H}^{k-1,k+1} \oplus \mathcal{F},\ \mathcal{F} \cong \mathcal{H}^{2k}/F^{k-1}\mathcal{H}^{2k},$$

where $\mathcal{H}^{k-1,k+1} := F^{k-1}\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$. Let $U'_\alpha \subset U$ be defined by the vanishing of the projection of $\tilde{\alpha}$ to the first summand. Clearly $U_\alpha \subset U'_\alpha$. The claim is that this inclusion is a scheme-theoretic isomorphism along U_α. At first order, this is exactly the content of the transversality property, which implies that along the Hodge locus U_α, the differentials of the equations

$$\{\tilde{\alpha} = 0 \ mod.\ F^k\mathcal{H}^{2k}\}$$

and

$$\{\tilde{\alpha} = 0 \ mod.\ F^k\mathcal{H}^{2k} \oplus \mathcal{F}\}$$

are the same. The general study is done by finite order expansion (cf. [40, II, 5.3.1]).

■

The following gives an explicit description of the differential of the above local equations. We refer to [40, II, 5.3.2] for the proof, which follows in a straightforward way from the definition of the map ${}^t\overline{\nabla}$.

Proposition 3.2. *The Zariski tangent space to U_α at t is described as*

$$T_{U_\alpha,t} = \mathrm{Ker}\,({}^t\overline{\nabla}\alpha^{k,k} : T_{U,t} \to \mathcal{H}_t^{k-1,k+1}).$$

Here $t \in U_\alpha$ so that $\alpha \in F^k\mathcal{H}_t^{2k}$, and $\alpha^{k,k}$ is the projection of α to $\mathcal{H}_t^{k,k}$. The map ${}^t\overline{\nabla}$ has been introduced in section 2.2. Lemma 3.1 and Proposition 3.2 imply:

Corollary 3.3. *The analytic subset U_α is smooth of codimension $h^{k-1,k+1}$ at a point t where the map ${}^t\overline{\nabla}\alpha^{k,k}$ is surjective.*

The second description we are going to give now is closer in spirit to the point of view we will adopt in section 3.3. We observe that each U_α has a natural lift to the total space F^kH^{2k} of the bundle $F^k\mathcal{H}^{2k}$. Indeed, by definition, along U_α, the section $\tilde{\alpha}$ of $\mathcal{H}^{2k}$ takes values in $F^k\mathcal{H}^{2k}$ and this gives the desired lift. Let us now describe in another way the corresponding analytic subset V_α of F^kH^{2k}. Using the trivialization of the bundle $\mathcal{H}^{2k}$ on U, we get a holomorphic map

$$\Phi : F^kH^{2k} \to H^{2k}(\mathcal{X}_0, \mathbb{C}),$$

which sends a element $\gamma_t \in F^kH_t^{2k} \subset H_t^{2k}$ to its parallel transport to 0. Thinking a little, we get the following:

Lemma 3.4. *The analytic subset V_α identifies schematically to the fiber $\Phi^{-1}(\alpha)$.*

This lemma gives another way of considering the Hodge locus: it is the projection to U of $\Phi^{-1}(H^{2k}(\mathcal{X}_0,\mathbb{Q}))$ or, rather, of the union of components of $\Phi^{-1}(H^{2k}(\mathcal{X}_0,\mathbb{Q}))$ which do not dominate U.

3.2. Weight 2: a density criterion

Let us use Lemma 3.4 to give a simple criterion, due to M. Green, for the topological density of the Hodge locus in weight 2 (it is then usually called the Noether-Lefschetz locus by reference to the Noether-Lefschetz theorem). We assume below that there is no locally constant rational class which is of type $(1,1)$ everywhere, which we can always do by splitting off a locally trivial subvariation of Hodge substructure purely of type $(1,1)$. We will put everywhere a subscript *var* (for "varying part") to indicate that we made this operation.

Theorem 3.5. *Assume B is connected and there exist $t\in B$, and $\lambda\in\mathcal{H}^{1,1}_{var,t}$ such that*

$$^t\overline{\nabla}\lambda : T_{B,t}\to\mathcal{H}^{0,2}_t$$

is surjective. Then the Hodge locus is dense (for the Euclidean topology) in B.

Here, the map $^t\overline{\nabla}$ has been introduced in section 2.2.

Theorem 3.5 (or rather a variant of it also based on Lemma 3.6 below) has various applications to the existence of 1-cycles in threefolds. For example, it is used in [37] to prove that a very general deformation of a non rigid Calabi-Yau threefolds has a non finitely generated Griffiths group, a result which was proved by Clemens for quintic threefolds [9]. It is used in [36] to show that integral Hodge classes of degree 4 on uniruled or Calabi-Yau threefolds are algebraic, a result which is not true for many smooth projective threefolds by [22].

Proof of Theorem 3.5. The condition is Zariski open on $t\in B$ and also on $\lambda\in\mathcal{H}^{1,1}_{var,t}$. Thus it is satisfied over a Zariski open set of B. As a non empty Zariski open set is dense for the classical topology, it suffices to show density near a point t_0 satisfying the condition above. Next observe that by Hodge symmetry, the complex vector spaces $\mathcal{H}^{1,1}_{var,t}$, $t\in B$ have a real structure,

$$\mathcal{H}^{1,1}_{var,t}=\mathcal{H}^{1,1}_{var,t,\mathbb{R}}\otimes\mathbb{C},$$

as they can be identified to $F^1\mathcal{H}^2_{var,t}\cap\overline{F^1\mathcal{H}^2_{var,t}}$ so that complex conjugation acts on them. So if the condition is satisfied for one $\lambda\in F^1\mathcal{H}^2_{var,t_0}$, it is satisfied as well for one $\lambda\in\mathcal{H}^{1,1}_{var,t_0,\mathbb{R}}$. Lifting $\mathcal{H}^{1,1}_{var,t_0}$ to $F^1\mathcal{H}^2_{var,t_0}\subset\mathcal{H}^2_{var,t_0}$ via the Hodge decomposition, λ is then lifted to a real element $\tilde{\lambda}\in\mathcal{H}^2_{var,t_0,\mathbb{R}}\cap F^1\mathcal{H}^2_{var,t_0}$. The following lemma 3.6, for which we refer to [40, II, 5.3.4], tells then that the map Φ is of maximal rank at $\tilde{\lambda}$.

Lemma 3.6. *The map $\Phi : F^1H^2_{var} \to H^2(\mathcal{X}_{var,t_0}, \mathbb{C})$ is a submersion at $\tilde{\lambda}$ if and only if the map ${}^t\overline{\nabla}\lambda : T_{B,t_0} \to \mathcal{H}^{0,2}_{t_0}$ is surjective, where λ is the projection of $\tilde{\lambda}$ in $\mathcal{H}^{1,1}_{var,t_0}$.*

We then conclude as follows: Under the assumptions of the theorem, we know that Φ is a submersion at $\tilde{\lambda}$. Hence Φ is open near $\tilde{\lambda}$, and so is the restricted map:

$$\Phi_{\mathbb{R}} : \Phi^{-1}(H^2(\mathcal{X}_{t_0}, \mathbb{R})_{var}) = F^1H^2_{var} \cap H^2_{var,\mathbb{R}} \to H^2(\mathcal{X}_{t_0}, \mathbb{R})_{var}.$$

The left hand side is the total space of the real analytic vector bundle $H^{1,1}_{var,\mathbb{R}}$ with fiber $\mathcal{H}^{1,1}_{var,t',\mathbb{R}}$ over $t' \in B$. The right hand side contains $H^2(\mathcal{X}_{t_0}, \mathbb{Q})_{var}$ as a topologically dense subset. Hence, as $\Phi_{\mathbb{R}}$ is open, $\Phi_{\mathbb{R}}^{-1}(H^2(\mathcal{X}_{t_0}, \mathbb{Q})_{var})$ is topologically dense in $H^{1,1}_{var,\mathbb{R}}$ near $\tilde{\lambda}$. But the projection to U of $\Phi_{\mathbb{R}}^{-1}(H^2(\mathcal{X}_{t_0}, \mathbb{Q})_{var})$ is nothing but the Hodge locus, as shown by Lemma 3.4, and because by definition of the "varying part", no component of $\Phi_{\mathbb{R}}^{-1}(H^2(\mathcal{X}_{t_0}, \mathbb{Q})_{var})$ dominates U. ∎

3.3. Locus of Hodge classes

None of the local descriptions given in section 3.1 can be made global, due to the fact that they use a local trivialization of the bundle $\mathcal{H}^{2k}$, which is highly transcendental. It is not even clear from these descriptions that the Hodge locus is a countable union of closed analytic subsets.

In [8], Cattani, Deligne and Kaplan consider rather the *locus of Hodge classes*, a locus that will lead to a better definition of the Hodge locus and its components, and a local version of which has been already considered in the previous section.

Definition 3.7. *Given a variation of Hodge structure of even weight $2k$ over a basis B, the locus of Hodge classes is the subset of F^kH^{2k} consisting of pairs (t, α_t), $t \in B$, $\alpha_t \in F^k\mathcal{H}^{2k}_t$, such that $\alpha_t \in H^{2k}_{t,\mathbb{Q}}$.*

This definition was already implicit in our second description of the Hodge locus in previous section. In fact that description gave as well a local description of the locus of Hodge classes. Later on, for arithmetic purposes, we will be led to slightly change this definition, and to consider rather $(2\iota\pi)^k$ times the above locus. The reason is simply the fact (cf. Theorem 5.2) that the algebraic cycle class, which takes value in algebraic de Rham cohomology and is defined over the definition field of the variety and the cycle, is equal to $(2\iota\pi)^k$ times the topological cycle class (which provides rational Betti cohomology classes).

The components of the Hodge locus can then be globally defined as the projections to B of the connected components of the locus of Hodge classes. Note that, even with the local description given in Lemma 3.4, it is still not clear that these components have a good global structure. One of our main goals in section 7 will be to sketch the proof of the following result, (where the variation of Hodge structure is assumed to come from a smooth projective family over a quasiprojective basis B):

Theorem 3.8. *(see [8]) The connected components of the locus of Hodge classes are closed algebraic subsets of F^kH^{2k}. In particular, the components of the Hodge locus are algebraic.*

This theorem makes sense using the algebraic structure of the bundle F^kH^{2k} described in section 2.3.

4. Monodromy

4.1. Monodromy and Hodge classes

We come back to the general setting of a variation of Hodge structure of weight $2k$ coming from a smooth projective family $\pi : \mathcal{X} \to B$, where B is connected. For any point $t \in B$, there is the monodromy representation

$$\rho : \pi_1(B, t) \to \mathrm{GL}\, H^{2k}(\mathcal{X}_t, \mathbb{Q})$$

associated to the local system $R^{2k}\pi_*\mathbb{Q}$. Let $t \in B$ be a *very general* point in the analytic sense (that means that locally on B, t can be taken away from a countable union of closed analytic subsets). Consider the subspace $Hdg^{2k}(\mathcal{X}_t) \subset H^{2k}(\mathcal{X}_t, \mathbb{Q})$.

Theorem 4.1. *The monodromy group* $\mathrm{Im}\, \rho$ *acting on* $H^{2k}(\mathcal{X}_t, \mathbb{Q})$ *leaves stable* $Hdg^{2k}(\mathcal{X}_t)$ *and acts via a finite group on* $Hdg^{2k}(\mathcal{X}_t)$.

Proof. The fact that $Hdg^{2k}(\mathcal{X}_t)$ is stable under monodromy is a consequence of analytic continuation. Indeed, consider a simply connected open set $U \subset B$ and take t in U to be away from the Hodge locus (see section 3.1). By definition then, any rational cohomology class $\alpha \in H^{2k}(\mathcal{X}_t, \mathbb{Q})$ which is Hodge (that is of type (k, k)) at t is Hodge everywhere on U. But then, extending α by parallel transport along real analytic paths in B, it is Hodge everywhere on B by analytic continuation along these paths.

Let us show that the monodromy action is finite on $Hdg^{2k}(\mathcal{X}_t, \mathbb{Q})$. We first of all remark that everything has an integral structure which is preserved by the monodromy action. Namely we could have worked with the local system of integral cohomology modulo torsion instead of rational cohomology. Next we show that there is a positive definite monodromy invariant rational intersection form on $Hdg^{2k}(\mathcal{X}_t, \mathbb{Q})$ (which of course can be made integral by scaling). This clearly implies the claim since then the monodromy group acts via the finite orthogonal group of a lattice endowed with a definite intersection form.

The intersection form is simply obtained by choosing a relatively ample line bundle H on $\mathcal{X}$. The class $h = c_1(H)$ of this line bundle provides for each fiber $\mathcal{X}_t$ a monodromy invariant Lefschetz decomposition of $H^{2k}(\mathcal{X}_t, \mathbb{Q})$, and as h is a Hodge class, the Lefschetz isomorphisms are isomorphisms of Hodge structures, hence preserve the set of Hodge classes in all degrees, which implies that there is an induced Lefschetz decomposition on the subspace (which is monodromy invariant for very general t) $Hdg^{2k}(\mathcal{X}_t)$. Assume first that $k \leq n = \dim \mathcal{X}_t$ and write this decomposition as

$$Hdg^{2k}(\mathcal{X}_t) = \oplus_{r\leq k} h^{k-r} Hdg^{2k-2r}(\mathcal{X}_t)_{prim}. \tag{4.2}$$

We finally have the rational monodromy invariant intersection pairing on $H^{2k}(\mathcal{X}_t, \mathbb{Q})$ given by

$$(\alpha, \beta) = \int_{\mathcal{X}_t} h^{n-k} \cup \alpha \cup \beta.$$

This pairing is not positive definite on $Hdg^{2k}(\mathcal{X}_t)$, but the monodromy invariant Lefschetz decomposition (4.2) is orthogonal for this pairing, and furthermore this pairing is of a definite sign on each term $h^{k-r}Hdg^{2k-2r}(\mathcal{X}_t)_{prim}$ (cf. [40, I, 6.3.2]). Modifying the pairing above by changing its sign where needed on certain pieces of the decomposition (4.2) gives the desired positive intersection pairing.

The case $k \geq n$ follows then from the case $k \leq n$ and from the Lefschetz isomorphism

$$h^{2k-n} : H^{2n-2k}(\mathcal{X}_t, \mathbb{Q}) \cong H^{2k}(\mathcal{X}_t, \mathbb{Q}).$$

∎

Remark 4.3. Theorem 4.1 is coherent with the Hodge conjecture. Indeed, monodromy also acts via a finite group on cycle classes at a very general point, by the following standard argument : There are countably many relative Hilbert schemes, which are proper over B and parameterize subschemes in the fibers of π. Thus there are countably many smooth varieties $\phi_i : B_i \to B$ mapping by a proper morphism to B, and parameterizing cycles $Z_i \subset \mathcal{X}_i := \mathcal{X} \times_B \mathcal{X}$, such that any cycle z_t in a fiber $\mathcal{X}_t$ of π is of the form $z_t = Z_{i,b}$ for some point $b \in B_i$ such that $r_i(b) = t$. Assume now that $t \in B$ is very general. Then it follows by definition of "'very general" that for any cycle $z \subset \mathcal{X}_t$, and any (i, b) with $Z_{i,b} = z$, the corresponding morphism $\phi_i : B_i \to B$ is surjective. In other words a cycle z on $\mathcal{X}_t$ with t very general in b is the restriction to $\mathcal{X}_t$ of a relative cycle Z_i defined on a variety B_i, where $\phi_i : B_i \to B$ is proper and surjective. The same is true for the corresponding cycle classes. The relative cycle class map being constant on the connected components of the fibers of ϕ_i, it follows that we have $[z] = [Z'_{i|\mathcal{X}_t}]$, where we replaced B_i by $B'_i \subset B_i$, generically finite over B, and Z'_i by $Z_{i|\mathcal{X}'_i}$, $\mathcal{X}'_i := B'_i \times_B \mathcal{X}$. It follows that the monodromy acting on the class $[z]$ becomes trivial on the generically finite cover B'_i, hence that monodromy acts via finite groups on cycle classes at a very general point of the base.

4.2. Mumford-Tate groups

This is a very important notion in Hodge theory, although this may seem quite abstract. This plays a crucial role in the study of Hodge loci for abelian varieties. We refer to the notes [26] for more detailed material. Let $H_{\mathbb{Q}}$, $H^{p,q}$, $p+q = r$ be a rational Hodge structure of weight r. The Hodge decomposition satisfying Hodge symmetry provides an algebraic action of the group $\mathbb{C}^*$, *seen as real algebraic group*, on $H_{\mathbb{R}}$, given by:

$$\mu(z)\cdot h^{p,q} := z^p\overline{z}^q h^{p,q},\ \forall\, h^{p,q}\in H^{p,q}. \tag{4.4}$$

Definition 4.5. *The Mumford-Tate group $MT(H)$ of the considered Hodge structure H is the smallest algebraic subgroup of* $\mathrm{GL}(H_{\mathbb{R}})$ *which is defined over* $\mathbb{Q}$ *and contains* $\mathrm{Im}\,\mu$. *The special Mumford-Tate group $SMT(H)$ of H is the smallest algebraic subgroup of* $\mathrm{GL}(H_{\mathbb{R}})$ *which is defined over* $\mathbb{Q}$ *and contains* $\mu(\mathbb{S}^1)$.

Remark 4.6. Being defined over $\mathbb{Q}$, the Mumford-Tate groups could be defined as well as subgroups of $\mathrm{GL}\,H_{\mathbb{Q}}$.

Remark 4.7. The interest of the special Mumford-Tate group is that it leaves pointwise invariant Hodge classes (assuming of course $r = 2k$). Indeed, $\mathbb{S}^1 = \{z\in\mathbb{C}^*,\ z\overline{z} = 1\}$ so that $\mu(\mathbb{S}^1)$ acts by Id on $H^{k,k}$.

Remark 4.7 quickly leads in fact to the following characterization of the Mumford-Tate group:

Theorem 4.8. *If H is a polarized Hodge structure, the Mumford-Tate group of H is the subgroup of* $\mathrm{GL}(H_{\mathbb{Q}})$ *fixing up to a scalar finitely many Hodge classes in (even weights) tensor powers $H^{\otimes I}$ of H (where the index set I may involve positive and negative exponents). Similarly, the special Mumford-Tate group of H is the subgroup of* $\mathrm{GL}(H_{\mathbb{Q}})$ *leaving invariant finitely many Hodge classes in (even weights) tensor powers of H.*

Proof. This is a consequence of Theorem 4.9 below. Having this, the general theory of reductive groups tells that there are finitely many tensor powers $H_{\mathbb{Q}}^{\otimes I}$ (where the index set I involves positive and negative numbers, with the convention that $H^{\otimes -1} = H^*$) of $H_{\mathbb{Q}}$ and elements h_I of $H_{\mathbb{Q}}^{\otimes I}$ such that our special Mumford-Tate group is the subgroup of $\mathrm{GL}\,H_{\mathbb{Q}}$ leaving h_I invariant up to a scalar. To conclude we now observe that a rational element h_I as above is invariant under $\mu(\mathbb{S}^1)$ up to a scalar if and only if it is Hodge for the Hodge structure on $H_{\mathbb{Q}}^{\otimes I}$, if and only if it is in fact actually invariant under $\mu(\mathbb{S}^1)$. This immediately gives both statements.

■

Theorem 4.9. *Mumford-Tate groups of polarized Hodge structures are reductive.*

Corollary 4.10. *The Mumford-Tate group of $H^k(X,\mathbb{Q})$, where X is a smooth projective complex variety, is reductive.*

Proof. We use for this the Lefschetz decomposition with respect to some ample divisor class $l = c_1(L)\in H^2(X,\mathbb{Q})$. As it is a decomposition into rational Hodge substructures, the definition of $MT(H^k(X,\mathbb{Q}))$ shows that it is the direct product of the Mumford-Tate groups of each Lefschetz summand. But the intersection form $(a,b) = \int_X l^{n-k} \cup a \cup b$, $n = \dim X$, polarizes (up to sign) each Lefschetz summand by [40, I, 6.3.2]. ■

Proof of Theorem 4.9. We just show the result for the special Mumford-Tate group $SMT(H)$, where H is polarized. As both differ by a factor $\mathbb{G}_m$, this is in fact equivalent. We use the following characterization of a reductive complex algebraic group G: G is reductive, if it admits a real form $G_\mathbb{R}$ for which $G(\mathbb{R})$ is compact. The trick is that we will not use for this the natural real structure given by the given intersection pairing on H and its associated Hermitian intersection pairing, but a twisted form of it. Namely, in $G = SMT(H)$, we have the element $C = \mu(\iota)$, where μ is defined in (4.4). As C^2 is a homothety, conjugation by C is an involution acting on G. We take the involution $g \mapsto C^{-1}\overline{g}C$ as the one defining a real structure on G. Then real elements in G for this real structure are those which satisfy

$$g = C^{-1}\overline{g}C. \tag{4.11}$$

But elements of G leave invariant the rational intersection pairing $<,>$ on H given by the polarization, (because $\mu(\mathbb{S}^1)$ does,) and thus elements of $G(\mathbb{R})$ also leave invariant the Hermitian intersection pairing $(\ ,\ )_C$ on $H_\mathbb{C}$ given by

$$(a,b)_C := < Ca, \overline{b} >$$

because they satisfy equality (4.11). By definition of a polarization (cf. [40, I, 7.1.2]), this pairing is definite. Hence the real part of $G_\mathbb{C}$ for this real structure is compact.

■

We have the following corollary:

Corollary 4.12. *Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. The special Mumford-Tate group of the Hodge structure on $H^r(\mathcal{X}_t, \mathbb{Q})$ at a very general point contains a finite index subgroup of the monodromy group $\mathrm{Im}\,(\pi_1(B,t) \to \mathrm{GL}(H^r(\mathcal{X}_t, \mathbb{Q})))$.*

Proof. Indeed, we know by Theorem 4.8 that the special Mumford-Tate group is characterized as leaving invariant a finite number of Hodge classes h_I in some tensor powers of $H^r(\mathcal{X}_t, \mathbb{Q})$. But t being very general, by Theorem 4.1, the monodromy preserves the spaces of Hodge classes on each of these tensor powers and acts via a finite group on them. Thus a finite index subgroup of the monodromy group leaves invariant all the h_I's and thus is contained in the special Mumford-Tate group. ■

Remark 4.13. In [1], André proves, using Theorem 5.10, the following important refinement of the above result: Let $H_t \subset \mathrm{GL}(H^r(\mathcal{X}_t, \mathbb{C}))$ be the connected component of the Zariski closure of the monodromy group at t. Then H_t is *normal* in the Mumford-Tate group G_t if t is a very general point of B.

Theorem 4.8 tells that Hodge loci for a given variation of Hodge structure have a natural refinement, namely loci with jumping Mumford-Tate groups (which exist as well for variations of Hodge structure of odd weight). These loci are then obtained as the union of Hodge loci for all tensor powers of the considered variation of Hodge structure. A priori, there could be no bound for the multidegree of the considered tensor powers. The following result due to Deligne shows in particular that there is in fact such a bound, for a given variation of Hodge structure:

Theorem 4.14. *(Deligne, [13]) Given a smooth projective family $\pi : \mathcal{X} \to B$, and an integer r, there are only finitely many conjugacy classes of subgroups of $\mathrm{GL}(H^r(\mathcal{X}_t, \mathbb{C}))$ arising as Mumford-Tate groups of Hodge structures on $H^r(\mathcal{X}_{t'}, \mathbb{Q})$, $t' \in B$.*

Proof. Notice that, as one can see by considering the case of $\mathbb{C}^*$ actions on a given complex vector space, it is not true that there are in general only finitely many conjugacy classes of complex reductive subgroups of a given complex reductive group. However, Mumford-Tate groups are special, due to their definition starting from a given representation

$$\mu_{t'} : \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{R}})$$

of the real algebraic group $\mathbb{C}^*$. Here, $H \cong H^r(\mathcal{X}_{t'}, \mathbb{Q})$ for any $t' \in B$, the isomorphism depending on a choice of path from t to t', so that $\mu_{t'}$ is defined only up to conjugacy by monodromy. Instead of $\mu_{t'}$, consider the algebraic action of the complex algebraic group $\mathbb{C}^*$ on $H_{\mathbb{C}}$ given by

$$\mu'_{t'}(z) \cdot h^{p,q} = z^{-p},\ \forall h^{p,q} \in H^{p,q}(\mathcal{X}_{t'}) \subset H^r(\mathcal{X}_{t'}, \mathbb{C}).$$

Then $MT(t')$ is also the smallest algebraic subgroup of $\mathrm{GL}(H)$ containing $\mathrm{Im}\,\mu'_{t'}$. Another convenient way to describe it is then as follows: Any field automorphism $\tau \in \mathrm{Aut}\,\mathbb{C}$ acts on $H_{\mathbb{C}}$, and thus on the 1-parameter subgroups of $\mathrm{GL}(H_{\mathbb{C}})$. As $SMT(t')$ is defined over $\mathbb{Q}$ and contains $\mathrm{Im}\,\mu'_{t'}$, it has to contain $\mathrm{Im}\,\mu'_{t',\tau}$ as well. Consider the subgroup $G' \subset \mathrm{GL}(H_{\mathbb{C}})$ generated by the 1-parameter subgroups $\mathrm{Im}\,\mu'_{t',\tau}$, for $\tau \in \mathrm{Aut}\,\mathbb{C}$. Then G' is defined over $\mathbb{Q}$ and contains $\mathrm{Im}\,\mu'_{t'}$, hence must contain $MT(t')$. It has thus to be equal to $MT(t')$ as we already noticed that it is contained in $MT(t')$.

Finally, notice that the 1-parameter subgroups $\mathrm{Im}\,\mu'_{t',\tau}$ belong to a finite set, independent of t', of conjugacy classes of 1-parameter subgroups of $\mathrm{GL}(H_{\mathbb{C}})$. Indeed, only the characters $z \mapsto z^{-p}$, $0 \leq p \leq r$ can appear non trivially.

The theorem is then a consequence of the following more general result concerning finiteness of conjugacy classes of reductive subgroups $G' \subset G$ over $\mathbb{C}$ satisfying an extra condition as in next Proposition. ■

Proposition 4.15. *Let G be a reductive group over $\mathbb{C}$. Fix a maximal torus $T \subset G$ and choose a finite set M of 1-parameter subgroups $\mu : \mathbb{C}^* \to T$. Then there are only finitely many reductive subgroups H of G which are generated by images of morphisms $\nu : \mathbb{C}^* \to G$ conjugate under G to some $\mu \in M$.*

Proof. (Taken from [13]) One first proves that G contains finitely many conjugacy classes of semi-simple subgroups S. Indeed, there are only finitely many isomorphisms types of complex semi-simple groups of given dimension and each of them has only finitely many isomorphism classes of representations of given dimension. Now consider G as embedded in some $\mathrm{GL}(V)$. Then if we fix the isomorphism type of S, the set of embeddings $S \hookrightarrow G$ forms a limited family, because it is closed algebraic in the set of embeddings in $\mathrm{GL}(V)$. On the other hand, the embeddings of S into G are rigid up to conjugacy by G, which proves finiteness.

The derived group $H_1 = [H, H]$ of H is semi-simple hence there are only finitely conjugacy classes of such subgroups. Now assume H_1 is given. H_1 is contained in the connected component H_2 of its normalizer in G. Let T_2 be a maximal torus in H_2 and let $H_3 = H_1 T_2$. We have $H_1 \subset H$ and may assume $H \subset H_3$. Then H is determined by its image in the quotient H_3/H_1, which is a subtorus T_H in the torus H_3/H_1. But under our assumptions, there are only finitely many choices for such subtorus. Indeed, it has to be generated by one-parameter subgroups $\nu : \mathbb{C}^* \to H_3/H_1$ which are the images of one-parameter subgroups of H_3 conjugate under G to one of the finitely many one-parameter subgroups of T belonging to M. There are only finitely many such ν's. ■

4.3. Noether-Lefschetz type theorems

Noether-Lefschetz type theorems concern particular variations of Hodge structures obtained as follows: Start from a smooth projective variety Y of dimension $n+1$, $n = 2k$, and choose a sufficiently ample line bundle H on Y (the result will depend on making precise "sufficiently ample"). Consider the linear system $|H|$ and let $B \subset |H|$ be the Zariski open set parameterizing smooth hypersurfaces $X_f = \{f = 0\} \subset Y$. Let

$$\mathcal{X} := \{(y, f) \in Y \times B,\, y \in X_f\},$$

with projection $\pi = pr_2 : \mathcal{X} \to B$. For $t \in B$, let $j_t : \mathcal{X}_t \hookrightarrow Y$ be the inclusion. Lefschetz theorem on hyperplane sections tells that the restriction maps $j_t^* : H^i(Y, \mathbb{Z}) \to H^i(\mathcal{X}_t, \mathbb{Z})$ are isomorphisms for $i \leq n-1$ and are injective with torsion free cokernel for $i = n$. These maps are morphisms of Hodge structures. We will be interested in the variation of Hodge structure on the cohomology groups $H^n(\mathcal{X}_t, \mathbb{Q})_{van}$, $t \in B$, which are defined as the cokernel of the restriction map above, or as the orthogonal complement of its image with respect to the intersection pairing on $\mathcal{X}_t$. The classical Noether-Lefschetz theorem concerns the case where $Y = \mathbb{P}^3$.

Theorem 4.16. *A very general surface S of degree $d \geq 4$ in $\mathbb{P}^3$ has no non zero vanishing Hodge class. Hence its Picard group is cyclic, generated by $c_1(\mathcal{O}_S(1))$.*

In general, let us assume $n = 2k$. The same proof will give as well:

Theorem 4.17. *Assume H is sufficiently ample. Then for a very general $t \in B$, there are no non zero Hodge classes in $H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$.*

Sketch of proof. There are two types of proofs for such statements. The first type is a direct extension of the Lefschetz proof of Theorem 4.16. The only ampleness condition on H is then the following:

a) H is very ample and there should be at least one hypersurface $\mathcal{X}_t \subset Y$, $t \in |H|$, with only ordinary double points as singularities.

b) The Hodge number $h^{k-1,k+1}$ for the Hodge structure on $H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$ is non zero.

That b) is satisfied for H ample enough follows from Griffiths' description of the Hodge structure on the vanishing cohomology of an hypersurface (see [17] and [40, II, 6.1.2]). In fact, an assumption weaker than b) is necessary for the Lefschetz proof, but b) is needed for the infinitesimal proof and also for Terasoma's Theorem in next section.

The second proof is infinitesimal and needs a Macaulay type statement for the infinitesimal variation of Hodge structure on $H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$ (see [40, II, 6.2.2]). We will focus on the first proof. The argument is then as follows: as t is very general, we know by Theorem 4.1 that the monodromy group $\operatorname{Im}\rho$ acting on $H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$ leaves stable $Hdg^{2k}(\mathcal{X}_t)_{van}$. Assumption b) makes sure that $Hdg^{2k}(\mathcal{X}_t)_{van} \subset H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$ is a proper subspace. Then we conclude using the fact that under assumption a), the monodromy action on $H^{2k}(\mathcal{X}_t, \mathbb{Q})_{van}$ is irreducible (cf. [40, II, 3.2.3]). This follows from Picard-Lefschetz formula and from the irreducibility of the discriminant hypersurface which makes all the vanishing cycles conjugate (up to sign) under monodromy (cf. [40, II, 3.2.2]). Thus the proper globally invariant subspace

$$Hdg^{2k}(\mathcal{X}_t)_{van} \subset H^{2k}(\mathcal{X}_t)_{van}$$

has to be 0. ∎

Remark 4.18. We have not been completely careful in stating Theorem 4.17. Indeed, it is clear that the result holds away from the Hodge locus, and we still do not know that the Hodge locus is a countable union of closed algebraic subsets. This will be proved later on.

4.4. Monodromy versus Galois; Terasoma theorem

We consider the same situation as before, but assume (Y, H) is defined over a number field K. There are then only countably many points of B defined over $\overline{\mathbb{Q}}$, and Theorem 4.17 a priori does not apply to them, since it concerns very general points in B. In this section, we would like to explain the following beautiful result due Terasoma [35], which proves, unconditionally for $n = 2$ and under some extra hypothesis which we explain below and would be implied by the Hodge conjecture for the considered hypersurfaces, that there are points defined over $\overline{\mathbb{Q}}$ and not contained in the Hodge locus:

Theorem 4.19. *(Terasoma) Assume that for smooth hypersurfaces defined over a number field in Y, Hodge classes are Hodge-Tate classes. Under the same assumption as in Theorem 4.17, there exists a point (in fact many) $t \in B(\overline{\mathbb{Q}})$ such that the conclusion of Theorem 4.17 holds for $\mathcal{X}_t$.*

Before sketching the proof, let us explain the assumption. In this section, to avoid confusion, we will denote for any complex point $t \in B$ the Betti cohomology of the corresponding fiber $\mathcal{X}_t$ by $H^i_B(\mathcal{X}_t)$. Assume t is defined over a subfield $K(t) \subset \mathbb{C}$, we will also have the étale cohomology groups $H^i_{et}(\mathcal{X}_{\bar{t}})$ with value in adequate coefficients, where $\mathcal{X}_{\bar{t}}$ is the corresponding closed geometric fiber defined over $\overline{K(t)} \subset \mathbb{C}$.

If X is a smooth projective variety over a finitely generated subfield K of $\mathbb{C}$ (say a number field), a degree $2k$ *Tate class* on X is a cohomology class in $H^{2k}_{et}(X_{\overline{K}}, \mathbb{Q}_l(k))$ whose orbit under $\mathrm{Gal}(\overline{K}/K)$ is finite. Given a codimension-k cycle Z on $X_{\overline{K}}$, its étale cohomology class is a Tate class, because Z can be defined over a finite extension of K. The Tate conjecture asserts the converse, that Tate classes are classes of algebraic cycles.

A degree $2k$ *Hodge-Tate class* is a Hodge class whose image under the comparison isomorphism $H^{2k}_B(X, \mathbb{Q}) \otimes \mathbb{Q}_l \cong H^{2k}_{et}(X_{\overline{K}}, \mathbb{Q}_l(k))$ is a Tate class.

If the Hodge conjecture is true, then Hodge classes are Hodge-Tate. In particular, if $k = 1$, where the Hodge conjecture is known as the Lefschetz theorem on $(1,1)$-classes, Hodge classes are Hodge-Tate.

Theorem 4.19 is an immediate consequence of the fact that Hodge classes on hypersurfaces $\mathcal{X}_t \subset Y$, $t \in B(\overline{\mathbb{Q}})$, are supposed to be Tate, and of the following theorem :

Theorem 4.20. *With the same assumptions as in Theorem 4.17, for many points $\bar{t} \in B(\overline{\mathbb{Q}})$, any Hodge-Tate class in $H^{2k}_{et}(\mathcal{X}_{\bar{b}}, \mathbb{Q}_l(k))$ comes from the image of the restriction map $j^*_{\bar{t}}$ in étale cohomology.*

Proof. The proof is obtained by combining a similar statement with Galois group replaced by monodromy group (Lemma 4.21) and a comparison lemma between Galois groups of special fiber and l-adic monodromy groups (Lemma 4.22).

First of all, the profinite completion of the fundamental group $\widehat{\pi_1(B,b)}$ acts on $H^{2k}_B(\mathcal{X}_b,\mathbb{Q}_l)_{van}$, this action, which we denote by ρ_l, being induced by the classical monodromy action.

Lemma 4.21. *Under the same assumptions as in Theorem 4.17, and assuming both Hodge numbers $h^{k,k}$ and $h^{k-1,k+1}$ for the variation of Hodge structure on $H^{2k}_B(\mathcal{X}_t,\mathbb{Q})_{van}$ are non zero, the l-adic monodromy group*

$$\mathrm{Im}\,(\rho_l:\widehat{\pi_1(B,b)}\to \mathrm{Aut}\,H^{2k}_B(\mathcal{X}_b,\mathbb{Q}_l)_{van})$$

acts with no non zero finit orbit.

Proof. It clearly suffices to prove the same result for the usual monodromy group

$$\mathrm{Im}\,\pi_1(B^{an},b)\to H^{2k}(\mathcal{X}_b^{an},\mathbb{Q})_{van}.$$

Observe now that the set of classes $\alpha\in H^{2k}(\mathcal{X}_b^{an},\mathbb{Q})_{van}$ with finite orbit under $\pi_1(B^{an},b)$ is a $\mathbb{Q}$-vector subspace of $H^{2k}(\mathcal{X}_b^{an},\mathbb{Q})_{van}$ stable under $\pi_1(B^{an},b)$. By irreducibility of the monodromy action, one concludes that if there is a non zero finite orbit, the monodromy action is finite on $H^{2k}(\mathcal{X}_b^{an},\mathbb{Q})_{van}$. But the monodromy group is generated by Picard-Lefschetz reflections with respect to (integral) vanishing classes, and furthermore it acts irreducibly on the set of vanishing classes, which generate $H^{2k}_B(\mathcal{X}_b,\mathbb{Q})_{van}$ (cf. [40, II, 2.3.3]). One uses then a classical result in group theory, which says that this is possible only when the intersection pairing on the lattice $H^{2k}_B(\mathcal{X}_b,\mathbb{Z})_{van}$ is definite.

By the second Hodge-Riemann bilinear relations, using the fact that $H^{2k}_B(\mathcal{X}_b,\mathbb{Q})_{van}$ is contained in primitive cohomology of $\mathcal{X}_b$, the intersection pairing on $H^{2k}_B(\mathcal{X}_b,\mathbb{Q})_{van}$ cannot be of a definite sign if both Hodge numbers $h^{k,k}$ and $h^{k-1,k+1}$ for the Hodge structure on $H^{2k}_B(\mathcal{X}_b,\mathbb{Q})_{van}$ are non zero. This leads to a contradiction which concludes the proof. ■

The next step is to pass from the l-adic monodromy to the Galois group $\mathrm{Gal}\,(\overline{K(b)}/K(b))$, where b is supposed to be defined over a finite extension of K.

To do this, we first go to the generic point and introduce much bigger groups:

- The Galois group $\mathrm{Gal}\,(\overline{K(B)}/K(B))$ of the generic geometric point acts naturally on the étale cohomology group $H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k))_{van}$ of the geometric generic fiber of $\mathcal{X}\to B$, where the subscript "*van*" means as before that we consider the cokernel of the natural map

$$j_{\overline{\eta}}^*:H^{2k}_{et}(Y\times_K Spec\,\overline{K(B)},\mathbb{Q}_l(k))\to H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k)).$$

- On the other hand, $\mathrm{Gal}\,(\overline{K(B)}/K(B))$ contains the subgroup $\mathrm{Gal}\,(\overline{\overline{\mathbb{Q}}(B)}/\overline{\mathbb{Q}}(B))$ defined as the kernel of the natural map $\mathrm{Gal}\,(\overline{K(B)}/K(B))\to\mathrm{Gal}\,(\overline{\mathbb{Q}}/K)$. The group $\mathrm{Gal}\,(\overline{\overline{\mathbb{Q}}(B)}/\overline{\mathbb{Q}}(B))$, which can be understood as the group of birational deck transformations of the projective limit of ramified covers of B, admits now as a quotient a group conjugate to $\widehat{\pi_1(B,\overline{b})}$, acting on the projective limit of unramified covers of B. The action of $\mathrm{Gal}\,(\overline{\overline{\mathbb{Q}}(B)}/\overline{\mathbb{Q}}(B))$ on $H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k))_{van}$ factors through this quotient and induces the l-adic monodromy representation on

$$H^{2k}_B(\mathcal{X}_b,\mathbb{Q}_l)_{van}=H^{2k}_{et}(\mathcal{X}_{\overline{b}},\mathbb{Q}_l(k))$$

for an adequately chosen isomorphism

$$H^{2k}_{et}(\mathcal{X}_{\overline{b}},\mathbb{Q}_l(k))_{van}\cong H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k))_{van}.$$

We conclude from this that the image of the Galois representation of $\mathrm{Gal}\,(\overline{K(B)}/K(B))$ on $H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k))_{van}$ contains (via the isomorphisms above) $\mathrm{Im}\,\rho_l$ and in particular still satisfies the conclusion of Lemma 4.21.

Finally, note that for any point $t\in B(\overline{\mathbb{Q}})$, there is also the group $\mathrm{Gal}\,(\overline{K(t)}/K(t))$ acting on $H^{2k}_{et}(\mathcal{X}_{\overline{t}},\mathbb{Q}_l(k))$. The last step which concludes the proof of Theorem 4.20 is Lemma 4.22 due to Terasoma, which compares the action of the Galois group of the generic geometric point on étale cohomology of the geometric generic fiber of $\mathcal{X}\to B$ and the action of the Galois group of the closed geometric point $\overline{t}$ on étale cohomology of the closed geometric fiber $\mathcal{X}_{\overline{t}}$.

Lemma 4.22. *(see [35]) For many points $t\in B(\overline{\mathbb{Q}})$, the image of the Galois representation*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K(t))\to \mathrm{Aut}\,H^{2k}_{et}(\mathcal{X}_{\overline{t}},\mathbb{Q}_l(k))$$

contains the image of the Galois representation

$$\mathrm{Gal}(\overline{K(B)}/K(B))\to \mathrm{Aut}\,H^{2k}_{et}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_l(k)).$$

■

Remark 4.23. Terasoma's result concerns the particular situation of a family of hypersurfaces or complete intersections in a given variety. Using the Deligne global invariant cycles theorem 5.10, one can prove a similar Noether-Lefschetz type result for points defined over a small field in a much more general situation. We refer to [2], section 5, for this.

Remark 4.24. In any case, such statements remain subject to the assumption that Hodge classes in fibers should be Hodge-Tate. This is true in degree 2 (by the Lefschetz theorem on $(1,1)$-classes, and the cases considered by Terasoma are Fano fourfolds, for which the Hodge conjecture is known to hold (cf. [5]). André considers more general Hodge classes for which the Hodge conjecture is not necessarily known, but which are clearly Hodge-Tate (eg those classes appearing in the standard conjectures, cf. [21]).

To conclude this section, I mention the recent work of Maulik and Poonen [23], which provides a new approach to the Noether-Lefschetz locus (for general families $\mathcal{X} \to B$), via crystalline cohomology. They are then able to strengthen Terasoma's type theorems by proving that the Noether-Lefschetz locus is nowhere p-adically dense, under adequate assumptions on p.

5. Absolute Hodge classes

5.1. Algebraic cycle class and absolute Hodge classes

Let X be a smooth projective variety defined over K and $Z \subset X$ be a local complete intersection closed algebraic subset of X, also defined over K. Following Bloch [4], we construct a cycle class

$$[Z]_{alg} \in H^{2k}_{dR}(X/K)$$

which by construction lies in fact in

$$F^k H^{2k}_{dR}(X/K) := \mathrm{Im}\,(\mathbb{H}^{2k}(X, \Omega^{\bullet \geq k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^{\bullet}_{X/K})).$$

Being a local complete intersection, Z can be defined, locally in the Zariski topology, by k equations $f_1, \ldots, f_k$. On a Zariski open set U where these k equations define $Z \cap U$, we have a covering of $U \setminus (Z \cap U)$ by k open sets

$$U_i := \{x \in U,\ f_i(x) \neq 0\}.$$

On the intersection $U_1 \cap \ldots \cap U_k$, the closed degree k algebraic differential form $\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_k}{f_k}$ has no poles hence defines a section of $\Omega^k_{U/K}$ which is obviously a closed form. We can see it as a Čech cocycle on $U \setminus (Z \cap U)$ relative to the open cover above, with value in $\Omega^{k,c}_{U/K}$, where the superscript c stands for "closed". We thus get an element e of $H^{k-1}(U \setminus (Z \cap U), \Omega^{k,c}_{U/K})$. Observe now that there is an obvious map of complexes

$$\Omega^{k,c}_{X/K}[-k] \to \Omega^{\bullet \geq k}_{X/K},$$

where the left hand side is the complex consisting of the sheaf $\Omega^{k,c}_{X/K}$ put in degree k. Applying this morphism to the class e, we thus get a class in

$$\mathbb{H}^{2k-1}(U \setminus (Z \cap U), \Omega^{\bullet \geq k}_{U/K}) \cong \mathbb{H}^{2k}_{Z \cap U}(U, \Omega^{\bullet \geq k}_{U/K}).$$

This class can be shown to be independent of the choice of equations f_i. These locally defined classes thus provide a global section of the sheaf $\mathcal{H}^{2k}_Z(\Omega^{\bullet \geq k}_{X/K})$ of hypercohomology with support on Z. Examining the local to global spectral sequence for $\mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geq k}_{X/K})$, one finds now that it is very degenerate, so that

$$H^0(Z, \mathcal{H}^{2k}_Z(\Omega^{\bullet \geq k}_{X/K})) = \mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geq k}_{X/K})$$

which provides us with a class in $\mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geq k}_{X/K})$. Using the natural map

$$\mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geq k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^{\bullet \geq k}_{X/K}),$$

we finally get the desired cycle class $[Z]_{alg}$.

Let now X be a smooth projective variety defined over $\mathbb{C}$ and $Z \subset X$ be as above. Recall from section 2.3 (applied to the case where B is a point) that there is a natural isomorphism:

$$\mathbb{H}^{2k}(X, \Omega^{\bullet \geq k}_{X/\mathbb{C}}) \cong F^k H^{2k}_B(X, \mathbb{C}), \tag{5.1}$$

where the subscript B on the right denotes Betti cohomology of the corresponding complex manifold. The following comparison result can be verified to hold as a consequence of Cauchy formula or rather multiple residue formula:

Theorem 5.2. *Via the isomorphism (5.1) in degree $2k$, one has*

$$[Z]_{alg} = (2\iota\pi)^k [Z],$$

where $[Z] \in Hdg^{2k}(X)$ is the topological or Betti cycle class of Z.

This leads us to the definition of *absolute cycle classes* (in the de Rham sense), as introduced by Deligne [12]. Let $X \subset \mathbb{P}^N$ be a smooth algebraic variety, defined by equations $P_i = 0$, $i \in I$. Let $\tau \in \operatorname{Aut}\mathbb{C}$ be any field automorphism. Let $X_\tau \subset \mathbb{P}^N$ be the smooth complex algebraic variety defined by the equations $P_i^\tau = 0$, where P_i^τ is deduced from P_i by letting τ act on the coefficients of P_i.

It follows from the definition of algebraic de Rham cohomology that there is a natural τ-linear isomorphism, preserving the Hodge filtrations on both sides:

$$\mathbb{H}^{2k}(X, \Omega^{\bullet}_{X/\mathbb{C}}) \cong \mathbb{H}^{2k}(X_\tau, \Omega^{\bullet}_{X_\tau/\mathbb{C}}). \tag{5.3}$$

Let now $\alpha \in Hdg^{2k}(X)$ and consider

$$\alpha' := (2\iota\pi)^k \alpha \in F^k H^{2k}_B(X, \mathbb{C}) \cong \mathbb{H}^{2k}(X, \Omega^{\bullet \geq k}_{X/\mathbb{C}}).$$

Then by the isomorphism above, we get a class

$$\alpha'_\tau \in \mathbb{H}^{2k}(X_\tau, \Omega^{\bullet \geq k}_{X_\tau/\mathbb{C}}) \cong F^k H^{2k}_B(X_\tau, \mathbb{C}).$$

Definition 5.4. *The Hodge class α is said to be absolute Hodge if for any τ, the class α'_τ is of the form $(2\iota\pi)^k \beta$ where $\beta \in H^{2k}_B(X_\tau, \mathbb{Q})$.*

Theorem 5.2 immediately implies:

Corollary 5.5. *Cycle classes $[Z]$ are absolute Hodge. Hence, if the Hodge conjecture is true, Hodge classes have to be absolute Hodge.*

Proof. Let $\alpha = [Z] \in Hdg^{2k}(X)$. From the explicit description of the algebraic cycle class $[Z]_{alg} = (2\iota\pi)^k\alpha$, we get that $\alpha_\tau = [Z_\tau]_{alg}$, where $Z_\tau \subset X_\tau$ is the cycle obtained by applying τ to the defining equations of the components of Z. But we also have $[Z_\tau]_{alg} = (2\iota\pi)^k[Z_\tau]$, and the class $[Z_\tau]$ is a Betti rational cohomology class on the complex variety X_τ. ∎

5.2. Hodge loci and absolute Hodge classes

We want to spell-out in this section a geometric interpretation of the notion of absolute Hodge class, as being a property of the associated component of the locus of Hodge classes. We follow here [38].

Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold. Using the Hilbert scheme of $\mathbb{P}^N$, we can construct a (non-necessarily geometrically irreducible) smooth quasi-projective variety B defined over $\mathbb{Q}$, and a smooth projective morphism $\pi : \mathcal{X} \to B$ defined over $\mathbb{Q}$, such that X identifies to a fiber $\mathcal{X}_t$ for some complex point $t \in B(\mathbb{C})$.

Over B, we have the algebraic vector bundle F^kH^{2k}, which is also defined over $\mathbb{Q}$ (see section 2.3).

The locus of degree $2k$ Hodge classes for the family above is naturally a subset of the set of complex points of the algebraic variety F^kH^{2k}. We remark that it follows from the definition that for $\tau \in \mathrm{Aut}\,\mathbb{C}$ and for $\alpha \in F^kH^{2k}_B(\mathcal{X}_t, \mathbb{C}) \cong F^kH^{2k}_t$, the class $\alpha_\tau \in F^kH^{2k}B(\mathcal{X}_{t,\tau}, \mathbb{C}) = F^kH^{2k}_{\tau(t)}$ is simply obtained by letting τ act on the complex points of the variety F^kH^{2k} defined over $\mathbb{Q}$.

We introduce the following terminology to make more digest the rest of this section : the twisted locus of Hodge classes is the image of the locus of Hodge classes by the multiplication map $\alpha \mapsto (2\iota\pi)^k\alpha$. It thus consists of pairs (t, α_t) with $\alpha_t \in (2\iota\pi)^kH^{2k}_B(\mathcal{X}_t, \mathbb{Q}) \cap F^kH^{2k}_B(\mathcal{X}_t, \mathbb{C})$. We have the following:

Proposition 5.6. [38] *To say that Hodge classes on fibers of the family $\pi : \mathcal{X} \to B$ are absolute Hodge is equivalent to saying that the twisted locus of Hodge classes is a countable union of closed algebraic subsets of F^kH^{2k} which are defined over $\mathbb{Q}$.*

Proof. Indeed, the definition of absolute Hodge and the remark above tell that the Hodge classes for the family $\pi : \mathcal{X} \to B$ are absolute Hodge if and only if the twisted locus of Hodge classes is invariant under the action of $\mathrm{Aut}\,\mathbb{C}$. The result follows then from the local descriptions in section 3.1 and from the following:

Claim. *Let Z be a subset of (the set of complex points of) a complex algebraic variety defined over $\mathbb{Q}$, which is locally in the classical topology a countable union of closed analytic subsets. Then Z is invariant under the action of* $\mathrm{Aut}\,\mathbb{C}$ *if and only if it is a countable union of closed algebraic subsets defined over $\mathbb{Q}$.*

In one direction, this is obvious. To prove that the condition is necessary, observe that given a complex point x in an algebraic variety Y defined over $\mathbb{Q}$, its orbit under $\mathrm{Aut}\,\mathbb{C}$ is the dense subset of the $\mathbb{Q}$-Zariski closure of $\{x\}$ consisting of all complex points in this closure not satisfying any further equation with coefficients in a number field. In other words, this is the complementary set, in this $\mathbb{Q}$-Zariski closure, of a countable union of proper closed algebraic subsets. We now assume our Z is invariant under $\mathrm{Aut}\,\mathbb{C}$. Let $z \in Z$. We know that a subset of the the set of complex points of the $\mathbb{Q}$-Zariski closure of $\{z\}$, which is the complementary set of a countable union of proper closed algebraic subsets, is contained in Z. On the other hand, Z is locally a countable union of closed analytic subsets, and it immediately follows from a Baire argument that the full $\mathbb{Q}$-Zariski closure of $\{z\}$ is contained in Z. For each local analytic component $Z_{i,loc}$ of Z, it is obvious that we can choose a point $z_{i,loc}$ in sufficiently general position so that the $\mathbb{Q}$-Zariski closure of $\{z\}$ contains $Z_{i,loc}$. As Z can be written locally as a countable union of $Z_{i,loc}$, we conclude that it is also a countable union of closed algebraic subsets defined over $\mathbb{Q}$.

Thus the claim is proved and also the proposition. ■

In view of Proposition 5.6, the algebraicity theorem 3.8 can be seen as proving the geometric part of the prediction that Hodge classes are absolute Hodge, which is a strong evidence for the Hodge conjecture (cf. Corollary 5.5).

If we use the algebraicity theorem, we can also make proposition 5.6 more precise:

Proposition 5.7. *(see [38]) A degree $2k$ Hodge class α on X is absolute Hodge if and only if the connected component Z_α of the twisted locus of Hodge classes passing through $(2\iota\pi)^k\alpha$ is a closed algebraic subset of F^kH^{2k} which is defined over a number field and its $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ transforms are still contained in the twisted locus of Hodge classes.*

Proof. The "if" is obvious from the definition of "absolute Hodge". We thus concentrate on the direct implication. We first establish the following lemma.

Lemma 5.8. *(cf. "Principle B" in [12]) Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism, with B quasiprojective and connected. Let $\tilde{\alpha}$ be a global section of $R^{2k}\pi_*\mathbb{Q} \subset \mathcal{H}^{2k}$ which is everywhere in $F^k\mathcal{H}^{2k}$. If the Hodge class $\tilde{\alpha}_0 \in Hdg^{2k}(\mathcal{X}_0)$ is absolute Hodge at some point $0 \in B$, then the Hodge class $\tilde{\alpha}_t \in Hdg^{2k}(\mathcal{X}_t)$ is absolute Hodge at every point $t \in B$.*

Proof. We will heavily use here the construction of the Hodge bundles $F^k\mathcal{H}^{2k}$ as algebraic bundles and the algebraic construction of the Gauss-Manin connection made in section 2.3. In fact, we also need the global invariant theorem 5.10 stated in next section to guarantee that our section $\tilde{\alpha}$ is also a flat *algebraic* section of the bundle $F^k\mathcal{H}^{2k} \subset \mathcal{H}^{2k}$.

Let now $\tau \in \mathrm{Aut}\,\mathbb{C}$. We want to prove that for any $t \in B$, the class $\frac{1}{(2\iota\pi)^k}((2\iota\pi)^k\alpha_t)_\tau$ belongs to $H_B^{2k}(\mathcal{X}_{t,\tau},\mathbb{Q})$.

Consider the family $\pi_\tau : \mathcal{X}_\tau \to B_\tau$. The flat algebraic section $(2\iota\pi)^k\tilde{\alpha} \in F^k\mathcal{H}^{2k}$ provides a flat algebraic section $\beta := ((2\iota\pi)^k\tilde{\alpha})_\tau \in F^k\mathcal{H}_\tau^{2k}$. Because $\tilde{\alpha}_0$ is absolute Hodge, the class $\frac{1}{(2\iota\pi)^k}\beta_{\tau(0)}$ is a rational cohomology class on $\mathcal{X}_{0,\tau}$. But by flatness of β and connexity of B, it follows as well that the class $\frac{1}{(2\iota\pi)^k}\beta_{\tau(t)} = \frac{1}{(2\iota\pi)^k}((2\iota\pi)^k\alpha_t)_\tau$ is a rational Betti cohomology class on $\mathcal{X}_{t,\tau}$.

∎

Coming back to the prof of Proposition 5.7, it follows from the above lemma, by algebraicity of Z_α and by considering the base change $Z_\alpha \to B$, that if α is absolute Hodge, so is any class $\alpha' \in Z_\alpha$. On the other hand, we have seen that this implies that for every $\alpha' \in Z_\alpha$, the $\mathbb{Q}$-Zariski closure of $(2\iota\pi)^k\alpha'$ is contained in the twisted locus of Hodge classes. Taking a point α' in sufficiently general position in Z_α allows to conclude as in the previous proof that the $\mathbb{Q}$-Zariski closure of $(2\iota\pi)^k Z_\alpha$ is contained in the twisted locus of Hodge classes. As Z_α is the connected component of this locus passing through $(2\iota\pi)^k\alpha$, it follows easily that Z_α is in fact defined over $\overline{\mathbb{Q}}$ and its $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ transforms are still contained in the twisted locus of Hodge classes.

∎

5.3. The global invariant cycles theorem and its applications

In this section, we want to describe the consequences of a property which is much weaker than the property of being absolute. Let X be a smooth complex projective variety, and let α be a degree $2k$ Hodge class on X. As in the previous section, introduce a smooth projective family $\pi : \mathcal{X} \to B$ over a quasiprojective family B, everything being defined over $\mathbb{Q}$, and such that X is one fiber $\mathcal{X}_t$ for some $t \in B(\mathbb{C})$. We have the connected component Z_α of the locus of Hodge classes, which is algebraic in F^kH^{2k} by Theorem 3.8. Let $B_\alpha := p(Z_\alpha)$, where $p : F^kH^{2k} \to B$ is the structural projection. B_α is the *Hodge locus of* α. It is closed algebraic by Theorem 3.8 completed with the fact that the map $Z_\alpha \to B$ is finite (see section 7). The weaker property of α to be considered is whether B_α is defined over a number field. It is satisfied if α is absolute Hodge by Proposition 5.7. Its interest lies in the following:

Proposition 5.9. [38] *If B_α is defined over a number field, the Hodge conjecture for (X, α) can be deduced from the Hodge conjecture for Hodge classes on varieties defined over a number field.*

For the proof, we will need the "global invariant cycle theorem" of Deligne (or "Théorème de la partie fixe", cf. [14]) and some precisions of it.

Let Y be a smooth complex projective variety, and $\mathcal{X} \subset Y$ a Zariski open set. Let $\pi : \mathcal{X} \to B$ be a smooth proper algebraic morphism, where B is quasi-projective. Thus the fibers of π are smooth complex projective varieties and there is a monodromy action

$$\rho : \pi_1(B, 0) \to \mathrm{Aut}\, H^l(\mathcal{X}_0, \mathbb{Q}),\ 0 \in B.$$

Theorem 5.10. *The space of invariant classes*

$$H^l(\mathcal{X}_0,\mathbb{Q})^\rho := \{\alpha \in H^l(\mathcal{X}_0,\mathbb{Q}),\ \rho(\gamma)(\alpha)=\alpha,\ \forall\gamma\in\pi_1(B,0)\}$$

is equal to the image of the restriction map (which is a morphism of Hodge structures) :

$$i_0^*: H^l(Y,\mathbb{Q})\to H^l(\mathcal{X}_0,\mathbb{Q}),$$

where i_0 is the inclusion of $\mathcal{X}_0$ in Y. In particular it is a Hodge substructure of $H^l(\mathcal{X}_0,\mathbb{Q})$.

We can make this result more precise by combining it with the semi-simplicity of the category of polarized rational Hodge structures. In fact, we have the following:

Theorem 5.11. *Under the same assumptions as in Theorem 5.10, for any monodromy invariant rational Hodge class $\alpha\in Hdg^{2k}(\mathcal{X}_0)^\rho$, there exists a Hodge class $\tilde{\alpha}\in Hdg^{2k}(Y)$ such that $i_0^*\tilde{\alpha}=\alpha$.*

Proof. Indeed, choose an ample line bundle H on Y. It allows to construct a Lefschetz decomposition on $H^{2k}(Y,\mathbb{Q})$ which is a decomposition into Hodge substructures. Furthermore the intersection pairing $(a,b)_H := \int_Y h^{N-2k}\cup a\cup b$ on $H^{2k}(Y,\mathbb{Q})$, where $N=\dim Y$ and $h=c_1(H)$, polarizes up to a sign each of the Lefschetz summands, which are pairwise orthogonal for $(\ ,\ )_H$. Changing the sign of this pairing where needed endows the Hodge structure $H^{2k}(Y,\mathbb{Q})$ with a polarization $<,>_H$ (cf. [40, I, 7.1.2]). But for such a polarization, the intersection form $<,>_H$ remains non degenerate on any Hodge substructure. Applying this to the Hodge substructure $\operatorname{Ker} i_0^*$, we conclude that

$$H^{2k}(Y,\mathbb{Q}) = \operatorname{Ker} i_0^* \oplus (\operatorname{Ker} i_0^*)^{\perp_{<,>_H}},$$

where the summand $(\operatorname{Ker} i_0^*)^{\perp_{<,>_H}}$ is also a Hodge substructure of $H^{2k}(Y,\mathbb{Q})$, isomorphic (as a Hodge structure) via i_0^* to $H^{2k}(\mathcal{X}_0,\mathbb{Q})^\rho$ by Theorem 5.10. But then, any invariant Hodge class on $\mathcal{X}_0$ has a unique lift in $(\operatorname{Ker} i_0^*)^{\perp_{<,>_H}}$ which is also a Hodge class. ∎

Proof of Proposition 5.9. B_α is defined over $\overline{\mathbb{Q}}$, and one easily shows, using the flatness of the section $\tilde{\alpha}$ introduced below, that Z_α is an étale cover of B_α over the normal locus of B_α. It thus follows that a desingularization T of Z_α can be defined over $\overline{\mathbb{Q}}$, and the morphism $T\to B_\alpha$ too. Consider the variety obtained by base change : $\pi:\mathcal{X}_T\to T$. This variety is smooth, defined over $\overline{\mathbb{Q}}$. Choose a smooth projective compactification Y of $\mathcal{X}_T$ defined over $\overline{\mathbb{Q}}$.

Observe now that the class α is monodromy invariant on the fibers of $\mathcal{X}_T\to T$. Indeed, by definition, there is a morphism $T\to Z_\alpha$ and $Z_\alpha\subset F^kH^{2k}$. This provides on Z_α, hence on T, a section $\tilde{\alpha}$ of the bundle F^kH^{2k}. As the class $\tilde{\alpha}_t$ is a rational cohomology class everywhere along Z_α, it follows that $\tilde{\alpha}$ is in fact flat. Hence $\tilde{\alpha}$ is a flat section of $R^{2k}\pi_{T*}\mathbb{Q}$ on T extending α.

We can thus apply Theorem 5.11 to α, which provides a Hodge class β on Y, restricting to α on X. If the Hodge conjecture is true for Hodge classes on varieties defined over $\overline{\mathbb{Q}}$, it is true for β on Y, hence also for α on X. ∎

6. Deligne's theorem for abelian varieties

This section overlaps with the paper [25] in this volume and the interested reader will find there many more precisions on CM abelian varieties and Mumford-Tate groups of abelian varieties. However, we will also see that one can in fact avoid the construction of Shimura varieties if one only wants to prove theorem 6.1, which is the main result presented here.

6.1. Deligne's theorem

The main result discussed in this section is the following theorem due to Deligne [12]:

Theorem 6.1. *Hodge classes on abelian varieties are absolute Hodge.*

Let us first outline the strategy of the proof. A crucial role is played by Lemma 5.8. The first step (see subsection 6.2) below is the study of actions of CM fields on abelian varieties by isogenies. Under a certain sign condition, such an action forces them to carry Hodge classes of Weil type. The first step is to prove that these Hodge classes are absolute Hodge.

The second step extends this result to prove that all Hodge classes on CM abelian varieties are absolute Hodge. Technically, this is the most difficult part of the proof, although not the most conceptual one. We will not explain it here.

The last step (see 6.3) is the following : One deduces that Hodge classes on abelian varieties are absolute Hodge by proving that for any pair consisting of an abelian variety and a Hodge class on it, the Hodge locus associated to this Hodge class contains the isogeny class of a CM abelian variety. The previous step combined with Lemma 5.8 concludes the proof.

6.2. CM abelian varieties and Weil Hodge classes

The interested reader will find more detailed material on the subject of CM abelian varieties in the paper [28] in this volume, which has a large part devoted to this subject in char. $p > 0$. We restrict here (as everywhere in this paper) to the situation over the complex numbers.

Definition 6.2. *A CM field is a number field E which is a quadratic extension of a totally real field F satisfying the following property: the quadratic extension is of the form $y^2 = f$, $f \in F$, and under all (by definition real) embeddings of F into $\mathbb{R}$, f is sent to a negative number.*

Definition 6.3. *A simple CM abelian variety is a simple abelian variety A such that a CM field E is contained in* $(\mathrm{End}\, A) \otimes \mathbb{Q}$ *and the E-vector space* $H^1(A, \mathbb{Q})$ *has rank* 1*. (In particular,* $2\dim A = \deg(E : \mathbb{Q})$*.)*

We consider below more generally an abelian variety A such that a CM field E is contained in $(\mathrm{End}\, A) \otimes \mathbb{Q}$ and denote $d := \dim_E H^1(A, \mathbb{Q})$. We will assume for simplicity that we are in the simplest Weil case, where $E = \mathbb{Q}[I]$, $I^2 = -1$, so that in particular $d = \dim A$. We want to show that under a certain sign condition (implying in particular that $d = 2n$ is even), there are interesting Hodge classes (the Weil classes) in $H^d(A, \mathbb{Q})$.

We start with a $\mathbb{Z}[I]$-action on $\Gamma := \mathbb{Z}^{4n}$, where $I^2 = -1$, which makes

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$$

into a E-vector space, where E is the quadratic field $\mathbb{Q}[I]$.

Let

$$\Gamma_{\mathbb{C}} = \Gamma \otimes \mathbb{C} = \mathbb{C}^{2n}_{\iota} \oplus \mathbb{C}^{2n}_{-\iota}$$

be the associated decomposition into eigenspaces for I. A $2n$-dimensional complex torus X with underlying lattice Γ and inheriting the I-action is determined by a $2n$ dimensional complex subspace W of $\Gamma_{\mathbb{C}}$, which has to be stable under I, hence has to be the direct sum

$$W = W_{\iota} \oplus W_{-\iota}$$

of its intersections with $\mathbb{C}^{2n}_{\iota}$ and $\mathbb{C}^{2n}_{-\iota}$. It has furthermore to satisfy the condition that

$$W \cap \Gamma_{\mathbb{R}} = \{0\}. \tag{6.4}$$

Given W, X is given by the formula $X = \Gamma_{\mathbb{C}}/(W \oplus \Gamma)$.

We will choose W so that

$$\dim W_{\iota} = \dim W_{-\iota} = n. \tag{6.5}$$

Then W, hence X, is determined by the choice of the n-dimensional subspaces

$$W_{\iota} \subset \mathbb{C}^{2n}_{\iota},\, W_{-\iota} \subset \mathbb{C}^{2n}_{-\iota},$$

which have to be general enough so that condition (6.4) is satisfied.

We have isomorphisms

$$H^{2n}(X, \mathbb{Q}) \cong H_{2n}(X, \mathbb{Q}) \cong \bigwedge^{2n} \Gamma_{\mathbb{Q}}. \tag{6.6}$$

Consider the subspace (the inclusion is described below)

$$\bigwedge_E^{2n} \Gamma_{\mathbb{Q}} \subset \bigwedge^{2n} \Gamma_{\mathbb{Q}}.$$

Since $\Gamma_{\mathbb{Q}}$ is a $2n$-dimensional E-vector space, $\bigwedge_E^{2n} \Gamma_{\mathbb{Q}}$ is a one dimensional E-vector space, and its image under this inclusion is thus a 2 dimensional $\mathbb{Q}$-vector space.

Claim. *Under the assumption (6.5), $\bigwedge_E^{2n}\Gamma_{\mathbb{Q}}$ is made of Hodge classes, that is, is contained in the subspace $H^{n,n}(X)$ of the Hodge decomposition.*

Proof. (See also [27]) Notice that under the isomorphisms (6.6), tensored by $\mathbb{C}$, $H^{n,n}(X)$ identifies with the image of $\bigwedge^n W \otimes \bigwedge^n \overline{W}$ in $\bigwedge^{2n}\Gamma_{\mathbb{C}}$. To prove the claim, note that we have the decomposition

$$\Gamma_E := \Gamma_{\mathbb{Q}} \otimes E = \Gamma_{E,\iota} \oplus \Gamma_{E,-\iota}$$

into eigenspaces for the I action (where we see E as contained in $\mathbb{C}$ via $I \mapsto \iota$). Then $\bigwedge_E^{2n}\Gamma_{\mathbb{Q}} \subset \bigwedge^{2n}\Gamma_{\mathbb{Q}}$ is defined as the image of $\bigwedge_E^{2n}\Gamma_{E,\iota} \subset \bigwedge_E^{2n}\Gamma_E$ via the trace map

$$\bigwedge_E^{2n}\Gamma_E = \bigwedge_{\mathbb{Q}}^{2n}\Gamma_{\mathbb{Q}} \otimes E \to \bigwedge^{2n}\Gamma_{\mathbb{Q}}.$$

We have now the inclusion

$$\Gamma_E \subset \Gamma_{\mathbb{C}},$$

with $\Gamma_{\mathbb{C}} = \Gamma_E \otimes_{\mathbb{Q}} \mathbb{R}$, (because $\mathbb{C} \cong E \otimes_{\mathbb{Q}} \mathbb{R}$,) and the equality

$$\Gamma_{E,\iota} = \Gamma_E \cap \mathbb{C}_\iota^{2n}.$$

The space $\Gamma_{E,\iota}$ is a $2n$ dimensional E-vector space which generates over $\mathbb{R}$ the space $\mathbb{C}_\iota^{2n}$. It follows that the image of $\bigwedge_E^{2n}\Gamma_{E,\iota}$ in $\bigwedge^{2n}\Gamma_{\mathbb{C}}$ generates over $\mathbb{R}$ the complex line $\bigwedge^{2n}\mathbb{C}_\iota^{2n}$.

But we know that $\mathbb{C}_\iota^{2n}$ is the direct sum of the two spaces W_ι and $\overline{W_{-\iota}}$ which are n-dimensional. Hence

$$\bigwedge^{2n}\mathbb{C}_\iota^{2n} = \bigwedge^n W_\iota \otimes \bigwedge^n \overline{W_{-\iota}}$$

is contained in $\bigwedge^n W \otimes \bigwedge^n \overline{W}$, that is in $H^{n,n}(X)$. ∎

The main first step in the proof of Theorem 6.1 is the following (or rather its generalization to any CM field E).

Theorem 6.7. *Weil Hodge classes on Weil abelian varieties are absolute Hodge.*

Proof. The first step is the following lemma:

Lemma 6.8. *If $A = A_0 \otimes \mathbb{Z}[I]$ with the natural action of E, Weil classes on A are absolute Hodge.*

Proof. Indeed, with the notations above, we have

$$\Gamma = \Gamma_0 \otimes \mathbb{Z}[I],\ \operatorname{rank}\Gamma_0 = 2n$$

and for some $W_0 \subset \Gamma_{0,\mathbb{C}}$ such that $A_0 = \Gamma_{0,\mathbb{C}}/(W_0 \oplus \Gamma_0)$,

$$W = W_0 \otimes_{\mathbb{Z}} \mathbb{Z}[I].$$

If we follow the above description of $\bigwedge_E^{2n} H^1(A,\mathbb{Q})$, we find it to be equal to $\bigwedge^{2n} H^1(A_0,\mathbb{Q}) \otimes_{\mathbb{Q}} E$ which is naturally contained in

$$\bigwedge^{2n}(H^1(A_0,\mathbb{Q}) \otimes_{\mathbb{Q}} E) = \bigwedge^{2n} H^1(A,\mathbb{Q}).$$

The key point is that the inclusion

$$\bigwedge^{2n} H^1(A_0,\mathbb{Q}) \otimes_{\mathbb{Q}} E \hookrightarrow \bigwedge^{2n} H^1(A,\mathbb{Q})$$

is canonically determined by the E-structure of $H^1(A,\mathbb{Q})$. Hence, passing to cohomology with complex coefficients, this inclusion will be preserved by the transport map $\alpha \mapsto \alpha_\tau$, $\tau \in \operatorname{Aut}\mathbb{C}$. Furthermore, the space $\bigwedge^{2n} H^1(A_0,\mathbb{Q})$ consists of absolute Hodge classes on A_0, as it is generated by the class $[p]$ of a point in A_0. This means by definition that passing to cohomology with complex coefficients, the rational structure of $\bigwedge^{2n} H^1(A_0,\mathbb{C})$ given by the generator $(2\iota\pi)^n[p]$ is also preserved by the transport map $\alpha \mapsto \alpha_\tau$, $\tau \in \operatorname{Aut}\mathbb{C}$. This immediately implies that $\bigwedge_E^{2n} H^1(A,\mathbb{Q}) \subset \bigwedge^{2n} H^1(A,\mathbb{Q})$ consists of absolute Hodge classes. ∎

Remark 6.9. In the case we have been considering, where E is just a quadratic field, there is in fact a much easier proof. Indeed, by results of Tankeev [34] or Ribet [29], the Hodge classes on a self-product $A_0 \times A_0$, where A_0 is a very general abelian variety (hence with maximal Mumford-Tate group), are generated by products of divisors. Thus they are algebraic, hence absolute Hodge by Corollary 5.5. This argument does not adapt however to the general case where F is different from $\mathbb{Q}$.

The second step is the following, which concludes the proof of Theorem 6.7 by Lemma 5.8:

Proposition 6.10. *Under the sign conditions (6.5), there is a connected algebraic family of deformations of A, where all members admit an action of $\mathbb{Z}[I]$, and the special member is isogenous to $A_0 \otimes \mathbb{Z}[I]$.*

Proof. We observe that because A is projective, there is a polarization θ on A which is invariant under I. We use again the previous notations and note that the Betti class $\theta \in H^2(A,\mathbb{C})$ must be by I-invariance in $(\mathbb{C}_\iota^{2n})^* \wedge (\mathbb{C}_{-\iota}^{2n})^*$. The subspaces $W = W_\iota \oplus W_{-\iota}$ determines a Hodge structure polarized by θ if W is Lagrangian for θ and furthermore $\iota\theta(a,\overline{b})$ gives a positive definite Hermitian pairing on W. The first condition says that W_ι and $W_{-\iota}$ are orthogonal with respect to $\theta \in (\mathbb{C}_\iota^{2n})^* \wedge (\mathbb{C}_{-\iota}^{2n})^*$. So by the condition (6.5), $W_{-\iota}$ must be the orthogonal complement of W_ι with respect to θ. The second condition defines then a bounded symmetric Hermitian domain X with two connected components, contained in $Grass(n,\mathbb{C}_\iota^{2n})$, on which $\operatorname{Aut}_I(\Gamma,\theta)$ acts naturally and permutes the components. This X parameterizes a family of polarized complex tori with action by $\mathbb{Z}[I]$. Fixing a sufficiently high level structure, we will get the desired family of abelian varieties by descending this family to the quotient $X/\mathrm{Aut}_I(\Gamma,\theta,N)$. That the resulting family is algebraic follows from Baily-Borel theory [3]. The quotient is irreducible by the above description of X and it remains to prove that some points correspond to abelian varieties isogenous to abelian varieties of the form $A_0\otimes\mathbb{Z}[I]$. Such abelian varieties correspond to choices

$$W_\iota = W_0\otimes_{\mathbb{C}} E_{\mathbb{C},\iota},\ W_{-\iota} = W_0\otimes_{\mathbb{C}} E_{\mathbb{C},-\iota}$$

for some isomorphism

$$\Gamma_{\mathbb{Q}}\cong\Gamma_0\otimes\mathbb{Q}[I]$$

and some $W_0\subset\Gamma_{0,\mathbb{C}}$ of dimension n. There is an interesting sign subtlety here. Indeed, it appears that the sign condition (6.5) can be translated as follows: consider the I-invariant polarization $\theta\in\bigwedge^2 H^1(A,\mathbb{Q})$. (6.5) implies that the corresponding Hermitian form $\phi(a,b):=\iota\theta(a,\bar b)$ on the ι-eigenspace

$$H_1(A,\mathbb{C})_\iota$$

of I has signature $(n,-n)$. Noticing that ϕ induces a E-Hermitian form ϕ_E on the E-vector space $H_1(A,\mathbb{Q})$, it follows that there is a totally ϕ_E-isotropic E-vector subspace of rank n of the E-vector space $H_1(A,\mathbb{Q})$.

Conversely, start from a $\mathbb{Q}$-vector space V_0 of rank $2n$ with a non degenerate skew-symmetric bilinear form $\theta_0\in\bigwedge^2 V_0^*$. Introduce $V=V_0\otimes E$ endowed with the obvious I-action. θ_0 extends to an I-invariant 2-form θ' on V in an obvious way and the E-Hermitian form

$$\psi(a,b)=I\theta(a,\bar b)$$

on $V_0\otimes E$ (where complex conjugation acts on E by $I\mapsto -I$) also has the property that there exists a totally ϕ-isotropic E-vector subspace of rank n. One deduces from this that there is an isomorphism of E-vector spaces

$$V_0\otimes E\cong H_1(A,\mathbb{Q}) \tag{6.11}$$

identifying θ' to θ. Any abelian variety $A_0\times A_0=A_0\otimes\mathbb{Z}[I]$ with $H_1(A_0,\mathbb{Q})\cong V_0$ which is polarized by θ_0 is then in the isogeny class of a fiber of the above family. ∎

6.3. Hodge loci for abelian varieties

We come back to CM abelian varieties. They can be characterized as follows.

Proposition 6.12. *A simple abelian variety is CM if and only if the Mumford-Tate group of the Hodge structure on $H^1(A,\mathbb{Q})$ is abelian.*

Proof. Since we are in characteristic 0, the simplicity of A implies that $(\mathrm{End}\,A)\otimes\mathbb{Q}$ is a division algebra. Note that $(\mathrm{End}\,A)\otimes\mathbb{Q}$ is also the endomorphism algebra of the Hodge structure on $H^1(A,\mathbb{Q})$, determined by the representation $h:\mathbb{C}^*\to\mathrm{Aut}\,H^1(A,\mathbb{R})$ of the real algebraic group $\mathbb{C}^*$ considered in (4.4). These endomorphisms clearly are the endomorphisms of $H^1(A,\mathbb{Q})$ commuting with $\operatorname{Im} h$. Recalling that the Mumford-Tate group $MT(A)$ is the smallest algebraic subgroup of $\operatorname{Aut} H^1(A,\mathbb{R})$ defined over $\mathbb{Q}$ and containing $\operatorname{Im} h$, we also identify $(\operatorname{End} A)\otimes\mathbb{Q} \subset \operatorname{End} H^1(A,\mathbb{Q})$ with the commuting algebra of $MT(A)$. If A is CM, $(\operatorname{End} A)\otimes\mathbb{Q}$ contains a CM field E such that $\dim_E H^1(A,\mathbb{Q}) = 1$. As $MT(A)$ commutes with E acting on $H^1(A,\mathbb{Q})$, it is contained in $\operatorname{Aut}_E H^1(A,\mathbb{Q})$ which is commutative. Conversely, if $MT(A)$ is commutative, the algebra $(\operatorname{End} A)\otimes\mathbb{Q}$ tensored by $\mathbb{C}$ is a sum over characters χ of $MT(A)$, decomposing the $MT(A)$-module $H^1(A,\mathbb{C})$, of the algebras $\operatorname{End} H^1(A,\mathbb{C})_\chi$. It follows that $\dim_{\mathbb{Q}} (\operatorname{End} A)\otimes\mathbb{Q} \geq 2\dim A$, and that in case of equality $\operatorname{End} A$ is commutative. The strict inequality is not possible in the simple case by elementary reasons. So we know that $E := \operatorname{End} A\otimes\mathbb{Q}$ is a commutative field of rank $2\dim A$ and, again by elementary reasons, we must have $\dim_E H^1(A,\mathbb{Q}) = 1$. It remains to see that E is a CM field. This is very classical (see [11]). Let us for completeness recall the argument: A admits a polarization $\theta \in NS(A)$ that we will see as a 2-form on $H_1(A,\mathbb{Q})$. There is the Rosati involution i_θ acting on $\operatorname{End} A$ associated to θ, given by $\psi \mapsto \theta^{-1}\circ {}^t\psi\circ\theta$, where we see θ as giving an isogeny between A and its dual $\widehat{A}$. We have to show that the fixed field $F \subset E$ of this involution is totally real, and that for any embedding of F into $\mathbb{R}$, the tensor product $E\otimes_F \mathbb{R}$ is isomorphic to $\mathbb{C}$. In other words, let $\phi \in \operatorname{End} A$ be fixed by the Rosati involution and let λ be an eigenvalue of ϕ acting on $H^1(A,\mathbb{Q})$. Then we have to show that λ is real. Recall that ϕ acts on $H_1(A,\mathbb{C})$ preserving $H_{1,0}(A)$. Thus λ is an eigenvalue of ϕ on either $H_{1,0}(A)$ or $H_{0,1}(A)$. We may thus assume $\phi(\eta) = \lambda\eta$ for some $\eta \in H_{1,0}(A)$. Now we have

$$\iota\theta(\eta,\phi(\overline{\eta})) = \overline{\lambda}\iota\theta(\eta,\overline{\eta})$$

$$= \iota\theta(\phi(\eta),\overline{\eta}) = \lambda\iota\theta(\eta,\overline{\eta}).$$

As we know that $\theta(\eta,\overline{\eta}) \neq 0$, this implies that λ is real.

As λ is real, it also appears as an eigenvalue of ϕ on $H_{0,1}(A)$ and it follows that elements of F have only multiplicity ≥ 2 eigenvalues, which is not the case for E because $\dim_E H_1(A,\mathbb{Q}) = 1$. The Rosati involution is thus non trivial on E, and the same computation as above shows that for the element σ of E such that $i_\theta(\sigma) = -\sigma$, the eigenvalues of σ are pure imaginary numbers.

■

The final step in the proof of theorem 6.1 is the following proposition, which reduces the statement to CM abelian varieties, by Lemma 5.8.

Proposition 6.13. *Let A be an abelian variety, and α a Hodge class on A. Then there exists a quasi-projective family $\mathcal{B} \to W$ of abelian varieties, where W is irreducible, such that for some point $t_0 \in W$, $\mathcal{B}_{t_0} \cong A$, W is contained in the Hodge locus of α and for some point $t_1 \in W$, $\mathcal{B}_{t_1}$ is a CM abelian variety.*

Sketch of proof. This is an application of proposition 6.12 and in the proof given in [12], one can avoid the construction of Shimura varieties (for which we refer to [25], [15]) using the algebraicity theorem 7.1. Let G be the Mumford-Tate group of A. Then we know by Theorem 4.8 that G is the subgroup of $\mathrm{Aut}\,(H^1(A,\mathbb{Q}))$ fixing finitely many Hodge classes p_i in tensor products of copies of $H^1(A,\mathbb{Q})$ and its dual. The small deformations A_t of A for which $MT(A_t) \subset MT(A)$ are thus parameterized by the intersection of Hodge loci for self-products of A. More precisely, introducing a polarization θ on A, there is a moduli space $\mathcal{A}_{g,\theta,N}$ of abelian varieties with polarization of the same numerical type as θ, and with level N structure. For N large enough, there is an universal abelian variety $\mathcal{B} \to \mathcal{A}_{g,\theta,N}$, and for some point $t_0 \in \mathcal{A}_{g,\theta,N}$, $\mathcal{B}_{t_0} \cong A$ with a certain choice of level N structure. This family induces as well the families given by relative self-products, and we can take for variety W the intersection inside $\mathcal{A}_{g,\theta,N}$ of the Hodge loci associated to the p_i's, or rather its connected component passing through t_0. This locus is algebraic by Theorem 7.1 and parameterizes abelian varieties for which the Mumford-Tate group is contained in G (or rather in some conjugate if one takes monodromy into account). It remains to prove that there is a point $t_1 \in W$ for which $\mathcal{B}_{t_1}$ is a CM abelian variety. For this we use the following alternative description of W as a set, which does not give its algebraicity: G contains by definition the image $\mathrm{Im}\, h_A$, where $h_A : \mathbb{C}^* \to \mathrm{Aut}\, H^1(A,\mathbb{R})$ is the representation determining the Hodge structure on $H^1(A)$. Then W can also be described as the set of 1-parameter subgroups $h : \mathbb{C}^* \to G(\mathbb{R})$ which are conjugate under $G(\mathbb{R})$ to h_A. If for some parameter t_1, h_{t_1} is contained in a subtorus of G, defined over $\mathbb{Q}$, then the Mumford-Tate group of the abelian variety A_{t_1} is contained in T, and thus A_{t_1} is CM by proposition 6.12. The conclusion then comes by saying that $\mathrm{Im}\, h_A$ is contained in a maximal subtorus T of $G(\mathbb{R})$ and that there is a maximal subtorus T' of $G(\mathbb{R})$, conjugate to T under $G(\mathbb{R})$ and defined over $\mathbb{Q}$. ∎

7. Deligne-Cattani-Kaplan theorem

Let $f : \mathcal{X} \to B$ be a smooth projective morphism, where $\mathcal{X}$ and B are quasiprojective. Fix an integer k. Recall from section 2.3 that the Hodge bundle F^kH^{2k} is an algebraic bundle over B. We introduced in section 3.3 the locus of Hodge classes of degree $2k$ for this family. The next two subsections are devoted to a sketch of the proof of the following major result, already stated in section 3.3.

Theorem 7.1. *(see [8]) The connected components of the locus of Hodge classes are closed algebraic subsets of F^kH^{2k}. In particular, the components of the Hodge locus are algebraic.*

Actually, the result of [8] is not stated this way but it implies the above theorem whose formulation is more coherent with our presentation. The locus of Hodge classes considered in [8] parameterizes pairs consisting of a variety, fiber of a certain family, and a degree $2k$ integral cohomology class of type (k,k) on it. This locus is not seen inside F^kH^{2k} but inside the local constant system of integral cohomology. In order to better state their result, one needs first to recall that by Lefschetz decomposition induced by a relative polarization of $f:\mathcal{X}\to B$, and by looking individually at each component of this decomposition, which provides a subvariation of Hodge structures, one immediately reduces the statement to the case of a polarized variation of Hodge structures. Associated to such a polarized variation of Hodge structures, there is the so-called Hodge metric h on the bundle H^{2k}. This metric is not flat and not defined over $\mathbb{Z}$, but it has the advantage of being a metric. It is obtained by multiplying the natural Hermitian intersection pairing on $H^{2k}(\mathcal{X}_t,\mathbb{Q})_{prim}$ given by

$$(a,b)_l=(-1)^k\int_{\mathcal{X}_t} l^{n-2k}\cup a\cup\overline{b},\ n=\dim\mathcal{X}_t\geq k,$$

where l is the class of a relatively ample line bundle L on $\mathcal{X}$, by $(-1)^p$ on $H^{p,q}_{prim}$, $p+q=2k$.

The main theorem can be formulated as follows. Introduce for any positive number A the following set :

$$S_A=\{(t,\alpha),\,t\in B,\,\alpha\in H^{2k}(\mathcal{X}_t,\mathbb{Z})_{prim}\cap F^kH^{2k}(\mathcal{X}_t),\ h(\alpha)\leq A\}.$$

Theorem 7.2. *The image S of the natural map $o:S_A\to B$ is a closed algebraic subset and the map $S_A\to S$ is finite.*

Remark 7.3. This theorem in fact proves a result which is stronger than what is predicted by the Hodge conjecture. Indeed, it implies that, when working with primitive degree $2k$ cohomology of a family as above, and denoting by q_l the natural integral intersection pairing given by

$$q_l(a,b)=\int_{\mathcal{X}_t} l^{n-2k}\cup a\cup b,$$

there are only finitely many components of the Hodge locus associated to integral Hodge classes with $|q_l(a)|\leq A$. In the degree 2 case, this is well-known because one can reduce to the surface case, apply Riemann-Roch formula and verify that if a is a primitive divisor class on $\mathcal{X}_t$ and $|q_l(a)|\leq A$, then $a+NL$ is effective on $\mathcal{X}_t$, N being independent of t. One then applies finiteness results for the relative Hilbert scheme. In the case of higher degree, this finiteness statement is intriguing, as it is not what is predicted by geometry. Indeed, it is not true that the locus in B where an *integral* Hodge class is algebraic is a closed algebraic subset of B. One can find a counterexample to this in [33]. It is proved there that the Kollár counterexamples to the integral Hodge conjecture are examples where an integral Hodge class is non algebraic at the very general point of a certain parameter space, but becomes algebraic on a countable union of closed algebraic subsets, dense for the Euclidean topology.

7.1. Schmid's theorem

The main tool for the proof of Theorem 7.2 is Schmid's nilpotent orbit theorem [30]. This theorem gives a very good asymptotic estimate of the variation of Hodge structure near a degeneration point $b \in \overline{B} \setminus B$. As usual, one assumes that the base B has a (projective) compactification $\overline{B}$ such that $\overline{B} \setminus B$ is a divisor with normal crossings. The local situation over B is thus essentially $(\Delta^*)^p \times \Delta^{N-p} \subset \Delta^N$, $N = \dim B$. By the semi-stable reduction theorem, one can even (maybe after a finite base change) assume that there is a compactification $f : \overline{\mathcal{X}} \to \overline{B}$, where the divisor over $\overline{B} \setminus B$ is *reduced* with normal crossings. This last condition implies the following result first proved by Borel (see [18]):

Theorem 7.4. *(Monodromy theorem) The monodromy T_i on $H^{2k}(\mathcal{X}_t, \mathbb{Z})$ around each branch B_i of $\overline{B} \setminus B$ is unipotent.*

Without this condition, one would only obtain quasiunipotency. For each $i \leq p$, denote $N_i := \log T_i$, which is an operator with rational coefficients, which can be written as a polynomial expression in $Id - T_i$, acting on $H^{2k}(\mathcal{X}_t, \mathbb{Q})$.

Consider the universal cover $\mathbb{H} \to \Delta^*$, $z \mapsto e^{2\iota\pi z}$. The inverse image of the local system $(R^{2k} f_* \mathbb{Z})_{prim}$ is trivial on $\mathbb{H}^p \times \Delta^{N-p}$. We will denote its trivialized fiber by V. The variation of Hodge structure on $H^{2k}(\mathcal{X}_t, \mathbb{Z})_{prim}$ is described on $\mathbb{H}^N$ by a holomorphic map

$$\Phi : \mathbb{H}^p \times \Delta^{N-p} \to \mathcal{D},$$

where $\mathcal{D}$ is the corresponding period domain, contained in a flag manifold of V, and Φ has to satisfy the property that

$$\Phi(z + e_j) = T_j \Phi(z) = \exp(N_j)\Phi(z), \; j \leq p,$$

and it follows that the map $z \mapsto \exp(\sum_{i \leq p} -N_i z_i)\Phi(z)$ is invariant under translation by the various e_j's (here we denoted by e_i the 1 of $\mathbb{H}$ put in i-th position). It thus descends to a map Ψ from $(\Delta^*)^p \times \Delta^{N-p}$ to $\mathcal{D}$.

The result of [30] is the following:

Theorem 7.5. *i) The holomorphic map Ψ extends holomorphically over* 0.

ii) Let $\Phi_0(z) = \exp(\sum_i N_i z_i)\Psi(0)$. Then there exists a constant M such that when $y_i := \operatorname{Im} z_i \geq M$, $\Phi_0(z)$ defines a polarized Hodge structure on V.

iii) Φ_0 is an excellent asymptotic approximation of Φ: this makes sense using an invariant metric on the period domain $\mathcal{D}$. The distance between $\Phi_0(z)$ and $\Phi(z)$ is $\leq Ce^{-C' \mathrm{Inf}(y_i)} \mathrm{Inf}(y_i)^{C''}$ for some positive constants C, C', C''.

7.2. Strategy of the proof of Theorem 7.2

The algebraicity result is essentially an extension result for the Hodge loci, since the basis is quasi-projective, so that by Chow's theorem, proving they are algebraic amounts to proving they extend over the boundary.

The proof is completely analytic, and involves the following steps. In the local situation, with the same notations as above, one goal is to prove the following:

Theorem 7.6. *For any $A > 0$, there exists a constant $A_1 > 0$ such that there are finitely many integral points $\alpha \in V$ with the property that for some point $z = (z_i)$, $z_i = x_i + \iota y_i \in \mathbb{H}$ with $y_i > A_1$, α is in F^kV at z and the Hodge norm of α is $\leq A$. Furthermore, these α lie in F_0^kV for a limiting Hodge filtration at some point of the boundary divisor.*

This finiteness statement implies the main theorem 7.2 by a delicate argument involving among other things Lemma 3.1 which gives a smaller set of local holomorphic equations for the Hodge loci. It of course also gives the finiteness part of that theorem.

The proof of Theorem 7.6 involves the nilpotent orbit theorem. In fact, as the nilpotent orbit theorem only gives an asymptotic approximation of the period map near the boundary divisor, what is proved is a generalization of theorem 7.6, involving integral Hodge classes approximately of type (k, k).

Theorem 7.7. *For any $A > 0$, there exists a constant $A_1 > 0$ such that there are finitely many integral points $\alpha \in V$ with the property that for some point $z = (z_i)$, $z_i = x_i + \iota y_i \in \mathbb{H}$ with $y_i > A_1$, α is close to F^kV in the Hodge metric at z and the Hodge norm of α is $\leq A$. Furthermore, these α lie in F_0^kV for a limiting Hodge filtration at some point of the boundary divisor.*

7.3. Some arithmetic improvements

As above, X being given, we introduce a "spread" $\pi : \mathcal{X} \to B$ of X defined over $\mathbb{Q}$. As we explained in section 5, a Hodge class on X is absolute Hodge exactly when the corresponding component $(2\iota\pi)^k Z_\alpha$ of the twisted locus of Hodge classes is defined over $\overline{\mathbb{Q}}$ and its translates under $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ are again components of the twisted locus of Hodge classes. We explained in Proposition 5.9 the interest of studying the analogous property for the corresponding component $B_\alpha = p(Z_\alpha) \subset B$ of the Hodge locus.

We have the following criterion for a component of the Hodge locus to be defined over $\overline{\mathbb{Q}}$:

Theorem 7.8. [38] *Let $\alpha \in F^kH^{2k}(X, \mathbb{C})$ be a Hodge class. Suppose that any locally constant Hodge substructure $L \subset H^{2k}(\mathcal{X}_t, \mathbb{Q})$, $t \in B_\alpha$, is purely of type (k, k). Then B_α is defined over $\overline{\mathbb{Q}}$, and its translates under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are again components B_γ of the Hodge locus.*

Proof. We shall show the following :

Claim. *Under the assumptions of the Theorem, for any $\tau \in \mathrm{Aut}\,\mathbb{C}$, $B_{\alpha,\tau}$ is also a component B_γ of the Hodge locus.*

This implies that B_α is defined over $\overline{\mathbb{Q}}$ because this implies that there at most countably many distinct translates $B_{\alpha,\tau}$, $\tau \in \operatorname{Aut}\mathbb{C}$, and this property characterizes algebraic subsets defined over $\overline{\mathbb{Q}}$. This also obviously implies the statement concerning the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ translates.

To prove the claim, observe the following: as already explained in section 5, we can choose a desingularization T of Z_α and introduce a projective compactification Y of $\mathcal{X}_T$. Furthermore, over T, α extends as a flat section $\tilde{\alpha}$ of the local system $R^{2k}\pi_{T*}\mathbb{Q}$ and thus, by Theorem 5.11, there exists a Hodge class $\beta \in Hdg^{2k}(Y,\mathbb{Q})$ such that $\alpha = i_0^*\beta$ on $\mathcal{X}_0$, where $X = \mathcal{X}_0 \stackrel{i_0}{\hookrightarrow} Y$.

Now we apply our assumption: any locally constant Hodge substructure of $H^{2k}(\mathcal{X}_t,\mathbb{Q})$, $t \in T$ must be of type (k,k). It thus follows that in the above situation, the constant Hodge substructure $i_t^*H^{2k}(Y,\mathbb{Q})$ is purely of type (k,k).

Let now $\tau \in \operatorname{Aut}\mathbb{C}$. Consider the family

$$\mathcal{X}_{T,\tau} \subset Y_\tau,\ \pi_\tau : \mathcal{X}_{T,\tau} \to T_\tau.$$

Then it is also true that the restriction map $i_{0,\tau}^* : H_B^{2k}(Y_\tau,\mathbb{Q}) \to H_B^{2k}(X_\tau,\mathbb{Q})$ has an image which is purely of type (k,k), hence made of Hodge classes, because this property can be seen on the restriction map in algebraic de Rham cohomology. We now conclude by proving that for a generic element $\gamma \in i_{0,\tau}^*H_B^{2k}(Y_\tau,\mathbb{Q})$, $p_\tau(T_\tau) = B_{\alpha,\tau}$ equals the component B_γ of the Hodge locus associated to γ. This is done as follows: If we take $\gamma' = \frac{1}{(2\iota\pi)^k} i_{0\tau}^*(2\iota\pi)^k\beta_\tau$, then we do not know that γ' is a rational cohomology class, because we do not know that α is absolute Hodge. But we know, because this is a purely algebraic statement, that $B_{\alpha,\tau}$ can be defined locally schematically by the fact that γ' remains in F^kH^{2k}. As $H_B^{2k}(Y_\tau,\mathbb{Q})$ is Zariski dense in $H_B^{2k}(Y_\tau,\mathbb{C})$, it follows that for a generic element γ of $i_{0,\tau}^*H_B^{2k}(Y_\tau,\mathbb{Q})$, $B_{\alpha,\tau}$ can be defined locally set theoretically by the fact that γ remains in F^kH^{2k}, that is $B_{\alpha,\tau} = B_\gamma$. ■

References

[1] Yves André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part, Compositio Math. 82 (1992), no. 1, 1–24. 17

[2] Yves André. Pour une théorie inconditionnelle des motifs, Publications Mathématiques de l'IHÉS, 83 (1996), p. 5-49. 5, 22

[3] W. Baily, A. Borel. Compactifications of arithmetic quotients of bounded symmetric domains, Ann. of Math. (2) 84, 442-528 (1966). 33

[4] S. Bloch. Semi-regularity and de Rham cohomology, Inventiones Math. 17, 51-66 (1972). 23

[5] S. Bloch et V. Srinivas, Remarks on correspondences and algebraic cycles, Amer. J. of Math. 105 (1983) 1235-1253. 22

[6] J. Carlson, Ph. Griffiths. Infinitesimal variation of Hodge structure and the global Torelli problem, in *Géométrie Algébrique*, Angers, 1980, (A. Beauville, ed.), Sijthoff-Noordhoff, 51-76.

[7] J. Carlson, M. Green, Ph. Griffiths, J. Harris. Infinitesimal variations of Hodge structure (I), Compositio Mathematica 50 (1983) 109-205.

[8] E. Cattani, P. Deligne, A. Kaplan, On the locus of Hodge classes, J. Amer. Math. Soc. 8 (1995), 2, 483-506. 1, 2, 4, 12, 13, 35

[9] H. Clemens. Homological equivalence, modulo algebraic equivalence, is not finitely generated. Inst. Hautes Études Sci. Publ. Math. No. 58 (1983), 19-38 (1984). 11

[10] J.-L. Colliot-Thélène, C. Voisin. Cohomologie non ramifiée et conjecture de Hodge entière, preprint 2010. 2

[11] O. Debarre. *Tores et variétés abéliennes complexes*, Cours Spécialisés 6, Société Mathématique de France, EDP Sciences, (1999). 34

[12] P. Deligne. Hodge cycles on abelian varieties (notes by J. S. Milne), in [15], 9-100. 4, 5, 24, 26, 29, 35

[13] P. Deligne, Letter to Bjorn Poonen, August 2009. 6, 17, 18

[14] P. Deligne. Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5-57. 27

[15] P. Deligne, J. S. Milne, A. Ogus, and K.-Y. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. 35, 40

[16] Ph. Griffiths. Periods of integrals on algebraic manifolds, I, II, Amer. J. Math. 90, 568-626, 805-865 (1968). 1, 7

[17] Ph. Griffiths. On the periods of certain rational integrals I,II, Ann. of Math. 90, 460-541 (1969). 19

[18] Ph. Griffiths, L. Tu. Curvature properties of the Hodge bundles, in *Topics in transcendental algebraic geometry* (Princeton, N.J., 1981/1982), 29–49, Ann. of Math. Stud., 106, Princeton Univ. Press, Princeton, NJ, 1984. 37

[19] A. Grothendieck. On the de Rham cohomology of algebraic varieties, Inst. Hautes Etudes Sci. Publ. Math. No. 29, 1966, 95-103. 8

[20] N. Katz, T. Oda. On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. 8 (1968), 199–213. 7

[21] S. Kleiman. Algebraic cycles and the Weil conjectures. Dix exposés sur la cohomologie des schémas, pp. 359-386. North-Holland, Amsterdam; Masson, Paris, (1968). 22

[22] J. Kollár. Lemma p. 134 in *Classification of irregular varieties*, edited by E. Ballico, F. Catanese, C. Ciliberto, Lecture Notes in Math. 1515, Springer (1990). 11

[23] D. Maulik, B. Poonen. Néron-Severi groups under specialization, arXiv:0907.4781. 22

[24] J. Milne. *Étale cohomology*, Princeton Mathematical Series 33, Princeton University Press (1980).

[25] J. Milne. Shimura Varieties and Moduli, this volume. 4, 29, 35

[26] B. Moonen. Notes on Mumford-Tate groups, available at 14
http://staff.science.uva.nl/ bmoonen/NotesMT.pdf.

[27] B. Moonen, Yu. Zarhin. Weil classes on abelian varieties, J. Reine Angew. Math. 496 (1998), 83-92. 31

[28] Frans Oort. Moduli of abelian varieties, this volume. 29

[29] K. Ribet. Hodge classes on certain types of abelian varieties, Amer. J. Math. 105 (1983), 523-538. 32

[30] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. Invent. Math. 22 (1973), 211–319. 37

[31] J.-P. Serre. Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42. 8

[32] T. Shioda. On the Picard number of a complex projective variety, Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 3, 303-321. 5

[33] C. Soulé, C. Voisin. Torsion cohomology classes and algebraic cycles on complex projective manifolds. Adv. Math. 198 (2005), no. 1, 107–127. 36

[34] S. G. Tankeev. Cycles on simple abelian varieties of prime dimension Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), n0 1, 155-170. 32

[35] T. Terasoma. Complete intersections with middle Picard number 1 defined over $\mathbb{Q}$, Math. Z. 189 (1985), no. 2, 289-296. 5, 20, 22

[36] C. Voisin. On integral Hodge classes on uniruled or Calabi-Yau threefolds. Moduli spaces and arithmetic geometry, 43-73, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, (2006). 11

[37] C. Voisin. The Griffiths group of a general Calabi-Yau threefold is not finitely generated. Duke Math. J. 102 (2000), no. 1, 151-186. 11

[38] C. Voisin. Hodge loci and absolute Hodge classes, Compositio Mathematica, Vol. 143 Part 4, 945-958, 2007. 2, 25, 26, 27, 38

[39] C. Voisin. On integral Hodge classes on uniruled and Calabi-Yau threefolds, in *Moduli Spaces and Arithmetic Geometry*, Advanced Studies in Pure Mathematics 45, 2006, pp. 43-73. 2

[40] C. Voisin. *Hodge Theory and Complex Algebraic Geometry* I, II, Cambridge studies in advanced Mathematics 76, 77, Cambridge University Press 2002, 2003. 5, 6, 7, 8, 9, 10, 11, 14, 16, 19, 21, 28

[41] C. Voisin. Transcendental methods in the study of algebraic cycles, in *Algebraic cycles and Hodge theory*, Lecture Notes in Mathematics, Springer verlag (1994), 153-222. 9

CNRS, Institut de Mathématiques de Jussieu, Case 247, 4 Place Jussieu, 75005 PARIS

E-mail address: `voisin@math.jussieu.fr`