Remarks on curve classes on rationally connected varieties

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To Joe Harris, on his 60th birthday

1 Introduction

Let $X$ be a smooth complex projective variety. Define

$$Z^{2i}(X) = \frac{\text{Hdg}^{2i}(X, \mathbb{Z})}{H^{2i}(X, \mathbb{Z})_{\text{alg}}},$$

where $\text{Hdg}^{2i}(X, \mathbb{Z})$ is the space of integral Hodge classes on $X$ and $H^{2i}(X, \mathbb{Z})_{\text{alg}}$ is the subgroup of $H^{2i}(X, \mathbb{Z})$ generated by classes of codimension $i$ closed algebraic subsets of $X$.

These groups measure the defect of the Hodge conjecture for integral Hodge classes, hence they are trivial for $i = 0, 1$ and $n = \dim X$, but in general they can be nonzero by [1]. Furthermore they are torsion if the Hodge conjecture for rational Hodge classes on $X$ of degree $2i$ holds. In addition to the previously mentioned case, this happens when $i = n - 1$, $n = \dim X$, due to the Lefschetz theorem on $(1, 1)$-classes and the hard Lefschetz isomorphism (cf. [23]). We will call classes in $\text{Hdg}^{2n-2}(X, \mathbb{Z})$ “curve classes”, as they are also degree 2 homology classes.

Note that the Kollár counterexamples (cf. [14]) to the integral Hodge conjecture already exist for curve classes (that is degree 4 cohomology classes in this case) on projective threefolds, unlike the Atiyah-Hirzebruch examples which work for degree 4 integral Hodge classes in higher dimension.

It is remarked in [21], [23] that the two groups $Z^4(X), Z^{2n-2}(X), n := \dim X$ are birational invariants. (For threefolds, this is the same group, but not in higher dimension.) The nontriviality of these birational invariants for rationally connected varieties is asked in [23]. Still more interesting is the nontriviality of these invariants for unirational varieties, having in mind the Lüroth problem (cf. [3], [2], [4]).

Concerning the group $Z^4(X)$, Colliot-Thélène and the author proved in [8], building on the work of Colliot-Thélène and Ojanguren [5], that it can be nonzero for unirational varieties starting from dimension 6. What happens in dimensions 5 and 4 is unknown (the four dimensional case being particularly challenging in our mind), but in dimension 3, there is the following result proved in [22]:

**Theorem 1.1. (Voisin 2006)** Let $X$ be a smooth projective threefold which is either uniruled or Calabi-Yau (meaning that $K_X$ is trivial and $H^1(X, \mathcal{O}_X) = 0$). Then the group $Z^4(X)$ is equal to 0.

This result, and in particular the Calabi-Yau case, implies that the group $Z^6(X)$ is also 0 for a Fano fourfold $X$ which admits a smooth anticanonical divisor. Indeed, a smooth anticanonical divisor $j : Y \hookrightarrow X$ is a Calabi-Yau threefold, so that we have $Z^4(Y) = 0$.
by Theorem 1.1 above. As \( H^2(Y, \mathcal{O}_Y) = 0 \), every class in \( H^4(Y, \mathbb{Z}) \) is a Hodge class, and it follows that \( H^4(Y, \mathbb{Z}) = H^4(Y, \mathbb{Z})_{\text{alg}} \). As the Gysin map \( j_* : H^4(Y, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \) is surjective by the Lefschetz theorem on hyperplane sections, it follows that \( H^4(X, \mathbb{Z}) = H^4(X, \mathbb{Z})_{\text{alg}} \), and thus \( Z^6(X) = 0 \).

In the paper [11], it was proved more generally that if \( X \) is any Fano fourfold, the group \( Z^6(X) \) is trivial. Similarly, if \( X \) is a Fano fivefold of index 2, the group \( Z^8(X) \) is trivial.

These results have been generalized to higher dimensional Fano manifolds of index \( n - 3 \) and dimension \( \geq 8 \) by Enrica Floris [9] who proves the following result:

**Theorem 1.2.** Let \( X \) be a Fano manifold over \( \mathbb{C} \) of dimension \( n \geq 8 \) and index \( n - 3 \). then the group \( Z^{2n-2}(X) \) is equal to 0: Equivalently, any integral cohomology class of degree \( 2n - 2 \) on \( X \) is algebraic.

The purpose of this note is to provide a number of evidences for the vanishing of the group \( Z^{2n-2}(X) \), for any rationally connected variety over \( \mathbb{C} \). Note that in this case, since \( H^2(X, \mathcal{O}_X) = 0 \), the Hodge structure on \( H^2(X, \mathbb{Q}) \) is trivial, and so is the Hodge structure on \( H^{2n-2}(X, \mathbb{Q}) \), so that \( Z^{2n-2}(X) = H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{\text{alg}} \). We will first prove the following two results.

**Proposition 1.3.** The group \( Z^{2n-2}(X) \) is locally deformation invariant for rationally connected manifolds \( X \).

Let us explain the meaning of the statement. Consider a smooth projective morphism \( \pi : \mathcal{X} \to B \) between connected quasi-projective complex varieties, with \( n \) dimensional fibers. Recall from [15] that if one fiber \( X_b := \pi^{-1}(b) \) is rationally connected, so is every fiber. Let us endow everything with the usual topology. Then the sheaf \( R^{2n-2}\pi_*\mathbb{Z} \) is locally constant on \( B \). On any Euclidean open set \( U \subset B \) where this local system is trivial, the group \( Z^{2n-2}(X_b), b \in U \) is the finite quotient of the constant group \( H^{2n-2}(X_b, \mathbb{Z}) \) by its subgroup \( H^{2n-2}(X_b, \mathbb{Z})_{\text{alg}} \). To say that \( Z^{2n-2}(X_b) \) is locally constant means that on open sets \( U \) as above, the subgroup \( H^{2n-2}(X_b, \mathbb{Z})_{\text{alg}} \) of the constant group \( H^{2n-2}(X_b, \mathbb{Z}) \) does not depend on \( b \).

It follows from the above result that the vanishing of the group \( Z^{2n-2}(X) \) for \( X \) a rationally connected manifold reduces to the similar statement for \( X \) defined over a number field.

Let us now define an \( l \)-adic analogue \( Z^{2n-2}(X)_l \) of the group \( Z^{2n-2}(X) \) (cf. [6], [7]). Let \( X \) be a smooth projective variety defined over a field \( K \) which in the sequel will be either a finite field or a number field. Let \( \overline{K} \) be an algebraic closure of \( K \). Any cycle \( Z \in CH^*(X_{\overline{K}}) \) is defined over a finite extension of \( K \). Let \( l \) be a prime integer different from \( p = \text{char} \, K \) if \( K \) is finite. It follows that the cycle class

\[
cl(Z) \in H^{2n}_{et}(X_{\overline{K}}, \mathbb{Q}_l(s))
\]

is invariant under an open subgroup of \( \text{Gal}(\overline{K}/K) \).

Classes satisfying this property are called Tate classes. The Tate conjecture for finite fields asserts the following:

**Conjecture 1.4.** (cf. [18] for a recent account) Let \( X \) be smooth and projective over a finite field \( K \). The cycle class map gives for any \( s \) a surjection

\[
cl : CH^*(X_{\overline{K}}) \otimes \mathbb{Q}_l \to H^{2n}(X_{\overline{K}}, \mathbb{Q}_l(s))_{\text{Tate}}.
\]

Note that the cycle class defined on \( CH^*(X_{\overline{K}}) \) takes in fact values in \( H^{2n}(X_{\overline{K}}, \mathbb{Z}_l(s)) \), and more precisely in the subgroup \( H^{2n}(X_{\overline{K}}, \mathbb{Z}_l(s))_{\text{Tate}} \), of classes invariant under an open subgroup of \( \text{Gal}(\overline{K}/K) \). We thus get for each \( i \) a morphism

\[
cl^i : CH^*(X_{\overline{K}}) \otimes \mathbb{Z}_l \to H^{2i}(X_{\overline{K}}, \mathbb{Z}_l(i))_{\text{Tate}}.
\]
We can thus introduce the following variant of the groups \( Z_{\text{et}}^2(X) \):

\[
Z_{\text{et}}^2(X)_t := H^2_{\text{et}}(X_{\overline{K}}, \mathbb{Z}_l(i))_{\text{Tate}}/\text{Im} \text{cl}^t.
\]

An argument similar to the one used for the proof of Proposition 1.3 will lead to the following result:

**Proposition 1.5.** Let \( X \) be a smooth rationally connected variety defined over a number field \( K \), with ring of integers \( \mathcal{O}_K \). Assume given a projective model \( X' \) of \( X \) over \( \text{Spec} \mathcal{O}_K \). Fix a prime integer \( l \). Then except for finitely many \( p \in \text{Spec} \mathcal{O}_K \), the group \( Z_{\text{et}}^{2n-2}(X)_t \) is isomorphic to the group \( Z_{\text{et}}^{2n-2}(X'_p)_t \).

In the course of the paper, we will also consider variants \( Z_{\text{rat}}^{2n-2}(X) \), resp. \( Z_{\text{et,rat}}^{2n-2}(X)_t \), of the groups \( Z_{\text{et}}^{2n-2}(X) \), resp. \( Z_{\text{et,rat}}^{2n-2}(X)_t \), obtained by taking the quotient of the group of integral Hodge classes (resp. integral \( l \)-adic Tate classes) by the subgroup generated by classes of rational curves. This variant is suggested by Kollár’s paper (cf. [16, Question 3, (1)]). By the same arguments, these groups are also deformation and specialization invariants for rationally connected varieties.

Our last result is conditional but it strongly suggests the vanishing of the group \( Z_{\text{et}}^{2n-2}(X) \) for \( X \) a smooth rationally connected variety over \( \mathbb{C} \). Indeed, we will prove using the main result of [19] and the two propositions above the following consequence of Theorem 1.5:

**Theorem 1.6.** Assume Tate’s conjecture 1.4 holds for degree 2 Tate classes on smooth projective surfaces defined over a finite field. Then the group \( Z_{\text{et}}^{2n-2}(X) \) is trivial for any smooth rationally connected variety \( X \) over \( \mathbb{C} \).

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It is a pleasure to dedicate this note to Joe Harris, whose influence on the subject of rational curves on algebraic varieties (among other topics!) is invaluable.

### 2 Deformation and specialization invariance

**Proof of Proposition 1.3.** We first observe that, due to the fact that relative Hilbert schemes parameterizing curves in the fibers of \( B \) are a countable union of varieties which are projective over \( B \), given a simply connected open set \( U \subset B \) (in the classical topology of \( B \)), and a class \( \alpha \in \Gamma(U, R^{2n-2} \pi_* \mathbb{Z}) \) such that \( \alpha_t \) is algebraic for \( t \in V \), where \( V \) is a smaller nonempty open set \( V \subset U \), then \( \alpha_t \) is algebraic for any \( t \in U \).

To prove the deformation invariance, we just need using the above observation to prove the following:

**Lemma 2.1.** Let \( t \in U \subset B \), and let \( C \subset X_t \) be a curve and let \( [C] \in H^{2n-2}(X_t, \mathbb{Z}) \cong \Gamma(U, R^{2n-2} \pi_* \mathbb{Z}) \) be its cohomology class. Then the class \([C]_s\) is algebraic for \( s \) in a neighborhood of \( t \) in \( U \).

**Proof of Lemma 2.1.** By results of [15], there are rational curves \( R_t \subset X_t \) with ample normal bundle which meet \( C \) transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number \( D \) of such curves with generically chosen tangent directions at the attachment points. We then know by [10, §2.1] that the curve \( C' = C \cup_{i \in D} R_t \) is smoothable in \( X_t \) to a smooth unobstructed curve \( C'' \subset X_t \), that is \( H^1(C'', N_{C''/X_t}) = 0 \). This curve \( C'' \) then deforms with \( X_t \) (cf. [12], [13, II.1]) in the sense that the morphism from the deformation of the pair \( (C'', X_t) \) to \( B \) is smooth, and in particular open. So there is a neighborhood of \( V \) of \( t \) in \( U \) such that for \( s \in V \), there is a curve \( C''_s \subset X_t \) which is a deformation of \( C'' \subset X_t \). The class \([C''_s] = [C'']_s\) is thus algebraic on \( X_s \). On the other hand, we have

\[
[C''_s] = [C''] = [C] + \sum_i [R_t].
\]
As the $R_t$'s are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_t]_s$ also are algebraic on $X_t$ for $s$ in a neighborhood of $t$ in $U$. Thus $[C]_s = [C']_s - \sum [R_t]_s$ is algebraic on $X_s$ for $s$ in a neighborhood of $t$ in $U$. The lemma, hence also the proposition, is proved.

\[\text{Remark 2.2.}\] There is an interesting variant of the group $Z^{2n-2}(X)$, which is suggested by Kollár (cf. [16]) given by the following groups:

$$Z^{2n-2}_{\text{rat}}(X) := H^{2n-2}(X, \mathbb{Z})/(\langle C \rangle, C \text{ rational curve in } X).$$

Here, by a rational curve, we mean an irreducible curve whose normalization is rational. These groups are torsion for $X$ rationally connected, as proved by Kollár ([13, Theorem 3.13 p 206]). It is quite easy to prove that they are birationally invariant.

The proof of Proposition 1.3 gives as well the following result (already noticed by Kollár [16]):

\[\text{Variant 2.3.}\] If $X \to B$ is a smooth projective morphism with rationally connected fibers, the groups $Z^{2n-2}_{\text{rat}}(X_t)$ are local deformation invariants.

Let us give one application of Proposition 1.3 (or rather its proof) and/or its variant 2.3.

\[\text{Theorem 2.4.}\] 1) Assume that the sheaves $\mathcal{E} \otimes \mathcal{I}_{C_i}$ are generated by global sections for $i = 1, \ldots, k$. Then if $X_\sigma$ is smooth rationally connected for general $\sigma$, the group $Z^{2n-2}(X_\sigma)$ vanishes for any $\sigma$ such that $X_\sigma$ is smooth of dimension $n$.

2) Under the same assumptions as in 1), assume the curves $C_i \subset X$ are rational. Then if $X_\sigma$ is smooth rationally connected for general $\sigma$, the group $Z^{2n-2}_{\text{rat}}(X_\sigma)$ vanishes for any $\sigma$ such that $X_\sigma$ is smooth of dimension $n$.

\[\text{Proof.}\] 1) Let $j_\sigma : X_\sigma \to X$ be the inclusion map. Since $n \geq 3$ and $\mathcal{E}$ is ample, by Sommese's theorem [20], the Gysin map $j_{\sigma*} : H^{2n-2}(X_\sigma, \mathbb{Z}) \to H^{2n+2r-2}(X, \mathbb{Z})$ is an isomorphism. It follows that the group $H^{2n-2}(X_\sigma, \mathbb{Z})$ is a constant group. In order to show that $Z^{2n-2}(X_\sigma)$ is trivial, it suffices to show that the classes $(j_{\sigma*})^{-1}(\langle C_i \rangle)$ are algebraic on $X_\sigma$ since they generate $H^{2n-2}(X_\sigma, \mathbb{Z})$. Since the $X_\sigma$'s are rationally connected, Theorem 1.3 tells us that it suffices to show that for each $i$, there exists a $\sigma(i)$ such that $X_{\sigma(i)}$ is smooth $n$-dimensional and that the class $(j_{\sigma(i)*})^{-1}(\langle C_i \rangle)$ is algebraic on $X_{\sigma(i)}$.

It clearly suffices to exhibit one smooth $X_{\sigma(i)}$ containing $C_i$, which follows from the following lemma:

\[\text{Lemma 2.5.}\] Let $X$ be a variety of dimension $n + r$ with $n \geq 2$, $C \subset X$ be a smooth curve, $\mathcal{E}$ be a rank $r$ vector bundle on $X$ such that $\mathcal{E} \otimes \mathcal{I}_{C}$ is generated by global section. Then for a generic $\sigma \in H^0(X, \mathcal{E} \otimes \mathcal{I}_{C})$, the zero set $X_\sigma$ is smooth of dimension $n$.

\[\text{Proof.}\] The fact that $X_\sigma$ is smooth of dimension $n$ away from $C$ is standard and follows from the fact that the incidence set $(\sigma, x) \in P(H^0(X, \mathcal{E} \otimes \mathcal{I}_{C})) \times (X \setminus C)$, $\sigma(x) = 0$ is smooth of dimension $n + N$, where $N := \dim P(H^0(X, \mathcal{E} \otimes \mathcal{I}_{C}))$. It thus suffices to check the smoothness along $C$ for generic $\sigma$.

This is checked by observing that since $\mathcal{E} \otimes \mathcal{I}_{C}$ is generated by global sections, its restriction $\mathcal{E} \otimes N^*_C/X$ is also generated by global sections. This implies that for each point $c \in C$, the condition that $X_\sigma$ is singular at $c$ defines a codimension $n$ closed algebraic subset $P_c$ of $P := P(H^0(X, \mathcal{E} \otimes \mathcal{I}_{C}))$, determined by the condition that $\sigma|_c : N^*_C/X.c \to \mathcal{E}_c$ is not surjective. Since $\dim C = 1$, the union of the $P_c$’s cannot be equal to $P$ if $n \geq 2$.

This concludes the proof of 1) and the proof of 2) works exactly in the same way.
Proof of Proposition 1.5. Let \( p \in \text{Spec} \mathcal{O}_K \), with residue field \( k(p) \). Assume \( X_p \) is smooth. For \( l \) prime to char \( k(p) \), the (adequately constructed) specialization map

\[
H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1)) \to H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))
\]

is then an isomorphism (cf. [17, Chapter VI, 4]).

Observe also that since \( X_p \) is rationally connected, the rational étale cohomology group \( H_{ct}^{2n-2}(X_p, \mathbb{Q}_l(n-1)) \) is generated over \( \mathbb{Q}_l \) by curve classes. Hence the same is true for \( H_{ct}^{2n-2}(X_p, \mathbb{Q}_l(n-1)) \). Thus the whole cohomology groups

\[
H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1)), H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))
\]

consist of Tate classes, and (2) gives an isomorphism

\[
H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))_{\text{Tate}} \to H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))_{\text{Tate}}.
\]

In order to prove Proposition 1.5, it thus suffices to prove the following:

**Lemma 2.6.** 1) For almost every \( p \in \text{Spec} \mathcal{O}_K \), the fiber \( X_p \) is smooth and separably rationally connected.

2) If \( X_p \) is smooth and separably rationally connected, for any curve \( C_p \subset X_p \), the inverse image \( [C_p]_{\mathbb{P}^1} \in H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1)) \) of the class \( [C_p] \in H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1)) \) via the isomorphism (3) is the class of a 1-cycle on \( X_p \).

**Proof.** 1) When the fiber \( X_p \) is smooth, the separable rational connectedness of \( X_p \) is equivalent to the existence of a smooth rational curve \( C_p \cong \mathbb{P}^1_{k(p)} \) together with a morphism \( \phi : C_p \to X_p \) such that the vector bundle \( \phi^* T_{X_p} \) on \( \mathbb{P}^1_{k(p)} \) is a direct sum \( \oplus_i \mathcal{O}_{\mathbb{P}^1_{k(p)}}(a_i) \) where all \( a_i \) are positive. Equivalently

\[
H^1(\mathbb{P}^1_{k(p)}, \phi^* T_{X_p}(-2)) = 0.
\]

The smooth projective variety \( X_p \) being rationally connected in characteristic 0, it is separably rationally connected, hence there exists a finite extension \( K' \) of \( K \), a curve \( C \) and a morphism \( \phi : C \to X \) defined over \( K' \), such that \( C \cong \mathbb{P}^1_{K'} \) and \( H^1(\mathbb{P}^1_{K'}, \phi^* T_{X_{K'}}(-2)) = 0 \).

We choose a model

\[
\Phi : C \cong \mathbb{P}^1_{\mathcal{O}_{K'}} \to X'
\]

doing defined over a Zariski open set of \( \text{Spec} \mathcal{O}_{K'} \). By upper-semi-continuity of cohomology, the vanishing (4) remains true after restriction to almost every closed point \( p \in \text{Spec} \mathcal{O}_{K'} \), which proves 1).

2) The proof is identical to the proof of Proposition 1.3: we just have to show that the curve \( C_p \subset X_p \) is algebraically equivalent in \( X_p \) to a difference \( C_p - \sum_i R_i, p \), where each curve \( C'_{p_i} \) resp. \( R_i, p \) (they are in fact defined over a finite extension \( k(p)'' \) of \( k(p) \)), lifts to a curve \( C''_{p_i} \) resp. \( R_i \) in \( X_{K'} \), for some finite extension \( K' \) of \( K \).

Assuming the curves \( C''_{p_i}, R_i, p \) are smooth, the existence of such a lifting is granted by the condition \( H^1(C''_{p_i}, N_{C''_{p_i}/X_p}) = 0 \), resp. \( H^1(R_i, p, N_{R_i, p/X_p}) = 0 \).

Starting from \( C \subset X_p \) where \( X_p \) is separably rationally connected over \( p \), we obtain such curves \( C''_{p_i}, R_i, p \) as in the previous proof, applying [10, §2.1].

The proof of Proposition 1.5 is finished.

Again, this proof leads as well to the proof of the specialization invariance of the \( l \)-adic analogues \( Z_{ct, rad}^{2n-2}(X) \) of the groups \( Z_{ct}^{2n-2}(X) \) introduced in Remark 2.2.

**Variant 2.7.** Let \( X \) be a smooth rationally connected variety defined over a number field \( K \), with ring of integers \( \mathcal{O}_K \). Assume given a projective model \( X \) of \( X \) over \( \text{Spec} \mathcal{O}_K \). Fix a prime integer \( l \). Then for any \( p \in \text{Spec} \mathcal{O}_K \) such that \( X_p \) is smooth separably connected, the group \( Z_{ct, rad}^{2n-2}(X)_l \) is isomorphic to the group \( Z_{ct, rad}^{2n-2}(X_p)_l \).
3 Consequence of a result of Chad Schoen

In [19], Chad Schoen proves the following theorem:

**Theorem 3.1.** Let $X$ be a smooth projective variety of dimension $n$ defined over a finite field $k$ of characteristic $p$. Assume that the Tate conjecture holds for degree 2 Tate classes on smooth projective surfaces defined over a finite extension of $k$. Then the étale cycle class map:

$$\text{cl} : CH^{n-1}(X_{\overline{k}}) \otimes \mathbb{Z}_l \rightarrow H^{2n-2}(X_{\overline{k}}, \mathbb{Z}_l(n-1))_{\text{Tate}}$$

is surjective, that is $Z^{2n-2}_\text{et}(X)_{l} = 0$.

In other words, the Tate conjecture 1.4 for degree 2 rational Tate classes implies that the groups $Z^{2n-2}_\text{et}(X)_{l}$ should be trivial for all smooth projective varieties defined over finite fields. This is of course very different from the situation over $\mathbb{C}$ where the groups $Z^{2n-2}(X)$ are known to be possibly nonzero.

**Remark 3.2.** There is a similarity between the proof of Theorem 3.1 and the proof of Theorem 1.1. Schoen proves that given an integral Tate class $\alpha$ on $X$ (defined over a finite field), there exist a smooth complete intersection surface $S \subset X$ and an integral Tate class $\beta$ on $S$ such that $j_S^! \alpha = \alpha$ where $j_S$ is the inclusion of $S$ in $X$. The result then follows from the fact that if the Tate conjecture holds for degree 2 rational Tate classes on $S$, it holds for degree 2 integral Tate classes on $S$.

I prove that for $X$ a uniruled or Calabi-Yau, and for $\beta \in Hdg^2(X, \mathbb{Z})$ there exists surfaces $S_i \hookrightarrow X$ (in an adequately chosen linear system on $X$) and integral Hodge classes $\beta_i \in Hdg^2(S_i, \mathbb{Z})$ such that $\alpha = \sum_i j_{S_i}^! \beta_i$. The result then follows from the Lefschetz theorem on $(1,1)$-classes applied to the $\beta_i$.

We refer to [7] for some comments on and other applications of Schoen’s theorem, and conclude this note with the proof of the following theorem (cf. Theorem 1.6 of the introduction).

**Theorem 3.3.** Assume Tate’s conjecture 1.4 holds for degree 2 Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety $X$ over $\mathbb{C}$.

**Proof.** We first recall that for a smooth rationally connected variety $X$, the group $Z^{2n-2}(X)$ is equal to the quotient $H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{\text{alg}}$, due to the fact that the Hodge structure on $H^{2n-2}(X, \mathbb{Q})$ is trivial. In fact, we have more precisely

$$H^{2n-2}(X, \mathbb{Q}) = H^{2n-2}(X, \mathbb{Q})_{\text{alg}}$$

by hard Lefschetz theorem and the fact that

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{\text{alg}}$$

by the Lefschetz theorem on $(1,1)$-classes.

Next, in order to prove that $Z^{2n-2}(X)$ is trivial, it suffices to prove that for each $l$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\text{Im cl}) \otimes \mathbb{Z}_l$ is trivial.

We apply Proposition 1.3 which tells as well that over $\mathbb{C}$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}$ is locally deformation invariant for families of smooth rationally connected varieties. Note that our smooth projective rationally connected variety $X$ is the fiber $X_t$ of a smooth projective morphism $\phi : X \rightarrow B$ defined over a number field, where $X$ and $B$ are quasiprojective, geometrically connected and defined over a number field. By local deformation invariance, the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is equivalent to the vanishing of $Z^{2n-2}(X_t) \otimes \mathbb{Z}_l$ for any point $t' \in B(\mathbb{C})$. Taking for $t'$ a point of $B$ defined over a number field, $X_{t'}$ is defined over a number field. Hence it suffices to prove the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ for $X$ rationally connected defined over a number field $L$. 

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We have
\[ Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/\left(\text{Im } \text{cl} \otimes \mathbb{Z}_l \right), \]
and by the Artin comparison theorem (cf. [17, Chapter III, §3]), this is equal to
\[ \frac{H^{2n-2}(X, \mathbb{Z}_l(n-1))}{\left(\text{Im } \text{cl} \otimes \mathbb{Z}_l \right)} = Z^{2n-2}_{et}(X)_l \]
since \( H^{2n-2}_c(X, \mathbb{Z}_l(n-1)) \) consists of Tate classes. Hence it suffices to prove that for \( X \) rationally connected defined over a number field and for any \( l \), the group \( Z^{2n-2}_{et}(X)_l \) is trivial.

We now apply Proposition 1.5 to \( X \) and its reduction \( X_p \) for almost every closed point \( p \in \text{Spec } \mathcal{O}_L \). It follows that the vanishing of \( Z^{2n-2}_{et}(X)_l \) is implied by the vanishing of \( Z^{2n-2}_{et}(X_p)_l \). According to Schoen’s theorem 3.1, the last vanishing is implied by the Tate conjecture for degree 2 Tate classes on smooth projective surfaces.

Remark 3.4. This argument does not say anything on the groups \( Z^{2n-2}_{rat}(X) \), since there is no control on the 1-cycles representing given degree 2 \( n-2 \) Tate classes on varieties defined over finite fields. Similarly, Theorem 1.1 does not say anything on \( Z^4_{rat}(X) \) for \( X \) a rationally connected threefold.

References


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