

ON THE CHOW RING OF A K3 SURFACE

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ABSTRACT. We show that the Chow group of 0-cycles on a K3 surface contains a class of degree 1 with remarkable properties: any product of divisors is proportional to this class, and so is the second Chern class c_2 .

1. INTRODUCTION

An important algebraic invariant of a projective manifold X is the Chow ring $\mathrm{CH}(X)$ of algebraic cycles on X modulo rational equivalence. It is graded by the codimension of cycles; the ring structure comes from the intersection product. For a surface we have

$$\mathrm{CH}(X) = \mathbf{Z} \oplus \mathrm{Pic}(X) \oplus \mathrm{CH}_0(X) ,$$

where the group $\mathrm{CH}_0(X)$ parametrizes 0-cycles on X . While the structure of the Picard group $\mathrm{Pic}(X)$ is well understood, this is not the case for $\mathrm{CH}_0(X)$: if X admits a nonzero holomorphic 2-form, it is a huge group, which cannot be parametrized by an algebraic variety [M].

Among the simplest examples of such surfaces are the K3 surfaces, which carry a nowhere vanishing holomorphic 2-form. In this case $\mathrm{Pic}(X)$ is a lattice, while $\mathrm{CH}_0(X)$ is very large; the following result is therefore somewhat surprising:

Theorem 1. *Let X be a K3 surface.*

- a) *All points of X which lie on some (possibly singular) rational curve have the same class c_X in $\mathrm{CH}_0(X)$.*
- b) *The image of the intersection product*

$$\mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \rightarrow \mathrm{CH}_0(X)$$

is contained in $\mathbf{Z}c_X$.

- c) *The second Chern class $c_2(X) \in \mathrm{CH}_0(X)$ is equal to $24c_X$.*

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The proof is elementary in the sense that it only appeals to simple geometric constructions, based on the existence of sufficiently many rational and elliptic curves on X . We prove a) and b) in section 2; the proof of c), which is more involved, is given in section 3. If we represent the class c_X by a point c of X , a key property of this class is the formula

$$(x, x) - (x, c) - (c, x) + (c, c) = 0 \quad \text{in} \quad \text{CH}_0(X \times X),$$

valid for any $x \in X$. In section 4 we discuss the importance of this formula and its relation with property b) of the Theorem. We prove that an analogous formula holds when X is replaced by a hyperelliptic curve, but that it cannot hold for a generic curve C of genus ≥ 3 – we show that this would imply that C is algebraically equivalent to $-C$ in its Jacobian, contradicting a result of Ceresa.

2. THE IMAGE OF THE INTERSECTION PRODUCT

We work over the complex numbers. By a *rational curve* on a surface we mean an irreducible (but possibly singular) curve of geometric genus zero. If V is an algebraic variety and $p \in \mathbf{N}$, we denote by $\text{CH}_p(V)$ the group of p -dimensional cycles on V modulo rational equivalence; we put $\text{CH}_p(V)_{\mathbf{Q}} = \text{CH}_p(V) \otimes \mathbf{Q}$.

2.1. *Proof of a) and b).* Let R be a rational curve on X ; it is the image of a generically injective map $j : \mathbf{P}^1 \rightarrow X$. Put $c_R = j_*(p) \in \text{CH}_0(X)$, where p is an arbitrary point of \mathbf{P}^1 . For any divisor D on X , we have in $\text{CH}_0(X)$

$$R \cdot D = j_* j^* D = j_*(n p) = n c_R, \quad \text{with } n = \deg(R \cdot D).$$

Let S be another rational curve. If $\deg(R \cdot S) \neq 0$, the above equality applied to $R \cdot S$ gives $c_S = c_R$ (recall that $\text{CH}_0(X)$ is torsion free [R]). If $\deg(R \cdot S) = 0$, choose an ample divisor H ; by a theorem of Bogomolov and Mumford [M-M], H is linearly equivalent to a sum of rational curves (this is proved in [M-M] assuming that the class of H in $\text{Pic}(X)$ is primitive; but any ample class is a multiple of an ample primitive class). Since H is connected, we can find a chain R_0, \dots, R_k of distinct rational curves such that $R_0 = R$, $R_k = S$ and $R_i \cap R_{i+1} \neq \emptyset$ for $i = 0, \dots, k-1$. We conclude from the preceding case that $c_R = c_{R_1} = \dots = c_S$. Thus the class c_R does not depend on the choice of R : this is assertion a) of the Theorem. Let us denote it by c_X .

We have $R \cdot D = \deg(R \cdot D) c_X$ for any divisor D and any rational curve R on X . Since the group $\text{Pic}(X)$ is spanned by the classes of rational curves (again by the Bogomolov-Mumford theorem), assertion b) follows. \square

Remark 2.2. The result (and the proof) hold more generally for any surface X such that:

- a) The Picard group of X is spanned by the classes of rational curves;
- b) There exists an ample divisor on X which is a sum of rational curves.

This is the case when X admits a non-trivial elliptic fibration over \mathbf{P}^1 with a section, or for some particular surfaces like Fermat surfaces in \mathbf{P}^3 with degree prime to 6 [S].

Remark 2.3. Let A be an abelian surface. According to [Bl], the image of the product map $\text{Pic}(A) \otimes \text{Pic}(A) \rightarrow \text{CH}_0(A)$ has finite index, so the situation looks rather different from the K3 case. There is however an analogue to the Theorem. Let $\text{Pic}^+(A)$ be the subspace of $\text{Pic}(A)_{\mathbf{Q}}$ fixed by the action of the involution $a \mapsto -a$. We have a direct sum decomposition

$$\text{Pic}(A)_{\mathbf{Q}} = \text{Pic}^+(A) \oplus \text{Pic}^{\circ}(A)_{\mathbf{Q}} ,$$

so that $\text{Pic}^+(A)$ is canonically isomorphic to the image of $\text{Pic}(A)_{\mathbf{Q}}$ in $H^2(A, \mathbf{Q})$. Now we claim that *the image of the map $\mu : \text{Pic}^+(A) \otimes \text{Pic}^+(A) \rightarrow \text{CH}_0(A)_{\mathbf{Q}}$ is $\mathbf{Q}[0]$* , where $[0] \in \text{CH}_0(A)$ denotes the class of the origin $0 \in A$. This is a direct consequence of the decomposition of $\text{CH}(A)_{\mathbf{Q}}$ described in [B]: let k be an integer ≥ 2 , and let \mathbf{k} be the multiplication by k in A . We have $\mathbf{k}^*D = k^2D$ for any element D of $\text{Pic}^+(A)$, thus $\mathbf{k}^*c = k^4c$ for any element c in the image of μ ; but the latter property characterizes the multiples of $[0]$. \square

2.4. The cycle class c_X has some remarkable properties that we will investigate in the next section. Let us observe first that for any irreducible curve C on X , there is a rational curve $R \neq C$ which intersects C ; thus we can represent c_X by the class of a point $c \in C$ (namely any point of $C \cap R$).

We will need a more subtle property of c_X . Let us first prove a lemma:

Lemma 2.5. *Let E be an elliptic curve, x, y two points of E . Then*

$$(x, x) - (x, y) - (y, x) + (y, y) = 0 \quad \text{in} \quad \text{CH}_0(E \times E) .$$

Since the divisors $[x] - [y]$ generate the group $\text{Pic}^{\circ}(E)$, this is equivalent to the formula $\text{pr}_1^*D \cdot \text{pr}_2^*D = 0$ in $\text{CH}_0(E \times E)$ for every D in $\text{Pic}^{\circ}(E)$.

Proof: Put $\xi = (x, x) - (x, y) - (y, x) + (y, y)$. Then 2ξ is the pull-back of a 0-cycle $\eta = (x, x) + (y, y) - 2(x, y)$ on the second symmetric product \mathbf{S}^2E . The addition map $a : \mathbf{S}^2E \rightarrow E$ is a \mathbf{P}^1 -fibration; this implies that the push-down map $a_* : \text{CH}_0(E \times E) \rightarrow \text{CH}_0(E)$ is an isomorphism. Since $a_*\eta = 0$, we have $\eta = 0$, hence $2\xi = 0$. On the other hand ξ has degree 0 and its image in the Albanese variety of $E \times E$ is zero, so $\xi = 0$ by Rojzman's result. \square

Proposition 2.6. *Let Δ be the diagonal embedding of X into $X \times X$.*

- a) *For every $\alpha \in \text{CH}_1(X)$, we have*

$$\Delta_*\alpha = \text{pr}_1^*\alpha \cdot \text{pr}_2^*c_X + \text{pr}_1^*c_X \cdot \text{pr}_2^*\alpha \quad \text{in} \quad \text{CH}_1(X \times X) ;$$

b) For every $\xi \in \text{CH}_0(X)$, we have

$$\Delta_*\xi = \text{pr}_1^*\xi \cdot \text{pr}_2^*c_X + \text{pr}_1^*c_X \cdot \text{pr}_2^*\xi - (\deg \xi) \Delta_*c_X \quad \text{in} \quad \text{CH}_0(X \times X) .$$

Proof: a) Since both sides are additive in α , it is enough to check this relation when α is the class of a rational curve; in that case it follows from the fact that the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$ is linearly equivalent to $\mathbf{P}^1 \times \{0\} + \{0\} \times \mathbf{P}^1$.

b) Again both sides are additive in ξ , so we may assume that ξ is the class of a point $x \in X$. The Bogomolov-Mumford theorem tells us that x lies on the image of a curve E of genus ≤ 1 ; by 2.4 we can represent c_X by a point $c \in E$. We have $(x, x) - (x, c) - (c, x) + (c, c) = 0$ in $\text{CH}_0(E \times E)$ by lemma 2.5 (the case when E is rational is trivial); by push-down this gives the same formula in $\text{CH}_0(X \times X)$. \square

3. THE FORMULA $c_2(X) = 24 c_X$

3.1. Let c be a point of X lying on some rational curve. We will denote by (x, x, x) , (x, x, c) , (x, c, c) , etc... the classes in $\text{CH}_2(X \times X \times X)$ of the image of X by the maps $x \mapsto (x, x, x)$, $x \mapsto (x, x, c)$, $x \mapsto (x, c, c)$, etc... With this notation we have the following key result:

Proposition 3.2. *The cycle*

$$x = (x, x, x) - (c, x, x) - (x, c, x) - (x, x, c) + (x, c, c) + (c, x, c) + (c, c, x)$$

is zero in $\text{CH}_2(X \times X \times X)_{\mathbf{Q}}$.

Corollary 3.3. *Let Δ , i_c and j_c be the maps of X into $X \times X$ defined by $\Delta(x) = (x, x)$, $i_c(x) = (x, c)$ and $j_c(x) = (c, x)$. For every ξ in $\text{CH}_2(X \times X)$, we have an equality in $\text{CH}_0(X)$*

$$\Delta^*\xi = i_c^*\xi + j_c^*\xi + n c, \quad \text{with} \quad n = \deg(\Delta^*\xi - i_c^*\xi - j_c^*\xi) .$$

From this formula we recover part b) of the Theorem by taking $\xi = \text{pr}_1^*\alpha \cdot \text{pr}_2^*\beta$, with $\alpha, \beta \in \text{Pic}(X)$, and we get part c) by taking for ξ the class of the diagonal, so that $\Delta^*\xi = c_2(X)$.

3.4. *Proof of the Corollary.* We will denote by p_i , for $1 \leq i \leq 3$, the projection of $X \times X \times X$ onto the i -th factor, and by p_{ij} , for $1 \leq i < j \leq 3$, the projection $(x_1, x_2, x_3) \mapsto (x_i, x_j)$.

Let us compute $p_{3*}(x \cdot p_{12}^*\xi)$. Let $\delta : X \rightarrow X \times X \times X$ be the map $x \mapsto (x, x, x)$. We have $p_{3*}\delta = \text{Id}_X$ and $p_{12*}\delta = \Delta$, hence

$$p_{3*}((x, x, x) \cdot p_{12}^*\xi) = p_{3*}\delta_*(\delta^*p_{12}^*\xi) = \Delta^*\xi .$$

The same argument applied to the maps $x \mapsto (c, x, x)$, $x \mapsto (x, c, x)$, ... , gives

$$\begin{aligned} p_{3*}((c, x, x) \cdot p_{12}^* \xi) &= i_c^* \xi & p_{3*}((x, c, x) \cdot p_{12}^* \xi) &= j_c^* \xi \\ p_{3*}((x, x, c) \cdot p_{12}^* \xi) &= \deg(\Delta^* \xi) \cdot c & p_{3*}((x, c, c) \cdot p_{12}^* \xi) &= \deg(i_c^* \xi) \cdot c \\ p_{3*}((c, x, c) \cdot p_{12}^* \xi) &= \deg(j_c^* \xi) \cdot c & p_{3*}((c, c, x) \cdot p_{12}^* \xi) &= 0, \end{aligned}$$

hence our formula. \square

Remark 3.5. One also recovers Proposition 2.6 b) by restricting the class x to the slices $X \times X \times \{x\} \subset X \times X \times X$ corresponding to all $x \in X$.

For the proof of the proposition we will need two results on products of elliptic curves. Let F be an elliptic curve over an arbitrary field. We denote by $\text{Pic}(F^3)^{\text{inv}}$ the subgroup of elements of $\text{Pic}(F^3)_{\mathbf{Q}}$ which are invariant under permutations of the factors and under the involution (-1_{F^3}) . We keep the notation of 3.1.

Lemma 3.6. a) *The cycle class*

$$\mathbf{v} = (u, u, u) - (0, u, u) - (u, 0, u) - (u, u, 0) + (u, 0, 0) + (0, u, 0) + (0, 0, u)$$

in $\text{CH}_1(F^3)_{\mathbf{Q}}$ is zero.

b) *The divisors $\alpha_F = \sum_i p_i^* 0$ and $\beta_F = \sum_{i < j} p_{ij}^* \Delta$ form a basis of $\text{Pic}(F^3)^{\text{inv}}$.*

Proof: a) The class \mathbf{v} is symmetric, hence comes from a cycle class $\bar{\mathbf{v}}$ in the third symmetric product $\mathbf{S}^3 F$. This variety is a \mathbf{P}^2 -bundle over F , through the addition map $a : \mathbf{S}^3 F \rightarrow F$. Thus we have $\text{CH}_1(\mathbf{S}^3 F) = a^* \text{Pic}(F) \cdot h \oplus \mathbf{Z}h^2$, where h is any divisor class on $\mathbf{S}^3 F$ which induces on a fibre $a^{-1}(u) \cong \mathbf{P}^2$ the class of a line.

Write $\bar{\mathbf{v}} = (a^* d) \cdot h + nh^2$. We have $n = \deg(\bar{\mathbf{v}} \cdot a^* 0) = 3^2 - 3 \cdot 2^2 + 3 \cdot 1 = 0$, hence $d = a_*(\bar{\mathbf{v}} \cdot h)$. We can represent h by the image of the divisor $p_1^* 0$ in $F \times F \times F$; since $\mathbf{v} \cdot p_1^* 0 = 0$, we get $d = 0$ and finally $\mathbf{v} = 0$.

b) As above we have $\text{Pic}(\mathbf{S}^3 F) = a^* \text{Pic}(F) \oplus \mathbf{Z}h$. Taking the invariants under (-1_{F^3}) we see that $\text{Pic}(F^3)^{\text{inv}}$ has rank 2. Thus it suffices to prove that the divisors α_F and β_F are not proportional in $\text{Pic}(F^3)$; but their restriction to F^2 (embedded in F^3 by $(u, v) \mapsto (u, v, 0)$) are $p_1^* 0 + p_2^* 0$ and $\Delta + p_1^* 0 + p_2^* 0$, which are clearly non-proportional. \square

3.7. Proof of Proposition 3.2. It will make our life easier to assume that $\text{Pic}(X)$ is generated by an ample divisor class H ; the general case will follow by specialization (see [SGA6], X.7.14). By the Bogomolov-Mumford theorem, we can find in the linear system $|H|$ a one-dimensional family $(E'_b)_{b \in B}$ of (singular) elliptic curves; that is, we can find a surface E with a fibration $p : E \rightarrow B$ onto a smooth curve, with general fibre a smooth curve E_b of genus

1, and a generically finite map $\pi : E \rightarrow X$ which maps each fibre E_b of p birationally onto the singular curve E'_b . Passing to a covering of B if necessary, we may assume that:

- a) p has a section $0 : B \rightarrow E$;
- b) The curve $\pi(0_B)$ is *rational*.

(To see b), replace B by a component of $\pi^{-1}(R)$, where R is a rational curve on X not contained in any E'_b .)

Note that because of the assumption on $\text{Pic}(X)$ every fibre E_b is irreducible.

3.8. Using again the notation of 3.1, we consider on the fibre product $E_B^3 = E \times_B E \times_B E$ the cycle class

$$\mathbf{u} = (u, u, u) - (0_{pu}, u, u) - (u, 0_{pu}, u) - (u, u, 0_{pu}) + (u, 0_{pu}, 0_{pu}) + (0_{pu}, u, 0_{pu}) + (0_{pu}, 0_{pu}, u) .$$

For $b \in B$, the class in $\text{CH}_2(X \times X \times X)$ of the cycle

$$\{c\} \times E'_b \times E'_b + E'_b \times \{c\} \times E'_b + \{c\} \times E'_b \times E'_b$$

does not depend on b , since the curves E'_b all belong to the same linear system $|H|$; let us denote it by \mathfrak{z} . Let $\pi^3 : E_B^3 \rightarrow X^3$ be the morphism deduced from π .

Lemma 3.9. *The class $\pi_*^3(\mathbf{u})$ is proportional to \mathfrak{z} .*

Proof: By lemma 3.6.a), the restriction of \mathbf{u} to the generic fibre of the fibration $E_B^3 \rightarrow B$ is zero. It follows that \mathbf{u} is a sum of cycles of the form $i_{b*}D_b$, where i_b is the inclusion of E'_b into E_B^3 and D_b a (Weil) divisor on E'_b [Bl-S].

The involution σ of E which coincides with $u \mapsto -u$ on each smooth fibre gives rise to an involution σ^3 of E_B^3 which commutes with the action of \mathfrak{S}_3 by permutations of the factors. The cycle \mathbf{u} is invariant by this action of $\mathfrak{S}_3 \times \mathbf{Z}/2$. By averaging on this group we may choose the above divisor classes D_b in the invariant subgroup of $\text{CH}_2(E'_b)_{\mathbf{Q}}$. We want to prove that each cycle class $i_{b*}D_b$ is pushed down to a multiple of \mathfrak{z} by π^3 .

Assume first that the curve E_b is smooth. By lemma 3.6.b) the class D_b is a \mathbf{Q} -linear combination of $\alpha_{E'_b}$ and $\beta_{E'_b}$. By 3.7.b) $\pi(0_b)$ is linearly equivalent to c , thus we have $\pi_*^3(i_{b*} \alpha_{E'_b}) = \mathfrak{z}$. The cycle $\pi_*^3(i_{b*} \beta_{E'_b})$ is the sum of $(\Delta_* E'_b) \times E'_b$ and of the two cycles obtained by permutation of the factors. Now using lemma 2.4 this class is equivalent to $2\mathfrak{z}$, hence the result in this case.

If E_b is singular, its normalization \tilde{E}_b is a smooth rational curve, and we have a surjective homomorphism $\text{CH}_2(\tilde{E}_b)_{\mathbf{Q}} \rightarrow \text{CH}_2(E'_b)_{\mathbf{Q}}$. The \mathfrak{S}_3 -invariant part of $\text{CH}_2(\tilde{E}_b)_{\mathbf{Q}}$ is spanned by the divisor $\alpha_{\tilde{E}_b} = \sum p_i^* 0$ which again maps to a cycle linearly equivalent to \mathfrak{z} under π^3 . \square

Lemma 3.10. *Let $d = \deg \pi$, and let \mathfrak{x} be the cycle class defined in 3.2. Then*

$$\pi_*^3(\mathbf{u}) = d \mathfrak{x} .$$

Proof: We compute the images under π_*^3 of the cycles which appear in the definition of \mathfrak{x} .

a) We have $\pi_*^3(u, u, u) = d(x, x, x)$ in $\mathrm{CH}_2(X \times X \times X)$.

b) Let $\Gamma \subset X \times X$ be the image of the surface $(u, 0_{pu})$ (that is, the graph of $0 \circ p$) in $E \times E$. We have $\pi_*^3(u, u, 0_{pu}) = p_{12}^* \Delta \cdot p_{23}^* \Gamma$. The normalization \tilde{R} of $R = \pi(0_B)$ is a smooth rational curve (3.7.b). Since our cycle Γ is supported by $X \times R$, it comes from a divisor Γ_0 in $X \times \tilde{R}$. Such a divisor is of the form $D \times \tilde{R} + mX \times \{r\}$ for some divisor D on X , some point $r \in \tilde{R}$ and some integer m ; this integer is equal to the degree of Γ_0 over X , that is d . Therefore Γ is linearly equivalent to $d(X \times c) + D \times R$; since we assume $\mathrm{Pic}(X) = \mathbf{Z}$ we have $D \times R = a E'_b \times E'_b$ in $\mathrm{CH}_2(X \times X)$ for some integer a and any $b \in B$. Intersecting with $p_{12}^* \Delta$ we get

$$\pi_*^3(u, u, 0_{pu}) = d(x, x, c) + a (\Delta_* E'_b) \times E'_b .$$

c) We have $\pi_*^3(u, 0_{pu}, 0_{pu}) = p_{12}^* \Gamma \cdot p_{23}^* \Delta$; reasoning as in b) we find

$$\pi_*^3(u, 0_{pu}, 0_{pu}) = d(x, c, c) + a E'_b \times (\Delta_* E'_b) .$$

d) The lemma follows by permuting and summing. □

3.11. Therefore $\mathfrak{x} = d^{-1} \pi_*^3(\mathbf{u})$ is proportional to the effective cycle \mathfrak{z} (Lemma 3.9). On the other hand \mathfrak{x} is homologically trivial: this follows from the Künneth formula and the fact that the cycles $p_{ij*} \mathfrak{x}$ are identically zero. Thus we obtain $\mathfrak{x} = 0$, which concludes the proof of the Proposition. □

4. 0-CYCLES ON A PRODUCT

4.1. The cycle c_X has two remarkable properties, namely the intersection property b) of the Theorem, and the diagonal property $(x, x) - (x, c) - (c, x) + (c, c) = 0$ in $\mathrm{CH}_0(X \times X)$ for any $x \in X$. These two properties may seem unrelated. However we can rephrase them in the following way, which shows that they are in some sense dual to each other: since the Picard group of X is isomorphic to its Néron-Severi group, the degree 1 zero-cycle c_X provides a splitting of $\mathrm{CH}(X)$ as

$$\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{hom}} \oplus \mathrm{H},$$

where $\mathrm{CH}(X)_{\mathrm{hom}}$ is the subgroup of 0-cycles homologous to 0, and H the image of $\mathrm{CH}(X)$ into $\mathrm{H}^*(X, \mathbf{Z})$ via the cycle map. This splitting induces the splitting

$$\mathrm{CH}(X) \otimes \mathrm{CH}(X) =$$

$$(\mathrm{CH}(X)_{\mathrm{hom}} \otimes \mathrm{CH}(X)_{\mathrm{hom}}) \oplus (\mathrm{CH}(X)_{\mathrm{hom}} \otimes \mathrm{H}) \oplus (\mathrm{H} \otimes \mathrm{CH}(X)_{\mathrm{hom}}) \oplus (\mathrm{H} \otimes \mathrm{H})$$

of $\mathrm{CH}(X) \otimes \mathrm{CH}(X)$. It is immediate to see that it induces one on the image $\mathrm{CH}(X \times X)_{\mathrm{dec}}$ of $\mathrm{CH}(X) \otimes \mathrm{CH}(X)$ in $\mathrm{CH}(X \times X)$. We can see these decompositions as giving gradings on $\mathrm{CH}(X)$ and $\mathrm{CH}(X \times X)_{\mathrm{dec}}$. (Here it is natural from the point of view of the Bloch-Beilinson conjectures to assign the degree 0 to H and the degree 2 to $\mathrm{CH}(X)_{\mathrm{hom}}$ since our surface is regular.) Then the intersection property b) says that if $\Delta : X \rightarrow X \times X$ is the diagonal embedding, the homomorphism

$$\Delta^* : \mathrm{CH}(X \times X)_{\mathrm{dec}} \rightarrow \mathrm{CH}(X)$$

is compatible with the gradings, while the diagonal relations a) and b) of Proposition 2.6 say that for $p > 0$ the homomorphism

$$\Delta_* : \mathrm{CH}_p(X) \rightarrow \mathrm{CH}_p(X \times X)$$

takes values in $\mathrm{CH}_p(X \times X)_{\mathrm{dec}}$ and is also compatible with the gradings.

We are now going to investigate the corresponding diagonal property for a curve.

Proposition 4.2. *Let C be a hyperelliptic curve, and w a Weierstrass point of C . For any $x \in C$, we have*

$$(x, x) - (x, w) - (w, x) + (w, w) = 0 \quad \text{in} \quad \mathrm{CH}_0(C \times C) .$$

(Note that the class of w is well-defined in $\mathrm{CH}_0(C)_{\mathbf{Q}}$.)

Proof: Let J be the Jacobian variety of C ; choose an Abel-Jacobi embedding $C \hookrightarrow J$. The induced map $C \times C \rightarrow J \times J$ is an Albanese map for $C \times C$.

The subgroup of degree 0 cycles in $\mathrm{CH}_0(C \times C)$ maps onto the Albanese variety $J \times J$; let $T(C \times C)$ be the kernel of this map. The surjective map

$$\mathrm{CH}_0(C) \otimes \mathrm{CH}_0(C) \longrightarrow \mathrm{CH}_0(C \times C)$$

induces a surjective map

$$J \otimes J \longrightarrow T(C \times C) .$$

Let ι be the hyperelliptic involution of C ; since ι acts as (-1) on J , we see that *the involution (ι, ι) of $C \times C$ acts trivially on $T(C \times C)$.*

Let $\mathfrak{c} : J \rightarrow \mathrm{CH}_0(C \times C)$ be the homomorphism defined by

$$\mathfrak{c}(\alpha) = \Delta_* \alpha - \mathrm{pr}_1^* \alpha \cdot \mathrm{pr}_2^* w - \mathrm{pr}_1^* w \cdot \mathrm{pr}_2^* \alpha .$$

The cycle $\mathfrak{c}(\alpha)$ is of degree zero, and its image in $J \times J$ is $(\alpha, \alpha) - (\alpha, 0) - (0, \alpha) = 0$, hence it is invariant under (ι, ι) . On the other hand we have

$$(\iota, \iota)^* \mathfrak{c}(\alpha) = \mathfrak{c}(\iota^* \alpha) = \mathfrak{c}(-\alpha) = -\mathfrak{c}(\alpha) .$$

Therefore $2\mathbf{c}(\alpha) = 0$, and actually $\mathbf{c}(\alpha) = 0$ by Rojtman's result. Applying this to $\alpha = [x] - [w]$ gives the result. \square

In contrast, we have now :

Proposition 4.3. *Let C be a general curve of genus ≥ 3 . There exists no divisor c on C such that the 0-cycle¹ $(x, x) - (x, c) - (c, x)$ in $\mathrm{CH}_0(C \times C)$ is independent of $x \in C$.*

Proof: As above the hypothesis on c is equivalent to the relation

$$\Delta_*\alpha = \mathrm{pr}_1^* \alpha \cdot \mathrm{pr}_2^* c + \mathrm{pr}_1^* c \cdot \mathrm{pr}_2^* \alpha$$

for all α in J . Applying pr_1^* we observe that this formula implies $\deg c = 1$.

Put $c' = (x, c) + (c, x) - (x, x)$, and assume that this class in $\mathrm{CH}_0(C \times C)$ is independent of x . With the notation of 3.1, we consider in $\mathrm{CH}_1(C \times C \times C)_{\mathbf{Q}}$ the cycle

$$\mathfrak{z} = (x, x, x) - (c, x, x) - (x, c, x) - (x, x, c) + (c, c, x) + (c, x, c) + (x, c')$$

Our hypothesis ensures that the restriction of \mathfrak{z} to the generic fibre of p_1 is zero. As in [Bl-S] we conclude that \mathfrak{z} is a sum of 1-cycles of the form $i_{b*}D_b$, where $i_b : C \times C \rightarrow C \times C \times C$ is the embedding $(x, y) \mapsto (b, x, y)$ and D_b is a divisor on $C \times C$.

Let us now work in the group $A_1(C \times C \times C)_{\mathbf{Q}}$ of cycles modulo algebraic equivalence. In this group the class of $i_{b*}D$, for $D \in \mathrm{CH}_1(C \times C)_{\mathbf{Q}}$, is independent of $b \in C$; thus we can write $\mathfrak{z} = i_{b*}D$ for some fixed $b \in C$ and some divisor D in $C \times C$. Since $p_{12} \circ i_b = \mathrm{Id}_{C \times C}$ we have $D = p_{12*}\mathfrak{z}$.

Now the cycle \mathfrak{z} is homologically trivial: as in 3.11 it suffices to check this for the projections $p_{ij*}\mathfrak{z}$ on $C \times C$, and this is straightforward. Thus, the divisor D is homologically, and therefore algebraically, trivial in $C \times C$; we conclude that \mathfrak{z} is zero in $A_1(C \times C \times C)_{\mathbf{Q}}$.

Now let J be the Jacobian variety of C , and $\alpha : C \rightarrow J$ the Abel-Jacobi map which maps a point x of C to the divisor class $[x] - c$; we will identify C with its image under α . Let $\alpha^3 : C^3 \rightarrow J$ be the map deduced from α . We have

$$(\alpha^3)_*(\mathfrak{z}) = \mathbf{3}_*C - 3(\mathbf{2}_*C) + 3C = 0 \quad \text{in } A_1(J)_{\mathbf{Q}},$$

where \mathbf{k} denotes the multiplication in J by the integer k .

According to [B] we have a decomposition

$$A_1(J)_{\mathbf{Q}} = A_1(J)_0 \oplus \cdots \oplus A_1(J)_{g-1},$$

where \mathbf{k}_* acts by multiplication by k^{2+s} on $A_1(J)_s$. Since $3^\ell - 3 \cdot 2^\ell + 3 > 0$ for $\ell \geq 3$, the above equality implies that the components of the 1-cycle C in $A_1(J)_i$ are zero for $i \geq 1$,

¹Here (x, c) stands for the 0-cycle $\mathrm{pr}_1^* x \cdot \mathrm{pr}_2^* c$

that is, $[C] \in A_1(J)_0$. Taking $k = -1$ we see that C is algebraically equivalent to $-C$; this contradicts the result of Ceresa [C]. \square

Remark 4.4. The cycle class \mathfrak{z} is studied in [G-S].

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