On the universal $\text{CH}_0$ group of cubic hypersurfaces

Claire Voisin
CNRS, École Polytechnique

Abstract

We study the existence of a Chow-theoretic decomposition of the diagonal of a smooth cubic hypersurface, or equivalently, the universal triviality of its $\text{CH}_0$-group. We prove that for odd dimensional cubic hypersurfaces or for cubic fourfolds, this is equivalent to the existence of a cohomological decomposition of the diagonal, and we translate geometrically this last condition. For cubic threefolds $X$, this turns out to be equivalent to the algebraicity of the minimal class $\theta^4/4!$ of the intermediate Jacobian $J(X)$. In dimension 4, we show that a special cubic fourfold with discriminant not divisible by 4 has universally trivial $\text{CH}_0$ group.

1 Introduction

Let $X$ be a smooth rationally connected projective variety over $\mathbb{C}$. Then $\text{CH}_0(X) = \mathbb{Z}$, as all points of $X$ are rationally equivalent. However, if $L$ is a field containing $\mathbb{C}$, e.g. a function field, the group $\text{CH}_0(X_L)$ can be different from $\mathbb{Z}$. As explained in [4], the group $\text{CH}_0(X_L)$ is equal to $\mathbb{Z}$ for any field $L$ containing $\mathbb{C}$ if and only if, for $L = \mathbb{C}(X)$, the diagonal (or generic) point $\delta_L$ is rationally equivalent over $L$ to a constant point $x_L$ for some (in fact any) point $x \in X(\mathbb{C})$. Following [4], we will then say that $X$ has universally trivial $\text{CH}_0$ group. Observe that, on the other hand, the equality

$$\delta_L = x_L \text{ in } \text{CH}_0(X_L) = \text{CH}^n(X), \ n = \dim X$$

is, by the localization exact sequence applied to Zariski open sets of $X \times X$ of the form $U \times X$, equivalent to the vanishing in $\text{CH}^n(U \times X)$ of the restriction of $\Delta_X - X \times x$, where $U$ is a sufficiently small dense Zariski open set of $X$ and $\Delta_X \subset X \times X$ is the diagonal of $X$. This provides a Bloch-Srinivas decomposition of the diagonal

$$\Delta_X = X \times x + Z \text{ in } \text{CH}^n(X \times X), \ \text{(1)}$$

where $Z$ is supported on $D \times X$ for some proper closed subset $D$ of $X$. As in [27], we will call an equality (1) a Chow-theoretic decomposition of the diagonal. So, having universally trivial $\text{CH}_0$ group is equivalent to admitting a Chow-theoretic decomposition of the diagonal, but the second viewpoint is much more geometric, and leads to the study of weakened properties, like the existence of a cohomological decomposition of the diagonal, which is the cohomological counterpart of (1), studied for threefolds in [27]:

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}), \ \text{(2)}$$

where $Z$ is supported on $D \times X$ for some proper closed algebraic subset $D$ of $X$. In these notions, integral coefficients are essential in order to make the property restrictive, as the existence of decompositions as above with rational coefficients already follows from the assumption that $\text{CH}_0(X) = \mathbb{Z}$ (see [7]).

Projective space has universally trivial $\text{CH}_0$ group. It follows that rational or stably rational varieties admit a Chow-theoretic (and a fortiori cohomological) decomposition of the
diagonal. More generally, if $X$ is a unirational variety admitting a unirational parametrization $\mathbb{P}^n \dashrightarrow X$ of degree $N$, then there is a decomposition

$$N\Delta_X = NX \times x + Z \in CH^n(X \times X),$$

with $Z$ supported on $D \times X$ for some $D \subsetneq X$.

On the other hand, the existence of a decomposition of the diagonal is certainly not a sufficient condition for stable rationality, as there are surfaces of general type (hence very far from being rational or stably rational) which admit a Chow-theoretic decomposition of the diagonal (see Corollary 2.2). It could be also the case that a smooth projective variety $X$ admits unirational parametrizations of coprime degrees $N_i$ without being stably rational (although we do not know such examples). Nevertheless, the existence of a decomposition of the diagonal is a rather strong condition, and there are now a number of unirational examples where the non-existence provides an obstruction to rationality or stable rationality:

1) Examples of rationally connected varieties with no cohomological decomposition of the diagonal include varieties with non-trivial Artin-Mumford invariant (this is the torsion in $H^2(X, \mathbb{Z})$ or the second unramified cohomology group with torsion coefficients, or non-trivial third unramified cohomology group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ (see [10] for examples), as both of these groups have to be 0 when $X$ has a cohomological decomposition of the diagonal, see [27]).

2) Examples of rationally connected varieties with no Chow-theoretic decomposition of the diagonal include varieties with non-trivial unramified cohomology groups $H^i_{nr}(X, \mathbb{Q}/\mathbb{Z})$ with $i \geq 4$, as these groups have to be 0 when $X$ has a Chow-theoretic decomposition of the diagonal, see [11]. We refer to [21] for such examples and to [31] for the cycle-theoretic interpretation of this group in degree 4.

3) Furthermore, we proved in [26] that the non-existence of a decomposition of the diagonal is a criterion for (stable) irrationality which is actually stronger than those given by the nontriviality of unramified cohomology: For example, we show in loc. cit. that very general smooth quartic double solids do not admit a Chow-theoretic or even a cohomological decomposition of the diagonal while their unramified cohomology vanishes in all positive degrees. We prove similar results for very general nodal quartic double solids with $k \leq 7$ nodes and in the case of very general double solids with exactly 7 nodes, we prove in [26] that the non-existence of a cohomological decomposition of the diagonal is equivalent to the non-existence of a universal codimension 2 cycle $Z \in CH^2(J(X) \times X)$, where $J(X)$ is the intermediate Jacobian of $X$. ($J(X)$ is also known to be isomorphic to the group $H^2(X)$ hom of codimension 2 cycles homologous to 0 on $X$.) Thus, in this case, the study of the decomposition of the diagonal led to the discovery of new stable birational invariants which are possibly nontrivial for some unirational varieties.

The purpose of this paper is to investigate the existence of a decomposition of the diagonal for cubic hypersurfaces. Motivations for this study are the following problems:

1) It is well-known that 3-dimensional smooth cubics are irrational (see [9]), but they are not known not to be stably rational.

2) In dimension 4, some cubics are known to be rational and it is a famous open problem to prove that there exist irrational cubic fourfolds. Some precise conjectures concerning the rationality of cubic fourfolds have been formulated and compared (see [14], [17], [2], [12], [1]), the strongest (conjecturally) necessary condition has been recently described by Galkin and Shinder [12], who prove that, assuming the cancellation conjecture saying that the class of $\mathbb{A}^3$ is not a zero divisor in the Grothendieck ring of complex varieties, a rational cubic fourfold has to satisfy the property that its variety of lines is birational to Hilb$^2(S)$ for some $K3$ surface $S$.

In general, the existence of a cohomological decomposition is much weaker than the existence of a Chow-theoretic one. Our first result, which is unconditional for cubic fourfolds and for odd-dimensional cubics, is the following:

**Theorem 1.1.** Let $X$ be a smooth cubic hypersurface. If $n = \dim X$ is even, assume that $H^n(X, \mathbb{Z})/H^n(X, \mathbb{Z})_{alg}$ has no 2-torsion. Then $X$ admits a Chow-theoretic decomposition
of the diagonal (equivalently, \( \text{CH}_0(X) \) is universally trivial) if and only if it admits a cohomological decomposition of the diagonal.

The assumption on \( X \) is satisfied when \( X \) is a cubic fourfold, since we proved in [30] that they satisfy the Hodge conjecture for degree 4 integral Hodge classes. This implies that \( H^4(X,\mathbb{Z})_{\text{alg}} \) is the set of integral Hodge classes of degree 4, so that \( H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{\text{alg}} \) is torsion free. In higher (even) dimension, the analogous statement is unknown, except for the very general cubic hypersurface, which has \( H^n(X,\mathbb{Z})_{\text{alg}} \) generated by \( h^m, n = 2m, \ h = c_1(O_X(1)) \).

The proof of Theorem 1.1 uses the fact that \( \text{Hilb}^2(X) \) is birationally a projective bundle over \( X \), a property which is also crucially used in the recent beautiful paper [12].

One consequence of this result, established in Section 5, concerns the following notion: Instead of having universally trivial \( \text{CH}_0 \) group, \( X \) may have its \( \text{CH}_0 \) group universally supported on a given \( \Sigma \subset X \), in the sense that \( \text{CH}_0(\Sigma_L) \to \text{CH}_0(X_L) \) is surjective for any field \( L \) containing \( \mathbb{C} \). We will prove (see Theorem 5.3) that for a smooth cubic hypersurface of dimension \( n \) with no 2-torsion in \( H^n(X,\mathbb{Z})/H^n(X,\mathbb{Z})_{\text{alg}} \) and such that \( \text{End}_{\hat{H}}H^n(X,\mathbb{Q})_{\text{prim}} = \mathbb{Q} \), its \( \text{CH}_0 \) group is universally supported on a given \( \Sigma \not\subset X \) if and only if its \( \text{CH}_0 \) group is universally trivial. In dimension 4, we get further precise consequences, for example we prove that a cubic fourfold which is special in the sense of Hassett [14], with discriminant not divisible by 4 has universally trivial \( \text{CH}_0 \) group.

The rest of the paper focuses on the cohomological decomposition of the diagonal. We first investigate the existence of a cohomological decomposition of the diagonal for varieties whose non-algebraic cohomology is supported in middle degree, like complete intersections. We prove the following result:

**Theorem 1.2.** Let \( X \) be a smooth projective variety such that \( H^*(X,\mathbb{Z}) \) has no torsion. Assume that \( H^2(X,\mathbb{Z}) \) is generated over \( \mathbb{Z} \) by algebraic cycles for \( 2i \neq n = \dim X \). Then \( X \) admits a cohomological decomposition of the diagonal if and only if there exist varieties \( Z_i \) of dimension \( n - 2 \), correspondences \( \Gamma_i \in \text{CH}^{n-1}(Z_i \times X) \), and integers \( n_i \) with the property that for \( \alpha, \beta \in H^n(X,\mathbb{Z}) \),

\[
\sum_i n_i < \Gamma_i^* \alpha, \Gamma_i^* \beta >_{\mathbb{Z}} = < \alpha, \beta >_X .
\]

Note that the condition (3) presents obvious similarities with the one considered in [22] by Shen, who studied the case of cubic fourfolds. It is however weaker in several respects: the integers \( n_i \) do not need to be positive, and the correspondences \( \Gamma_i \) do not need to factor through the variety of lines. Finally, the condition formulated by Shen is only conjecturally a necessary condition for rationality, while our condition is actually a necessary condition for the triviality of the universal \( \text{CH}_0 \) group, hence a fortiori for (stable) rationality.

**Remark 1.3.** Concerning the second assumption in Theorem 1.2, it is satisfied by uniruled threefolds by [29] and by cubic fourfolds by [30], but it is not clear that it is satisfied by Fano complete intersections in any dimension. The group \( H^2(X,\mathbb{Z}) \), \( 2i \neq n \), is equal to \( \mathbb{Z} \) by Lefschetz hyperplane restriction theorem, but Kollár [16] exhibits examples of hypersurfaces where this group is not generated by an algebraic class for \( 2i > n \). It is not known if such Fano examples can be constructed.

The case of rationally connected threefolds \( X \) is also particularly interesting. In this case, we complete the results of [27] by proving the following result. Let \( J(X) \) be the intermediate Jacobian of \( X \). It is isomorphic as a group to \( \text{CH}^2(X)_{\text{hom}} \) via the Abel-Jacobi map. It is canonically a principally polarized abelian variety, the polarization being determined by the intersection pairing on \( H^2(X,\mathbb{Z})/\text{torsion} \cong H^2(J(X),\mathbb{Z}) \). Let \( \theta \in H^2(J(X),\mathbb{Z}) \) be the class of the Thêta divisor of \( J(X) \).

**Theorem 1.4.** (See also Theorem 4.1) Let \( X \) be a rationally connected threefold. Then \( X \) admits a cohomological decomposition of the diagonal if and only if the following three conditions are satisfied:

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1. $H^3(X, \mathbb{Z})$ has no torsion.

2. There exists a universal codimension 2 cycle in $X \times J(X)$.

3. The minimal class $\theta^{g-1}/(g-1)!$ on $J(X)$, $\dim J(X) = g$, is algebraic, that is, the class of a 1-cycle in $J(X)$.

The main new result in this theorem is the fact that condition 3 above is implied by the existence of a cohomological decomposition of the diagonal. In particular, it is a necessary condition for stable rationality. This can be seen as a variant of Clemens-Griffiths criterion, which can be stated as saying that a necessary criterion for rationality of a threefold is the fact that the minimal class $\theta^{g-1}/(g-1)!$ on $J(X)$ is the class of an effective curve in $J(X)$.

Note that examples of unirational threefolds not satisfying 1 were constructed by Artin and Mumford [3], and examples of unirational threefolds not satisfying 2 were constructed in [26]. It is not known if examples not satisfying 3 exist. More generally, it is not known if there exists any principally polarized abelian variety $(A, \Theta)$ such that the minimal class $\theta^{g-1}/(g-1)!$ is not algebraic on $A$, where $g = \dim A$. Notice that for many Fano threefolds, the intermediate Jacobian $J(X)$ is a Prym variety, so the class $2\theta^{g-1}/(g-1)!$ is known to be algebraic. In the case of cubic threefolds, the algebraicity of $\theta^4/4!$ is a classical completely open problem. Combining the theorems above, we get in this case:

**Theorem 1.5.** Let $X$ be a smooth cubic threefold. Then $X$ has universally trivial $\text{CH}_0$ group if and only if the class $\theta^4/4!$ on $J(X)$ is algebraic. This happens (at least) on a countable union of closed subvarieties of codimension $\leq 3$ of the moduli space of $X$.

The paper is organized as follows: Theorem 1.1 is proved in Section 2. Theorem 1.2 is proved in Section 3 and Theorem 1.4 is proved in Section 4. In Section 5, we come back to the case of cubic hypersurfaces, where we prove Theorem 5.3 and establish further results, particularly in dimension 4.

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## 2 Chow-theoretic and cohomological decomposition of the diagonal

We prove in this section Theorem 1.1. In the case of cubic hypersurfaces of dimension $\leq 4$, a shorter proof will be given, which uses the following result of independent interest:

**Proposition 2.1.** If $X$ admits a decomposition of the diagonal modulo algebraic equivalence, that is

$$\Delta_X - X \times x = Z \text{ in } \text{CH}(X \times X)/\text{alg},$$

with $Z$ supported on $D \times X$ for some $D \subseteq X$, then $X$ admits a Chow-theoretic decomposition of the diagonal.

**Proof.** We use the fact proved in [24], [25] that cycles algebraically equivalent to 0 are nilpotent for the composition of self-correspondences. We write (4) as

$$\Delta_X - X \times x - Z = 0 \text{ in } \text{CH}(X \times X)/\text{alg}$$

and apply the nilpotence result mentioned above. This provides

$$(\Delta_X - X \times x - Z)^{\circ N} = 0 \text{ in } \text{CH}(X \times X)$$
for some large $N$. As $\Delta_X - X \times x$ is a projector and $Z \circ X \times x = 0$, this gives
\[ \Delta_X - X \times x - W \circ Z = 0 \text{ in } \text{CH}(X \times X), \] (7)
for some cycle $W$ on $X \times X$. As $W \circ Z$ is supported on $D \times X$, for some $D \not\subseteq X$, this concludes the proof.

**Corollary 2.2.** Let $S$ be a surface of general type with $\text{CH}_0(S) = \mathbb{Z}$ and $\text{Tors}(H^*(S, \mathbb{Z})) = 0$ (for example the Barlow surface ([5])). Then $S$ has universally trivial $\text{CH}_0$ group.

**Proof.** (See [4] for a different proof.) As $p_g(S) = q(S) = 0$ by Mumford theorem [20], the cohomology $H^*(S, \mathbb{Z})$ is generated by classes of algebraic cycles. As $H^*(S, \mathbb{Z})$ has no torsion, the cohomology of $S \times S$ admits a Künneth decomposition with integral coefficients, so that we can write the class of the diagonal of $S$ as
\[ [\Delta_S] = \sum_i [\alpha_i] \otimes [\beta_i] \text{ in } H^4(S \times S, \mathbb{Z}), \] (8)
where $\alpha_i, \beta_i$ are algebraic cycles on $S$ with $\dim \alpha_i + \dim \beta_i = 2$. Clearly, $[\alpha_i] \otimes [\beta_i] = [\alpha_i \times \beta_i]$ is supported over $D \times S$ with $D \not\subseteq S$ when $\dim \alpha_i < 2$, so that (8) provides in fact a cohomological decomposition of $\Delta_S$:
\[ [\Delta_S] = [S \times s] + [Z] \text{ in } H^4(S \times S, \mathbb{Z}), \] (9)
where $Z$ is a cycle supported over $D \times S$, for some $D \not\subseteq S$. Next, as $\text{CH}_0(S \times S) = \mathbb{Z}$, codimension 2 cycles on $S \times S$ which are cohomologous to 0 are algebraically equivalent to 0 by [7]. Thus the cycle $\Gamma := \Delta_S - S \times s - Z$ is algebraically equivalent to 0 on $S \times S$. The surface $S$ thus admits a decomposition of the diagonal modulo algebraic equivalence, and we then apply Proposition 2.1.

For a smooth projective variety $X$, we denote by $X^{[2]}$ the second punctual Hilbert scheme of $X$. It is smooth, obtained as the quotient of the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal by its natural involution. Let $\mu : X \times X \dashrightarrow X^{[2]}$ be the natural rational map and $r : X \times X \rightarrow X^{[2]}$ be the quotient morphism. We start with the following result:

**Lemma 2.3.** Let $X$ be a smooth projective variety of dimension $n$. Then there exists a codimension $n$ cycle $Z$ in $X^{[2]}$ such that $\mu^*Z = \Delta_X$ in $\text{CH}^n(X \times X)$.

**Proof.** Let $E_\Delta$ be the exceptional divisor over the diagonal of the blow-up map $r : \widetilde{X \times X} \rightarrow X \times X$. The key point is the fact that there is a (non effective) divisor $\delta$ on $X^{[2]}$ such that $r^*\delta = E_\Delta$. It follows that
\[ r^*\delta^n = E_\Delta^n \text{ in } \text{CH}^n(\widetilde{X \times X}). \]
Now we use the fact that
\[ \tau_*((-1)^{n-1}E_\Delta^n) = \Delta_X \text{ in } \text{CH}^n(X \times X), \]
which gives
\[ \mu^*((-1)^{n-1}\delta^n) = \tau_*(r^*((-1)^{n-1}\delta^n)) = \Delta_X \text{ in } \text{CH}^n(X \times X). \]

**Corollary 2.4.** Any symmetric codimension $n$ cycle on $X \times X$ is rationally equivalent to $\mu^*\Gamma$ for a codimension $n$ cycle $\Gamma$ on $X^{[2]}$.
Indeed, we can write \( Z = Z_1 + Z_2 \) where \( Z_1 \) is a combination of irreducible subvarieties of \( X \times X \) invariant under the involution \( i \) of \( X \times X \), and \( Z_2 \) is of the form \( Z_2^i + i(Z_2^i) \), where the diagonal does not appear in \( Z_2^i \). Write \( Z_1 = n_1 \Delta_X + Z_1' \), with \( Z_1' = \sum_j n_j Z_{1,j} \), the \( Z_{1,j} \) being invariant under \( i \) but different from \( \Delta_X \). Then \( Z_{1,j}^i \) is the inverse image of a subvariety \( Z_{1,j}'' \) of \( X \). Let \( Z_{1,j}'' \) be the proper transform of \( Z_{1,j} \) under the Hilbert-Chow map \( X^2 \to X^2 \). Then clearly \( \mu^*(Z_{1,j}''') = Z_{1,j}'' \) in \( CH^n(X \times X) \) so
\[
\mu^*(\sum_j n_j Z_{1,j}''') = Z_1 \text{ in } CH^n(X \times X).
\]

Next, let \( Z_2' \) be the image of \( Z_2^i \) in \( X \) by \( \mu \). Then clearly
\[
\mu^*(Z_2') = Z_2' + i(Z_2^i) = Z_2 \text{ in } CH^n(X \times X).
\]
Thus
\[
Z = n_1 \Delta_X + Z_1' + Z_2 = n_1 \Delta_X + \mu^*(\sum_j n_j Z_{1,j}''') + \mu^*(Z_2').
\]

Finally, we use Lemma 2.3 to conclude.

We have next:

Lemma 2.5. Suppose \( X \) admits a cohomological decomposition of the diagonal:
\[
[\Delta_X - x \times X] = [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
where \( Z \) is a cycle supported on \( D \times X \) for some proper closed algebraic subset \( D \) of \( X \). Then \( X \) admits a cohomological decomposition of the diagonal
\[
[\Delta_X - x \times X - x \times x] = [W] \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
where \( W \) is a cycle supported on \( D \times X \) for some \( D \not\subseteq X \), and \( W \) is invariant under the involution \( i \).

Proof. Let us denote by \( i' \Gamma \) the image of a cycle \( \Gamma \) under the involution \( i \) of \( X \times X \). Formula (10) gives as well
\[
[\Delta_X - X \times x] = [i' Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
and
\[
[(\Delta_X - X \times x) \circ (\Delta_X - x \times x)] = [i' Z \circ Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}).
\]
The cycle \( i' Z \circ Z \) is invariant under the involution and supported on \( D \times X \). On the other hand, the left hand side in (12) is equal to \( [\Delta_X - X \times x - x \times x] \) (we assume here \( n > 0 \)).

The following proposition is a key point in our proof of Theorem 1.1.

Proposition 2.6. Let \( X \) be a smooth projective variety such that the groups \( H^*(X, \mathbb{Z}) \) and \( H^{even}(X, \mathbb{Z})/H^*(X, \mathbb{Z})_{alg} \) have no 2-torsion. If \( X \) admits a cohomological decomposition of the diagonal, there exists a cycle \( \Gamma \in CH^n(X[2]) \) with the following properties:
(i) \( \mu^* \Gamma = \Delta_X - x \times X - X \times x - W \) in \( CH^n(X \times X) \), with \( W \) supported over \( D \times X \), for some closed algebraic subset \( D \not\subseteq X \).
(ii) \( [\Gamma] = 0 \) in \( H^{2n}(X[2], \mathbb{Z}) \).

Proof. We know by Lemma 2.5 that there is a symmetric cycle \( W \) supported on \( D \times X \), \( D \not\subseteq X \), such that \( [\Delta_X - x \times X - X \times x] = [W] \) in \( H^{2n}(X \times X) \). Corollary 2.4 provides a cycle \( \Gamma \in CH^n(X[2]) \) such that \( \mu^* \Gamma = \Delta_X - x \times X - X \times x - W \) in \( CH^n(X \times X) \), which is property (i). It remains to see that we can impose condition (ii), namely
\[
[\Gamma] = 0 \text{ in } H^{2n}(X[2], \mathbb{Z}).
\]
We know that $\mu^*[\Gamma] = 0$ in $H^{2n}(X \times X, \mathbb{Z})$, which implies that $r^*[\Gamma]$ vanishes in $H^{2n}(\overline{X \times X} \setminus E_\Delta, \mathbb{Z})$. Thus there is a cohomology class $\beta \in H^{2n-2}(E_\Delta, \mathbb{Z})$ whose image in $H^{2n}(X^{[2]}, \mathbb{Z})$ is equal to $r^*[\Gamma]$. This implies that there is a cycle $\gamma \in \text{CH}^{n-1}(E_\Delta)$ such that

$$i_{E_\Delta} [\gamma] = r^*[\Gamma],$$

where $i_{E_\Delta} : X \times X \rightarrow \overline{X \times X}$ is the inclusion of the exceptional divisor $E_\Delta$ over $\Delta$. Indeed, write

$$\beta = \sum_{k=0}^{n-1} [e]^k \sim \tau_E^k \beta_k,$$

with $\beta_k \in H^{2n-2-2k}(X, \mathbb{Z})$, where $\tau_E : E \rightarrow X \subset X \times X$ is the restriction of $\tau$ to $E$, $e = E_{\Delta \setminus E_\Delta} \in \text{CH}^1(E_\Delta)$ and $[e] \in H^2(E_\Delta, \mathbb{Z})$ is its cohomology class. Then $\beta_{n-1} \in H^n(X, \mathbb{Z})$ is equal to 0 because $\mu^*[\Gamma] = \tau_r (r^*[\Gamma]) = 0$ in $H^{2n}(X \times X, \mathbb{Z})$ and $r^*[\Gamma] = i_{E_\Delta} [\beta]$, which implies that

$$\mu^*[\Gamma] = i_{E_\Delta} (\tau_{E_\Delta} (\beta)) = \beta_{n-1} [\Delta_X] \in H^{2n}(X \times X, \mathbb{Z}),$$

where $i_\Delta$ is the inclusion of the diagonal $\Delta_X$ in $X \times X$. Next we have

$$\tau_{E_\Delta} ([e]^l) \sim i_{E_\Delta} (i_{E_\Delta} ([e]^k \sim \tau_E^k \beta_k))) = 0$$

(13) for $l + k + 1 < n - 1$ and

$$\tau_{E_\Delta} ([e]^l) \sim i_{E_\Delta} (i_{E_\Delta} ([e]^k \sim \tau_E^k \beta_k))) = \pm \beta_k$$

(14) for $l + k + 1 = n - 1$. Using (13), (14) and $i_{E_\Delta} (\beta) = [r^* \Gamma]$, one concludes that $\beta_i$ is also algebraic for any $i > 0$.

Write now $\gamma = \sum_{k=0}^{n-2} e^k \tau_E \gamma_k$ in $\text{CH}(E_\Delta)$, with $\gamma_k \in \text{CH}^{n-k-1}(X)$.

Lemma 2.7. We have

$$i_{E_\Delta} [\gamma] = r^* [\Gamma']$$

in $H^{2n}(\overline{X \times X}, \mathbb{Z})$, for some cycle $\Gamma' \in \text{CH}^n(X^{[2]})$ which has the property that $\mu^* \Gamma' \in \text{CH}^n(X \times X)$ is rationally equivalent to a cycle supported over $D' \times X$, for some $D' \subset \not\subset X$.

Admitting the lemma, we then have

$$r^* [\Gamma - \Gamma'] = 0$$

in $H^{2n}(\overline{X \times X}, \mathbb{Z})$, which implies that $[\Gamma - \Gamma'] = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z})$ because $\tau_r \circ r^* = 2Id : H^{2n}(X^{[2]}, \mathbb{Z}) \rightarrow H^{2n}(X^{[2]}, \mathbb{Z})$, and $H^*(X^{[2]}, \mathbb{Z})$ has no 2-torsion under our assumptions. Furthermore, by (i) and the stated property of $\Gamma'$, we get

$$\mu^* (\Gamma - \Gamma') = \Delta_X - x \times X \times X - x - W - W'$$

in $\text{CH}^n(X \times X)$, with $W + W'$ supported over $(D \cup D') \times X$, $D \cup D'$ being a proper closed algebraic subset of $X$. Thus the cycle $\Gamma_1 := \Gamma - \Gamma'$ satisfies properties (i) and (ii) with $D$ replaced by $D_1 := D \cup D'$.

Proof of Lemma 2.7. First of all, observe that if $z \in \text{CH}^{n-1}(E_\Delta)$, we have

$$2i_{E_\Delta} z = r^* (i_E^* z)$$

in $\text{CH}^n(\overline{X \times X})$, where $i_E^*$ is the inclusion of $E_\Delta$ in $X^{[2]}$, and that, if $2z = \sum_{k=0}^{n-2} e^k \tau^* z_k$ is annihilated by $\tau_{E_\Delta}$, then $\mu^* (i_E^* z) = 0$ in $\text{CH}(X \times X)$. In other words, the conclusion of the lemma is satisfied by all $\gamma \in 2 \text{CH}^{n-1}(E_\Delta)$ such that $\tau_{E_\Delta} \gamma = 0$ in $\text{CH}^0(X)$. 7
Next, we use the fact that
\[ i_{E^*}\gamma = r^*\Gamma \in H^{2n}(X \times \tilde{X}). \]

The papers [19], [8, Theorems 2.4, 2.5] analyze (among other things) the 2-torsion in \( H^*(X^{[2]} \setminus E_{\Delta}, \mathbb{Z}) \), and, conveniently reinterpreted, this tells exactly which integral cohomology classes of \( X \times X \) which are supported on \( E_{\Delta} \) are pull-backs via \( r^* \) of integral cohomology classes on \( X^{[2]} \). The conclusion is that a class
\[ [\gamma] = \sum_{k=0}^{n-2} [\gamma_k] \]
is a pull-back \( r^* \alpha \) if and only if \( [\gamma_k] \in 2H^*(X, \mathbb{Z}) \) when \( k \leq n - 2 - k \). We now use the assumption that \( H^{even}(X, \mathbb{Z})/H^*(X, \mathbb{Z})_{alg} \) has no 2-torsion. Hence if \( [\gamma_k] \in 2H^*(X, \mathbb{Z}) \), there exists a cycle \( \gamma'_k \) in \( CH^{n-1-k}(X) \) such that \( 2[\gamma'_k] = [\gamma_k] \) in \( H^{2n-2-2k}(X, \mathbb{Z}) \). Thus we have
\[ [\gamma] = \sum_{k > n-2-k, k \leq n-2} e^k \tau_{E^*}[\gamma_k] + 2 \sum_{k \leq n-2-k} e^k \tau_{E^*}[\gamma_k]. \]

As explained above, the class \( [\gamma''] = 2 \sum_{k \leq n-2-k} e^k \tau_{E^*}[\gamma_k] \) satisfies the conclusion of the lemma. It remains to deal with the classes \( e^k \tau_{E^*}[\gamma_k] \) with \( k \geq n - 1 - k \). Let \( Y \subset X \) be a smooth locally closed subvariety of dimension \( k + 1 \), with Zariski closure \( \overline{Y} \subset X \). Then we have \( Y^{[2]} \subset X^{[2]} \) with Zariski closure \( \overline{Y}^{[2]} \subset X^{[2]} \). Let us denote by \( E_Y \) the inverse image \( \tau_{E^*}[\gamma] \subset E_{\Delta}. \) Then \( E_Y \) contains the exceptional divisor \( E_{\Delta,Y} \) of \( Y \) as a codimension \( n - k - 1 \) projective bundle over \( Y \). Replacing \( Y \) by a Zariski open set of \( Y \) if necessary, we can assume that \( E_{\Delta,Y} \subset E_Y \) has class
\[ E_{\Delta,Y} = e^{n-k-1} \text{ in } CH^{n-k-1}(E_Y). \]

On the other hand, consider the divisor class \( \delta_Y := \delta_{|Y^{[2]}} \) on \( Y^{[2]} \) such that
\[ r_Y^*\delta_Y = E_{\Delta_{|\overline{Y} \times \overline{Y}}} = E_{\Delta,Y} \text{ in } CH^1(\overline{Y} \times \overline{Y}), \]
where \( r_Y : \overline{Y} \times \overline{Y} \rightarrow Y^{[2]} \) is the restriction of \( r \). It follows that
\[ E_{\Delta,Y} \cdot e^l_{E_Y} = r_Y^*(\delta^{l+1}) \]
in \( CH(\overline{Y} \times \overline{Y}) \) for any \( l \geq 0 \), which provides for any \( l \geq 0 \), using (15) and the localization exact sequence:
\[ i_{E^*}(\tau_{E^*}[\gamma]) \cdot e^{l+n-k-1} = r^*(\overline{Y}^{[2]} \cdot \delta^{l+1}) + z \text{ in } CH(\overline{X} \times \overline{X}), \]
where \( z \in CH(E_{\Delta}) \) is supported over a closed algebraic subset of \( X \) of dimension \( \leq k \). On the other hand, it is clear that \( \mu^*\overline{Y}^{[2]} \cdot \delta^{l+1} \) is supported on \( Y \times Y \), so the cycles \( \gamma \) of the form \( \tau_{E^*}[\gamma] \cdot e^{l+n-k-1} \) almost satisfy the conclusion of the lemma.

Using (16), we now prove by induction on \( i \) that a cycle class
\[ [\gamma] = \sum_{k \leq i} [\gamma_k] \tau_{E^*}[\gamma_k] \in H^{2n-2}(E_{\Delta}, \mathbb{Z}) \]
which has the property that \( i_{E^*}[\gamma] = r^*\Gamma \) for some cycle \( \Gamma \) on \( X^{[2]} \) and \( \tau_{E^*}[\gamma] = 0 \) in \( H^{2n}(X, \mathbb{Z}) \), also has the property that \( i_{E^*}[\gamma] = r^*\Gamma' \) for some cycle \( \Gamma' \) on \( X^{[2]} \) such that \( \mu^*\Gamma' \) is supported on \( D \times X \), \( D \not\subseteq X \). Indeed, when \( i \) is small, we have \( k \leq n - 2 - k \) for \( k \leq i \), hence \( [\gamma] \) is divisible by 2, a case that we already treated. For the induction step, let
The rational map \( \Phi : X^{[2]} \rightarrow P_X \) which to an unordered pair of points \( (x, y) \in X \) not contained in a common line of \( X \) associates the line \( l_{x,y} \) is the line in \( \mathbb{P}^{n+1} \) generated by \( x \) and \( y \), and \( z \in X \) is the residual point of the intersection \( l_{x,y} \cap X \), is desingularized by the blow-up of \( P_2 \) in \( X^{[2]} \).

The induced morphism \( \Phi : X^{[2]} \rightarrow P_X \) identifies \( X^{[2]} \) to the blow-up \( \widetilde{P}_X \) of \( P \) in \( P_X \).

(iii) The exceptional divisors of the two maps \( X^{[2]} \rightarrow X^{[2]} \) and \( \widetilde{P}_X \rightarrow P_X \) are identified by the isomorphism \( \Phi^* : X^{[2]} \cong \widetilde{P}_X \) of (ii).

**Proof.** (i) There is a morphism from \( X^{[2]} \) to the Grassmannian \( G(1, n+1) \) of lines in \( \mathbb{P}^{n+1} \), which to \( x + y \) associates the line \( (l_{x,y}, z) \), where \( l_{x,y} \) is the line in \( \mathbb{P}^{n+1} \) generated by \( x \) and \( y \), and \( z \in X \) is the residual point of the intersection \( l_{x,y} \cap X \), is desingularized by the blow-up of \( P_2 \) in \( X^{[2]} \).

(ii) The induced morphism \( \Phi : X^{[2]} \rightarrow P_X \) identifies \( X^{[2]} \) to the blow-up \( \widetilde{P}_X \) of \( P \) in \( P_X \).

(iii) The exceptional divisors of the two maps \( X^{[2]} \rightarrow X^{[2]} \) and \( \widetilde{P}_X \rightarrow P_X \) are identified by the isomorphism \( \Phi^* : X^{[2]} \cong \widetilde{P}_X \) of (ii).
is thus proved. On the other hand, the application $\Phi$ clearly pulls-back to a morphism on $D$, so (i) is proved.

(ii) and (iii) If the length 2 subscheme $Z = x + y$ (or $Z = (2x, v)$ with $v$ tangent to $X$ at $x$) does not belong to $P_2$, then the line $l_{x,y}$ (or $l_{x,v}$) is not contained in $X$, the morphism $\Phi$ is well defined at $Z$ and its image is a pair $(\llbracket l \rrbracket, u)$ where $l$ is a line passing through $u$ and is not contained in $X$. At such point $(\llbracket l \rrbracket, u)$ of $P_X$, $\Phi^{-1}$ is well-defined, and associates to $(\llbracket l' \rrbracket, u')$ in a neighborhood of $(\llbracket l \rrbracket, u)$ in $P_X$, the residual scheme of $u'$ in $l' \cap X$. This proves (iii). It remains to understand what happens along the exceptional divisor $Q_{P_2}$ of $D$. Now we have $Q_{P_2} = \{(u, x + y, [l]), l \subset X, x + y \in I^2(2), u \in l\}$. By definition, $\tilde{\Phi}$ maps such a triple to the pair $(u, [l])$, which by definition belongs to $P_X$. Furthermore, the fiber of $\tilde{\Phi}$ over $(u, [l])$ when $l \subset X$, that is when $(u, [l]) \in P$, identifies to the plane $I^2(2) \cong \mathbb{P}^2$. Thus $\Phi^{-1}(P) = \Delta$ is equal to the smooth irreducible hypersurface $Q_{P_2}$ in the smooth variety $D$ and this implies that $\tilde{\Phi}$ factors through a morphism $f : D \to P_X$ which has to be an isomorphism, since it cannot contract any curve; indeed, otherwise a contracted curve would be a curve in a fiber $\mathbb{P}^2$ as described above, so the whole corresponding $\mathbb{P}^2$ would be contracted by $f$, hence also all deformations of this $\mathbb{P}^2$ in $D = X^{[2]}$. But then the divisor $Q_{P_2}$ would be contracted by $f$, while its image has to be the exceptional divisor of $P_X \to X$.

\[\square\]

We now first give the proof of Theorem 1.1 in the case of cubics of dimension $\leq 4$, because the argument is shorter in this case.

**Proof of Theorem 1.1 in the case $n \leq 4$.** Let $X$ be a smooth cubic hypersurface of dimension $\leq 4$ and assume $X$ admits a cohomological decomposition of the diagonal. The assumptions of Proposition 2.6 are satisfied by $X$, since the integral cohomology of a smooth cubic hypersurface has no torsion and the integral Hodge conjecture is proved in [30] for cubic fourfolds. Using the notation introduced previously, there exists by Proposition 2.6 a cycle $\Gamma \in CH^n(X^{[2]})$ such that

$$
\mu^* \Gamma = \Delta_X - x \times X - X \times x - W \text{ in } CH^n(X \times X),
$$

with $W$ supported over $D \times X$, $D \subset X$, and $[\Gamma] = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z})$.

By Proposition 2.8, the blow-up $\sigma : X^{[2]} \to X^{[2]}$ of $X^{[2]}$ along $P_2$ identifies via $\tilde{\Phi}$ to a blow-up of the projective bundle $P_X$ over $X$. Furthermore, the exceptional divisor of $\tilde{\Phi} : X^{[2]} \to P_X$ is also the exceptional divisor of $\sigma : X^{[2]} \to X^{[2]}$, hence maps via $\sigma$ to $P_2 \subset X^{[2]}$. It follows that the pull-back $\sigma^*(\Gamma)$ of the cycle $\Gamma$ to $X^{[2]}$ can be written as

$$
\sigma^*(\Gamma) = \Gamma_1 + \Gamma_2,
$$

where $\Gamma_1$ and $\Gamma_2$ are cohomologous to 0, $\Gamma_1$ is a cycle cohomologous to 0 on the exceptional divisor of $\tilde{\Phi}$, $\Gamma_2$ is the pull-back of a cycle $\Gamma'_2$ cohomologous to 0 on $P_X$. As the exceptional divisor of $\tilde{\Phi}$ equals the exceptional divisor of $\sigma$, it follows from (19), by applying $\sigma_*$, that

$$
\Gamma = i_{P_2*}(\Gamma'_1) + \Phi^*(\Gamma'_2) \text{ in } CH(X^{[2]}),
$$

where $\Gamma'_1$ is a cycle cohomologous to 0 on $P_2$. Here $i_{P_2}$ denotes the inclusion map of $P_2$ in $X^{[2]}$. It is known that for a smooth cubic hypersurface of dimension $\leq 4$, cycles homologous to 0 are algebraically equivalent to 0. For codimension 2 cycles, this is proved by Bloch and Srinivas [7] and is true more generally for any rationally connected variety; for 1-cycles on cubic fourfolds, this is proved in [23], and this is true more generally for 1-cycles on Fano complete intersections. The result then also holds for cycles on a projective bundle over a cubic of dimension $\leq 4$. Thus $\Gamma'_2$ is algebraically equivalent to 0 and we thus conclude that

$$
\Gamma = i_{P_2*}(\Gamma'_1) \text{ in } CH(X^{[2]})/\text{alg},
$$

for some $n$-cycle $\Gamma'_1$ homologous to 0 on $P_2$. We apply now the following result:
Lemma 2.9. Let $X$ be a $n$-dimensional smooth cubic hypersurface and $Z$ be a $n$-cycle homologous to 0 on $P_2 \overset{\iota_2}{\to} X^{[2]}$. Then $\mu^*(\iota_{P_2*}Z) \in CH^\circ(X \times X)$ is supported on $D \times X$ for some proper closed algebraic subset $D$ of $X$.

Proof. Recall that $P$ is the union of the symmetric products $L^{(2)}$ over all lines $L \subset X$. As before, denote by

$$q: P \to X, \ p: P \to F$$

the natural maps, and by $q_2$ the natural map $P \times_F P \to X \times X$ induced by $q$. We will also denote by $\pi: P \times_F P \to F$ the map induced by $p$. Via $\pi$, $P \times_F P$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $F$. Let $H \coloneqq c_1(\mathcal{O}_X(1)) \in CH^1(X)$ and $h = q^*H \in CH^1(P)$ be its pull-back to $P$. For a cycle $Z$ supported on $P_2$, we have

$$\mu^*(\iota_{P_2*}Z) = q_2*(r^*Z) \in CH(X \times X)$$

where $r^*$ is the quotient map $P \times_F P \to P_2$. Let $T := r^*Z \in CH(P \times_F P)$. This is a cycle homologous to 0 on $P \times_F P$, and thus it can be written as

$$T = h_1 h_2 \pi^*\alpha + h_1 \pi^*\beta + h_2 \pi^*\gamma + \pi^*\zeta \in CH(P \times_F P),$$

for some cycles $\alpha, \beta, \gamma, \zeta$ homologous to 0 on $F$, with $h_i = pr_i^*h$, for $i = 1, 2$, $pr_i: P \times_F P \to P$ being the $i$-th projection.

We now push-forward these cycles to $X \times X$ via $q_2$ and observe that the three cycles

$$q_2*(\pi^*\zeta), \ q_2*(h_1 h_2 \pi^*\alpha), \ q_2*(h_1 \pi^*\beta)$$

are cycles supported on $D \times X$ for some $D \not\subseteq X$. Indeed, for the two last ones, this is due to the projection formula (and the equality $h_1 = q_2^*(H_1)$, where $H_1 := pr_1^*H \in CH^1(X \times X)$), and for the first one, this is because $q_2*(\pi^*\zeta)$ is supported on $q(p^{-1}(\text{Supp } \zeta)) \times X$. Now $\zeta$ is an $n - 2$-cycle, so $q(p^{-1}(\text{Supp } \zeta))$ is a proper closed algebraic subset of $X$. It remains to examine the cycle $q_2*(h_2 \pi^*\gamma)$. We observe now that the diagonal $\Delta_P \subset P \times_F P \subset P \times P$ is a divisor $d$ in $P \times_F P$ whose class is of the form $d = h_1 h_2 + \pi^*\lambda$ for some divisor class $\lambda \in CH^1(F)$. Furthermore, we obviously have $q_2(\Delta_P) \subset \Delta_X$. Thus we can write

$$h_2 \pi^*\gamma = (d - h_1 - \pi^*\lambda)\pi^*\gamma$$

in $CH(P \times_F P)$. We thus have

$$q_2*(h_2 \pi^*\gamma) = q_2*(d\pi^*\gamma) - q_2*(h_1 \pi^*\gamma) - q_2*(\pi^*(\lambda \gamma)) \in CH(X \times X).$$

As already explained, the terms $q_2*(h_1 \pi^*\gamma), \ q_2*(\pi^*(\lambda \gamma))$ are supported over $D \times X$ for some $D \not\subseteq X$. Finally, the last term $q_2*(d\pi^*\gamma)$ has to be 0 in $CH(X \times X)$. Indeed, this is a $n$-cycle of $X \times X$ which is supported on the diagonal, hence proportional to it, and also cohomologous to 0.

Combining (18), (21) and Lemma 2.9, we conclude that

$$\Delta_X = x \times X + X \times x - W' \text{ in } CH(X \times X)/\text{alg},$$

where $W'$ is supported on $D' \times X$ for some $D' \not\subseteq X$. In conclusion, $X$ admits a decomposition of the diagonal modulo algebraic equivalence, and we can now apply Proposition 2.1 to conclude that $X$ admits a Chow-theoretic decomposition of the diagonal.

We now turn to the general case.

Proof of Theorem 1.1 for general $n$. If we examine the proof above, we see that the only place where we used the fact that dim $X \leq 4$ is in the analysis of the term $\Phi^*\langle \Gamma'_2 \rangle$, where $\Gamma'_2$ is cohomologous to 0 on $P_X$. In the above proof, we directly used the fact that this term is algebraically equivalent to 0, which we do not know in higher dimension. The following provides an alternative argument which works also in higher dimension:
Lemma 2.10. Let $X$ be a smooth cubic hypersurface of dimension $n \geq 2$. Let $Z$ be a $n$-cycle cohomologous to 0 on $P_X$. Then $3\tau^*(\Phi^*(Z)) \in CH^n(X \times X)$ is rationally equivalent to a cycle supported on $D \times X$, for some $D \not\subseteq X$.

Proof. Recall that $p_X : P_X \to X$ is the $\mathbb{P}^n$-bundle over $X$ with fiber over $x \in X$ the set of lines in $\mathbb{P}^{n+1}$ through $x$. Let us denote by $l \in CH^1(P_X)$ the class of $O_{P_X}(1)$ (we choose for $O_{P_X}(1)$ the pull-back of the Plücker line bundle on the Grassmannian of lines $G(1, \mathbb{P}^{n+1})$).

The cycle $Z$ can be written as
\[ Z = p_X^* Z_0 + lp_X^* Z_1 + \ldots + l^i p_X^* Z_n, \tag{24} \]
where $Z_i$ are cycles of codimension $n-i$ on $X$. Note that the cycles $Z_i$ are all homologous to 0. As dim $X \geq 2$, $\text{CH}_0(X)_{\text{hom}} = 0$ and thus $Z_0 = 0$ in $\text{CH}_0(X)$. Hence (24) shows that $Z = l \cdot Z'$ for some $Z' \in \text{CH}(P_X)$. Let $\Psi := \Phi \circ \mu : X \times X \dashrightarrow P_X$ and let $\tilde{\Psi} : \tilde{X} \times \tilde{X} \to P_X$ be the desingularization of $\Psi$ obtained by blowing-up first the diagonal of $X$, and then the inverse image of $P_2 \subset X^{[2]}$ (see Proposition 2.8). Let $\tau : \tilde{X} \times \tilde{X} \to X \times X$ be the composition of the two blow-ups. There are two exceptional divisors of $\tau$, namely $E_\Delta$ and $E_{P_2}$. We thus have (using the fact that $\tilde{\Psi}$ factors through $X^{[2]}$)
\[ \tilde{\Psi}^*(k) = \alpha \tau^*(H_1 + H_2) + \beta E_\Delta + \gamma E_{P_2}, \]
where the explicit computation of the coefficients $\alpha, \beta, \gamma$ can be done but is not useful here.

It follows that
\[ \Psi^*(Z) = \Psi^*(l \cdot Z') = \tau_*(\tilde{\Psi}^*(l \cdot Z')) = \tau_*(\tilde{\Psi}^*(l) \cdot \tilde{\Psi}^*(Z')) = \tau_*(\alpha \tau^*(H_1 + H_2) + \beta E_\Delta + \gamma E_{P_2}) \cdot \tilde{\Psi}^*(Z')) \]
\[ = \tau_*(\gamma E_{P_2} \cdot \tilde{\Psi}^*(Z')) \]
\[ = q_2^*(Z''), \tag{25} \]
where $q_2 : P \times F P \to X \times X$ is introduced above. We can then apply Lemma 2.9 to conclude that the cycle $\tau_*(\gamma E_{P_2} \cdot \tilde{\Psi}^*(Z'))$ is supported on $D \times X$ for some $D \not\subseteq X$. Thus we conclude from (25), the projection formula, and the analysis above that
\[ \Psi^*(Z) = H_1 \cdot W_1 + H_2 \cdot W_2 + W \text{ in } \text{CH}(X \times X), \tag{26} \]
where $W$ is supported on $D \times X$ for some proper closed algebraic subset of $X$, and $W_1, W_2$ are cycles homologous to 0 on $X \times X$. It thus suffices to show that cycles $\Gamma$ on $X \times X$ of the form $H_1 \cdot W_1$ and $H_2 \cdot W_2$ with $W_1$ homologous to 0 on $X \times X$ have the property that $3\Gamma$ is rationally equivalent to a cycle supported on $D \times X$ for some proper closed algebraic subset of $X$. For $H_1 \cdot W_1$ this is obvious, and the coefficient 3 is not needed. For $H_2 \cdot W_2$, we observe that if $i_2 : X \times X \to X \times \mathbb{P}^{n+1}$ denotes the natural inclusion, we have
\[ 3H_2 \cdot W_2 = i_2^* i_2^* (W_2) \text{ in } \text{CH}(X \times X). \]
Now the cycle $i_2^*(W_2)$ of $\text{CH}_{n+1}(X \times \mathbb{P}^{n+1})$ is homologous to 0. Using its decomposition
\[ i_2^*(W_2) = \sum_{i} pr_1^* \gamma_i \cdot pr_2^* L^{n-i}, \]
with $\gamma_i \in \text{CH}_i(X)_{\text{hom}}$ and $L = c_1(O_{\mathbb{P}^{n+1}}(1)) \in \text{CH}_1(\mathbb{P}^{n+1})$, we thus conclude that $\gamma_0 = 0$, hence that $i_2^*(W_2) = \sum_{i \geq 0} pr_1^* \gamma_i \cdot pr_2^* L^{n-i}$ in $\text{CH}^n(X \times \mathbb{P}^{n+1})$. Hence $i_2^*(W_2)$ is rationally equivalent to a cycle supported on $D \times \mathbb{P}^{n+1}$ for some proper closed algebraic subset $D$ of $X$, and thus $3H_2 \cdot W_2 = i_2^* i_2^* (W_2)$ is rationally equivalent to a cycle supported on $D \times X$ for some proper closed algebraic subset $D$ of $X$. \qed
The rest of the proof goes as before, using again Lemma 2.9, and this allows to conclude that, if X admits a cohomological decomposition of the diagonal, then $3(\Delta_X - X \times x)$ is rationally equivalent to a cycle supported on $D \times X$, for some $D \not\subseteq X$. On the other hand, as X admits a unirational parametrization of degree 2 (see [9]), we also know that $2(\Delta_X - X \times x)$ is rationally equivalent to a cycle supported on $D' \times X$, for some $D' \not\subseteq X$. It follows that X admits a Chow-theoretic decomposition of the diagonal. 

\section{Criteria for the cohomological decomposition of the diagonal}

This section is devoted to the existence of cohomological decomposition of the diagonal. Our main result here is the following criterion for such a decomposition to exist:

\begin{theorem}
(\textit{cf. Theorem 1.2}) Let X be smooth projective of dimension n. Assume that

(i) $H^{2i}(X, \mathbb{Z})$ is algebraic for $2i \neq n$ and $H^{2i+1}(X, \mathbb{Z}) = 0$ for $2i + 1 \neq n$.

(ii) $H^*(X, \mathbb{Z})$ has no torsion.

Then X admits a cohomological decomposition of the diagonal if and only if the following condition (*) is satisfied:

(*) There exist smooth projective varieties $Z_i$ of dimension $n - 2$, correspondences $\Gamma_i \in \text{CH}^{n-1}(Z_i \times X)$, and integers $n_i$, such that for any $\alpha, \beta \in H^n(X, \mathbb{Z})$,

$$< \alpha, \beta >_X = \sum_i n_i < \Gamma_i^* \alpha, \Gamma_i^* \beta >_{Z_i}. $$

(28)

Here $\Gamma_i^*: H^n(X, \mathbb{Z}) \rightarrow H^{n-2}(Z_i, \mathbb{Z})$ is as usual defined by

$$\Gamma_i^*(\alpha) := pr_{Z_i, \ast}(pr_X^* \alpha \sim [\Gamma_i]),$$

where $pr_{Z_i}, pr_X$ are the projections from $Z_i \times X$ to its factors. If X is a Fano complete intersection of dimension n, the integral cohomology of X has no torsion and the groups $H^{2i}(X, \mathbb{Z})$ are cyclic generated by $h^i$, $h = c_1(O_X(1))$ for $i < n/2$. For $i > n/2$, they are also cyclic but the generator is now $\frac{1}{2} h^i$, where $d = \text{deg} X$ and it is not known in general that they are generated by a cycle class, except for $n = 4$ (resp. $n = 3$), where $H^6(X, \mathbb{Z})$ (resp. $H^4(X, \mathbb{Z})$) is generated by the class of a line in X, and some sporadic cases. Note that in any case the class $h^i$ is algebraic for any i, and in some cases this can be used as a substitute assumption in Theorem 3.1, like smooth cubic hypersurfaces (see Corollary 3.2).

Another interesting class of varieties which satisfy these two properties, needed in order to apply Theorem 3.1 below, is the class of rationally connected threefolds with trivial Artin-Mumford invariant, for which it is proved in [29] that $H^4(X, \mathbb{Z})$ is algebraic. In this case, one gets Theorem 4.1 (see also [27, Theorems 4.4 and 4.9]).

\textbf{Proof of Theorem 3.1.} Let us first prove that assuming (i) and (ii), condition (*) implies that X admits a cohomological decomposition of the diagonal. So let $Z_i, \Gamma_i$ be as above and satisfy (28). As $\dim Z_i = n - 2$, and $\text{codim} \Gamma_i = n - 1$, the $\Gamma_i$’s are $n - 1$-cycles in $Z_i \times X$. We denote by $(\Gamma_i, \Gamma_i) \in \text{CH}^{2n-2}(Z_i \times Z_i \times X \times X)$ the correspondence $p_{13}^* \Gamma_i \cdot p_{24}^* \Gamma_i$ between $Z_i \times Z_i$ and $X \times X$, where the $p_{rs}$ are the projectors from $Z_i \times Z_i \times X \times X$ to the product of two of its factors. Observe that $(\Gamma_i, \Gamma_i), \Delta_{Z_i}$ is supported on $D_i \times D_i$, where $D_i \not\subseteq X$ is defined as the image of $\text{Supp} \Gamma_i$ in X by the second projection. Let

$$\Gamma := \sum_i n_i (\Gamma_i, \Gamma_i), \Delta_{Z_i} \in \text{CH}^n(X \times X).$$

Equation (28) can be written as

$$< \alpha, \beta >_X = \int_{X \times X} pr_1^* \alpha \sim pr_2^* \beta \sim [\Gamma]$$

(29)

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for any degree $n$ classes $\alpha, \beta$ on $X$. It follows that the class

$$[\Delta_X] - [\Gamma] = \sum_i pr_i^* [W_i] \sim pr_2^* [W_i']$$

(32)

for some cycles $W_i, W_i'$ of $X$ with $\dim W_i + \dim W_i' = n$. The right hand side of (32) is of the form

$$[X \times x] + \sum_{i, \dim W_i' > 0} [pr_i^* W_i \cdot pr_2^* W_i']$$

and clearly $\sum_{i, \dim W_i' > 0} pr_i^* W_i \cdot pr_2^* W_i'$ is supported on $D' \times X$ for some proper closed algebraic subset $D' \subset X$. Hence we get

$$[\Delta_X] - [X \times x] = [\Gamma] + \sum_{i, \dim W_i' > 0} [pr_i^* W_i \cdot pr_2^* W_i'] = [Z],$$

where the cycle $Z = \Gamma + \sum_{i, \dim W_i' > 0} pr_i^* W_i \cdot pr_2^* W_i'$ is supported on $(\cup D_i \cup D') \times X$.

We now prove conversely that condition (*) is implied by the existence of a cohomological decomposition of the diagonal of $X$. Let $D \subset X$ be a divisor and $Z \subset D \times X$ be an $n$-cycle such that

$$[Z] = [\Delta_X] - [X \times x] \in H^{2n}(X \times X, \mathbb{Z}).$$

Let $\tau : X' \to X$ be a blow-up of $X$ such that a global normal crossing divisor $D' \subset X'$ dominates $D$. Enlarging $D$ if necessary, we can assume that $Z$ lifts to a $n$-cycle $Z' \in CH^n(D' \times X')$, where $D' = \cup D_i$ is the normalization of $D'$. It follows easily that the cycle class

$$[\Delta_X] - [X' \times x] \in H^{2n}(X' \times X')$$

is the class of a cycle $Z_i$ supported on $(D' \cup E) \times X'$, where $E$ is the exceptional divisor of $\tau$. The divisor $D'_i = D' \cup E$ can be also assumed to have global normal crossings and the cycle $Z_i$ lifts to a $n$-cycle $Z_i' \in CH^n(D'_i \times X')$, where $D'_i = \cup D_i$ is the normalization of $D'_i$. On the other hand, as we have $< \alpha, \beta >_X = < \tau^* \alpha, \tau^* \beta >_{X'}$, for $\alpha, \beta \in H^n(X, \mathbb{Z})$, it suffices to prove (*) for $X'$. In other terms, up to replacing $X$ by $X'$, we can assume that our original divisor $D$ has global normal crossings. So from now on, we assume $X = X'$ and $D$ a global normal crossing divisor in $X$ with normalization $\cup D_i$, so that $Z$ lifts to a cycle $Z$ in $(\cup D_i) \times X$.

Let us denote by $\Gamma_i \in CH^n(D_i \times X)$ the restriction of $Z$ to the connected component $D_i \times X$. Let $k_i : D_i \to X$ be the inclusion map. We have

$$\sum_i (k_i, Id_X)_* [\Gamma_i] = [Z] = [\Delta_X] - [X \times x] \in H^{2n}(X \times X),$$

(33)

which implies the following (because we can of course assume $n > 0$, which gives $[Z]^* \alpha = \alpha$ for any $\alpha \in H^n(X, \mathbb{Z})$):

$$\forall \alpha, \beta \in H^n(X, \mathbb{Z}), < \alpha, \beta >_X = < [Z]^* (\alpha), [Z]^* (\beta) >_X$$

(34)

$$= \sum_i (k_i, Id_X)_* [\Gamma_i]^* (\alpha) = \sum_i (k_i, Id_X)_* [\Gamma_i]^* (\beta) >_X.$$
We now develop the last expression, which gives for all \( \alpha, \beta \in H^n(X, \mathbb{Z}) \):
\[
< \alpha, \beta >_X = \sum_{i,j} < (k_i, Id_X)_*, [\Gamma_i] >^*(\alpha), (k_j, Id_X)_*[\Gamma_j] >^*(\beta) >_X .
\]  

(35)

Note now that
\[
((k_i, Id_X)_*, [\Gamma_i] ) >^*(\alpha) = k_\ast([\Gamma_i] ) >^*(\alpha) \text{ in } H^n(X, \mathbb{Z})
\]
and similarly for \((k_j, Id_X)_*[\Gamma_j] >^*(\beta)\). Hence (35) becomes
\[
< \alpha, \beta >_X = \sum_{i,j} < k_\ast([\Gamma_i] ) >^*(\alpha), k_\ast([\Gamma_j] ) >^*(\beta) >_X,
\]

(36)

where \([\Gamma_i] >^*(\alpha) \in H^{n-2}(D_i, \mathbb{Z})\) and similarly \([\Gamma_j] >^*(\beta) \in H^{n-2}(D_j, \mathbb{Z})\). Let \( \delta_i = k_\ast(D_i) \in CH^1(D_i) \) and \([\delta_i] \in H^2(D_i, \mathbb{Z})\) its cohomology class. As \( k_i \) is an embedding, we have \( k_\ast \circ k_\ast = [\delta_i] \sim H^{n-2}(D_i, \mathbb{Z}) \to H^n(D_i, \mathbb{Z}) \).
\[
< k_\ast([\Gamma_i] ) >^*(\alpha), k_\ast([\Gamma_j] ) >^*(\beta) >_X = < [\delta_i] \sim [\Gamma_i] >^*(\alpha), [\Gamma_i] >^*(\beta) >_{D_i} .
\]

(37)

Write \( \delta_i = \sum_{l} n_{i,l} Z_{i,l} \) where \( n_{i,l} \in \mathbb{Z} \) and \( Z_{i,l} \) is a smooth \( n-2 \)-dimensional subvariety of \( D_i \). Then letting \( \Gamma_{i,l} \in CH^{n-1}(Z_{i,l} \times X) \) be the pull-back of \( \Gamma_i \) to \( Z_{i,l} \times X \), (37) can be written as
\[
< k_\ast([\Gamma_i] ) >^*(\alpha), k_\ast([\Gamma_j] ) >^*(\beta) >_X = \sum_{l} n_{i,l} < [\Gamma_{i,l}] >^* \alpha, [\Gamma_{i,l}] >^* \beta >_{Z_{i,l}} .
\]

(38)

The right hand side of this equation is exactly of the form allowed in (28) and it remains to analyze in (36) the terms
\[
< k_\ast([\Gamma_i] ) >^*(\alpha), k_j([\Gamma_j] ) >^*(\beta) >_X + < k_j([\Gamma_j] ) >^*(\alpha), k_\ast([\Gamma_i] ) >^*(\beta) >_X
\]
for \( i \neq j \). Denote by \( W_{i,j} \) the intersection \( D_i \cap D_j \). It admits two correspondences \( \Gamma_{i,j}, \Gamma_{j,i} \in CH^{n-1}(W_{i,j} \times X) \), namely the restriction to \( W_{i,j} \times X \) of \( \Gamma_i \in CH^{n-1}(D_i \times X) \) and the restriction to \( W_{i,j} \times X \) of \( \Gamma_j \in CH^{n-1}(D_j \times X) \) respectively. With this notation, we have
\[
< k_\ast([\Gamma_i] ) >^*(\alpha), k_j([\Gamma_j] ) >^*(\beta) >_X = < [\Gamma_{i,j}] >^*(\alpha), [\Gamma_{i,j}] >^*(\beta) >_{W_{i,j}},
\]

(39)

\[
< k_j([\Gamma_j] ) >^*(\alpha), k_\ast([\Gamma_i] ) >^*(\beta) >_X = < [\Gamma_{j,i}] >^*(\alpha), [\Gamma_{j,i}] >^*(\beta) >_{W_{i,j}},
\]

which provides
\[
< k_\ast([\Gamma_i] ) >^*(\alpha), k_j([\Gamma_j] ) >^*(\beta) >_X + < k_j([\Gamma_j] ) >^*(\alpha), k_\ast([\Gamma_i] ) >^*(\beta) >_X
\]
\[
= < [\Gamma_{i,j}] >^*(\alpha), [\Gamma_{i,j}] >^*(\beta) >_{W_{i,j}} + < [\Gamma_{i,j}] >^*(\alpha), [\Gamma_{i,j}] >^*(\beta) >_{W_{i,j}}
\]
\[
< ([\Gamma_{i,j}] + [\Gamma_{j,i}]) >^*(\alpha), ([\Gamma_{i,j}] + [\Gamma_{j,i}]) >^*(\beta) >_{W_{i,j}} = < [\Gamma_{i,j}] >^*(\alpha), [\Gamma_{i,j}] >^*(\beta) >_{W_{i,j}}
\]
\[
- < [\Gamma_{i,j}] >^*(\alpha), [\Gamma_{i,j}] >^*(\beta) >_{W_{i,j}} .
\]

(40)

Each of the terms appearing in the final expression of (40) is of the form allowed in (28), which concludes the proof.

\[\square\]

**Corollary 3.2.** Let \( X \) be a smooth cubic hypersurface. Then \( X \) admits a cohomological (or equivalently Chow-theoretic) decomposition of the diagonal if and only if \( X \) satisfies condition (*).

**Proof.** The necessity of the condition (*) is proved above and the proof does not use the assumptions (i) and (ii) of the Theorem, so it works in our case. For the converse, it is in fact not a direct corollary of the theorem, since the proof uses these assumptions and we do not know that cubic hypersurfaces satisfy assumption (ii), but we can make a small variant of the proof, using the following observation: As a smooth cubic hypersurface of
dimension ≥ 2 admits a unirational parametrization of degree 2, twice its diagonal admits a decomposition
\[ 2\Delta_X = 2X \times x + Z \in \text{CH}(X \times X), \]
with Z supported on \( D \times X, D \not\subseteq X \). So X admits a cohomological (or Chow-theoretic) decomposition of the diagonal if there is such a decomposition for 3\( \Delta_X \). But we know that \( h^i \) is algebraic for any i, and thus each class \( 3pr^*_1 \alpha_i \sim pr^*_2 \alpha_{n-i} \) for \( 2i \not= n, 0 \), is the class of a cycle supported on \( D \times X, D \not\subseteq X \), where \( \alpha_i \) is a generator of \( H^{2i}(X,\mathbb{Z}) \). The proof of the existence of a decomposition of 3\( \Delta_X \) assuming condition (*) thus works as in the proof of the theorem.

Let us conclude this section with the following variant of (part of) Theorem 3.1.

**Theorem 3.3.** Let \( X \) be a smooth projective variety of dimension \( n \), and let \( N \) be an integer. Assume there is a decomposition
\[ N[\Delta_X] = N[X \times x] + [Z] \in H^{2n}(X \times X,\mathbb{Z}), \]
where \( Z \) is supported on \( D \times X, D \not\subseteq X \). Then there exist smooth projective varieties \( Z_i \) of dimension \( n-2 \), correspondences \( \Gamma_i \in \text{CH}^{n-1}(Z_i \times X) \), and integers \( n_i \), such that for any \( \alpha, \beta \in H^n(X,\mathbb{Z}) \),
\[ N^2 < \alpha, \beta >_X = \sum_i n_i < \Gamma_i^* \alpha, \Gamma_i^* \beta >_{Z_i}. \] (41)

**Proof.** We look at the proof of condition (*) assuming the existence of a cohomological decomposition of the diagonal and we repeat it replacing everywhere \( < \alpha, \beta >_X \) by \( N^2 < \alpha, \beta >_X \). The main point is the fact that with the same notation as in the proof of the theorem, letting \( Z = \sum_i(k_i, \text{Id}_X)_* \Gamma_i^* \), we have by assumption
\[ [Z]^*(\alpha) = N\alpha \]
for \( \alpha \in H^n(X,\mathbb{Z}) \) (with \( n > 0 \)), and thus
\[ N^2 < \alpha, \beta >_X = [Z]^*(\alpha), [Z]^*(\beta) >_X. \] □

# 4 Rationally connected threefolds

In the case of rationally connected threefolds, we have the following result, which was partially proved in [27]:

**Theorem 4.1.** (cf. Theorem 1.4) Let \( X \) be a rationally connected 3-fold. Then \( X \) admits a cohomological decomposition of the diagonal if and only if
(a) \( H^3(X,\mathbb{Z}) \) has no torsion.
(b) There exists a universal codimension 2 cycle \( \Gamma \) on \( J(X) \times X \).
(c) The intermediate Jacobian \( J(X) \) contains a 1-cycle \( Z \) with class \( \theta^{g-1}/(g-1)! \in H^{2g-2}(J(X),\mathbb{Z}) \), where \( g = \dim J(X) \).

**Proof.** Indeed, it is proved in [27] that for a rationally connected 3-fold, conditions (a) and (b) are necessary for the existence of a cohomological decomposition of the diagonal, and that (a), (b) and (c) are sufficient for the existence of a cohomological decomposition of the diagonal. So it suffices to show that (c) is also necessary. Assume \( X \) admits a cohomological decomposition of the diagonal, and let \( Z_i, \Gamma_i \) and \( n_i \) be as in Theorem 3.1. Then the Abel-Jacobi map \( \Phi_X \) of \( X \) of \( X \) induces (after choosing a reference point in \( Z_i \)) a morphism
\[ \gamma_i = \Phi_X \circ \Gamma_i^* : Z_i \to \text{CH}^2(X)_{\text{hom}} \to J(X) \]
with image $Z'_i \subset J(X)$. We claim that (28) is equivalent to the equality
\[
\sum_i n_i[Z'_i] = \theta^{q-1}/(g-1)!. \tag{42}
\]
Indeed, as
\[
2^\bigwedge H^1(J(X), \mathbb{Z}) \cong H^2(J(X), \mathbb{Z}) = H^{2q-2}(J(X), \mathbb{Z})^*,
\]
(42) is equivalent to the fact that for $\alpha, \beta \in H^1(J(X), \mathbb{Z})$,
\[
\sum_i n_i < \gamma_i^* \alpha, \gamma_i^* \beta >_{Z_i} = < \frac{\theta^{q-1}}{(g-1)!} \alpha, \beta >_{J(X)}. \tag{43}
\]
Now, the right hand side is equal, by definition of the polarization $\theta$, to $< \alpha', \beta'>_X$, where we use the canonical isomorphism
\[
H^1(J(X), \mathbb{Z}) \cong H^3(X, \mathbb{Z})
\]
to identify $\alpha, \beta$ to classes $\alpha', \beta'$ of degree 3 on $X$. Finally, using again this canonical isomorphism, we have :
\[
\gamma_i^* \alpha = \Gamma_i^* \alpha', \quad \gamma_i^* \beta = \Gamma_i^* \beta'
\]
so that the left hand side in (43) is equal to $\sum_i n_i < \Gamma_i^* \alpha', \Gamma_i^* \beta' >_{Z_i}$. Hence (43) is equivalent to the fact that for any $\alpha', \beta' \in H^3(X, \mathbb{Z})$,
\[
\sum_i n_i < \Gamma_i^* \alpha', \Gamma_i^* \beta' >_{Z_i} = < \alpha', \beta' >_X,
\]
which is equality (28).
\[
\square
\]
The following variant is proved as above, using Theorem 3.3 instead of Theorem 3.1. It answers a question asked to us by A. Beauville.

**Theorem 4.2.** Let $X$ be a rationally connected threefold with no torsion in $H^3(X, \mathbb{Z})$. Assume that, for some integer $N$, $N\Delta_X$ admits a decomposition
\[
[N\Delta_X] = N[X \times x] + Z, \tag{44}
\]
with $Z$ supported on $D \times X$, $D \not\subset X$. Then
\begin{enumerate}
\item There exists a codimension 2 cycle $\Gamma \in \text{CH}^2(J(X) \times X)$ such that
\[
\Phi_X(\Gamma_t) = N t \text{ in } J(X)
\]
for any $t \in J(X)$.
\item The intermediate Jacobian $J(X)$ contains a 1-cycle $Z$ with class $N^2 \theta^{q-1}/(g-1)! \in H^{2q-2}(J(X), \mathbb{Z})$, where $g = \dim J(X)$.
\end{enumerate}

**Proof.** (i) Let $\tilde{D} \rightarrow X$ be a desingularization with a cycle $\tilde{Z} \in \text{CH}(\tilde{D} \times X)$ such that $(j, \text{Id}_X)_* (\tilde{Z}) = N \Delta_X$. It follows that for any $\alpha \in H^3(X, \mathbb{Z})$,
\[
N \alpha = j_* (\tilde{Z}^* \alpha),
\]
and similarly, looking at the induced morphisms of complex tori,
\[
N \text{Id}_{J(X)} = j_* \circ \tilde{Z}^*: J(X) \rightarrow J(X),
\]
where $\tilde{Z}^*$ here gives a morphism
\[
\psi : J(X) \rightarrow J^1(\tilde{D}) \cong \text{Pic}^0(\tilde{D}).
\]
Now we use the existence of a universal divisor \( \mathcal{D} \) on \( \text{Pic}^0(\tilde{D}) \times \tilde{D} \). Let
\[
\mathcal{Z} := (Id_{J(X)}, j)_* (\psi, Id_{\tilde{D}})^* (\mathcal{D}) \in \text{CH}^2 (J(X) \times X).
\]
Then for any \( t \in J(X) \),
\[
\Phi_X (Z_t) = j_* (\Phi_{\tilde{D}} (D_{\psi(t)})) \text{ in } J(X).
\]
The right hand side is equal to \( j_* (\psi(t)) = j_* (\tilde{Z}^* (t)) = N t \), proving (i).

(ii) We use Theorem 3.3. We thus have curves \( C_i \), correspondences \( \Gamma_i \in \text{CH}^2 (C_i \times X) \) and integers \( n_i \) such that for any \( \alpha, \beta \in H^3 (X, \mathbb{Z}) \),
\[
N^2 < \alpha, \beta > _X = \sum n_i < \Gamma_i^* \alpha, \Gamma_i^* \beta > _{C_i}.
\]
As in the proof of Theorem 1.4, this equality exactly says that the curves \( D_i := \text{Im} (\Phi_X \circ \Gamma_i : C_i \to J(X)) \) satisfy
\[
\sum n_i [D_i] = N^2 \frac{\theta^{g-1}}{(g - 1)!} \text{ in } H^{2g-2} (J(X), \mathbb{Z}).
\]

\[\square\]

**Remark 4.3.** When \( X \) is a unirational threefold admitting a degree \( N \) unirational parametrization
\[
\phi : \mathbb{P}^3 \dashrightarrow X,
\]
\( N \Delta_X \) admits a decomposition as in (44), simply because, denoting \( Y \) a blow-up of \( \mathbb{P}^3 \) on which \( \phi \) is desingularized to a true morphism \( \tilde{\phi} \), one has
\[
(\tilde{\phi}, \tilde{\phi})_*(\Delta_Y) = N \Delta_X
\]
and \( Y \) admits a decomposition of the diagonal. In this case, Theorem 4.2, (ii) has an immediate proof which provides the following stronger statement:

*There is an effective cycle of class \( N \theta^{g-1}/(g - 1)! \) in \( H^{2g-2} (J(X), \mathbb{Z}) \), where \( g = \dim J(X) \).*

To see this, recall from [9] that \( (J(Y), \theta_Y) \) is a direct sum of Jacobians of smooth curves. Thus there exists a (possibly reducible) curve \( C \subset J(Y) \) with class \( \theta_Y^{g' - 1}/(g' - 1)! \), where \( g' := \dim J(Y) \). Let \( \psi : J(Y) \to J(X) \) be the morphism induced by \( \tilde{\phi} : Y \to X \). We claim that \( \psi_*(C) \subset J(X) \) has class \( N \theta^{g-1}/(g - 1)! \) in \( J(X) \). Indeed, by definition of the \( \Theta \) divisor of \( J(X) \), this is equivalent to saying that for any \( \alpha, \beta \in H^3 (X, \mathbb{Z}) \), denoting by \( \alpha', \beta' \) the corresponding degree 1 cohomology classes on \( J(X) \) via the isomorphism \( H^3 (X, \mathbb{Z}) \cong H^1 (J(X), \mathbb{Z}) \),
\[
N < \alpha, \beta > _X = \int_{\psi_*(C)} (\alpha' \wedge \beta'). \tag{45}
\]
However,
\[
\int_{\psi_*(C)} (\alpha' \wedge \beta') = \int_C \psi^* \alpha' \wedge \psi^* \alpha',
\]
where \( \psi^* \alpha' \) identifies to \( \phi^* \alpha \in H^3 (Y, \mathbb{Z}) \) via the natural isomorphism \( H^1 (J(Y), \mathbb{Z}) \cong H^3 (Y, \mathbb{Z}) \), and similarly for \( \beta \). Finally, we get by definition of the \( \Theta \) divisor of \( J(Y) \):
\[
\int_C \psi^* \alpha' \wedge \psi^* \alpha' = < \phi^* \alpha, \phi^* \beta > _Y = N < \alpha, \beta > _X
\]
which proves (45).
In the case of a smooth cubic threefold, we get the following consequence of Theorem 4.1:

**Corollary 4.4.** (cf. Theorem 1.5) A cubic threefold admits a Chow-theoretic decomposition of the diagonal (that is, its $\text{CH}_0$ group is universally trivial) if and only if the class $\theta^4/4!$ is algebraic on $J(X)$.

**Proof.** Indeed, this is a necessary condition by Theorem 4.1. The fact that this is a sufficient condition is proved as follows: the cubic 3-fold has no torsion in $H^3(X, \mathbb{Z})$. It is not known if it admits a universal codimension 2 cycle, but it is known by work of Markushevich-Tikhomirov [18] that it admits a parametrization of the intermediate Jacobian with rationally connected fibers, that is, there exists a smooth projective variety $B$, and a codimension 2 cycle $Z \in \text{CH}^2(B \times X)$, such that the induced morphism

$$\Phi_Z : B \to J(X),$$

$$t \mapsto \Phi_X(Z_t)$$

is surjective with rationally connected general fiber. In [27, Theorem 4.1], it is proved that if such a parametrization exists for a given rationally connected 3-fold $X$, and if furthermore there exists a 1-cycle $\Gamma$ whose class is the minimal class $\theta^{g-1}/(g-1)!$ is algebraic on $J(X)$, then there exists a universal codimension 2 cycle on $J(X) \times X$. Let us sketch the argument for completeness, assuming for simplicity that the 1-cycle $\Gamma$ is effective, that is the class of a curve $C$: Let $B, Z$ be as above and consider our curve $C \to J(X)$ which we can put in general position by translating it if necessary. The morphism $\Phi_Z$ has then rationally connected general fiber over $C$, and thus by [13], there is a lift $\sigma : C \to B$ such that $\Phi_Z \circ \sigma = j$. Let $Z_C \in \text{CH}^2(C \times X)$ be the restriction of $Z$ to $C \times X$. Then $\Phi_{Z_C} = j : C \to J(X)$. Using $Z_C$, we now construct a codimension 2-cycle $Z_{C^{(g)}}$ on $C^{(g)} \times X$, defined by

$$Z_{C^{(g)}, \gamma_1 + \ldots + \gamma_g} = Z_{C, \gamma_1 + \ldots + \gamma_g}$$

for any $g$-uple $\gamma_1 + \ldots + \gamma_g \in C^{(g)}$. In other words, the pull-back $\Phi_{Z_{C^{(g)}}}^{-1}$ of $Z_{C^{(g)}}$ to $C^{(g)} \times X$ is the cycle $\sum_{i=1}^{g} (p_i, \text{Id}_X)^* Z_C$. The cycle $Z_{C^{(g)}}$ has for associated morphism

$$\Phi_{Z_{C^{(g)}}} : C^{(g)} \to J(X)$$

the sum morphism $j_{g} : C^{(g)} \to J(X)$ defined by

$$j_{g} (\gamma_1 + \ldots + \gamma_g) = j(\gamma_1) + \ldots + j(\gamma_g).$$

As $|C| = \theta^{g-1}/(g-1)!$, the morphism $j_{g}$ is birational. We then get a universal codimension 2 cycle $Z_J$ on $J(X) \times X$ by the formula

$$Z_J := (j_{g}, \text{Id}_X)^* (Z_{C^{(g)}}).$$

Coming back to the cubic threefold, we conclude that if the minimal class $\theta^4/4!$ is algebraic, then there exists a universal codimension 2 cycle on $J(X) \times X$. Thus Theorem 4.1 implies that $X$ admits a cohomological decomposition of the diagonal. By Theorem 1.1, $X$ admits then a Chow-theoretic decomposition of the diagonal. \qed

We conclude with the following result:

**Theorem 4.5.** There exists a non-empty countable union of proper subvarieties of codimension $\leq 3$ in the moduli space of smooth cubic threefolds parametrizing $X$ with universally trivial $\text{CH}_0$ group.
Proof. We first claim that if $(J(X), \theta)$ is isogenous to $(J(C), m\theta_C)$ via an odd degree isogeny, for some (possibly reducible) curve $C$, then $J(X)$ has a one-cycle whose class is an odd multiple of the minimal class $\theta^4/4!$. Indeed, we have the odd degree isogeny $\mu : J(C) \to J(X)$ with the property that $\mu^*\theta_X = m\theta_C$. As $\deg \mu$ is odd, $m$ is odd. Furthermore, we have $\mu_*((\theta_C^4/4!)) = m(\theta^4/4!)$ and $\theta_C^4/4!$ is algebraic on $J(C)$. As $2(\theta^4/4!)$ is an algebraic class because $J(X, \theta)$ is a Prym variety (see [9]), we conclude that $\theta^4/4!$ is algebraic, which proves the claim. Next we know that in the moduli space of principally polarized abelian varieties, the Jacobians of curves fill-in a codimension 3 subvariety, the Schottky locus. Hence, if we exhibit a cubic threefold $X$ a curve $C$, and an integer $m$ such that $(J(X), \theta)$ is isogenous to $(J(C), m\theta_C)$, we conclude that such an isogeny exists along a codimension $\leq 3$ subvariety in the moduli space of $X$.

An example is as follows: Consider a smooth cubic threefold defined by a homogeneous polynomial $P(x_0, \ldots, x_4)$, where $P$ is invariant under the automorphism $g$ of order 3 acting on coordinates by

$$g^*x_0 = x_0, \quad g^*x_1 = jx_1, \quad g^*x_2 = j^2x_2, \quad g^*x_3 = x_3, \quad g^*x_4 = x_4.$$ 

The invariant part $H^3(X, \mathbb{Q})^{\text{inv}}$ of $H^3(X, \mathbb{Q})$ under the action of $g$ has rank 6. Let $\pi = 1d + g^* + (g^2)^* \in \text{End}(H^3(X, \mathbb{Z}))$. Then $\pi/3$ is the orthogonal projector of $H^3(X, \mathbb{Q})^{\text{inv}}$ with respect to the intersection pairing on $H^3(X, \mathbb{Q})$. Hence over $\mathbb{Q}$ we have an orthogonal decomposition

$$H^3(X, \mathbb{Q}) = H^3(X, \mathbb{Q})^{\text{inv}} \oplus H^3(X, \mathbb{Q})^\perp,$$

where $H^3(X, \mathbb{Q})^\perp = \text{Im}(1d - \pi/3)$. Over $\mathbb{Z}$, we conclude that $H^3(X, \mathbb{Z})$ contains a sublattice $H^3(X, \mathbb{Z})^{\text{inv}} \oplus H^3(X, \mathbb{Z})^{\perp}$, where

$$H^3(X, \mathbb{Z})^{\perp} = H^3(X, \mathbb{Q})^\perp \cap H^3(X, \mathbb{Z}) = (H^3(X, \mathbb{Z})^{\text{inv}})^\perp.$$

The index of this sublattice is a power of 3, since for $a \in H^3(X, \mathbb{Z})$, we can write

$$3a = (a + g^*a + (g^2)^*a) + (3a - (a + g^*a + (g^2)^*a)),$$

with

$$a + g^*a + (g^2)^*a \in H^3(X, \mathbb{Z})^{\text{inv}}, \quad 3a - (a + g^*a + (g^2)^*a) \in H^3(X, \mathbb{Z})^{\perp}.$$

It follows that we can construct two sublattices

$$H_1 \subset H^3(X, \mathbb{Z})^{\text{inv}}, \quad H_2 \subset H^3(X, \mathbb{Z})^{\perp}$$

with the property that the restriction of the intersection form $<,>_X$ to $H_1$ and $H_2$ is $m$ times a unimodular intersection pairing, where $m$ is a power of 3. These sublattices determine principally polarized abelian varieties $A$ and $B$ of respective dimensions 3 and 2, together with an isogeny

$$A \oplus B \to J(X)$$

such that the pull-back of $\theta_X$ is $m(\theta_A, \theta_B)$. The ppav’s $A$ and $B$ are Jacobians of curves $C_A$, $C_B$, and $(A \oplus B, (\theta_A, \theta_B))$ is the Jacobian of the curve $C_A \cup_C C_B$.

Each choice of sublattices $H_1$, $H_2$ as above provides us with one subvariety of codimension $\leq 3$ in the moduli space of $X$ along which the class $\theta^4/4!$ is algebraic. So we get countably many such subvarieties and it is likely that they are Zariski dense in the moduli space of cubic threefolds, but we did not try to prove this.

5 More results on cubic hypersurfaces

Let $X$ be a smooth cubic hypersurface of dimension $n \geq 3$. The Hodge structure on $H^n(X, \mathbb{Q})_{\text{prim}}$ is a polarized nontrivial Hodge structure (that is, when $n = 2k + 1$, it is nonzero, and when $n = 2k$, it is not purely of type $(k, k)$). Let $\text{End}_{H^S}(H^n(X, \mathbb{Q})_{\text{prim}})$ be the space of endomorphisms of this Hodge structure.
Lemma 5.1. For the very general cubic hypersurface of dimension $n$, $\text{End}_{HS}(H^n(X, \mathbb{Q})_{\text{prim}}) = \mathbb{Q}\text{Id}$.

Proof. This follows from the fact that the Mumford-Tate group of the considered Hodge structures is the symplectic group if $n$ is odd and the orthogonal group if $n$ is even. This fact in turn follows from the fact that the Mumford-Tate group contains a finite index subgroup of the monodromy group (see [28]) and that by [6], the monodromy group of a smooth hypersurface is Zariski dense in the symplectic or orthogonal group of $H^n(X, \mathbb{Q})_{\text{prim}}$ except for even dimensional quadrics and cubic surfaces, for which it is finite. This immediately implies the lemma because by definition of the Mumford-Tate group $G := \text{MT}(H^n(X, \mathbb{Q})_{\text{prim}})$, any endomorphism $\phi$ of the Hodge structure on $H^n(X, \mathbb{Q})_{\text{prim}}$ has to commute with $G$, that is $\phi \circ g = g \circ \phi$ for $g \in G$. \hfill \square

We now have the following result, saying that for cubic hypersurfaces satisfying the conclusion of Lemma 5.1, the CH$_0$ group cannot be universally supported on a proper closed algebraic subset of $X$, unless it is trivial. First of all, let us make the following definition:

Definition 5.2. Let $Y \subset X$ be a proper closed algebraic subset. We say that $\text{CH}_0(Y) \to \text{CH}_0(X)$ is universally surjective, if the natural map

$$\text{CH}_0(Y_L) \to \text{CH}_0(X_L)$$

is surjective for any field $L$ containing $\mathbb{C}$.

When $Y$ is a point, the fact that $\text{CH}_0(Y) \to \text{CH}_0(X)$ is universally surjective is equivalent to the fact that $X$ has universally trivial CH$_0$ group.

Theorem 5.3. Let $X$ be a smooth cubic hypersurface such that $H^n(X, \mathbb{Z})/H^n(X, \mathbb{Z})_{\text{alg}}$ has no 2-torsion for $n = \dim X$, and $\text{End}_{HS}(H^n(X, \mathbb{Q})_{\text{prim}}) = \mathbb{Q}\text{Id}$. Assume there is a proper closed algebraic subset $Y \subset X$ such that $\text{CH}_0(Y) \to \text{CH}_0(X)$ is universally surjective, Then CH$_0(X)$ is universally trivial.

Proof. Let $L = \mathbb{C}(X)$. Then we have the diagonal point $\delta_L$ and the fact that it comes from a 0-cycle supported on $Y_L$ says, by taking the Zariski closure in $X \times X$ and using the localization exact sequence, that there is a decomposition of the diagonal of $X$ which takes the following form:

$$\Delta_X = Z_1 + Z_2 \text{ in } \text{CH}^n(X \times X),$$

where $Z_1$ is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$, and $Z_2$ is supported on $X \times Y$. This decomposition gives in particular a cohomological decomposition:

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

(46)

where $Z_1$ and $Z_2$ are as above. We then use Lemma 5.4 below which says that a decomposition as in (46) implies that $X$ admits a cohomological decomposition of the diagonal because we assumed $\text{End}_{HS}(H^n(X, \mathbb{Q})_{\text{prim}}) = \mathbb{Q}\text{Id}$, and Theorem 1.1 which says that $X$ admits then a Chow-theoretic decomposition of the diagonal because we assumed that $H^n(X, \mathbb{Z})/H^n(X, \mathbb{Z})_{\text{alg}}$ has no 2-torsion; hence we proved that CH$_0(X)$ is in fact universally trivial. \hfill \square

Lemma 5.4. Let $X$ be a smooth cubic hypersurface. Assume that there is a decomposition

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

(47)

where $Z_1$ is supported on $D \times X$ , and $Z_2$ is supported on $X \times Y$ for some proper closed algebraic subsets $D, Y \subset X$. Then if furthermore $\text{End}_{HS}(H^n(X, \mathbb{Q})_{\text{prim}}) = \mathbb{Q}\text{Id}$, $X$ admits a cohomological decomposition of the diagonal.
Indeed, recall that cubic hypersurfaces admit a unirational parametrization of degree 2. So \(2[\Delta_X]\) can be decomposed as \([X \times x] + [Z]\), where \(Z\) is supported on \(D \times X\) for a proper closed algebraic subset \(D \subset X\). Hence it suffices to show that for some odd integer \(m\) divisible by 3, we have a decomposition
\[
m[\Delta_X] = m[X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
where \(Z \in \text{CH}^n(X)\) is supported on \(D \times X\) for a proper closed algebraic subset \(D \subset X\). Consider the decomposition (47): each class appearing in this decomposition acts on \(H^n(X, \mathbb{Z})\) via K"unneth decomposition. We observe now that, because \(H^n(X, \mathbb{Z})\) is divisible by 3, we have a decomposition
\[
m[\Delta_X] = m_1Id, \quad [Z_1]^* = m_2Id,
\]
where \(m_1, m_2\) are two integers such that \(m_1 + m_2 = 1\). We may assume that \(m_1\) is odd, applying transposition to our cycles if necessary. It follows that \(m_2\) is even, \(m_2 = 2m_2's\), and then the cycle class \([Z_2] - 2m_2'[\Delta_X]\) acts trivially on \(H^n(X, \mathbb{Q})_{prim}\). Over \(\mathbb{Q}\), using the orthogonal decomposition
\[
H^*(X, \mathbb{Q}) = H^n(X, \mathbb{Q})_{prim} \oplus H^*(\mathbb{P}^{n+1}, \mathbb{Q})|_X,
\]
we conclude that
\[
[Z_2] - 2m_2'[\Delta_X] = \sum_{i} \alpha_i h_i^1 \otimes h_2^{n-i} \text{ in } H^{2n}(X \times X, \mathbb{Q}),
\]
where here \(h \in H^2(X, \mathbb{Z})\) is \(c_1(\mathcal{O}_X(1))\) and the right hand side makes sense in \(H^{2n}(X \times X, \mathbb{Q})\) via Künneth decomposition. We observe now that, because \(H^n(X, \mathbb{Z})\) has no torsion and the pairings \(h^i, h^{n-i}\) are equal to 3, the denominators in the coefficients \(\alpha_i\) of (48) are equal to 3 (or 1), so that we get
\[
3[Z_2] - 6m_2'[\Delta_X] = \sum_i \beta_i pr_1^i h^i \sim pr_2^i h^{n-i} \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
where now the \(\beta_i\) are integers. Combining (49) with (47), we get now:
\[
3[\Delta_X] = 3[Z_1] + 3[Z_2] = 3[Z_1] + 6m_2'[\Delta_X] + \sum_i \beta_i pr_1^i h^i \sim pr_2^i h^{n-i} \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
where \(Z_1\) is supported on \(D \times X\), for some proper closed algebraic subset \(D \subset X\). Hence we proved that
\[
(3 - 6m_2')[\Delta_X] = 3\beta_0[X \times x] = 3[Z_1] + \sum_{i>0} \beta_i pr_1^i h^i \sim pr_2^i h^{n-i} \text{ in } H^{2n}(X \times X, \mathbb{Z}),
\]
and clearly, the class \(\sum_{i>0} \beta_i pr_1^i h^i \sim pr_2^i h^{n-i}\) is the class of a cycle supported on \(D' \times X\), for some proper closed algebraic subset \(D' \subset X\). As \(3 - 6m_2'\) is odd, this finishes the proof. \(\square\)

In the case of cubic fourfolds, we can replace in Theorem 5.3 the assumption that \(\text{End}_{HS}(H^*(X, \mathbb{Q})_{prim}) = \mathbb{Q}Id\) by the following assumption which concerns the simple Hodge structure
\[
H^*(X, \mathbb{Q})_{tr} := H^*(X, \mathbb{Q})_{alg}^+,
\]
namely \(\text{End}_{HS}(H^*(X, \mathbb{Q})_{tr}) = \mathbb{Q}Id\). This is a weaker and more natural assumption because, when the Hodge structure on \(H^*(X, \mathbb{Q})_{prim}\) is not simple, or equivalently, when \(H^*(X, \mathbb{Q})_{prim}\) contains nonzero Hodge classes, the algebra \(\text{End}_{HS}(H^*(X, \mathbb{Q})_{prim})\) contains projectors associated to sub-Hodge structures, and this happens along codimension 1 loci paramerizing special cubic fourfolds. To the contrary, the non-existence of nontrivial endomorphisms of the Hodge structure on \(H^*(X, \mathbb{Q})_{tr}\) is satisfied in codimension 1, and in particular at the very general point of a Noether-Lefschetz locus. Our result is the following:
Let \( X \) be a smooth cubic fourfold such that \( \text{End}_{HS}(H^4(X, \mathbb{Q}))_{tr} = \mathbb{Q} \text{Id} \). Assume there is a proper closed algebraic subset \( Y \subset X \) such that \( \text{CH}_0(Y) \to \text{CH}_0(X) \) is universally surjective. Then \( \text{CH}_0(X) \) is universally trivial.

**Proof.** The proof is very similar to the previous proof. The assumption is that

\[
[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^8(X \times X, \mathbb{Z}),
\]

with \( Z_1 \) supported on \( D_1 \times X \), \( Z_2 \) supported on \( X \times D_2 \) for some closed proper algebraic subsets \( D_1, D_2 \not= X \). We consider the action of \([Z_i]^*\) on \( H^4(X, \mathbb{Z})_{tr} \). By the assumption, each of them must act as a multiple of the identity and these actions sum-up to \( Id_{H^4(X, \mathbb{Z})_{tr}} \). So one of them, say \([Z_1]^*\), must act as an odd multiple of the identity and the other as an even multiple of the identity. Let \([Z_2]^* = 2m Id_{H^4(X, \mathbb{Z})_{tr}}\). Then \((2m[\Delta_X] - [Z_2])^*\) acts as 0 on \( H^4(X, \mathbb{Z})_{tr} \). Note also that \((2m[\Delta_X] - [Z_2])^*\) maps \( Hdg^4(X, \mathbb{Z}) = H^*(X, \mathbb{Z})_2 \) to itself, and this implies that

\[
2m[\Delta_X] - [Z_2] \in Hdg^4(X, \mathbb{Z}) \otimes Hdg^4(X, \mathbb{Z}) \oplus \sum_{i<2} H^{2i}(X, \mathbb{Z}) \otimes H^{8-2i}(X, \mathbb{Z}) \subset H^8(X \times X, \mathbb{Z}).
\]

All classes in \( Hdg^4(X, \mathbb{Z}) \) are algebraic by the Hodge conjecture for integral Hodge classes on cubic fourfolds proved in [30] and all classes in \( H^{*+\frac{1}{2}}(X, \mathbb{Z}) \) are algebraic. Hence we conclude that

\[
2m[\Delta_X] - [Z_2] = \sum_i pr^*_1[W_i] - pr^*_2[W'_i]
\]

for some cycles \( W_i, W'_i \) on \( X \) satisfying \( \dim W_i + \dim W'_i = 4 \). The rest of the proof works as before, allowing to conclude that \( X \) admits a cohomological decomposition of the diagonal, hence also a Chow-theoretic one by Theorem 1.1.

We finally prove the following Theorem 5.6. Let \( X \) be a cubic fourfold. Assume \( X \) is special in the sense of Hassett, that is \( H^4(X, \mathbb{Z}) \) contains two independent Hodge classes (one being the class \( h^2 \), the other being denoted \( \sigma \)). Let \( P \subset H^4(X, \mathbb{Z}) \) be the sublattice generated by these two Hodge classes. The restriction of the intersection form \( \langle, \rangle_X \) to \( P \) has a discriminant \( D(\sigma) \). This number, which is always even, has been very much studied in conjunction with rationality properties of cubic fourfolds (see [14], [15], [17], [1]). Hassett conjectures that if a cubic fourfold is rational, then it is special and the discriminant \( D(\sigma) \) satisfies severe restrictions.

**Theorem 5.6.** If \( 4 \) does not divide \( D(\sigma) \), the \( \text{CH}_0 \) group of \( X \) is universally trivial (that is, \( X \) admits a Chow-theoretic decomposition of the diagonal).

**Proof.** The assumption on \( X \) being satisfied along a countable union of Noether-Lefschetz type divisors in the moduli space of cubic fourfolds (see [14]), it suffices to show that the conclusion holds for \( X \) general in each of these Noether-Lefschetz divisors. But as already mentioned, for a very general point \( X \) in a Noether-Lefschetz divisor, we have \( \text{End}_{HS}(H^4(X, \mathbb{Q}))_{tr} = \mathbb{Q} \text{Id} \). Theorem 5.5 then tells us that if there is a surface \( \Sigma \subset X \) such that \( \text{CH}_0(\Sigma) \to \text{CH}_0(X) \) is universally surjective, then \( \text{CH}_0(X) \) is universally trivial. The existence of a special Hodge class \( \sigma \) provides us with an algebraic cycle \( Z \) on \( X \) of class \( \sigma \), by the Hodge conjecture for integral Hodge classes proved in [30]. Adding to \( \sigma \) a high multiple of \( h^2 \), we can even assume that \( Z \) is the class of a smooth surface \( \Sigma \) in general position. Indeed, as we are working with codimension 2 cycles, their classes are generated by \( c_2 \) of vector bundles on \( X \). For any vector bundle \( E \) of rank \( r \), a twist of \( E \) is very ample and its \( c_2 \) is represented by a rank locus associated to a morphism \( \phi : \mathcal{O}^r \to E \), and this rank locus is smooth in codimension 5. Let \( \Sigma \) be as above. Recall the rational map

\[
\Phi' : X^2 \dashrightarrow X, \quad \Phi'(x, x') = x'' + x' = < x, x' > \cap X.
\]

We now have

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Lemma 5.7. Assume the restriction of $\Phi'$ to $\Sigma^2$ is dominant of degree $2N$ not divisible by 4. Then the map $\text{CH}_0(\Sigma) \to \text{CH}_0(X)$ is universally surjective.

Proof. Indeed, the rational map $\Phi': |\Sigma\times\Sigma| \to X$, and our assumption implies that the factored map $\psi$ has odd degree $N$. It follows that $\psi: \text{CH}_0(\Sigma^{(2)}) \to \text{CH}_0(X_L)$ is surjective for any field $L$ containing $\mathbb{C}$ because its cokernel is annihilated by $N$, with $N$ odd, and also by 2, since $X$ is a cubic of dimension $\geq 2$, hence admits a unirational parametrization of degree 2. On the other hand, if $z \in \text{CH}_0(\Sigma^{(2)})$ is of degree $k$, $z$ provides a 0-cycle $z'$ on $\Sigma_L$ of degree $2k$, and we obviously have

$$\psi_*(z) = k \cdot h^4 - j_*(z') \in \text{CH}_0(X_L),$$

where $j: \Sigma \to X$ is the inclusion map. It follows that the map $j_*: \text{CH}_0(\Sigma_L) \to \text{CH}_0(X_L)$ is surjective as well. □

The next lemma relates the degree of $\Phi'|_{\Sigma \times \Sigma}$ to the discriminant $D(\sigma)$.

Lemma 5.8. Let $\Sigma \subset X$ be a smooth surface in general position. Then the degree of the rational map $\Phi'_{|\Sigma \times \Sigma|}: \Sigma \times \Sigma \dashrightarrow X$ is congruent to $D(\sigma)$ modulo 4, where $\sigma = [\Sigma]$.

Proof. Let $x \in X$ be a general point of $X$ and let

$$\pi_x: X \dashrightarrow \mathbb{P}^4$$

be the linear projection from $x$. To say that $(z, z') \in \Sigma^2$ satisfies $\Phi'(z, z') = x$ is equivalent to say that $z, z'$ and $x$ are collinear, or that $\pi_x(z) = \pi_x(z')$. As $\Sigma$ is in general position, the restriction of $\pi_x$ to $\Sigma$ maps $\Sigma$ to a surface $\Sigma'$ which is smooth apart from finitely many double points corresponding to pairs $(z, z')$ as above. It follows that the degree of $\Phi'_{|\Sigma \times \Sigma|}$ is equal to twice the number $N$ of these double points (this argument appears in [15]). We now compare the geometry of the two immersions

$$\Sigma \subset X, \Sigma \to \mathbb{P}^4.$$

The two corresponding normal bundle exact sequences give

$$0 \to T_{\Sigma} \to T_{X|\Sigma} \to N_{\Sigma/X} \to 0, \quad 0 \to T_{\Sigma} \to T_{\mathbb{P}^4|\Sigma} \to N_{\Sigma/\mathbb{P}^4} \to 0,$$

which by Whitney formula provides

$$c_2(T_{\Sigma}) = c_2(T_{X|\Sigma}) - c_2(N_{\Sigma/X}) + K_{\Sigma} \cdot c_1(N_{\Sigma/X}),$$

$$c_2(T_{\Sigma}) = c_2(T_{\mathbb{P}^4|\Sigma}) - c_2(N_{\Sigma/\mathbb{P}^4}) + K_{\Sigma} \cdot c_1(N_{\Sigma/\mathbb{P}^4}).$$

We now use the equalities

$$c_2(T_{X|\Sigma}) = 6h^2_{\Sigma}, \ c_2(T_{\mathbb{P}^4|\Sigma}) = 10h^2_{\Sigma},$$

$$c_1(N_{\Sigma/X}) = K_{\Sigma} + 3h_{\Sigma}, \ c_1(N_{\Sigma/\mathbb{P}^4}) = K_{\Sigma} + 5h_{\Sigma}$$

together with

$$c_2(N_{\Sigma/X}) = \sigma^2, \ \sigma = [\Sigma] \in H^4(X, \mathbb{Z}),$$

$$c_2(N_{\Sigma/\mathbb{P}^4}) = (h^2_{\Sigma})^2 - 2N = (\sigma \cdot h^2)^2 - 2N.$$

Thus (53) becomes

$$c_2(T_{\Sigma}) = 6h^2 \cdot \sigma - \sigma^2 + K_{\Sigma} \cdot (K_{\Sigma} + 3h),$$

$$c_2(T_{\Sigma}) = 10h^2 \cdot \sigma - (\sigma \cdot h^2)^2 + 2N + K_{\Sigma} \cdot (K_{\Sigma} + 5h).$$

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We now add these two equalities and consider the result modulo 4, which gives
\begin{equation}
2c_2(T_\Sigma) = -\sigma^2 - (\sigma \cdot h^2)^2 + 2N + 2K_\Sigma^2 \mod 4. \tag{55}
\end{equation}
As $2c_2(T_\Sigma) - 2K_\Sigma^2$ is divisible by 4 by Noether’s formula, we conclude that
\begin{equation}
\sigma^2 + (\sigma \cdot h^2)^2 = 2N \mod 4. \tag{56}
\end{equation}
As $D(\sigma) = 3\sigma^2 - (\sigma \cdot h^2)^2$ is equal to $-\sigma^2 - (\sigma \cdot h^2)^2$ modulo 4, we proved that
\begin{equation}
2N = -D(\sigma) = D(\sigma) \mod 4.
\end{equation}

Combining Lemmas 5.8 and 5.7, we conclude that the map $\text{CH}_0(\Sigma) \to \text{CH}_0(X)$ is universally surjective, hence that $\text{CH}_0(X)$ is universally trivial.

**Remark 5.9.** If one looks at the proof of the integral Hodge conjecture for cubic fourfolds given in [30], one easily sees that it gives more, namely: the group of Hodge classes of degree 4 on a cubic fourfolds is generated by classes of rational surfaces. Thus the surface $\Sigma$ above can be chosen rational. However, Lemma 5.8 does not allow to conclude that if $D(\sigma)$ is not divisible by 4, $X$ admits a unirational parametrization of odd degree. Indeed, the rational surface produced by the construction of [30] will be presumably singular, and not in general position.

**References**


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Centre de mathématiques Laurent Schwartz
91128 Palaiseau Cédex
France
voisin@math.polytechnique.fr