Residue formulae for Verlinde sums, and for number of integral points in convex rational polytopes

Lectures by Michèle Vergne
Notes by Sylvie Paycha
Introduction

These two lectures are dedicated to two topics and their contents are independent. In the first lecture on Bernoulli series and Verlinde sums, I present some algebraic formulae concerning a (fascinating) subject of which I am rather ignorant. I believe I am more knowledgeable on the second topic: volume and number of integral points of rational polytopes. In both topics, we shall encounter polynomial functions $w(k)$ of $k$, where $k$ is a non-negative integer, and the fact that these functions are polynomials is not obvious at all from the definition of $w(k)$. There is a common geometric concept underlying this fact for both topics: compact symplectic manifolds. If $(M, \omega)$ is a compact symplectic manifold and if the form $\omega$ is integral, then we can associate to $M$ a quantized vector space $Q(M, \omega)$. The Riemann-Roch theorem asserts in particular that the dimension of $Q(M, k\omega)$ is a polynomial in $k$ with leading term $k^{\dim M/2} \text{vol}(M)$, where $\text{vol}(M)$ is the symplectic volume of $M$.

The underlying manifold $M$ in the first topic is the moduli space of flat connections on a Riemann surface with holonomy $t$ around one hole. Verlinde sums yield the dimension of $Q(M, k\omega)$ while Bernoulli series compute the volume of such manifolds.

Underlying manifolds in the second topic are toric manifolds. An integral polytope $P$ determines a toric manifold $M$ of dimension $2 \dim P$ together with a symplectic form $\omega$ on $M$. The volume of the polytope $P$ is the volume of $M$ and the number of integral points in $kP$ is the dimension of $Q(M, k\omega)$. It is a polynomial in $k$ with leading term $k^{\dim P} \text{vol}(P)$.

In both cases, inspired by the Riemann-Roch theorem, it is possible to develop a purely algebraic Riemann-Roch calculus, and to prove directly a beautiful relation between Bernoulli series and Verlinde sums, as well as between volumes of polytopes and number of integral points belonging to the polytope. The main idea of this purely algebraic relation in the first case is an idea of A. Szenes while in the second case it goes back to G. Khovanskii and A.V. Pukhlikov.
We have a sort of Riemann-Roch relation between two functions, one depending on a continuous parameter $t$, the volume, and the other on a discrete parameter (the integer $k$), the dimension of a vector space. What about showing directly the relation between these two quantities by ”computing” them explicitly? Thus my aim in both lectures is to give a hint of ”explicit residue formulae” for all these quantities (Bernoulli series, Verlinde sums, volume of polytopes, number of integral points in polytopes), formulae which allows both concrete calculations, and quick proofs of the Riemann-Roch relations.

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Lecture 1
Residue formulae for Bernoulli polynomials and Verlinde sums
1 Introduction

In this lecture, I will present some simple sum formulae, which relate Bernoulli series to Verlinde sums. I hope to show in particular that residue formulae for series or sums are very efficient tools to relate these sums and calculate them. The one dimensional case is presented here in detail and is already very amusing (and amazing).

These lectures are mainly based on the following articles by A. Szenes (all available on ArXiv):

–[Sz 3]: A residue formula for rational trigonometric sums and Verlinde’s formula (math CO/0109038)

I do not explain here the underlying geometry. These lectures may (also) serve as an introduction to the algebraic aspects (multi-dimensional residue formulae) of articles of Jeffrey-Kirwan and Bismut-Labourie on the Verlinde formulae. References for the geometry underlying this calculation are given at the end of this lecture.

2 Bernoulli’s theorem

We have all tried to work out formulae for sums of the $m$-th powers of the first $k+1$ numbers, and to compare them to the corresponding integral $\int_0^k x^m \, dx = \frac{k^{m+1}}{m+1}$.

For $m = 0$, we have:

$$0^0 + 1^0 + 2^0 + \cdots + k^0 = k + 1.$$  

For $m = 1$, we have:

$$0 + 1 + 2 + 3 + \cdots + k = \frac{k^2 + k}{2}.$$  

For $m = 2$, we have:

$$0^2 + 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}.$$
For $m = 3$, we have:

$$0^3 + 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^4}{4} + \frac{k^3}{2} + \frac{k^2}{4}.$$ 

\ldots

This polynomial behavior is a general feature of such sums as the following theorem shows:

**Theorem 1** \((Jakob BERNOULLI -(1654-1705))\)

For any positive integer $m$, the sum $S_m(k) = \sum_{a=0}^{k} a^m$ of the $m$th-powers of the first $k + 1$ integers is given by a polynomial formula in $k$.

We will prove this theorem and relate the sum $S_m(k)$ to the Bernoulli polynomials.

Given a real number $t$, the Bernoulli polynomials $B(m, t)$ are defined by the following relation:

$$ze^{tz} - 1 = \sum_{m=0}^{\infty} B(m, t) \frac{z^m}{m!}.$$ 

This means, we expand $\frac{ze^{tz}}{e^z - 1}$ in a Taylor series at $z = 0$. The coefficient of $\frac{z^m}{m!}$ in this Taylor series depends polynomially on $t$, and is defined to be the Bernoulli polynomial $B(m, t)$. The first ones are

\begin{align*}
B(0, t) &= 1, \\
B(1, t) &= t - \frac{1}{2}, \\
B(2, t) &= t^2 - t + \frac{1}{6}, \\
B(3, t) &= t^3 - \frac{3}{2} t^2 + \frac{1}{2} t, \\
B(4, t) &= t^4 - 2t^3 + t^2 - \frac{1}{30}, \\
\ldots
\end{align*}
It follows immediately from the definition of the Bernoulli polynomials that
\[
\frac{d}{dt} B(p, t) = pB(p - 1, t),
\]
\[
\int_0^1 B(p, t) dt = 0 \text{ if } p > 1.
\]
(Notice also that \((-1)^p B_{2p} = (-1)^p B(2p, 0)\) is positive.)

The Bernoulli number \(B_m\) is defined to be \(B(m, 0)\). Bernoulli numbers satisfy the following relation:
\[
z \frac{e^z - 1}{e^z - 1} = \sum_{m=0}^{\infty} B_m \frac{z^m}{m!}.
\]
Since
\[
\frac{1}{1 - e^{-z}} = \frac{1}{e^z - 1} + 1,
\]
we have \(\sum_{m=0}^{\infty} B_m \frac{z^m}{m!} + z = \sum_{m=0}^{\infty} B_m \frac{(-z)^m}{m!}\) so that the Bernoulli numbers are equal to 0 for \(m\) odd, except when \(m = 1\) in which case we have \(B_1 = -\frac{1}{2}\).

Now remark that
\[
\frac{1}{(m + 1)!} (B(m + 1, t + 1) - B(m + 1, t))
\]
is the coefficient of \(z^{m+1}\) in the Taylor series of
\[
z \frac{e^{(t+1)z}}{e^z - 1} - \frac{e^{tz}}{e^z - 1} = z \frac{e^{tz}(e^z - 1)}{e^z - 1} = ze^{tz}.
\]
Thus we obtain:
\[
B(m + 1, t + 1) - B(m + 1, t) = (m + 1)t^m.
\]
We write this equality for \(t = 0, 1, \ldots, k:\)
\[
B(m + 1, 1) - B(m + 1, 0) = (m + 1)0^m
\]
\[
B(m + 1, 2) - B(m + 1, 1) = (m + 1)1^m
\]
\[
\cdots
\]
\[
B(m + 1, k) - B(m + 1, k - 1) = (m + 1)(k - 1)^m,
\]
\[
B(m + 1, k + 1) - B(m + 1, k) = (m + 1)k^m.
\]
Adding up these equations, we obtain
\[ B(m + 1, k + 1) - B(m + 1, 0) = (m + 1)S_m(k). \]
As
\[ z \frac{e^{(t+1)z}}{e^z - 1} = (-z) \frac{e^{(-t)(-z)}}{e^{-z} - 1}, \]
it follows that \( B(m, t + 1) = (-1)^m B(m, -t). \)

Thus we obtain Bernoulli’s Theorem:

**Proposition 2** The function \( k \mapsto S_m(k) \) is given by the polynomial formula in \( k \):
\[ S_m(k) = \frac{1}{m + 1}((-1)^{m+1}B(m + 1, -k) - B_{m+1}). \]

### 3 Bernoulli series and residues

In order to describe the Bernoulli polynomials in terms of series, it is useful to consider rational functions of the type \( \phi(z) = \frac{E(z)}{z^p} \) where \( E(z) \) is a polynomial. The sum over all non-zero integers \( n \in \mathbb{Z}, n \neq 0 \):
\[ B(\phi)(t) = \sum_{n \neq 0} \phi(2i\pi n)e^{2i\pi nt} \]
converges absolutely at each real number \( t \) if \( p \) is a sufficiently large integer since \( \sum_{n \neq 0} \frac{1}{n^\alpha} \) converges for \( \alpha \) sufficiently large. It defines a periodic function of \( t \). It always converges as a generalized function of \( t \), which is smooth when \( t \notin \mathbb{Z} \). For \( 0 < t < 1 \) and \( p \) sufficiently large, the residue formula in the plane yields:
\[ B(\phi)(t) = \text{residue}_{x=0}(\phi(x) \frac{e^x}{1 - e^x}). \]
(For \(-1 < t < 0\), the formula is \( B(\phi)(t) = -\text{residue}_{x=0}(\phi(x) \frac{e^x}{1 - e^{-x}}). \))

In particular, \( t \mapsto B(\phi)(t) \) is given by a polynomial formula whenever \( 0 < t < 1 \).

From the definition of \( B(p, t) \), it follows that
\[ B(p, t) = -p! \text{residue}_{x=0}(x^{-p} \frac{e^x}{1 - e^x}) \]
so that setting \( \phi(z) = z^{-p} \), we obtain, for \( 0 < t < 1 \), the following formula for the Bernoulli polynomial \( B(p, t) \):
Proposition 3  Let $0 < t < 1$. Then, we have:

$$\frac{B(p, t)}{p!} = -\text{residue}_{x=0}(x^{-p} \frac{e^{xt}}{1-e^x}) = -\sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^p}.$$ 

It is important to notice that the expression on the right hand side is periodic with respect to $t$, while the left hand side is polynomial. We will call the right hand side the Bernoulli series. The Bernoulli polynomial and the Bernoulli series coincide ONLY on the interval $0 < t < 1$. If $p \geq 2$, the sum in the right hand side is absolutely convergent and the formula is valid for all $0 \leq t \leq 1$.

In particular, for $p = 2g$ and $t = 0$, we get:

$$B_{2g} = -(2g)! \sum_{n \neq 0} \frac{1}{(2i\pi n)^{2g}} = 2(-1)^g(2g)! (2\pi)^{-2g} \zeta(2g)$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the zeta function at point $k$.

For $g = 1$, as $B_2 = \frac{1}{6}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4 Trigonometric sums

We now consider a closely related sum, which will be a special case of a Verlinde sum. Let $p$ be an integer and let

$$F(z) = \frac{1}{(1-z)^p}.$$ 

Given a positive integer $t$, let us consider the expression:

$$W(p, t)(k) = \sum_{\omega^k = 1, \omega \neq 1} \omega^t F(\omega) = \sum_{1 \leq n \leq (k-1)} \frac{e^{2i\pi nt/k}}{1 - e^{2i\pi n/k}}.$$ 

It is very similar to the formula for $0 < t < k$: 

$$9$$
\[ k^p \frac{B(p, t/k)}{p!} = -\sum_{n\neq 0} \frac{e^{2i\pi nt/k}}{(2i\pi n/k)^p}. \]

In fact, when \( k \) tends to \( \infty \), it is not difficult to see that \( k^{-p} W(p, t)(k) \) tends to 0 if \( p \) is odd, while if \( p \) is even, \( k^{-p} W(p, t)(k) \) tends to \( -\frac{B_p}{p!} \).

It is not at all obvious from its definition that the function \( k \mapsto W(p, t)(k) \) is polynomial in \( k \). We will prove it now.

**Theorem 4** Assume that \( t \) and \( p \) are integers such that \( 0 \leq t \leq p \), and \( k \geq 1 \). Then the function \( k \mapsto W(p, t)(k) \) is given by a polynomial formula in \( k \). If \( p \) is even, or if \( p = 1 \), this polynomial is of degree \( p \) with highest degree term \( -k^p \frac{B_p}{p!} \). If \( p \) is odd and \( p \neq 1 \), this polynomial is of degree less or equal to \( p - 1 \).

For all \( p \geq 1 \), we have the residue formula:

\[ W(p, t)(k) = -k \text{residue}_{x=0} \left( \frac{e^{tx}}{(1-e^x)^p} \left( 1 - e^{kx} \right) dx \right). \]

**Proof.** Consider the 1-form \( k \frac{z^t}{(1-z)^p} \frac{z^k}{(1-z^k)} \frac{dz}{z} \). From the conditions \( 0 \leq t \leq p \), and \( k \geq 1 \), it follows that this form has no residues at 0 and \( \infty \). Poles of the factor \( \frac{1}{(1-z)} \) arise at \( z = e^{2i\pi n/k} \) (\( 0 \leq n \leq (k - 1) \)). They are simple when \( n \neq 0 \), and their residues add up to the sum \(-W(p, t)(k)\). As the sum of residues of this 1-form is equal to 0, we obtain from the residue theorem

\[ W(p, t)(k) = k \text{residue}_{z=1} \left( \frac{z^t}{(1-z)^p} \frac{z^k}{(1-z^k)} \frac{dz}{z} \right). \]

A change of variable \( z = e^x \) in the residue yields:

\[ W(p, t)(k) = -k \text{residue}_{x=0} \left( \frac{e^{tx}}{(1-e^x)^p} \left( 1 - e^{kx} \right) dx \right). \]

This last expression depends on \( k \) via the Laurent expression of \( \frac{k}{1-e^{-kx}}. \) Recall from the definition of the Bernoulli numbers that

\[ \frac{kx}{e^{kx} - 1} = \sum_{r=0}^{\infty} B_r \frac{(kx)^r}{r!}. \]
Hence

\[ W(p, t)(k) = - \sum_{r=0}^{\infty} B_r R_r(p, t) \frac{k^r}{r!} \]

where

\[ R_r(p, t) = (-1)^r \text{residue}_{x=0} \left( \frac{e^{tx}}{(1-e^x)^p} \right)^{r-1} dx \]

so that \( R_r(p, t) \) vanishes for large \( r \). In fact, as \( \frac{1}{(1-e^x)^p} \) has a pole at 0 of order \( p \), we see that \( R_r(p, t) \) vanishes for \( r > p \), so that \( W(p, t)(k) \) is a polynomial of degree less or equal than \( p \). More precisely, it is of degree \( p \) for even \( p \) and of degree less or equal to \( p - 1 \) for odd \( p \), \( p \neq 1 \). For \( p \) even or \( p = 1 \), the highest degree term is \( -B_p \frac{k^p}{p!} \). If \( p \) is odd, \( p \neq 1 \), the term of degree \( p \) in \( k \) is equal to 0 as \( B_p = 0 \), while the term of degree \( (p-1) \) is \( -(p/2-t)B_{p-1} \frac{k^{p-1}}{(p-1)!} \).

5 A relation between Bernoulli series and trigonometric sums

Define:

\[ V(q, k) = \sum_{n=1}^{k-1} \frac{1}{4^q (\sin(\pi n/k))^{2q}}. \]

As we will see later on, \( V(q, k) \) is a special case of a Verlinde sum. We have \( V(q, k) = (-1)^q W(2q, q)(k) \). Indeed

\[
(-1)^q W(2q, q)(k) = (-1)^q \sum_{1 \leq n \leq (k-1)} \frac{e^{2i\pi nq/k}}{(1-e^{2i\pi n/k})^{2q}}
\]

\[
= \sum_{n=1}^{k-1} \frac{1}{(1-e^{2i\pi n/k})q(1-e^{-2i\pi n/k})q}
\]

\[
= \sum_{n=1}^{k-1} \frac{1}{4^q (\sin(\pi n/k))^{2q}}.
\]

Thus, from Theorem 4, we obtain that \( k \mapsto V(q, k) \) is a polynomial in \( k \), of degree \( 2q \). Notice that \( V(q, k) \) is a sum of positive real numbers so that \( V(q, k) \) is positive.
The polynomial \( k \mapsto V(q, k) \) is of degree \( 2q \) and its highest degree term is
\[
(-1)^{q+1} \frac{B_{2q}}{(2q)!} k^{2q}.
\]
(In our conventions for Bernoulli numbers, \((-1)^{q+1} B_{2q}\) is positive.)

There is a more precise relation between the polynomial function \( t \mapsto B(2q, t) \) and the polynomial function \( k \mapsto V(q, k) \). Consider the Taylor series at the origin of the function of \( x \hat{A}(q, x) = (x/2)^{2q} \sinh(x/2) = 1 - \frac{q}{12}x^2 + \frac{q}{1440} + \frac{q^2}{288}x^4 + \cdots \).

Substitute \( x = \partial/\partial t \) in \( \hat{A}(q, x) \), and consider \( \hat{A}(q, \partial) \) as a series of differential operators in powers of \( \partial := \partial/\partial t \). The action of \( \hat{A}(q, \partial) \) on a polynomial function of \( t \) is well defined.

**Theorem 5**

\[
V(q, k) = (-1)^{(q+1)} \frac{k^{2q}}{2q!} (\hat{A}(q, \partial/k) \cdot B(2q, t))|_{t=0}.
\]

From this expression, we see again that the highest degree term of the polynomial function \( k \mapsto V(q, k) \) is \((-1)^{q+1} B_{2q} k^{2q} \).

(Remark. In the context of the Verlinde formula, this theorem is closely related to the Riemann-Roch theorem on the manifold \( M_g := M(SU(2), g) \) of flat connections on vector bundles of rank 2 on a Riemann surface of genus \( g \). This manifold is provided with a line bundle \( \mathcal{L} \). The above expression arises when calculating the integral \( \int_{M_g} \text{ch}(\mathcal{L}^{k-2}) \hat{A}(M_g) \) (where \( \text{ch} \) is the Chern character and \( \hat{A}(M_g) \) is the \( \hat{A} \) genus). This evaluates the dimension of the space of so-called generalized theta functions: the dimension of the space of holomorphic sections of the holomorphic line bundle \( \mathcal{L}^{k-2} \) over \( M_g \). We shall come back to this analogy later.)

**Proof.** How to prove this theorem: Consider the residue formula for the Bernoulli polynomial
\[
B(2q, t) = -(2q! \text{residue}_{x=0} \left( x^{-2q} \frac{e^{xt}}{(1 - e^x)} \right)).
\]
We can apply the series \( \hat{A}(q, \partial/k) \) to this expression. Under the residue, any analytic function is automatically replaced by its Taylor series. Thus we obtain

\[
(-1)^q k^{2q} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t) = (-1)^q k^{2q} \text{residue}_{x=0} \left( \frac{\hat{A}(q, x/k)x^{-2q} e^{xt}}{1 - e^x} \right)
\]

Thus at \( t = 0 \), we obtain

\[
(-1)^q k^{2q} \frac{k^{2q}}{2q!} \left( \hat{A}(q, \partial/k) B(2q, t) \right) |_{t=0} = \text{residue}_{x=0} \left( \frac{1}{1 - e^{x/k}q(1 - e^{-x/k})} \right)
\]

The change of variables \( x \mapsto -kx \) leads to

\[
(-1)^q k^{2q} \frac{k^{2q}}{2q!} \left( \hat{A}(q, \partial/k) B(2q, t) \right) |_{t=0} = -k \text{residue}_{x=0} \left( \frac{1}{1 - e^{x/k}q(1 - e^{-x/k})} \right)
\]

We recognize here the residue expression given in Theorem 4 for \((-1)^q W(2q, q)(k) = V(q, k)\). Thus we obtain

\[
(-1)^q k^{2q} \frac{k^{2q}}{2q!} \left( \hat{A}(q, \partial/k) B(2q, t) \right) |_{t=0} = V(q, k)
\]

This proves Theorem 5.

It is amusing to give a false proof of this theorem, by interverting differentiation and summations:
Indeed applying formally $\hat{A}(q, \partial/k)$ to the sum $-\sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^{2q}}$ expressing the Bernoulli polynomial $\frac{1}{2q} B(2q, t)$, we would obtain:

$$(-1)^{(q+1)} \frac{k^{2q}}{2^{2q}} \left( \hat{A}(q, \partial/k)B(2q, t) \right) \big|_{t=0} = (-1)^q k^{2q} \sum_{n \neq 0} \hat{A}(q, 2i\pi n/k) \frac{1}{(2i\pi n)^{2q}}$$

$$= \sum_{n \neq 0} \frac{1}{(1 - e^{2i\pi n/k})^q (1 - e^{-2i\pi n/k})^q}.$$

This last expression is highly divergent, for at least two reasons: first, for $n \neq 0$ multiple of $k$, the term to add to the sum is equal to $\infty$, second, all the terms in the arithmetic progression $n + kj$ give the same summand. However, it gives a hint of why this operator $\hat{A}(q, \partial/k)$ occurs in the comparison. The "renormalized" sum consists in restricting the sum to $0 < n < k$, a set of representatives of the non-zero elements of $\mathbb{Z}/k\mathbb{Z}$.

The function $V(q, k)$ has some remarkable integral property (which follows from the representation theory of $SL(2, \mathbb{C})$). Indeed, the function

$$Ver(q, k) := (2(k + 2))^q V(q, k + 2) = 2^{-q} (k + 2)^q \sum_{n=0}^{k} \frac{1}{\sin((n + 1)\pi/(k + 2))^{2q}}$$

takes positive integral values on integers $k$. Notice that for $k = 0$, we have $Ver(q, 0) = 1$.

We have

$$Ver(1, k) = \frac{1}{6} (k + 1)(k + 2)(k + 3),$$

$$Ver(2, k) = \frac{1}{180} (k + 2)^2 (k + 3)(k + 1)(k^2 + 4k + 15),$$

$$Ver(3, k) = \frac{1}{7560} (k + 2)^3 (k + 1)(k + 3)(2k^4 + 16k^3 + 71k^2 + 156k + 315),$$

... 

You can indeed verify on a few numbers $k$ the amazing fact that these functions take integral values on integers.
6 Preliminaries on semi-simple Lie algebras

The Verlinde sums we saw corresponded to the Lie group \( SL(2, \mathbb{C}) \) with Lie algebra \( sl(2) \) and compact form \( SU(2) \). Before introducing more general Verlinde sums, we need to recall some basic facts on the representation theory of semi-simple Lie algebras.

A Lie algebra \( \mathfrak{g} \) over a field \( \mathbb{C} \) is called semi-simple if its Cartan-Killing form \( \langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) defined by

\[
\langle X, Y \rangle = \text{tr}(\text{ad}(X)\text{ad}(Y))
\]

is non-degenerate. Any semi-simple Lie algebra is the direct product of simple Lie algebras (a Lie algebra \( \mathfrak{g} \) is called simple if it has no proper ideals). An example of simple Lie algebra is the algebra \( sl(n) \) of all \( n \times n \) matrices with trace equal to 0. Then, up to normalization, the Killing form is \( \langle X, Y \rangle = \text{tr}(XY) \) where \( \text{tr} \) is the ordinary trace.

In particular \( sl(2) := \{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} | x_1, x_2, x_3 \in \mathbb{C} \} \) is a simple Lie algebra.

In order to describe the finite dimensional representations of \( \mathfrak{g} \), we need to introduce a few preliminary definitions.

A Cartan subalgebra of a semi-simple Lie algebra \( \mathfrak{g} \) is a subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) such that:

i) \( \mathfrak{h} \) is abelian and every element \( X \in \mathfrak{h} \) is such that the transformation \( \text{ad}(X) \) is diagonalizable,

ii) \( \mathfrak{h} \) is its own normalizer in \( \mathfrak{g} \) i.e. \( \{ X \in \mathfrak{g} | [X, \mathfrak{h}] \subset \mathfrak{h} \} = \mathfrak{h} \).

Such an algebra \( \mathfrak{h} \) is a maximal abelian subalgebra of \( \mathfrak{g} \) and is unique up to conjugacy. The dimension \( r \) of \( \mathfrak{h} \) is called the rank of \( \mathfrak{g} \). In the case of \( sl(n) \), the algebra \( \mathfrak{h} \) is the set of diagonal matrices (with zero trace). The rank of \( sl(n) \) is \( n - 1 \).

For \( sl(2) \), we write an element \( t \) of \( \mathfrak{h} \) as

\[
t = \begin{pmatrix} t_1 & 0 \\ 0 & -t_1 \end{pmatrix}.
\]

The restriction of \( \langle \cdot, \cdot \rangle \) to \( \mathfrak{h} \times \mathfrak{h} \) is non-degenerate. We can then identify \( \mathfrak{h} \) and its dual \( \mathfrak{h}^* \).

A root is an element \( \alpha \in \mathfrak{h}^* \), \( \alpha \neq 0 \), such that the corresponding root space

\[
\mathfrak{g}_\alpha := \{ X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \}
\]
is non-zero.

Let $\Delta := \{ \alpha \in \mathfrak{h}^* | \alpha \text{ is a root} \}$. A positive system of roots is a set $\Delta^+ \subset \Delta$ such that
\[
\Delta^+ \cap (-\Delta^+) = \emptyset, \\
\Delta^+ \cup (-\Delta^+) = \Delta,
\]
and such that for any $\alpha, \beta \in \Delta^+$, $\alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta^+$.

A simple system of roots is a set $S \subset \Delta^+$, $S = \{\alpha_1, \ldots, \alpha_r\}$, such that given $\alpha \in \Delta^+$, there are uniquely defined non-negative integers $m_i (i = 1, \ldots, r)$ such that $\alpha = m_1 \alpha_1 + \cdots + m_r \alpha_r$.

Let $S = \{\alpha_1, \ldots, \alpha_r\}$ be a simple system of roots; we set $H_i$ to be the unique element of $\mathfrak{h}$ such that $\lambda(H_i) = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \langle \lambda, \alpha_i \rangle$, for any $\lambda \in \mathfrak{h}^*$. This element is well defined, since $\langle \cdot, \cdot \rangle_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. Thus we have $\alpha_i(H_i) = 2$.

We denote by $\mathfrak{h}_{\mathbb{R}}$ the real vector space of $\mathfrak{h}$ spanned by the elements $H_i$ of the complex Cartan subalgebra $\mathfrak{h}$. An element $\lambda \in \mathfrak{h}^*$ is called an integral weight if $\lambda(H_i)$ is an integer for all $1 \leq i \leq r$. It is called dominant if $\lambda(H_i)$ is real and non-negative for all $1 \leq i \leq r$.

We denote by $P \subset \mathfrak{h}_{\mathbb{R}}^*$ the lattice of integral weights. We denote by $Q$ its dual lattice in $\mathfrak{h}_{\mathbb{R}}$: the lattice $Q$ is exactly the set of elements $t \in \mathfrak{h}_{\mathbb{R}}$, where all integral weights take integral values. A weight $\lambda$ is called regular, if $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$. We denote by $P_{reg} \subset \mathfrak{h}_{\mathbb{R}}^*$ the set of regular integral weights.

The set
\[
\Gamma := \{m_1 \alpha_1 + \cdots + m_r \alpha_r | m_i \text{ non-negative integers}\}
\]
induces a partial ordering on the weights: $\lambda \leq \lambda'$ whenever $\lambda' - \lambda \in \Gamma$.

Given a finite-dimensional representation $U$ of $\mathfrak{g}$ in a complex vector space $V$, a weight $\lambda$ of $U$ is an element $\lambda \in \mathfrak{h}^*$ such that
\[
V_\lambda := \{ v \in V \mid U(H)v = \lambda(H)v \quad \text{for all } H \in \mathfrak{h} \}
\]
is not reduced to zero. We have $V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$. All weights of a finite-dimensional representation are integral weights.

If $V$ is an irreducible finite-dimensional representation of $\mathfrak{g}$, then $V$ has a unique highest weight $\lambda$ (all other weights $\lambda'$ of the representation $V$ satisfy
\( \lambda' < \lambda \). This weight \( \lambda \) is an integral and dominant weight. Reciprocally, consider the dominant cone

\[
D := \{ \lambda \in h_\mathbb{R}^* | \lambda(H_i) \text{ non-negative integer for } 1 \leq i \leq r \}
\]
of all dominant integral weights. Given \( \lambda \in D \), there is exactly one equivalence class of finite dimensional irreducible representations of \( \mathfrak{g} \) admitting \( \lambda \) as its highest weight. Let \( U_\lambda \) be a representative of this class. In other words, the representations \( U_\lambda, \lambda \in D \), exhaust all the irreducible representations of finite dimension of \( \mathfrak{g} \) up to equivalence.

7 Witten series

One interesting generalization of the Bernoulli series are the Witten series \( B(p, \mathfrak{g})(t) \). Here \( t \) is an element of \( h_\mathbb{R} \), and

\[
B(p, \mathfrak{g})(t) = \sum_{\lambda \in P_{reg}} \frac{e^{2i\pi \lambda, t}}{\prod_{\alpha \in \Delta^+} 2i\pi \langle \alpha, \lambda \rangle}.
\]

This series converges for \( p \) sufficiently large. For any \( p \), it is well defined as a generalized function of \( t \). The function of \( t \in h_\mathbb{R} \) defined by this series is periodic with respect to the lattice \( Q \). On \( h_\mathbb{R}/Q \) (represented by a domain in \( h_\mathbb{R} \)), the expression \( B(p, \mathfrak{g})(t) \) above is polynomial in sectors delimited by hyperplanes. On each sector, these series can in fact be expressed in terms of the Bernoulli polynomials in one variable.

In the case of \( sl(2) \) one recovers:

\[
B(p, t_1) = -B(p, sl(2))(t) = -\sum_{n \neq 0} \frac{e^{2i\pi t_1 n}}{(2i\pi n)^p}.
\]

Here

\[
t \in h_\mathbb{R} = \begin{pmatrix} t_1 & 0 \\ 0 & -t_1 \end{pmatrix}
\]

with \( t_1 \in ]0, 1[ \).

Similarly to the case of \( sl(2) \), a residue formula due to A. Szenes ([Sz1]) can be given for these infinite sums, a formula which allows to calculate them effectively and to prove their polynomial behavior in sectors.
Consider the example of \( sl(3) \): we get

\[
B(q, sl(3))(t_1, t_2) = \sum_{n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0} \frac{e^{2i\pi(n_1 t_1 + (t_1 + t_2)n_2)}}{(2i\pi n_1)^q(2i\pi n_2)^q(2i\pi(n_1 + n_2))^q}.
\]

Here the point \((t_1, t_2) \in \mathbb{R}^2\) represents the diagonal matrix in \( \mathfrak{h}_\mathbb{R} \)

\[
\begin{pmatrix}
t_1 & 0 & 0 \\
0 & t_2 & 0 \\
0 & 0 & -(t_1 + t_2)
\end{pmatrix}.
\]

Changing \( n_1 \) to \( n + m \), and \( n_2 \) to \(-m\) this is also equal to

\[
(-1)^q \sum_{n \neq 0, m \neq 0, n + m \neq 0} \frac{e^{2i\pi(n_1 t_1 - t_2 m)}}{(2i\pi n)^q(2i\pi m)^q(2i\pi(n + m))^q}.
\]

The residue formula depends on the position of \((t_1, t_2)\) in sectors (as in the one dimensional case, the residue formula for the similar sum \( \sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^q} \) was only valid for \( 0 < t < 1 \)). It reads

\[
(-1)^q B(q, su(3))(t_1, t_2) =
\]

\[
-\text{residue}_{x=0} \left( \text{residue}_{y=0} \frac{e^{(t_1)x-(t_2)y}}{x^q y^q (x + y)^q (1 - e^x)(1 - e^{-y})} \cdot 1 \right)
\]

\[
+\text{residue}_{x=0} \left( \text{residue}_{y=0} \frac{e^{(t_1 + t_2)x+(t_2)y}}{x^q y^q (x + y)^q (1 - e^x)(1 - e^{-y})} \cdot 1 \right).
\]

Here \( \{t\} \) denotes \( t - [t] \) where \([t]\) is the integral part of \( t \).

For example, we have \( B(2, su(3))(0, 0) = -\frac{1}{30240} \).

Very naively, looking at the sum of residues at the points \((x = 2i\pi n, y = 2i\pi m)\), the first iterated residue: \(-\text{residue}_{x=0}(\text{residue}_{y=0})\) should already lead to

\[
\sum_{n \neq 0, m \neq 0, n + m \neq 0} \frac{e^{2i\pi(n_1 t_1 - t_2 m)}}{(2i\pi n)^q(2i\pi m)^q(2i\pi(n + m))^q}
\]

and the second residue should lead to the sum

\[
\sum_{n \neq 0, m \neq 0, n + m \neq 0} \frac{e^{2i\pi(t_1 + t_2)n+(t_2)(n+m)}}{(2i\pi n)^q(2i\pi m)^q(2i\pi(n + m))^q}
\]
which is equal, as seen from the first formula for $B(q, \text{sl}(3))(t_1, t_2)$, after changing $n_1$ and $n_2$.

But, in fact the two iterated residues are not equal, and Szenes formula shows that the correct answer is the sum of both iterated residues. Furthermore we must be careful with the order in which we take residues.

( Remark: Let $G$ be the compact simply connected group whose Lie algebra is a compact form of $\mathfrak{g}$. Up to some normalization, for $p = 2g - 1$, the Witten series computes the symplectic volume of the manifold $M(G, g, t)$ of moduli space of flat connections on a $G$-bundle on a Riemann surface of genus $g$ with one hole (the variable $t \in \mathfrak{h}_\mathbb{R}$ parametrizes the holonomy of the flat connection around the hole). We describe this manifold in Section 9, and give some references for its geometric meaning.)

8 Verlinde sums

One interesting generalization of the sum

$$V(q, k) = \sum_{n=1}^{k-1} \frac{1}{4q(\sin(\pi n/k))^2}$$

considered in Section 5 is the Verlinde sum, that we are going to describe.

Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$. Let $S$ be a simple system of roots, $\Delta^+$ the corresponding positive system of roots and $\mathcal{D}$ the associated cone of dominant integral weights. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Let $\theta$ be the highest root. Let us also introduce the set (called an alcove):

$$A_k := \left\{ \lambda \in \mathcal{D} \mid 2\frac{\langle \lambda, \theta \rangle}{\langle \theta, \theta \rangle} \leq k \right\}$$

(the definition of $A_k$ does not depend of the choice of the scalar product.)

Let $h$ be the dual Coxeter number $h = 2\langle \rho, \theta \rangle + 1$. When $\mathfrak{g} = \text{sl}(n)$, then $h = n$.

Consider the scalar product $<\cdot, \cdot>$ such that $<\theta, \theta> = 2$.

Let $q$ be an integer. The Verlinde sum is
\[ \text{Ver}(q, g)(k) = c_G^q(k + h)^r \sum_{\alpha \in \Delta^+} \frac{1}{(1 - e^{-\frac{2\pi i \alpha \cdot u + \rho}{k+h}})^q(1 - e^{-\frac{2\pi i \alpha \cdot u + \rho}{k+h}})^q} \]
\[ = c_G^q(k + h)^r \sum_{\alpha \in \Delta^+} \frac{1}{(2 \sin(\frac{\pi \alpha \cdot u + \rho}{k+h}))^{2q}}. \]

Here \( c_G \) is an explicit constant depending on \( G \). It is such that \( \text{Ver}(1, g)(0) = 1 \).

When \( g = sl(2) \), \( \text{Ver}(q, g)(k) \) is the polynomial \( \text{Ver}(q, k) \) we considered in Section 4. Similarly to this case, there is a residue formula (due to Szenes) for the Verlinde sum, which allows to show its polynomial behavior in \( k \) and to compare it to the Witten series.

For \( g = sl(3) \), the above sum is

\[ \text{Ver}(q, sl(3))(k) = 3^q(k + 3)^2q \]
\[ \times \sum_{n_1 \geq 0, n_2 \geq 0, n_1 + n_2 \leq k} \left( 8 \sin\left(\frac{\pi (n_1 + 1)}{k + 3}\right) \sin\left(\frac{\pi (n_2 + 1)}{k + 3}\right) \sin\left(\frac{\pi (n_1 + n_2 + 2)}{k + 3}\right) \right)^{-2q}. \]

It is known (and amazing) that the function \( k \mapsto \text{Ver}(q, g)(k) \) takes integral values on integers: it is the dimension of a vector space \( V \) arising as the space of generalized theta functions which is the space of holomorphic sections of the holomorphic line bundle \( \mathcal{L} \) over the manifold \( M(G, g) = M(G, g, 0) \) of a Riemann surface of genus \( g = q + 1 \) with central charge \( k \). It was proved independently by Beauville-Laszlo, Faltings, Kumar-Narasimhan-Ramanathan that the space \( V \) is isomorphic to the space \( L \) of conformal blocks extensively studied by Tsuchiya-Ueno-Yamada. From their factorization theorem, one obtains the Verlinde formula giving the dimension of the space \( V \) as \( \text{Ver}(q, g)(k) \).

There is a relation between the Witten series and the Verlinde sum: the Verlinde sum is obtained from the Witten series by applying a series of differential operators \( \hat{A}(q, \partial/(k + h)) \) to the Witten series. The form of the wanted operator \( \hat{A}(q, \partial/(k + h)) \) can be guessed from the same intuitive argument of “differentiating” under the sum sign. The correct argument follows immediately from the explicit residue formula. It allows to compute the integral \( \int_{M(G, g)} \text{ch}(\mathcal{L}^k) \hat{A}(M(G, g)) \) (\( \text{ch} \) is the Chern character and \( \hat{A}(M(G, g)) \) is the \( \hat{A} \) genus), at least when \( k \) is sufficiently large.
It is known that there exists an integer $d > 0$ such that the function $k \mapsto \text{Ver}(q, \mathfrak{g})(dk)$ is polynomial in $k$. What is not known is the value of the smallest possible $d$. It is known that $d = 1$ for $\mathfrak{g} = \text{sl}(n)$.

The following conjecture on $d$ is natural in view of Kumar-Narasimhan work on the Picard group of $M(g, G, 0)$ (Math. Ann. 308 (1997) 155-173):

\[
\begin{align*}
    d &= 1 \text{ for } C_r, \\
    d &= 2 \text{ for } B_r (r \geq 3), D_r (r \geq 4), G_2, \\
    d &= 6 \text{ for } F_4, E_6, \\
    d &= 12 \text{ for } E_7, \\
    d &= 60 \text{ for } E_8,
\end{align*}
\]

where $C_r, B_r, D_r$ are the series of classical simple Lie algebras and $G_2, F_4, E_6, E_7, E_8$ are the exceptional Lie algebras. (From the above cited work of Kumar-Narasimhan, it follows that $d$ is smaller or equal to these values).
9 Geometry beyond. Very few references

The underlying geometric object to the theory of Witten series and Verlinde sums is the manifold $M(G, g, t)$. Here $G$ is a compact (simply connected) Lie group with Lie algebra $\mathfrak{g}$, $g$ is a positive integer and $t$ is an element of the Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ of the complex semi-simple Lie algebra $\mathfrak{g}_\mathbb{C}$. If $u_1, u_2 \in G$, we denote $[u_1, u_2] = u_1u_2u_1^{-1}u_2^{-1}$.

The manifold $M(G, g, t)$ is defined to be:

$$M(G, g, t) = \{(u_1, v_1, \ldots, u_g, v_g) | u_i, v_i \in G \text{ such that } \prod_{i=1}^{g} [u_i, v_i] = e^{2i\pi t}\}/T.$$

Here $T$ denotes the maximal compact torus of $G$ with Lie algebra $i\mathfrak{h}_\mathbb{R}$. The notation $/T$ means that we identify the $(2g)$-tuples $(u_1, v_1, \ldots, u_g, v_g)$ and $(u_1', v_1', \ldots, u_g', v_g')$ if there exists $t \in T$ such that $u_i' = tu_i t^{-1}$ and $v_i' = tv_i t^{-1}$.

- The main reference on the geometric properties of the manifold $M(G, g, t)$ for $G = U_n$ is:

- The computation of the volume of $M(G, g, t)$ when $G = SU(2)$ is due to M. Thaddeus, who related volumes and Bernoulli numbers.

In the general case, the volume of $M(G, g, t)$ is determined in the form of Witten series in:


- A simple description of the manifold $M(G, g, t)$ together with the computation of its symplectic form is in

A quick computation of Witten formulae for its symplectic volume is in:

The Verlinde formula was conjectured by:


It was proved using fusion rules, by


Results on computation of intersection numbers (and thus the Riemann-Roch formula) are obtained via multi-dimensional residues in:


A proof of the Verlinde formula for general compact simply connected groups and any number of holes (but with restrictions on $k$) is obtained via the Riemann-Roch theorem and multi-dimensional residue calculus in:


Another approach, which handles the general case, can be found in