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A PROOF OF BISMUT LOCAL INDEX THEOREM FOR A FAMILY OF DIRAC OPERATORS

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INTRODUCTION

IN [3], using probabilistic methods, Bismut generalized his heat kernel proof of Atiyah–Singer index theorem to give a local index theorem for a family of Dirac operators.

The main purpose of the present work is to give a proof of Bismut's theorem based on the classical expansion of heat kernels. We employ the method, introduced in [5], of expressing the heat kernel of the Laplacian of a vector bundle as an average over the holonomy group. In particular, as in [5], the $\hat{\mathscr{A}}$-genus is naturally related to the Jacobian of the exponential map on a frame bundle.

Let us summarize the content of this article.

In [9], Quillen introduces a Chern–Weil theory with superconnections which, in the finite dimensional case, is easily seen to produce differential form representatives for the Chern character of a difference bundle: let $\mathscr{W}=\mathscr{W}^+\oplus\mathscr{W}^-$ be a supervector bundle on a manifold B. A superconnection on $\mathscr{W}$ is an odd operator D on the space $\mathscr{A}(B,\mathscr{W})=\mathscr{A}(B)\otimes_{\mathscr{A}^0(B)}\Gamma(B,\mathscr{W})$ of $\mathscr{W}$-valued differential forms on B, which satisfies

0.1 $$D(\omega\phi)=(\mathrm{d}\omega)\phi+(-1)^{\deg\omega}\omega D\phi \text{ for } \omega\in\mathscr{A}(B),\ \phi\in\mathscr{A}(B,\mathscr{W}).$$

Thus, the square of the operator D is given by

$$(D^2\phi)(x)=R(x)\cdot\phi(x),$$

where $R\in\mathscr{A}(B,\operatorname{End}\mathscr{W})$ is a matrix with differential form entries. The supertrace noted $\operatorname{str}(e^{-R})$ of the matrix e^{-R} is a differential form which represents the (non-normalized) Chern character of the difference bundle $[\mathscr{W}^+]-[\mathscr{W}^-]$.

Let $u=\begin{vmatrix}0 & u^-\\ u^+ & 0\end{vmatrix}$ be an odd hermitian endomorphism of $\mathscr{W}$. We consider u as a family of odd endomorphisms $(u_x)_{x\in B}$. The index bundle $\operatorname{Ind} u=[\operatorname{Ker} u^+]-[\operatorname{Ker} u^-]$ is well defined as an element of $K(B)$. If B is compact, the equality $\operatorname{Ind} u=[\mathscr{W}^+]-[\mathscr{W}^-]$ holds in $K(B)$.

Any superconnection may be expanded according to the exterior degree on B as $D=D^{[0]}+D^{[1]}+D^{[2]}+\dots$ where $D^{[1]}$ is a connection on $\mathscr{W}$ preserving the grading and $D^{[j]}\in\mathscr{A}^j(B)\otimes\operatorname{End}(\mathscr{W})$, for $j\neq 1$. In particular $D^{[0]}$ is a family of odd endomorphisms. For $t>0$, let δ_t be the automorphism of $\mathscr{A}(B)$ such that $\delta_t\omega=t^{-j/2}\omega$ for $\omega\in\mathscr{A}^j(B)$. Then $D_t=t^{1/2}\delta_t\circ D\circ\delta_t^{-1}$ is again a superconnection and

$$D_t=t^{1/2}D^{[0]}+D^{[1]}+t^{-1/2}D^{[2]}+\dots.$$

If R is the curvature of D, then the curvature of R_t is $t\delta_t(R)$.

We prove (theorem I.1.9) that, when $u_x=D_x^{[0]}$ is hermitian and has constant rank, the matrix $e^{-t\delta_t(R)}$ has a limit when $t \to \infty$ and this limit is e^{-R_0} where R_0 is the curvature of the connection on Ker u given by the orthogonal projection of the connection component $D^{[1]}$ of D.

Bismut's theorem on the local index of a family of Dirac operators is based on an infinite dimensional analogue of Quillen's construction. Let $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$ be an infinite dimensional vector bundle and $u = (u_x)_{x \in B}$ be a family of odd self-adjoint Fredholm operators, so that the index bundle Ind$u$ is well defined as an element of $K(B)$. Let D be a superconnection on $\mathscr{W}$ with zero-exterior degree term equal to u, let R be the curvature of D. If the semigroup e^{-tR} can be defined, the differential form $\delta(t)(\mathrm{str}\, e^{-tR})$ should represent the Chern character of Ind u^+, for every $t>0$. This generalization of Mac-Kean-Singer formula was proven by Bismut for a family of Dirac operators. It is not difficult to see that our homotopy argument extends to this infinite dimensional situation, thus giving an easier and more transparent proof of the Bismut heat equation formula for a family.

The Atiyah-Singer index density [2] is obtained at the other end of the homotopy, when $t \to 0$, by asymptotic calculations which can be handled by probabilistic methods or, as well, as hopefully shown, by classical methods of analysis.

The situation is the following: Let $\pi\colon \tilde{M} \to B$ be a fibration where the fibers $(M_x)_{x \in B}$ are compact Riemannian manifolds of even dimension. Let $V\tilde{M}$ be the tangent bundle along the fibers. Let us assume that $V\tilde{M}$ has a spin structure. Consider the bundle $\mathscr{S} = \mathscr{S}^+ \oplus \mathscr{S}^-$ over $\tilde{M}$ whose fiber $\mathscr{S}_y$ at a point $y \in \tilde{M}$ is the space of vertical spinors. Let $\mathscr{E}$ be an auxiliary vector bundle over $\tilde{M}$ with an hermitian connection. For each $x \in B$, consider the Dirac operator $\mathbb{D}_x$ acting on $\mathscr{W}_x^{\pm} = \Gamma(M_x, \mathscr{S}^{\pm} \otimes \mathscr{E})$. The family $\mathbb{D} = (\mathbb{D}_x)_{x \in B}$ can be considered as an odd endomorphism of the infinite dimensional bundle $\mathscr{W} = (\mathscr{W}_x)_{x \in B}$. The operator $\mathbb{D}_x$ is elliptic, self-adjoint, so that the index bundle of the family $\mathbb{D} = (\mathbb{D}_x)_{x \in B}$ is well defined. The corresponding element of $K(B)$ was identified in topological terms by Atiyah-Singer [2]. In particular a cohomological formula was given for its Chern character.

In [3], Bismut constructs a remarkable superconnection on the infinite dimensional superbundle $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$ whose term of exterior degree 0 is the family of Dirac operators $(\mathrm{D}_x)_{x \in B}$. We denote the Bismut superconnection by $\mathbb{B}$ and its curvature by I. Thus, for $x \in B$, $I_x \in \Lambda T_x^* B \otimes \mathrm{End}(\mathscr{W}_x)$. Then I is given by a formula analogous to Lichnerowicz formula for the square of a Dirac operator. In particular I_x is a second order differential operator with principal symbol given by the Riemannian metric of M_x, so that the semigroup e^{-tI_x} can be constructed and is given by a kernel

$$\langle y_1 | e^{-tI_x} | y_2 \rangle \in \Lambda T_x^* B \otimes \mathrm{Hom}((\mathscr{S} \otimes \mathscr{E})_{y_2}, (\mathscr{S} \otimes \mathscr{E})_{y_1}).$$

By scaling the metric in the fibers M_x, Bismut obtains for each $t>0$ a superconnection $\mathbb{B}_t$ which in fact coincides with the deformation $\mathbb{B}_t = t^{1/2} \delta_t \circ \mathbb{B} \circ \delta_t^{-1}$. The supertrace $\delta_t \cdot (\mathrm{str}\, e^{-tI})$ is defined as a differential form on B by

$$\delta_t \cdot (\mathrm{str}\, e^{-tI})_x = \int_{M_x} \delta_t (\mathrm{str} \langle y | e^{-tI_x} | y \rangle)\, \mathrm{d}y.$$

It is a closed form, its cohomology class is independent of t, and it represents the Chern character of Ind $\mathbb{D}$ ([3] Theorem 2.6 or Theorem I.2.4. below).

The main result of Bismut is to show that when $t \to 0$ the index density $\delta_t (\mathrm{str} \langle y | e^{-tI_x} | y \rangle) \mathrm{d}y$ associated to this particular superconnection $\mathbb{B}$ has a limit and to identify this limit as the highest term with respect to the coordinates of the fiber M_x of $\hat{\mathscr{A}}(V\tilde{M})\, \mathrm{ch}\, \mathscr{E}$. In the present paper we obtain the existence and computation of the above limit by the following method: the heat kernel of an operator of the form $\Delta + V$, where Δ is the horizontal Laplacian of a vector bundle and V is a potential, is written as an average, over the structure group, of a heat kernel on the frame bundle, considered itself as a Riemannian manifold. This gives integral formulas for the coefficients of the Minakshisundaram–Pleijel expansion of the heat kernel $e^{-t(\Delta+V)}$ (Theorem II.2.20) which are of interest in a variety of computations. Recall that if $\mathscr{E}$ is a vector bundle with connection ∇ on a compact Riemannian manifold P, the heat kernel of the associated Laplacian has an expansion

$$\langle u|e^{-t\Delta}|u'\rangle = (4\pi t)^{-\frac{\dim P}{2}} e^{-\frac{d(y,y')^2}{4t}} \sum_{i=0}^{N} t^i U_i(u,u') + 0\left(t^{N-\frac{\dim P}{2}}\right) \quad \text{for} \quad u,u' \in P,$$

the first term is given by

$$U_0(u,u') = (\det \theta^P(u,u'))^{-1/2} \tau(u,u'),$$

where θ^P is the Jacobian of the Riemannian exponential map and τ is the geodesic parallel transport in $\mathscr{E}$ with respect to ∇. When P is a principal bundle over a manifold M, with structure group G, the Jacobian $\theta^P(u, u \exp A)$, for A in the Lie algebra of G, is given by an explicit formula involving the curvature of the fibration $P \to M$ (Proposition II.1.3). Applying this averaging method to the operator I_x on M_x, it is easy to see on the integral formula for $\delta_t(\operatorname{str}\langle y|e^{-tI_x}|y\rangle)$ that the limit when $t \to 0$ exists and can be computed using only the first term $U_0(u, u \exp A)$ on the frame bundle. The appearance in the limit of the product $\hat{\mathscr{A}}(V\tilde{M}) \operatorname{ch} \mathscr{E}$ is readily seen from the above factorization of U_0. The elimination of the singular part in the asymptotic expansion of $\delta_t(\operatorname{str}\langle y|e^{-tI_x}|y\rangle)$ is reduced to the following simple lemma in functional calculus: let ξ be a nilpotent element in an algebra $\mathscr{A}$ (here an exterior algebra), P a polynomial in one variable, then,

$$P(\xi) = \lim_{t\to 0} (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4t}} P(x) dx.$$

Our study is very much simplified by the description which we give of the Bismut superconnection (III.1.13): as in [3] we consider a connection for the fibration π: $\tilde{M} \to B$, ie., a family of horizontal tangent subspaces in $T\tilde{M}$. This defines an isomorphism $T\tilde{M} \simeq \pi^*TB \oplus V\tilde{M}$. We consider the vector bundle $\pi^*TB \oplus V\tilde{M} \oplus \pi^*T^*B$ and define on it a connection $\tilde{\nabla}$ (III.1.4) which is a natural generalization of the Levi–Civita connection on a Riemannian manifold. In particular its curvature satisfies a symmetry relation (III.1.8) which generalizes the symmetry $R_{ijkl} = R_{klij}$ of the Levi–Civita curvature. Then the Bismut superconnection $\mathbb{B}$ is constructed out of $\tilde{\nabla}$ as a generalization of the usual Dirac operator. The formula for $I = \mathbb{B}^2$ follows in a way quite similar to the computation of the square of the Dirac operator. The symmetry relation above comes in to relate the $\hat{\mathscr{A}}$-genus to the Jacobian of the exponential map on the frame bundle.

A noteworthy difficulty arises from the fact that the holonomy group of $\tilde{\nabla}$ is not compact: it appears as a parabolic subgroup of the orthogonal group of an indefinite metric. This "defect" is built in the requirement that $\mathbb{B}$ be a superconnection. In fact, using the deformation of the Clifford multiplication into the exterior one, $\mathbb{B}$ appears as the limit when $\varepsilon \to 0$ of a Dirac operator D_ε on the Clifford bundle $\mathscr{S} \otimes \pi^* \Lambda T^*B \otimes \mathscr{E}$ over $\tilde{M}$, when $\tilde{M}$ is given a Riemannian metric g_ε through the choice of a small metric on the base B. (This is how Bismut proves the formula for $\mathbb{B}^2$ ([3], theorem 3.5)). We use this deformation to write I as a limit of operators $I(\varepsilon)$ to which the averaging method can be applied.

I. SUPERCONNECTIONS AND THE CHERN CHARACTER

1. *The finite dimensional case*

1.1. Let C_1, C_2 be $\mathbb{Z}_2$-graded associative algebras. We denote by $C_1 \hat{\otimes} C_2$ the graded tensor product of C_1, C_2. ie., the tensor product $C_1 \otimes C_2$ with the multiplicative law:

$$(\phi_1 \otimes \phi_2)\cdot(\phi'_1 \otimes \phi'_2) = (-1)^{(\deg \phi_2)(\deg \phi'_1)} \phi_1 \phi'_1 \otimes \phi_2 \phi'_2.$$

If W_i $(i=1,2)$ is a graded module over C_i, then $C_1 \hat{\otimes} C_2$ operates on $W_1 \otimes W_2$ by:

$$(\phi_1 \otimes \phi_2)\cdot(\omega_1 \otimes \omega_2) = (-1)^{(\deg \phi_2)(\deg \omega_1)} \phi_1 \omega_1 \otimes \varphi_2 \omega_2.$$

These sign conventions on actions will hold throughout this article.

Let B be a manifold and $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$ be a finite dimensional super vector bundle over B. Denote by $\mathscr{A}(B, \mathscr{W})$ the space of sections of $\Lambda T^*B \otimes \mathscr{W}$. It has a $\mathbb{Z} \times \mathbb{Z}_2$-grading, thus a total $\mathbb{Z}_2$-grading. For $i \in \mathbb{Z}$, $\omega \in A(B, \mathscr{W})$, $\omega^{[i]}$ will denote the term of exterior degree i of ω.

Let $\mathscr{A}(B, \operatorname{End} \mathscr{W}) = \Gamma(\Lambda T^*B \hat{\otimes} \operatorname{End} \mathscr{W})$. It acts fiberwise on $\mathscr{A}(B, \mathscr{W})$.

A superconnection ∇, as defined by Quillen, [9], is an odd operator on $\mathscr{A}(B, \mathscr{W})$ such that

1.2. $$\nabla(\omega\, \phi) = d\omega\, \phi + (-1)^{(\deg \omega)} \omega \nabla \phi$$

for $\omega \in \mathscr{A}(B)$ and $\phi \in \mathscr{A}(B, \mathscr{W})$.

As $d^2 = 0$, we have

1.3. $$\nabla^2 \phi = R \cdot \phi,$$

where R is an even element of $\mathscr{A}(B, \operatorname{End} \mathscr{E})$, which it is natural to call the curvature of the superconnection ∇.

Define the Chern character form

$$\operatorname{ch}_Q(\mathscr{W}, \nabla) = \operatorname{str}(e^{-R}).$$

It is a closed even form on B and its de Rham cohomology class is independent of the choice of the superconnection ∇. More precisely, for $\lambda \in \mathbb{C}^*$, choose a square root of λ and let δ_λ be the automorphism of $\Lambda T^*(B)$ defined by $\delta_\lambda \omega^{[j]} = \lambda^{-j/2} \omega^{[j]}$. (For even forms δ_λ does not depend on the choice of $\lambda^{1/2}$). Extend δ_λ to an automorphism of $\mathscr{A}(B)$. Then $\delta_{2i\pi} \operatorname{ch}_Q(\mathscr{W}, \nabla)$ represents the Chern character of the difference bundle $[\mathscr{W}^+] - [\mathscr{W}^-]$.

1.4. For $t > 0$, let δ_t be the automorphism of ΛT^*B defined by $\delta_t \omega^{[j]} = t^{-j/2} \omega^{[j]}$. Extend δ_t to an automorphism of $\mathscr{A}(B, \mathscr{W})$. If ∇ is a superconnection on $\mathscr{W}$, then $\nabla_t = t^{1/2} \delta_t \circ \nabla \circ \delta_t^{-1}$ is again a superconnection on $\mathscr{W}$. We have

$$\nabla_t = t^{1/2} \nabla^{[0]} + \nabla^{[1]} + t^{-1/2} \nabla^{[2]} + \dots$$

where $\nabla^{[i]}: \Gamma(\Lambda^{\bullet} T^*B \otimes \mathscr{W}) \to \Gamma(\Lambda^{\bullet + i} T^*B \otimes \mathscr{W})$ raises the exterior degree by i. The operator $\nabla^{[1]}$ is a "usual" connection on $\mathscr{W}$, which preserves the decomposition $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$. The operator $\nabla^{[i]}$ $(i \neq 1)$ is given by the action of an element of exterior degree i of $\mathscr{A}(B, \operatorname{End} \mathscr{W})$. We will be mainly concerned by the term $\nabla^{[0]}$. It is an odd endomorphism of $\mathscr{W}$ which we denote by u.

It will be convenient in parts II and III to associate to a superconnection a differential form on a principal bundle. Thus, assume that $\mathscr{W}$ is associated to a principal bundle P over B with structure group G and a representation ρ of G in a vector space $W = W^+ \oplus W^-$. We suppose that ρ preserves this decomposition and we write $\rho = \rho^+ \oplus \rho^-$. Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$, we denote also by X the vertical vector field on P generated by the action of G on P and we denote by $i(X)$ the contraction on $\mathscr{A}(P)$. Let $(\mathscr{A}(P) \otimes W)^{b\,G}$ be the space of G-basic W-valued differential forms on P, that is the space of $\phi\in(\mathscr{A}(P)\otimes W)^G$ such that $i(X)\phi=0$ for every $X\in\mathfrak{g}$. Then $\mathscr{A}(B,\mathscr{W})$ identifies with $(\mathscr{A}(P)\otimes W)^{b\,G}$.

Let θ be an odd element of $(\mathscr{A}(P)\hat{\otimes}\operatorname{End} W)^G$ such that

1.5. $$i(X)\theta=\rho(X)\quad\text{for every } X\in\mathfrak{g}.$$

Such an element will be called a superconnection form. Define an operator on $\mathscr{A}(P)\otimes W$ by

1.6. $$\nabla\phi=\mathrm{d}\phi+\theta\phi.$$

Then ∇ preserves the space of basic forms and the restriction of ∇ to $(\mathscr{A}(P)\otimes W)^{b\,G}\simeq\mathscr{A}(B,\mathscr{W})$ is a superconnection in the sense of 1.2. It is easy to see that every superconnection on $\mathscr{W}$ is of this form. The usual notion of connection corresponds to a 1-form on P, while in this generalization θ can have arbitrary exterior degree. We write $\theta=\theta^{[0]}+\theta^{[1]}+\ldots$ for the decomposition of θ in homogeneous components. We have

1.7. $$\nabla^2\phi=R\phi$$

where

$$R=\mathrm{d}\theta+\theta\theta\in(\mathscr{A}(P)\hat{\otimes}\operatorname{End} W)^{bG}=\mathscr{A}(B,\operatorname{End}\mathscr{W}).$$

If

$$\nabla_t=t^{1/2}\delta_t\circ\nabla\circ\delta_t^{-1},$$

then

$$\nabla_t\phi=(\mathrm{d}+\theta_t)\phi$$

with

$$\theta_t=t^{1/2}\delta_t(\theta)=t^{1/2}\theta^{[0]}+\theta^{[1]}+t^{-1/2}\theta^{[2]}+\ldots.$$

The term $\theta^{[1]}$ is the connection form on P for the "usual" connection $\nabla^{[1]}$. The term $\theta^{[0]}\in(\mathscr{A}^0(P)\otimes\operatorname{End}^- W)^G$ identifies with the odd endomorphism u of $\mathscr{W}$.

1.8. Let us suppose that $\mathscr{W}^+$, $\mathscr{W}^-$ have hermitian structures, and that u is hermitian so that u^- is the adjoint of u^+. Let, for $t>0$, $R_t=\delta_t\circ R\circ\delta_t^{-1}$ be the curvature of the superconnection ∇_t. Then the cohomology class of $\operatorname{str}(\mathrm{e}^{-R_t})$ is independent of t, and represents the (non-normalized) Chern character of the difference bundle $[\mathscr{W}^+]-[\mathscr{W}^-]$. In [9], Quillen asks whether $\operatorname{str}(\mathrm{e}^{-R_t})$ has a limit as a current on B, when $t\to\infty$. This does not seem to be true without further conditions, at least when $x\to u_x$ is not analytic. However, when u has constant rank there is a limit in the space of differential forms and it is what it should be: consider then the vector bundle $\mathscr{W}_0=\operatorname{Ker} u=\mathscr{W}_0^+\oplus\mathscr{W}_0^-$ where $\mathscr{W}_0^+=\operatorname{Ker} u^+$, $\mathscr{W}_0^-=\operatorname{Ker} u^-$. Let P_0: $\mathscr{W}\to\mathscr{W}_0$ be the orthogonal projection of $\mathscr{W}$ on $\mathscr{W}_0$. Then $\nabla_0=P_0\circ\nabla^{[1]}\circ P_0$ defines a (usual) connection on $\mathscr{W}_0$ (preserving $\mathscr{W}_0^+,\mathscr{W}_0^-$), which we call the projection of the connection $\nabla^{[1]}$ on $\mathscr{W}_0$. We prove:

THEOREM 1.9. *Let $\mathscr{W}=\mathscr{W}^+\oplus\mathscr{W}^-$ be an hermitian vector bundle with a hermitian odd endomorphism u of constant rank. Let $\mathscr{W}_0^\pm=\operatorname{Ker} u^\pm$. Let ∇ be a superconnection on $\mathscr{W}$ whose term of exterior degree 0 is equal to u. Let, for $t>0$,*

$$\nabla_t=t^{1/2}\delta_t\circ\nabla\circ\delta_t^{-1}=t^{1/2}u+\nabla^{[1]}+\ldots$$

be the dilated connection, R_t its curvature. Let ∇_0 be the projection on $\mathscr{W}_0$ of the connection $\nabla^{[1]}$, R_0 its curvature. Then e^{-R_t} converges to e^{-R_0} in $\mathscr{A}(B,\operatorname{End}\mathscr{W})$ when $t\to\infty$.

Proof. Since the result is local, we may assume that $\mathscr{W}$ and $\mathscr{W}_0$ are trivial superbundles, $\mathscr{W}=B\times W$, $\mathscr{W}_0=B\times W_0$. Thus $\nabla=d+\theta$ with $\theta\in\mathscr{A}(B)\hat{\otimes}\operatorname{End} W$. Let $W_1=W_0^\perp$. In the decomposition $W = W_0 \oplus W_1$, we write

$$\theta = \begin{vmatrix} \theta_0 & \beta \\ \gamma & \theta_1 \end{vmatrix}$$

then $\theta_0^{[0]} = \beta^{[0]} = \gamma^{[0]} = 0$, $\theta_1^{[0]} = u|W_1$. Denote $\theta_0^{[1]}$ by ω, $u|V_1$ by v. Then $\theta_0^{[1]}$ is the connection form for ∇_0 and v is an invertible hermitian operator on W_1. Let us write the curvature $R = \mathrm{d}\theta + \theta \cdot \theta$ as $R = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$. We have

$$A^{[0]} = A^{[1]} = 0, \quad A^{[2]} = \mathrm{d}\omega + \omega \cdot \omega + \beta^{[1]}\gamma^{[1]},$$

$$B^{[0]} = 0, \quad B^{[1]} = \beta^{[1]}v,$$

$$C^{[0]} = 0,\ C^{[1]} = v\gamma^{[1]},$$

$$D^{[0]} = v^2.$$

Thus $A^{[2]} - B^{[1]}(D^{[0]})^{-1}\, C^{[1]} = \mathrm{d}\omega + \omega \cdot \omega$ is the curvature of ∇_0.

The Theorem will follow from Lemma 1.17 below.

1.10. Let W be a hermitian finite dimensional vector space. Let $A = \bigoplus_{j \geq 0} A_j$ be a $\mathbb{Z}$-graded finite dimensional algebra, with $A_0 = \mathbb{C}$. Consider the algebra $A \otimes \mathrm{End}\, W$ with its $\mathbb{Z}$-gradation induced by the gradation of A. For $t > 0$, we denote by δ_t the automorphism of $A \otimes \mathrm{End}\, W$ given by $\delta_t(a) = t^{-j/2}a$, for $a \in A^j \otimes \mathrm{End}\, W$. Assume that W is an orthogonal direct sum $W = W_0 \oplus W_1$.

LEMMA 1.11. *Let* $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A \otimes \mathrm{End}\, W$. *Assume that* $a^{[0]} = a^{[1]} = 0$, $b^{[0]} = c^{[0]} = 0$ *and that* $d^{[0]}$ *is a self-adjoint negative definite operator on* W_1. *Put* $r_0 = a^{[2]} - b^{[1]}\,(d^{[0]})^{-1}c^{[1]}$. *Then* $r_0 \in A^{[2]} \otimes \mathrm{End}\, W_0$. *We have*:

$$\lim_{t \to \infty} e^{t\delta_t(r)} = \begin{vmatrix} e^{r_0} & 0 \\ 0 & 0 \end{vmatrix}.$$

Proof. We use the expansion formula:

1.12. $$e^{A+B} = e^A + \int_0^1 e^{s_1 A} B e^{(1-s_1)A} \mathrm{d}s_1 + \dots + \int_{\Delta_n} e^{s_1 A} B e^{(s_2 - s_1)A} B \dots B\, e^{(s_n - s_{n-1})A} B\, e^{(1-s_n)A} \mathrm{d}s_1 \dots \mathrm{d}s_n$$

where Δ_n is the simplex $0 \leqslant s_1 \leqslant s_2 \leqslant \dots \leqslant s_n \leqslant 1$ in $\mathbb{R}^n$.

We put

$$A = t\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}, \quad B = t\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}.$$

Since b and c have no 0-degree term, the sum (1.12) is actually finite. Let us write

$$e^{tR} = \begin{vmatrix} X_t & Y_t \\ Z_t & U_t \end{vmatrix}.$$

We then have:

$$X_t = e^{ta} + X_2(t) + \dots + X_{2n}(t) + \dots$$
$$U_t = e^{td} + U_2(t) + \dots + U_{2n}(t) + \dots$$
$$Y_t = Y_1(t) + \dots + Y_{2n-1}(t) + \dots$$
$$Z_t = Z_1(t) + \dots + Z_{2n-1}(t) + \dots$$

with

$$X_{2n}(t) = t^{2n} \int_{\Delta_{2n}} e^{s_1 ta} b\, e^{(s_2 - s_1)td} c\, e^{(s_3 - s_2)ta} \dots c\, e^{(1 - s_{2n})ta} ds_1 \dots ds_{2n}.$$

$$U_{2n}(t) = t^{2n} \int_{\Delta_{2n}} e^{s_1 td} c\, e^{(s_2 - s_1)ta} b\, e^{(s_3 - s_2)td} \dots b\, e^{(1 - s_{2n})td} ds_1 \dots ds_{2n}.$$

$$Y_{2n+1}(t) = t^{2n+1} \int_{\Delta_{2n+1}} e^{s_1 ta} b e^{(s_2 - s_1)td} c \dots b\, e^{(1 - s_{2n+1})td} ds_1 \dots ds_{2n+1}.$$

$$Z_{2n+1}(t) = t^{2n+1} \int_{\Delta_{2n+1}} e^{s_1 td} c\, e^{(s_2 - s_1)ta} b \dots c\, e^{s_{2n+1} ta}\, ds_1 \dots ds_{2n+1}.$$

Let us show that

$$\lim_{t \to \infty} \delta_t(X_{2n}(t)) = \int_{\Delta_n} e^{s_1 a^{[2]}} f\, e^{(s_3 - s_1)a^{[2]}} f \dots f\, e^{(1 - s_{2n-1})a^{[2]}} ds_1 \dots ds_{2n-1}.$$

with

$$f = -b^{[1]} (d^{[0]})^{-1} c^{[1]},$$

so that (by 1.12) $\lim_{t \to \infty} \delta_t(X_t)$ will be equal to $e^{(a^{[2]} + f)}$. Consider the case where $n = 1$ first, ie.,

$$X_2(t) = t^2 \int_{\Delta_2} e^{s_1 ta} b\, e^{(s_2 - s_1)td} c\, e^{(1 - s_2)ta} ds_1\, ds_2.$$

Consider as new variables s_1, $s_2 - s_1 = x_2$, so that

$$X_2(t) = t^2 \int_0^1 e^{s_1 ta} \left(\int_0^{(1 - s_1)} b\, e^{x_2 td} c\, e^{-x_2 ta} dx_2 \right) e^{(1 - s_1)ta} ds_1.$$

Our hypotheses are

$$\lim_{t \to \infty} \delta_t(ta) = a^{[2]}$$

$$\lim_{t \to \infty} \delta_t(t^{1/2} b) = b^{[1]}$$

$$\lim_{t \to \infty} \delta_t(t^{1/2} c) = c^{[1]}$$

$$\lim_{t \to \infty} \delta_t(d) = d^{[0]}.$$

In particular $\lim_{t \to \infty} \delta_t\, e^{(s_1 ta)}$ exists and is $e^{s_1 a^{[2]}}$, $\lim_{t \to \infty} \delta_t\, e^{(1 - s_1)ta} = e^{(1 - s_1)a^{[2]}}$.

Consider

1.13. $$Q(t,x)=t^2 b\,e^{xtd} c\,e^{-xta}.$$

Let us see that for $\lambda>0$,

1.14. $$\lim_{t\to\infty}\delta_t\int_0^\lambda Q(t,x)\,dx=-b^{[1]}(d^{[0]})^{-1}c^{[1]}.$$

The change of variables $tx=y$ gives

$$\delta_t\int_0^\lambda Q(t,x)dx=\int_0^{t\lambda}\delta_t(t^{1/2}b)\,e^{y\delta_t(d)}\delta_t(t^{1/2}c)\,e^{-y\delta_t(a)}dy.$$

As $d^{[0]}$ is a negative operator, there exists $t_0>0$ and $\gamma>0$ such that:

1.15. $$\|\delta_t(e^{yd})\|<M\,e^{-y\gamma}\quad\text{for}\quad y>0\quad t>t_0.$$

(This follows for instance from the expansion formula applied to $d^{[0]}+t^{-1/2}d^{[1]}+\dots$). Therefore we obtain by dominated convergence,

$$\lim_{t\to\infty}\delta_t\int_0^\lambda Q(t,x)dx=\int_0^\infty b^{[1]}e^{yd[0]}c^{[1]}dy=-b^{[1]}(d^{[0]})^{-1}c^{[1]},$$

and the desired formula for $\lim_{t\to\infty}\delta_t X_2(t)$.

For arbitrary n, consider the map $\Delta_{2n}\to\Delta_n$ given by $s=(s_1,s_2,\dots,s_{2n})\to s'=(s_1,s_3,s_5,\dots,s_{2n-1})$. Write an element of Δ_{2n} as $(s_1,s_1+x_2,s_3,s_3+x_4,\dots,s_{2n-1}+x_{2n})$. Then the fiber of the map $s\to s'$ is the product of intervals

$$I_{s'}=\{(x_2,x_4,\dots x_{2n}),\ 0\leqslant x_2\leqslant s_3-s_1,\dots 0\leqslant x_{2n}\leqslant 1-s_{2n-1}\}.$$

The integral for $X_{2n}(t)$ is thus:

$$\int_{s'\in\Delta_n}\int_{I_{s'}}e^{s_1ta}Q(t,x_2)e^{(s_3-s_1)ta}Q(t,x_4)\,e^{(s_5-s_3)ta}\cdots Q(t,x_{2n})\,e^{(1-s_{2n-1})ta}dx\,ds'$$

so that by 1.14, we obtain the desired result for $\lim_{t\to\infty}\delta_t X_{2n}(t)$.

Let us prove that $\lim_{t\to\infty}\delta_t U_{2n}(t)=0$. We have

1.16. $$\|\delta_t(t^{1/2}c)\|<M_1,\quad \|\delta_t(t^{1/2}c\|<M_2$$

$$\|\delta_t(e^{std})\|<M_3\,e^{-st\gamma},\quad \|\delta_t(e^{sta})\|<M_4,\ \text{for}\ 0\leqslant s\leqslant 1,$$

so that

$$\|\delta_t U_{2n}(t)\|\leqslant Ct^n\int_{\Delta_{2n}}e^{-s_1t\gamma}e^{-(s_3-s_2)t\gamma}\dots e^{-(1-s_{2n})t\gamma}ds.$$

Let $I=[0,1]$. Write an element of Δ_{2n} as $s=(x_1,x_2,x_2+x_3,x_4,x_4+x_5,\dots,x_{2n})$. We obtain

$$\|\delta_t U_{2n}(t)\|\leqslant Ct^n\int_{I^{2n}}e^{-x_1t\gamma}e^{-x_3t\gamma}\dots e^{-x_{2n-1}t\gamma}e^{-(1-x_{2n})t\gamma}dx.$$

The integrand is a product of $(n+1)$ exponentials. As

$$t\int_0^1 e^{-xt}dx\leqslant\int_0^\infty e^{-x\gamma}dx=\gamma^{-1}$$

we have

$$\|\delta_t U_{2n}(t)\| < c(t\gamma)^{-1}.$$

Similarly for $\delta_t Y_{2n+1}(t)$, (or $\delta_t Z_{2n+1}(t)$), we use the estimates 1.16 to show

$$\|\delta_t Y_{2n+1}(t)\| < ct^{n+1/2}\int_{\Delta_{2n+1}} e^{-(s_2-s_1)t\gamma}e^{-(s_4-s_3)t\gamma}\ldots e^{-(1-s_{2n+1})t\gamma}ds_1 ds_2\ldots ds_{2n-1}.$$

The integrand is a product of $(n+1)$ exponentials and we obtain $\|\delta_t Y_{2n+1}(t)\| < c\gamma^{-1}t^{-1/2}$. This finishes the proof of the Lemma 1.11.

It is clear that a lemma similar to 1.11 holds for a graded tensor product $A\hat{\otimes}\operatorname{End} W$. Let W^+, W^- be two hermitian vector spaces. Consider $W=W^+\oplus W^-$. Let $A=\underset{j\geq 0}{\otimes} A_j$ be a $\mathbb{Z}$-graded finite dimensional algebra. Consider the graded tensor product $A\otimes\operatorname{End} W$, with its $\mathbb{Z}$-gradation induced by A. For $t>0$, denote by δ_t the automorphism of $A\hat{\otimes}\operatorname{End} W$ given by $\delta_t a = t^{-j/2}a$, for $a\in A_j\hat{\otimes}\operatorname{End} W$. Let $W=W_0\oplus W_1$, with $W^+=W_0^+\oplus W_1^+$, $W^-=W_0^-\oplus W_1^-$ decomposed in orthogonal direct sums.

LEMMA 1.17. *Consider* $r=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in A\hat{\otimes}\, End\ W$. *Suppose* $d^{[0]}\in End(W_1)$ *is an even endomorphism of* W_1, *and suppose that* $d^{[0]}$ *is hermitian negative definite on* W_1; *Suppose that* $a^{[0]}=a^{[1]}=0$, $b^{[0]}=0$, $c^{[0]}=0$. *Put*

$$r_0 = a^{[2]} - b^{[1]}\,(d^{[0]})^{-1}\,c^{[1]}\in A\hat{\otimes}\, End\ W_0.$$

Then

$$\lim_{t\to\infty}\delta_t e^{tr} = \begin{vmatrix} e^{r_0} & 0\\ 0 & 0\end{vmatrix}.$$

It is also clear that if $a(x)$, $b(x)$, $c(x)$ depend smoothly on parameters, an estimate as 1.16 can be obtained uniformly, as well as similar bounds for derivatives. So we obtain the Theorem 1.9.

2. *Superconnections and the Chern character of the index bundle of a family of elliptic operators*

We turn now to the situation where we have a family $(M_x)_{x\in B}$ of compact Riemannian manifolds, a family of superbundles $\mathscr{S}_x=\mathscr{S}_x^+\oplus\mathscr{S}_x^-$ over M_x and where $\mathscr{W}$ is the bundle over B with infinite dimensional fiber $\mathscr{W}_x^\pm=\Gamma(M_x,\mathscr{S}_x^\pm)$. To be more precise, we consider a fibration $\pi\colon \tilde{M}\mapsto B$ and a hermitian finite dimensional superbundle $\mathscr{S}=\mathscr{S}^+\oplus\mathscr{S}^-$ over $\tilde{M}$. For $x\in B$, let M_x be the fiber $\pi^{-1}(x)$ and let $\mathscr{S}_x$ be $\mathscr{S}|M_x$. Let g_x be a Riemannian metric on M_x and $\mathscr{W}_x=\Gamma(M_x,\mathscr{S}_x)$. Using the hermitian structure on $\mathscr{S}$ and the Riemannian volume on M_x, we define a prehilbert structure on $\mathscr{W}_x$. Let

$$u=(u_x)_{x\in B}=\begin{pmatrix} 0 & u_x^-\\ u_x^+ & 0\end{pmatrix}$$

be a smooth family of odd elliptic formally self-adjoint first order differential operators on $\mathscr{W}_x$. One can define the index bundle $\operatorname{Ind} u$ as an element of $K(B)$. When $\operatorname{Ker}(u_x)$ has constant dimension, then $\operatorname{Ind}(u)=[\mathscr{W}_0^+]-[\mathscr{W}_0^-]$, with $(\mathscr{W}_0^-)_x=\operatorname{Ker} u_x^+$, $(\mathscr{W}_0^-)_x=\operatorname{Ker} u_x^-$.

We define $\Gamma(B,\mathscr{W})$ as $\Gamma(\tilde{M},\mathscr{S})$ and

$$\mathscr{A}(B,\mathscr{W})=\mathscr{A}(B)\underset{\mathscr{A}^0(B)}{\otimes}\Gamma(B,\mathscr{W})=\Gamma(\tilde{M},\pi^*\Lambda T^*B\otimes\mathscr{S}).$$

2.1. A superconnection on $\mathscr{W}$ is therefore an odd operator D on $\Gamma(\tilde{M}, \pi^*\Lambda T^*B\otimes\mathscr{S})$ which satisfies I.1.2. In addition we will suppose that D is a first order differential operator. Let D be such a superconnection with u as its zero-degree term and let $I(x)$ be the curvature of D. Thus $I(x)\in\Lambda T_x^*B\otimes\mathrm{End}(\mathscr{W}_x)$ is a second order differential operator. We make the following assumptions:

2.2. The principal symbol of u_x^2 is scalar and given by the metric of M_x.

2.3. $I(x)$ has the same principal symbol as u_x^2.

Remark. Given the family u_x satisfying 2.2, we can always construct a superconnection which satisfies these conditions: choose a connection ∇ on $\mathscr{S}$ and a connection on the fibration π: $\tilde{M}\to B$, ie., a smooth family $T_y^h\tilde{M}\subset T_y\tilde{M}$ of horizontal subspaces. Define a connection $\bar{\nabla}$ on $\Gamma(B, \mathscr{W})=\Gamma(\tilde{M}, \mathscr{S})$ by $\bar{\nabla}_X\phi=\nabla_{X^h}\phi$ for $X\in TB$. Then $D=u+\bar{\nabla}$ is a superconnection with zero degree term equal to u. Its curvature is given by $D^2=u^2+\bar{\nabla}u+\bar{\nabla}^2$ and clearly satisfies 2.3. Bismut superconnection, which will be introduced in part III, is not of this form (one must add higher degree terms) but it also satisfies condition 2.3.

Using the scalar product on $\mathscr{W}_x$ denote by P_0 the orthogonal projection on $\mathscr{W}_0=\mathrm{Ker}\, u$. The following infinite dimensional generalization of theorem I.1.9. holds.

THEOREM 2.4. *Let $(u_x)_{x\in B}$ be a family of odd elliptic formally self adjoint first order differential operators, satisfying condition 2.2. Assume moreover, that $(\mathscr{W}_0)_x=Ker\, u_x$ has constant (finite) dimension. Let D be a superconnection on $\mathscr{W}$ as in 2.1, with zero degree term u, which satisfies condition 2.3. Let R_0 be the curvature of the connection on $\mathscr{W}_0$ defined by the orthogonal projection $P_0\circ D^{[1]}\circ P_0$ of the connection component $D^{[1]}$ of D. Then $\delta_t(e^{-tI(x)})$ has a limit (in the operator norm on the prehilbert space $\mathscr{W}_x$) when $t\to\infty$ which is equal to e^{-R_0}.*

The idea of the proof is the same as in the finite dimensional case. We need an expansional formula similar to 1.12, and this follows from the existence and unicity of the heat kernel for $I(x)$. We refer to [1] for details.

Let D be a superconnection satisfying conditions 2.1, 2.2, 2.3. Then by [3, Prop. 2.10] the differential form $\mathrm{str}\,\delta_t(e^{-tI(x)})$ is closed and its cohomology class does not depend on t. Moreover, [3, p. 121] one can construct by a standard argument in K-theory another family u'_x and a superconnection D' with zero degree term u'_x such that

(1) $\mathrm{Ker}\, u'_x$ has constant dimension.
(2) The index bundles of $(u_x)_{x\in B}$ and $(u'_x)_{x\in B}$ have the same Chern character (in cohomology).
(3) Letting $I'(x)$ be the curvature of D', the differential forms $\mathrm{str}\,\delta_t(e^{-tI(x)})$ and $\mathrm{str}\,\delta_t(e^{-tI'(x)})$ are equal in cohomology.

Therefore the following generalization to families of Mac–Kean–Singer heat equation Theorem, due to Bismut, can be more naturally deduced from 2.4.

THEOREM 2.5 [3]. *Let D be a superconnection with zero degree term $(u_x)_{x\in B}$. Under conditions 2.1, 2.2, 2.3, the differential form str $\delta_{2i\pi t}(e^{-tI(x)})$ represents for all $t>0$ the Chern character of the index bundle of $(u_x)_{x\in B}$.*

II. ASYMPTOTIC EXPANSION FOR LAPLACIANS ON VECTOR BUNDLES

In this section, G is a compact Lie group with Lie algebra $\mathfrak{g}$. M is a Riemannian manifold and $P\xrightarrow{\pi}M$ is a principal bundle over M with structure group G and connexion ω. We choose on $\mathfrak{g}$ a G-invariant positive inner product. For $u \in P$, the tangent space T_uP is the direct sum of the horizontal space T_u^hP identified with T_xM and the vertical space identified with $\mathfrak{g}$. This turns P into a Riemannian manifold, (the metric depends on ω).

1. *The Jacobian of the exponential map on a principal bundle*

For u and v in P sufficiently close, we denote by $\theta^P(u, v)$: $T_uP \to T_vP$ the derivative of the map $\exp_u$ at the point $\exp_u^{-1}(v)$. The object of this paragraph is the computation of $\theta^P(u, v)$ when u and $v = ug$ are in the same fiber.

1.1. Let V be a Euclidean vector space with orthonormal basis ψ_i. We consider on $\Lambda^2 V$ the inner product with orthonormal basis $\psi_i \wedge \psi_j$. We identify $\Lambda^2 V$ with $\mathfrak{so}(V)$ via the map τ: $\Lambda^2 V \to \mathfrak{so}(V)$ given by $\langle \tau(\omega).\psi_i, \psi_j \rangle = 2\langle \omega, \psi_i \wedge \psi_j \rangle$. We identify V and V^*.

We denote by R the curvature of the connection ω. It is a horizontal $\mathfrak{g}$-valued 2-form on P. Thus for $u \in P$, we consider R_u as an element of $\Lambda^2(T_u^hP)^* \otimes \mathfrak{g}$. For $a \in \mathfrak{g}$, the contraction $(Id \otimes i(a)) \cdot R_u$ is an element of $\Lambda^2(T_u^hP)^*$, we denote it by (R_u, a). Then $\tau(R_u, a)$ is an antisymmetric transformation of T_u^hP.

If ξ is a vector field on M, we denote by $\tilde{\xi}$ its horizontal lift to P. For $a \in \mathfrak{g}$, we denote also by a the corresponding vertical vector field on P. Denote by ∇^M (respectively ∇^P) the Levi-Civita connection on M (respectively P). From the equation $d\omega + 1/2[\omega, \omega] = R$, one easily verifies the following:

LEMMA 1.2. *Let ξ, ξ' be vector fields on M and a, b elements of $\mathfrak{g}$, then*

$$\nabla^P_{\tilde{\xi}} \tilde{\xi}' = (\nabla^M_{\xi} \cdot \xi')^{\sim} - \tfrac{1}{2} R(\xi, \xi') \tag{1}$$

$$\nabla^P_{a} \tilde{\xi} = \tfrac{1}{4} \tau(R_u, a) \cdot \tilde{\xi} = \nabla^P_{\tilde{\xi}} \cdot a \tag{2}$$

$$\nabla^P_a \cdot b = \tfrac{1}{2}[a, b]. \tag{3}$$

In particular, for $u \in P$ and $a \in \mathfrak{g}$, we have $\exp_u(a) = u \exp a$ and the fibers of $P \xrightarrow{\pi} M$ are totally geodesic.

Fix $x \in M$ and let $V = T_xM$. For $u \in P$ such that $\pi(u) = x$, identify T_u^hP with V and T_uP with $V \oplus \mathfrak{g}$. Let $a \in \mathfrak{g}$. Then the derivative $\theta^P(u, u \exp a)$ is identified with a linear map from $V \oplus \mathfrak{g}$ to itself which we denote by $J(u, a)$.

PROPOSITION 1.3. *$J(u, a)$ preserves V and $\mathfrak{g}$ and*

$$J(u, a)|_{\mathfrak{g}} = \frac{1 - e^{-ada}}{ada} \tag{1}$$

$$J(u, a)|_V = \frac{1 - e^{-\frac{1}{2}\tau(R_u, a)}}{\frac{1}{2}\tau(R_u, a)}. \tag{2}$$

Proof. (1) is well-known. For (2) we need to compute $d/d\varepsilon \exp_u(a + \varepsilon x)$ for $x \in V$. Introduce $u(s, t) = \exp_u s(a + tx)$, $u(s) = \exp_u(sa)$. Put $Y(s) = d/dt\, u(s, t)|_{t=0}$. Then $Y(s)$ is a vector field on P along the curve $u(s)$ and $J(u, a) \cdot x = Y(1)$. Let C be the curvature of the manifold P. The Jacobi vector field $Y(s)$ is determined by the differential equation:

$$\begin{aligned} &\nabla^P_{\frac{\partial}{\partial s}} \nabla^P_{\frac{\partial}{\partial s}} Y(s) = C\left(\frac{\partial}{\partial s}, Y(s)\right) \cdot \frac{\partial}{\partial s} \\ &Y(0) = 0 \\ &\nabla^P_{\frac{\partial}{\partial s}} \cdot Y(s)|_{s=0} = x. \end{aligned} \tag{1.4}$$

Let ξ be a vector field on M. By the G-invariance of the connection ω, we have

$$(\nabla^P_a\cdot\tilde{\xi})_{u(s)}=\tfrac{1}{4}(\tau(R_{u\exp sa},a)\cdot\xi)^{\sim}=\tfrac{1}{4}(\tau(R_u,a)\cdot\xi)^{\sim}.$$

As $\nabla^P_a\cdot a=0$, $[a,\tilde{\xi}]=0$, $C(a,\tilde{\xi})\cdot a=\nabla^P_a\nabla^P_{\tilde{\xi}}\cdot a=(\frac{1}{4}\tau(R_u,a))^2.\tilde{\xi}$. The relation (1.4) implies that $Y(s)$ remain horizontal, hence identifies with an element $y(s)$ of V; The differential equation (1.4) reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}+\tfrac{1}{4}\tau(R_u,a)\right)^2\cdot y(s)=(\tfrac{1}{4}\tau(R_u,a))^2\cdot y(s)$$

and we obtain (2).

2. *Asymptotic expansion for Laplacians on associated vector bundles*

Let $\mathscr{V}\mapsto M$ be a vector bundle. Consider a second order elliptic operator H on $\Gamma(\mathscr{V})$ with principal symbol $\|\xi\|^2 Id$. Then there exists a connection on $\mathscr{V}$ such that $H=\Delta^{\mathscr{V}}+F$ where F is a potential. When the holonomy group G of the connection is compact, we give an integral formula for the asymptotic expansion of the heat kernel of H on the diagonal. In the applications, $\mathscr{V}$ will be of the form $\mathscr{W}^{\pm}\otimes\mathscr{E}$, with $\mathscr{W}=\mathscr{W}^+\oplus\mathscr{W}^-$ a supervector bundle and $\mathscr{E}$ an auxiliary bundle. We begin by the case where $F=0$.

2.1. Let (ρ, W) be a representation of G. Let $\mathscr{W}=P\times_G W$ be the associated vector bundle on M. We identify the space $\Gamma(M,\mathscr{W})$ of sections of $\mathscr{W}$ with the space $(C^\infty(P)\otimes W)^G$ of functions ϕ: $P\to W$ such that

$$\phi(ug)=\rho(g)^{-1}\phi(u)\quad\text{for}\quad u\in P,\ g\in G.$$

We denote by ∇ the covariant differentiation on $\mathscr{W}$ associated to the connection ω. Let $\Delta^{\mathscr{W}}$ be the Laplacian on $\Gamma(\mathscr{W})$. If ξ_i is a local orthonormal frame of TM, then

$$\Delta^{\mathscr{W}}=-\sum_i\left(\nabla_{\xi_i}\nabla_{\xi_i}-\nabla_{\nabla^M_{\xi_i}\cdot\xi_i}\right).$$

Let Δ^P be the scalar Laplacian of P acting on $C^\infty(P)$. Let E_j be an orthonormal basis of $\mathfrak{g}$ and let $C=\sum_j\rho(E_j)^2$ be the Casimir operator (thus C is scalar when W is irreducible).

PROPOSITION 2.2. *$\Delta^{\mathscr{W}}$ coincides with the restriction to $(C^\infty(P)\otimes W)^G$ of the operator $\Delta^P\otimes 1+1\otimes C$.*

Proof. By (1) of Lemma 1.2, we have

$$\Delta^P=-\sum_i\tilde{\xi}_i^2-(\nabla^M_{\xi_i}\cdot\xi_i)^{\sim}-\sum_j E_j^2.$$

For $\phi\in\Gamma(\mathscr{W})=(C^\infty(P)\otimes W)^G$, we have

$$E_j\phi=-\rho(E_j)\phi.$$

Let us consider the semi-groups $\mathrm{e}^{-t\Delta^{\mathscr{W}}}$ on $\Gamma(\mathscr{W})$ and $\mathrm{e}^{-t\Delta^P}$on $C^\infty(P)$. The Schwartz kernel of $\mathrm{e}^{-t\Delta^{\mathscr{W}}}$ with respect to the Riemannian density of M is denoted by

$$\langle x|\mathrm{e}^{-t\Delta^{\mathscr{W}}}|y\rangle\in\mathrm{Hom}\,(\mathscr{W}_y,\mathscr{W}_x).$$

It can also be considered as an End W-valued function $(u,u')\mapsto\langle u|\mathrm{e}^{-t\Delta^{\mathscr{W}}}|u'\rangle$ on $P\times P$ which satisfies $\langle ug|\mathrm{e}^{-t\Delta^{\mathscr{W}}}|u'g'\rangle=\rho(g)^{-1}\langle u|\mathrm{e}^{-t\Delta^{\mathscr{W}}}|u'\rangle\rho(g')$.

From 2.2, we have

$$(e^{-t\Delta^{\mathscr{W}}}\cdot\phi)(u_0)=e^{-tC}\int_P\langle u_0|e^{-t\Delta^P}|u\rangle\phi(u)du=e^{-tC}\int_M\left(\int_G\langle u_0|e^{-t\Delta^P}|ug\rangle\phi(ug)dg\right)dx.$$

Thus:

PROPOSITION 2.3.

$$\langle u_0|e^{-t\Delta^{\mathscr{W}}}|u\rangle=e^{-tC}\int_G\langle u_0|e^{-t\Delta^P}|ug\rangle\rho(g)^{-1}dg.$$

This simple observation will be the starting point in the computation of the asymptotic of $\langle x|e^{-t\Delta^{\mathscr{W}}}|x\rangle$.

2.4. Let us now consider a second order operator $H=\Delta+V$ where V is a potential. We introduce a filtration on potentials, analogous to Getzler filtration [6].

Let $\mathscr{E}$ be an auxiliary vector bundle over M, with hermitian metric and hermitian connection $\nabla^{\mathscr{E}}$. The group G acts on the space $\Gamma(P,\pi^*\mathscr{E})$ and $\Gamma(M,\mathscr{E}\otimes\mathscr{W})$ identifies with the space $(\Gamma(P,\pi^*\mathscr{E})\otimes W)^G$ of sections ϕ: $u\mapsto\phi(u)\in\mathscr{E}_x\otimes W$, for $u\in P$ and $x=\pi(u)$ which satisfy $\phi(ug)=(1\otimes\rho(g^{-1}))\phi(u)$. In the same way, the space $\Gamma(M,\operatorname{End}(\mathscr{E}\otimes\mathscr{W}))$ identifies with $(\Gamma(P,\pi^*\operatorname{End}\mathscr{E})\otimes\operatorname{End}W)^G$. Let $\mathscr{U}(\mathfrak{g})=\bigcup_{j=0}^{\infty}\mathscr{U}^j(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ with its natural filtration. The elements F of $(\Gamma(P,\pi^*\operatorname{End}\mathscr{E})\otimes\mathscr{U}^j(\mathfrak{g}))^G$ will be called potentials of $\mathfrak{g}$-degree j. Extend ρ to a representation of $\mathscr{U}(\mathfrak{g})$. Then $(1\otimes\rho)(F)=\rho(F)$ is defined as an element of $\Gamma(M,\operatorname{End}(\mathscr{E}\otimes\mathscr{W}))$.

Let us consider the tensor product connection ∇ on $\mathscr{E}\otimes\mathscr{W}$ and form the corresponding Laplacian $\Delta^{\mathscr{E}\otimes\mathscr{W}}$ acting on $\Gamma(M,\mathscr{E}\otimes\mathscr{W})$. Let F be a potential of $\mathfrak{g}$-degree 1. We consider the operator $H=H(P,\omega,\rho,\mathscr{E},F)$ on $\Gamma(M,\mathscr{E}\otimes\mathscr{W})$ given by

2.5. $$H=\Delta^{\mathscr{E}\otimes\mathscr{W}}+\rho(F).$$

The main result of this part will be an integral formula for the asymptotic expansion of the heat kernel of H.

2.6. We introduce some notations. We have $\mathscr{U}^1(\mathfrak{g})=\mathbb{C}\oplus\mathfrak{g}$, hence $F=F^0+F^1$, where $F_u^0\in(\operatorname{End}\mathscr{E}_x)$ and $F_u^1\in(\operatorname{End}\mathscr{E}_x)\otimes\mathfrak{g}$, for $u\in P$ and $\pi(u)=x$. Thus for $a\in\mathfrak{g}$, the contraction $(F_u^1,a)\in\operatorname{End}\mathscr{E}_x$ is well defined.

2.7. Let ϕ be a C^∞ function on $\mathfrak{g}$ with compact support. The function

$$t\mapsto\int_{\mathfrak{g}}e^{-\frac{\|a\|^2}{4t}}\phi(a)da=t^{\frac{\dim\mathfrak{g}}{2}}\int_{\mathfrak{g}}e^{\frac{-\|a\|^2}{4}}\phi(t^{1/2}a)da$$

has an asymptotic expansion near $t=0$ in powers of $t^{1/2}$ which depends only on the Taylor series of ϕ at 0. We denote the corresponding formal power series in $t^{1/2}$ by

2.8. $$\oint_{\mathfrak{g}}e^{-\frac{\|a\|^2}{4t}}\phi(a)da.$$

When ϕ does not have compact support, we still make sense of 2.8 by cutting off ϕ. Let

$\Phi(t, a)=\sum_{i=0}^{\infty} t^i\phi_i(a)$ be a formal power series in t with coefficients in $C^\infty(\mathfrak{g})$. We denote by

$$\oint_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}}\Phi(t, a)da$$

the formal series $\sum_{i=0}^{\infty} t^i \oint_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}}\phi_i(a)da$.

2.9. Let E be a real vector space. For $X\in\mathrm{End}\, E$, we put $j_E(X)=\det_E\left(\frac{e^{X/2}-e^{-X/2}}{X}\right)$. We select an analytic square root $j_E^{1/2}(X)$ near $X=0$ by the condition $j_E^{1/2}(0)=1$. For $a\in\mathfrak{g}$, we write $j_{\mathfrak{g}}(a)$ for $j_{\mathfrak{g}}(\mathrm{ad}\, a)$.

2.10. As in 2.3, we view the kernel of e^{-tH} as a section $(u, u')\to\langle u|e^{-tH}|u'\rangle$ of the bundle $\pi^*\mathscr{E}\otimes\pi^*\mathscr{E}^*\otimes\mathrm{End}\, W$ over $P\times P$. Thus for $u\in P$ with $\pi(u)=x$, $\langle u|e^{-tH}|u\rangle\in\mathrm{End}\,\mathscr{E}_x\otimes\mathrm{End}\, W$. To simplify the statement of the theorem below, we suppose here that the Casimir C of the representation ρ of G in W is scalar.

THEOREM 2.11. *Let $H=\Delta^{\mathscr{E}\otimes\mathscr{W}}+\rho(F)$ where F is a potential of $\mathfrak{g}$-degree 1. Assume that C is scalar. Take $u\in P$ with $\pi(u)=x$. Put $V=T_xM$. There exist C^∞ functions ϕ_j on $\mathfrak{g}$ with values in $\mathrm{End}\,\mathscr{E}_x$ such that $\langle u|e^{-tH}|u\rangle$ has an asymptotic expansion equal to*

$$(4\pi t)^{-\frac{\dim M+\dim\mathfrak{g}}{2}}\oint_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}}\sum_{j=0}^{\infty} t^j(\phi_j(a)\otimes\rho(\exp a))da.$$

Furthermore

$$\phi_0(a)=j_{\mathfrak{g}}^{1/2}(a)\, j_V^{-1/2}\left(\tfrac{1}{2}\tau(R_u, a)\right)e^{-1/2(F_u^1, a)}.$$

Remark. In the application, ρ will be a super-representation and we will study the supertrace of $\langle u|e^{-tH}|u\rangle$. Then as $\mathrm{str}(\phi_j(a)\otimes\rho(\exp a))=tr\,\phi_j(a)\ \mathrm{str}\ \rho(\exp a)$, the integral above is an integral with respect to the measure $\mathrm{str}(\rho(\exp a))da$. This will eliminate poles in t. Furthermore the above factorization of $\phi_0(a)$ will lead to the factorization of the index density as the product of the $\hat{\mathscr{A}}$-class and the Chern character of $\mathscr{E}$.

Proof. The first step is to write H as the restriction to $\Gamma(M, \mathscr{E}\otimes\mathscr{W})=(\Gamma(P, \pi^*\mathscr{E})\otimes W)^G$ of a carefully chosen second-order elliptic operator H^P on P. We will thus obtain an integral formula similar to 2.3. Then we will apply the well-known asymptotic expansion of the heat kernel $\langle u|e^{-tH^P}|u'\rangle$. In order to define H^P we associate to the potential F a connection ∇' on $\pi^*\mathscr{E}$ as follows: First pull back $\nabla^{\mathscr{E}}$ to a connection $\nabla^{\mathscr{E}}$ on $\pi^*\mathscr{E}$. Then define ∇' by

2.12.
$$(\nabla'_\xi\phi)(u)=(\nabla^{\mathscr{E}}_\xi\phi)(u)\qquad\text{if } \xi\in T_u^hP,$$
$$(\nabla'_a\phi)(u)=(\nabla^{\mathscr{E}}_a\phi)(u)+\tfrac{1}{2}(F_u^1, a)\,\phi(u)\qquad\text{for } a\in\mathfrak{g}.$$

Let Δ' be the Laplacian of the bundle $\pi^*\mathscr{E}$ defined by the Riemannian structure of P and the connection ∇'.

LEMMA 2.13. *There exists $f\in\Gamma(P, \mathrm{End}\,\pi^*\mathscr{E})^G$ such that the operator H is the restriction to $\Gamma(M, \mathscr{E}\otimes\mathscr{W})=(\Gamma(P, \pi^*\mathscr{E})\otimes W)^G$ of $(\Delta'+f)\otimes 1+1\otimes C$.*

Proof. Let E_j be an orthonormal basis of $\mathfrak{g}$. Then

$$\Delta' = -\sum_i (\nabla'_{\tilde{\xi}_i})^2 - \nabla'_{(\nabla^M_{\xi_i}\cdot\xi_i)^\sim} - \sum_j (\nabla'_{E_j})^2.$$

The Babylonian method (completing squares) gives the lemma with

$$f = \tfrac{1}{4}\sum_j (F^1_u, E_j)^2 + F^0_u.$$

For $u_0, u \in P$ with $x_0 = \pi(u_0)$, $x = \pi(u)$, let us denote by $k(t, u_0, u) \in \mathrm{Hom}(\mathscr{E}_x, \mathscr{E}_{x_0})$ the kernel of $e^{-t(\Delta' - f)}$. As in 2.3, we have

2.14. $$\langle u_0 | e^{-tH} | u \rangle = \int_G k(t, u_0, ug) \otimes e^{-tC} \rho(g)^{-1} \mathrm{d}g.$$

We now recall the well known expansion (see [4], [8]).

2.15. $$k(t, u_0, u) = (4\pi t)^{-\frac{\dim P}{2}} e^{-\frac{d(u_0, u)^2}{4t}} \left(\sum_{i=0}^{N} t^i U_i(u_0, u) \right) + 0\left(t^{N - \frac{\dim P}{2}} \right),$$

where $\mathrm{d}(u_0, u)$ is the geodesic distance on P, U_j are smooth sections of $\pi^*\mathscr{E} \otimes \pi^*\mathscr{E}^*$. Furthermore the first term does not depend on f and is given by

2.16. $$U_0(u, v) = \det \theta^P(u, v)^{-1/2} \tau(u, v)^{-1},$$

where θ^P is the Jacobian of the Riemannian exponential map on P and $\tau(u, v) \in \mathrm{Hom}(\mathscr{E}_u, \mathscr{E}_v)$ is the geodesic parallel transport in $\pi^*\mathscr{E}$ with respect to the connection ∇'. We have

LEMMA 2.17. *For $u \in P$ and $a \in \mathfrak{g}$*

$$\tau(u, u \exp a) = e^{-1/2(F^1_u, a)}.$$

Proof. The function $t \to \tau(t) = \tau(u, u \exp ta)$ is the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(t) + \tfrac{1}{2}(F^1_{u \exp ta}, a) \circ \tau(t) = 0.$$

Now, for any $g \in G$, $(F^1_{ug}, a) = (F^1_u, g.a)$. Thus $(F^1_{u \exp ta}, a) = (F^1_u, a)$ for all t and we obtain the Lemma.

In order to complete the proof of the Theorem 2.11, we observe that we may use exponential coordinates in 2.14 to compute the asymptotics of $\langle u | e^{-tH} | u \rangle$. Let ϕ be a cut-off function on $\mathfrak{g}$ near 0. Since the heat kernel $k(t, u, u')$ is rapidly decreasing outside the diagonal, we have, for all N,

$$\langle u | e^{-tH} | u \rangle = \int_{\mathfrak{g}} k(t, u, u \exp a) \otimes e^{-tC} \rho(\exp -a)\, \phi(a) j_{\mathfrak{g}}(a) \mathrm{d}a + 0(t^N).$$

Thus the theorem follows from 2.15, using 2.17 and the computation of $\theta^P(u, u \exp a)$ (1.3).

2.18. In the application to the family index, the situation will be as follows: $\mathscr{W} \to M$ is a vector bundle over M, with connexion ∇. However the given structure group of $\mathscr{W}$ is smaller than the holonomy group of ∇. In other words consider a subgroup G_0 of a compact Lie group G. Let $P_0 \xrightarrow{\pi} M$ be a principal bundle with structure group G_0. Let ρ_0 be a representation of G_0 in a vector space W and consider $\mathscr{W} = P \times_{G_0} W$. Let $\omega \in (\mathscr{A}^1(P_0) \otimes \mathfrak{g})^{G_0}$ be a $\mathfrak{g}$-valued connexion form on P_0, (I.1.5), i.e.

$$(i(X_0) \cdot \omega) = X_0 \text{ for } X_0 \in \mathfrak{g}_0.$$

Let $\mathscr{E}$ be an auxiliary vector bundle over M with connection ∇. Consider

$$F^0 \in \Gamma(P_0, \pi^*(\operatorname{End} \mathscr{E}))^{G_0}$$

$$F^1 \in (\Gamma(P_0, \pi^*(\operatorname{End} \mathscr{E})) \otimes \mathfrak{g})^{G_0}.$$

Suppose that there exists a representation ρ_1 of G in W extending the representation ρ_0 of G_0. Then $\rho_1(\omega) \in \mathscr{A}^1(P_0) \otimes \operatorname{End} W$ defines a connexion form for $\mathscr{W}$, thus a covariant differentiation ∇ on $\mathscr{W}$. We also denote by ∇ the tensor product connection on $\mathscr{W} \otimes \mathscr{E}$. Denote by Δ the associated Laplacian. Put

$$\rho_1(F) = F^0 + \rho_1(F^1) \in \Gamma(M, \operatorname{End}(\mathscr{W} \otimes \mathscr{E})).$$

Consider the operator

$$H = \Delta + \rho_1(F) \text{ on } \Gamma(M, \mathscr{W} \otimes \mathscr{E}). \tag{2.19}$$

Indeed, this operator is of the type 2.5:
We can enlarge P_0 in $P = P_0 \times_{G_0} G$ which is now a principal bundle with structure group G. Then there exists a unique (ordinary) connection form $\omega_1 \in (\mathscr{A}^1(P) \otimes \mathfrak{g})^G$ such that $\omega_1 | P_0 = \omega$. Similarly, we can define uniquely $F_1^1 \in \Gamma(P, \pi^* \operatorname{End} \mathscr{E} \otimes \mathfrak{g})^G$ such that $F_1^1 | P_0 = F^1$ on P_0. Let $F_1 = F_1^1 + F^0$. Then $H = H(P, \omega_1, \rho_1, \mathscr{E}, F_1)$. Remark that for $u \in P_0$ above x, if we consider the curvature $R = d\omega + \frac{1}{2}[\omega, \omega]$ of the "small" connexion, then the form $R_u(\xi, \xi')$ on TP_0 depends only on the projection of ξ and ξ' on $T_x M = V$, thus defines an element $R_u \in \Lambda^2 V^* \otimes \mathfrak{g}$. We fix a G-invariant inner product on $\mathfrak{g}$, and define (R_u, a), (F^1, a). We can restate theorem 2.11 in terms of the "small" data ω and F^1.

THEOREM 2.20. *Let H be the operator defined in 2.19. Assume the Casimir $\rho_1(C)$ is scalar. Then for $u \in P_0$ with $\pi(u) = x$ there exist C^∞ functions ϕ_j on $\mathfrak{g}$ with values in* $\operatorname{End} \mathscr{E}_x$ *such that*

$$\langle u | e^{-tH} | u \rangle \sim (4\pi t)^{-\frac{\dim M - \dim \mathfrak{g}}{2}} \oint_{\mathfrak{g}} e^{\frac{-\|a\|^2}{4t}} \left(\sum_{k=0}^{\infty} t^k \Phi_k(a) \otimes \rho_1(\exp a) \right) da.$$

Furthermore

$$\Phi_0(a) = j_{\mathfrak{g}}^{1/2}(a) \, j_V^{-1/2}(\tfrac{1}{2}\tau(R_u, a)) \, e^{-1/2(F_u^1, a)}.$$

3. *Calculus with Grassmann and Clifford variables*

Let $\mathscr{A}$ be a finite dimensional supercommutative algebra with unit. (In application, $\mathscr{A}$ will be an exterior algebra). Let $\xi = (\xi_1, \ldots, \xi_n)$ be a n-tuple of even nilpotent elements of $\mathscr{A}$. Let ϕ be an $\mathscr{A}$-valued C^∞-function on $\mathbb{R}^n$. We define

$$\phi(x + \xi) = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} \partial^J \phi(x) \, \xi^J, \text{ for } x \in \mathbb{R}^n. \tag{3.1}$$

In particular

$$\phi(\xi) = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} \partial^J \phi(0) \xi^J$$

$$e^{-\|x - \xi\|^2} = e^{-\Sigma (x_j - \xi_j)^2}.$$

LEMMA 3.2. *Let ϕ be an $\mathscr{A}$-valued C^∞-function on $\mathbb{R}^n$ slowly increasing at infinity (as well as all its derivatives). Let $\xi=(\xi_1,\ldots,\xi_n)$ be an n-tuple of even nilpotent elements of $\mathscr{A}$. Then*

$$\lim_{t\to 0}(4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{\|x-\xi\|^2}{4t}}\phi(x)\mathrm{d}x=\phi(\xi).$$

Proof. If Φ is a function in the Schwartz space, we have

$$\int(\Phi(x+\xi)-\Phi(x))\mathrm{d}x=\sum_{J\neq 0}\left(\int\frac{1}{J!}\,\partial^J\phi(x)\,\mathrm{d}x\right)\xi^J=0.$$

Thus

$$\int_{\mathbb{R}^n} e^{-\frac{\|x-\xi\|^2}{4t}}\phi(x)\mathrm{d}x=\int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{4t}}\phi(x+\xi)\,\mathrm{d}x$$
$$=\sum_J\left(\int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{4t}}\frac{1}{J!}\,\partial^J\phi(x)\mathrm{d}x\right)\xi^J.$$

At the limit, we obtain 3.2.

3.3. Let V be an oriented Euclidean vector space with orthogonal basis $\psi_1,\ldots,\psi_n$. Consider on the exterior algebra ΛV the inner product such that the elements $\psi_I=\psi_{i_1}\wedge\ldots\wedge\psi_{i_p}$ form an orthonormal basis. For $\omega\in\Lambda V$, we denote by $T(\omega)$ the coefficient of the highest degree term $\omega^{[\max]}$ of ω. We denote by $\exp_\Lambda\omega$ the exponential of ω in ΛV. Identify $\mathfrak{g}=\Lambda^2 V$ with $\mathfrak{so}(V)$ via τ (1.1). Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the scalar product. By the universal property of the symmetric algebra, the injection $\mathfrak{g}\to\Lambda^+V$ extends to an algebra homomorphism from $S(\mathfrak{g}^*)$ to Λ^+V which we denote by A. We further extend A first to the algebra $\hat{S}(\mathfrak{g}^*)$ of formal power series on $\mathfrak{g}$, then to $C^\infty(\mathfrak{g})$ by composing with the Taylor series expansion at 0. Finally we extend A to a map from $C^\infty(\mathfrak{g})\otimes\Lambda V$ to ΛV by $A(\phi\otimes\psi_I)=A(\phi)\wedge\psi_I$. Let us describe A in coordinates. Let ψ_i be an orthonormal basis of V, we identify $\mathfrak{g}$ with $\mathbb{R}^{1/2n(n-1)}$ by writing $X=\sum_{i<j}x_{ij}\psi_i\wedge\psi_j$. An element f of $C^\infty(\mathfrak{g})\otimes\Lambda V$ can thus be identified with a C^∞ function $f(x_{ij})$ on $\mathbb{R}^{1/2n(n-1)}$. Then $A(f)=f(\psi_i\wedge\psi_j)$ is defined as in 3.1., ie., in the expression $f(x_{ij})=\Sigma f_I(x_{ij})\psi_I$ we replace the commutative variable x_{ij} by $\psi_i\wedge\psi_j$. Denote by ψ the column vector $(\psi_1,\ldots\psi_n)$. Then $\psi\psi^*$ is the antisymmetric matrix with coefficients $\psi_i\wedge\psi_j$. Thus we will also write

$$A(f)=f(\psi\psi^*).$$

3.4. The group $0(V)$ acts on V, ΛV, $S(\mathfrak{g})$, etc.... The map A: $C^\infty(\mathfrak{g})\to\Lambda^+V$ commutes with the action of $0(V)$. As constants are the only $0(V)$-invariant elements of ΛV, we have $P(\psi\psi^*)=P(0)$, if P is a scalar valued function on $\mathfrak{g}$ which is invariant by the full orthogonal group.

The following proposition is the crucial technical tool leading to the elimination of the singular part of the heat kernel index density.

PROPOSITION 3.5. *Let P be a smooth slowly increasing ΛV-valued function on $\mathfrak{g}$. Then*

$$\lim_{t\to 0}(4\pi t)^{-\frac{\dim\mathfrak{g}}{2}}\int_{\mathfrak{g}} e^{-\frac{\|X\|^2}{4t}}P(X)\exp_\Lambda\left(\frac{X}{2t}\right)\mathrm{d}X=A(P).$$

Proof. Let $X=\sum_{i<j} x_{ij}\psi_i\wedge\psi_j$. As $(\psi_i\wedge\psi_j)^2=0$, the Babylonian method gives:

$$e^{-\frac{\|X\|^2}{4t}}\exp_\Lambda\left(\frac{X}{2t}\right)=\exp_\Lambda\left(-\frac{1}{4t}\sum_{i<j}(x_{ij}-\psi_i\wedge\psi_j)^2\right)$$

and we apply lemma 3.2.

3.6. Let $C(V)$ be the Clifford algebra of V. Denote by C the representation of $C(V)$ in ΛV given by $C(x)\cdot\omega=x\wedge\omega-i(x)\cdot\omega$ for $x\in V$, $\omega\in\Lambda V$, where $i(x)$ is the contraction by x. We identify $C(V)$ as a vector space to ΛV by the map $y\to C(y)\cdot 1$. We denote by $x\circ y$ the product on ΛV inherited from $C(V)$. Then $\psi_i\circ\psi_i=-1$, $\psi_i\circ\psi_j+\psi_j\circ\psi_i=0$ if $i\neq j$. We denote by $C^+(V)$ the subalgebra of $C(V)$ generated by the products of an even number of elements in V. We denote by $\exp_C(\omega)$ the exponential of $\omega\in\Lambda V$ with respect to the Clifford product.

Suppose dim $V=n=2\ell$. Consider the spin representation ρ of $C(V)$ in the spinor space S. Recall that $S=S^+\oplus S^-$ is the sum of even and odd spinor space and that the algebra $C^+(V)$ leaves S^+ and S^- stable. We denote by $\rho^\pm$ the corresponding representation of $C^+(V)$ in $S^\pm$. Define the supertrace of an element $a\in C(V)$ by:

3.7. $$\mathrm{str}(a)=2^l i^{-l}T(a).$$

Then, for $a\in C^+(V)$, $\mathrm{str}(a)=\mathrm{tr}\,\rho^+(a)-\mathrm{tr}\rho^-(a)$.

3.8. The bijection τ: $\mathfrak{g}=\Lambda^2V\to\mathfrak{so}(V)$ can be interpreted in terms of the Clifford algebra. The space $\mathfrak{g}=\Lambda^2V$ is a Lie subalgebra of $(\Lambda V,\circ)$. The space V is invariant under the map $v\to a\circ v-v\circ a$, for $a\in\mathfrak{g}$. The endomorphism of V thus defined coincides with $\tau(a)\in\mathfrak{so}(V)$.

The universal covering group $G=\mathrm{Spin}(V)$ with Lie algebra $\mathfrak{g}$ can be realised as a subgroup of the group of invertible elements of $C^+(V)$. The map exp: $\mathfrak{g}\to G$ coincides with the exponential map in $C^+(V)$. As $(\psi_i\circ\psi_j)^2=-1$ $(i<j)$, if v is the restriction to G of a representation of $C^+(V)$, then the Casimir acts by the scalar $-(\dim\mathfrak{g})$ in the space of v. We denote by the same letters ρ, $\rho^\pm$ the restrictions to G or $\mathfrak{g}$ of the spin (even, odd) representation of $C^+(V)$.

In the next proposition, we compare the exponentials in the Clifford algebra and the exterior algebra.

Recall that $j_V(X)=\det_V\left(\frac{e^{X/2}-e^{-X/2}}{X}\right)$, for $X\in\mathrm{End}\,V$. For $a\in\mathfrak{g}$, we write $j_V(a)$ for $j_V(\tau(a))$. The function $(j_V(a))^{1/2}$ is analytic on $\mathfrak{g}$, $O(V)$-invariant and $j_V^{1/2}(0)=1$. For $X\in\mathrm{End}\,V$ sufficiently small, define $q(X)=(\det_V((e^{X/2}+e^{-X/2})/2))^{1/2}$. For $a\in\mathfrak{g}$, we write $q(a)$ for $q(\tau(a))$. Consider $g(X)=\left(\frac{th\,X/2}{X/2}\right)^{1/2}\in GL(V)$, for $X\in\mathrm{End}\,V$. For $X=\tau(a)$, we denote $g(\tau(a))$ by $g(a)$. For $g\in GL(V)$ we still denote by g the automorphism of the exterior algebra extending g.

PROPOSITION 3.9. $\exp_C(a)=q(a)\exp_\Lambda(g(a).a)=q(a)\,g(a)\cdot\exp_\Lambda(a)$.

Proof. It is enough to prove this in the case $n=2$ and $a=\theta\psi_1\wedge\psi_2$, where it is easily verified.

LEMMA 3.10. *Let $\omega\in\Lambda V$. Consider the ΛV-valued function on $\mathfrak{g}$ given by $f(X)=g(X)^{-1}.\omega$, then $f(\psi\psi^*)=\omega$.*

Proof. It is enough to verify that for $v \in V$, $\tau(\psi\psi^*).v=0$. But, if $\omega=\Sigma\omega_j\psi_j$ and $X=\Sigma x_{ij}\psi_i\wedge\psi_j$, $\tau(X).\omega=-2\sum_{i<j}\omega_j x_{ij}\psi_i$, hence $\tau(\psi\psi^*).\omega=-2\sum_{i<j}\omega_j\psi_i\wedge\psi_j\wedge\psi_i=0$.

Let δ_t be the automorphism of ΛV such that $\delta_t\cdot\omega=t^{-i/2}\omega$, for $\omega\in\Lambda^i V$.

PROPOSITION 3.11. *Let P be a ΛV-valued slowly increasing smooth function on $\mathfrak{g}$, then*

$$\lim_{t\to 0}(4\pi t)^{-\frac{\dim\mathfrak{g}}{2}}\int_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}}\delta_t\left(\exp_C\frac{a}{2}\right)P(a)\,\mathrm{d}a=A(P).$$

Proof. We prove that for any $\omega\in\Lambda V$,

$$\lim_{t\to 0}(4\pi t)^{-\frac{\dim\mathfrak{g}}{2}}\int_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}}T\left(\delta_t\left(\exp_C\frac{a}{2}\right)P(a)\omega\right)=T(A(P)\wedge\omega).$$

By 3.9,

$$\begin{aligned}T\left(\delta_t\left(\exp_C\frac{a}{2}\right)P(a)\omega\right)&=T\left(q\left(\frac{a}{2}\right)g\left(\frac{a}{2}\right)\cdot\exp_\Lambda\left(\frac{a}{2t}\right)P(a)\omega\right)\\&=q\left(\frac{a}{2}\right)\det g\left(\frac{a}{2}\right)T\left(\exp_\Lambda\left(\frac{a}{2t}\right)\wedge g\left(\frac{a}{2}\right)^{-1}\cdot(P(a)\omega)\right),\\&=j_V^{1/2}\left(\frac{a}{2}\right)T\left(\exp_\Lambda\left(\frac{a}{2t}\right)\wedge g\left(\frac{a}{2}\right)^{-1}\cdot(P(a)\omega)\right).\end{aligned}$$

Then we apply 3.5 and recall that $j_V^{1/2}\left(\frac{\psi\psi^*}{2}\right)=1$ and, by 3.10,

$$g\left(\frac{\psi\psi^*}{2}\right)^{-1}\cdot(P(\psi\psi^*)\omega)=P(\psi\psi^*)\omega.$$

Remark. Consider V with the given quadratic form Q. Consider, for $t\in\mathbb{R}$, the form tQ. Denote the corresponding Clifford algebra structure on ΛV by C_t. Then we have $\exp_{C_t}(a)=q(ta)\exp_\Lambda(g(ta)\cdot a)$, and $q(0)=1$, $g(0)=Id$. This relation is a refinement of the fact that the product on C_t tends to the exterior product, when t tends to zero. Proposition 3.11 shows that in the limit, we can replace Clifford variables by Grassmann variables. A similar approximation argument was used by Getzler to give a simple proof of Atiyah–Singer theorem for a single operator [7].

3.12. For applications to the family index, we consider the situation, where V is an orthogonal direct sum $V=V_1\oplus V_0$ and V_0 is even dimensional. Let $S_0=S_0^+\oplus S_0^-$ be a spinor space for V_0 and let ρ_0 be the representation of $C(V_0)$ in S_0. Denote by C_1 the representation (3.6) of $C(V_1)$ in ΛV_1. As $C(V_1\oplus V_0)$ is isomorphic to the graded tensor product $C(V_1)\hat{\otimes}C(V_0)$, it acts on $\Lambda V_1\otimes S_0$ by the representation ρ_1:

$$\begin{aligned}\rho_1(X_0)(\omega\otimes s)&=(-1)^{\deg\omega}\omega\otimes\rho_0(X_0)\cdot s,\\ \rho_1(X_1)(\omega\otimes s)&=C_1(X_1)\omega\otimes s,\end{aligned}$$

for $X_0\in V_0$, $X_1\in V_1$, $\omega\in\Lambda V_1$, $s\in S_0$. Denote by T^0: $\Lambda V_0\to\mathbb{C}$ the highest degree term with respect to V_0, and denote also by T^0 the map $Id\otimes T^0$: $\Lambda V_1\otimes\Lambda V_0\to\Lambda V_1$. Let str^0: $\mathrm{End}(S_0)\to\mathbb{C}$ be the supertrace relative to S_0 and denote also by str^0 the map

$$Id\otimes\mathrm{str}^0\colon \mathrm{End}(\Lambda V_1)\otimes\mathrm{End}(S_0)\to\mathrm{End}(\Lambda V_1).$$

For $\omega \in C(V) \simeq \Lambda V$ we have $\rho_1(\omega) \in \mathrm{End}(\Lambda V_1) \otimes \mathrm{End}\, S_0$. Note the formula:

3.13.
$$\mathrm{str}^0(\rho_1(\omega)) \cdot 1 = 2^{l_0} i^{-l_0} T^0(\omega).$$

We denote by $\delta_{1,t}$ the automorphisms of ΛV_1 and $\Lambda V_1 \otimes \Lambda V_0$ which extend the dilatation $\delta_{1,t}(f) = t^{-1/2} f$ for $f \in V_1$. As

$$t^{-1/2 \dim V_0} \delta_{1,t}(T^0(\omega)) = T^0(\delta_t(\omega)),$$

note the obvious corollary to 3.11.

COROLLARY 3.14. *Let P be a scalar-valued slowly increasing smooth function on* $\mathfrak{g}$, *then*

$$\lim_{t \to 0} (4\pi t)^{-\frac{\dim V_0 + \dim \mathfrak{g}}{2}} \int_{\mathfrak{g}} e^{\frac{-\|a\|^2}{4t}} P(a)\, \delta_{1,t} T^0\left(\exp_C \frac{a}{2}\right) da = (4\pi)^{-\frac{\dim V_0}{2}} T^0(P(\psi\psi^*)).$$

III. BISMUT SUPERCONNECTION FOR A FAMILY OF DIRAC OPERATORS. THE INDEX DENSITY OF A FAMILY

1. *Bismut superconnection*

1.1. We consider a fibration $\tilde{M} \to B$ where the typical fiber M is a compact C^∞-manifold. We denote by $V\tilde{M}$ (V as vertical) the tangent bundle along the fibers. We assume that the bundle $V\tilde{M}$ is oriented and is given a Euclidean metric, which we denote by g_0. For $x \in B$, denote the fiber $\pi^{-1}(x)$ by M_x. We will assume that a spin structure exists and has been chosen on $V\tilde{M}$. Put $n_0 = \dim M$, let V_0 be $\mathbb{R}^{n_0}$ with its canonical metric, basis, orientation. Denote by P_0 the principal bundle of oriented orthonormal frames of $V\tilde{M}$ and denote by P_0' the principal bundle with structure group $\mathrm{Spin}(V_0)$ which defines the spin structure. We assume that $n_0 = 2l_0$ and consider the space of spinors $S_0 = S_0^+ \oplus S_0^-$. Let $\mathcal{S}_0 = (P_0' \times S_0)/\mathrm{Spin}(V_0)$ be the associated bundle of spinors along the fibers. For $x \in B$, we denote by ∇^0 the Levi-Civita connection on TM_x and on $\mathcal{S}_0 | M_x = \mathcal{S}_x$.

We suppose given an auxiliary complex vector bundle $\mathcal{E}$ over M with hermitian metric and hermitian connection, and we also denote by ∇^0 the tensor product connection on $(\mathcal{S}_0 \otimes \mathcal{E})|M_x$. These data define a smooth family $\not{D} = (\not{D}_x)_{x \in B}$ of Dirac operators, where $\not{D}_x$ acts on $\mathcal{W}_x = \Gamma(M_x, \mathcal{S}_0 \otimes \mathcal{E})$.

In order to compute the Chern character of the bundle Ind $\not{D}$, Bismut constructed a particular superconnection $\mathbb{B}$ with zero exterior degree term equal to $\not{D}$. The construction of the Bismut superconnection $\mathbb{B}$ requires two more choices:

First choose a connection for the fibration $\pi: \tilde{M} \to B$ that is a smooth family of horizontal tangent spaces $T_y^h\tilde{M}$, $y \in \tilde{M}$ such that $T_y\tilde{M} = V_y\tilde{M} \oplus T_y^h\tilde{M}$. We will identify the horizontal tangent bundle $T^h\tilde{M}$ to the pull back π^*TB.

Secondly, we fix a torsion-free connection ∇^B on the basis B (Later, we will also assume that ∇^B is trace-free). We will however see that the superconnection $\mathbb{B}$ depends on the choice of $T^h\tilde{M}$ but not on the connection ∇^B.

1.2. The choice of a horizontal tangent bundle $T^h\tilde{M}$ gives rise to a canonical metric connection ∇^0 on $V\tilde{M}$ which extends the Levi-Civita connection on the fibers. It can be defined as follows: choose a metric $\tilde{g}$ on $T\tilde{M}$ which extends g_0 and for which $V\tilde{M}$ and $T_y^h\tilde{M}$ are orthogonal; then define ∇^0 as the projection on $V\tilde{M}$ of the Levi-Civita connection of $\tilde{g}$. One verifies easily that ∇^0 does not depend on the choice of $\tilde{g}$.

The formula for the curvature of the Bismut superconnection, as well as the final computation of the family index will be rooted in a crucial symmetry property of the curvature of ∇^0 [Proposition (1.7)] generalizing the relation $R_{ijkl}=R_{klij}$ of the Riemannian curvature. We describe this symmetry and the Bismut superconnection itself, by introducing the bundle

$$\mathscr{V}=\pi^*TB\oplus V\tilde{M}\oplus\pi^*T^*B=T\tilde{M}\oplus\pi^*T^*B$$

together with a symmetric bilinear form g and connection $\tilde{\nabla}$ on $\mathscr{V}$.

1.3. Take $y\in\tilde{M}$, $x=\pi(y)$. On $\mathscr{V}_y=T_xB\oplus T_yM_x\oplus T_x^*B$, g_y is defined by the following conditions: T_yM_x and $T_xB\oplus T_x^*B$ are orthogonal, the restriction of g_y to T_yM_x is g_0, the restriction of g_y to $T_xB\oplus T_x^*B$ is given by the canonical duality. Thus $g_y(X,f)=\langle X,f\rangle$ for $X\in T_xB, f\in T_x^*B$ while T_xB, T_x^*B are totally isotropic. Note that the orthogonal space to T_x^*B with respect to g_y is $T_x^*B\oplus T_yM_x$.

Recall that we have chosen a connection ∇^B on TB. It induces connections on T^*B and π^*T^*B which we also denote by ∇^B.

PROPOSITION 1.4. *There exists a unique connection $\tilde{\nabla}$ on $\mathscr{V}=T\tilde{M}\oplus\pi^*T^*B$ such that*

1.4.1. *$\tilde{\nabla}$ preserves π^*T^*B and coincides on it with ∇^B.*
1.4.2. *$\tilde{\nabla}$ preserves the bilinear form g*
1.4.3. *For X, $Y\in\Gamma(T\tilde{M})$, we have*

$$\tilde{\nabla}_XY-\tilde{\nabla}_YX=[X,Y].$$

Remark. We do not assume that $\tilde{\nabla}$ preserves $T\tilde{M}$. Thus condition 1.4.3 includes the condition that $\tilde{\nabla}_XY-\tilde{\nabla}_YX$ belongs to $T\tilde{M}$.

Proof. The proof is similar to the case of the usual Levi-Civita connection. For X_1, X_2, $X_3\in\Gamma(T\tilde{M})$, $Y\in\Gamma(\pi^*T^*B)$, conditions 1,2,3 imply

1.5. $$2g(\tilde{\nabla}_{X_1}X_2,X_3)=X_1\cdot g(X_2,X_3)+X_2\cdot g(X_3,X_1)-X_3\cdot g(X_1,X_2)$$
$$+g([X_1,X_2],X_3)-g([X_2,X_3],X_1)+g([X_3,X_1],X_2),$$

1.6. $$g(\tilde{\nabla}_{X_1}X_2,Y)=X_1\cdot g(X_2,Y)-g(X_2,\tilde{\nabla}_{X_1}\cdot Y).$$

This proves the unicity of $\tilde{\nabla}$. Conversely 1.5 and 1.6 determine $\tilde{\nabla}_{X_1}\cdot X_2$ and one verifies easily that the operator $\tilde{\nabla}$ is a connection on $\mathscr{V}$ which satisfies 1.4.1, 1.4.2, 1.4.3.

Let $\tilde{R}$ be the curvature of $\tilde{\nabla}$.

PROPOSITION 1.7.

1.7.1. *For $X_1,X_2,X_3\in\Gamma(T\tilde{M})$,*

$$\tilde{R}(X_1,X_2).X_3+\tilde{R}(X_2,X_3).X_1+\tilde{R}(X_3,X_1).X_2=0$$

1.7.2. For X_1, $X_2\in\Gamma(T\tilde{M})$, X_3, $X_4\in\Gamma(T\tilde{M}\oplus\pi^*T^*B)$

$$g(\tilde{R}(X_1,X_2)X_3,X_4)+g(X_3,\tilde{R}(X_1,X_2)X_4)=0$$

1.7.3. For X_1, X_2, X_3, $X_4\in\Gamma(T\tilde{M})$,

$$g(\tilde{R}(X_1,X_2)X_3,X_4)=g(\tilde{R}(X_3,X_4)X_1,X_2).$$

Proof. 1.7.1 follows from 1.4.3 and 1.7.2 follows from 1.4.2. The relation 1.7.3 is a consequence of 1.7.1 and 1.7.2.

From 1.5, it follows that the connection ∇^0 on $V\tilde{M}$ defined in 1.2 coincides with the projection of $\tilde{\nabla}$ on $V\tilde{M}$. Let R^0 be the curvature of ∇^0. We obtain the desired symmetry property of R^0:

COROLLARY 1.8. *For X_1, X_2 in $\Gamma(T\tilde{M})$, X_3, X_4 in $\Gamma(V\tilde{M})$*

$$g_0(R^0(X_1, X_2)X_3, X_4) = g(\tilde{R}(X_3, X_4)X_1, X_2).$$

Proof. This is 1.7.3, since the left hand side is equal to $g(\tilde{R}(X_1, X_2)X_3, X_4)$.

1.9. Let $n_1 = \dim B$.

Put $V_1 = \mathbb{R}^{n_1}$ and consider on $V = V_1 \oplus V_0 \oplus V_1^*$ the bilinear form Q defined similarly to g_y on $\mathscr{V}_y$: the spaces V_0 and $V_1 \oplus V_1^*$ are orthogonal, the restriction of Q to $\mathbb{R}^{n_0}$ is the Euclidean scalar product, the restriction of Q to $V_1 \oplus V_1^*$ is given by the canonical duality. Let $C(V, Q)$ be the Clifford algebra over (V, Q) generated by the vectors $x \in V$ and relations $x \cdot y + y \cdot x = -2Q(x, y)$. Let $\tilde{S} = \Lambda V_1^* \otimes S_0$. Let ρ_0 be the representation of $C(V_0)$ in S_0. The space $\tilde{S}$ is the spinor space for $C(V, Q)$, the Clifford multiplication being defined by

$$\tilde{\rho}(X)\cdot(\omega \otimes s) = (-1)^{\deg \omega}\omega \otimes \rho_0(X).s, \text{ for } X \in V_0,\ \omega \in \Lambda V_1,\ s \in S_0$$

$$\tilde{\rho}(X).(\omega \otimes s) = -2i(X).\omega \otimes s, \text{ for } X \in V_1$$

$$\tilde{\rho}(f).(\omega \otimes s) = f \wedge \omega \otimes s, \text{ for } f \in V_1^*.$$

Consider now the vector bundle $\tilde{\mathscr{S}} = \pi^*(\Lambda T^*B) \otimes \mathscr{S}_0$ over $\tilde{M}$ and the bundle of Clifford algebras $C(\mathscr{V})$ with fiber $C(\mathscr{V})_y = C(\mathscr{V}_y, g_y)$ for $y \in \tilde{M}$. Then, similarly, $\tilde{\mathscr{S}}_y$ is a module for $C(\mathscr{V}_y)$, ie., $\tilde{\mathscr{S}}$ is a Clifford module for $C(\mathscr{V})$. We also denote by $\tilde{\rho}$ the action of $\mathscr{V}$ on $\tilde{\mathscr{S}}$.

1.10. The connection $\tilde{\nabla}$ on $\mathscr{V}$ gives rise naturally to a connection on $\tilde{\mathscr{S}}$. We describe this: let $\mathfrak{so}(V, Q)$ be the Lie algebra of infinitesimally orthogonal transformations of (V, Q). Let $\mathfrak{h}$ be the subalgebra which preserves V_1^*. Then $\mathfrak{h}$ is the set of matrices:

$$A = \begin{vmatrix} b & 0 & 0 \\ w & a & 0 \\ z & -w^* & -b^* \end{vmatrix},$$

where $a \in \mathfrak{so}(V_0)$, $b \in \mathfrak{gl}(V_1)$, $z \in \Lambda^2 V_1^* \subset \mathrm{Hom}(V_1, V_1^*)$, $[w \in \mathrm{Hom}(V_1, V_0)]$ and $*$ denotes the transposition. We will also consider the subalgebra $\mathfrak{h}_0 \subset \mathfrak{h}$ of elements A such that $b = 0$.

As in II.3.6, we have a canonical identification of $C(V, Q)$ with the exterior algebra ΛV (as vector spaces). We denote by $\circ$ the Clifford multiplication on ΛV. Thus, if $X \in V$ and $X^* \in V^*$ is defined by $\langle Y, X^* \rangle = Q(Y, X)$, we have, for $\omega \in \Lambda V$, $X \in V$,

$$X \circ \omega = X \wedge \omega - i(X^*)\omega.$$

Let $\mathfrak{g} = \Lambda^2 V$. For $A \in \mathfrak{g}$, the map $\tau(A)x = A \circ x - x \circ A$ preserves $V \subset C(V, Q)$ and defines an isomorphism of $\mathfrak{g}$ with $\mathfrak{so}(V, Q)$. The restriction of $\tilde{\rho}$ to $\mathfrak{g}$ defines the spin representation of $\mathfrak{so}(V, Q)$ in $\tilde{S}$. The group $\mathrm{Spin}(V, Q)$ is also considered as a subset of $C(V, Q)$ and $\tilde{\rho}$ is its spin representation. We will consider in particular the restriction of the representation $\tilde{\rho}$ to

$$\mathrm{Spin}(V_0) \times GL(V_1) \subset \mathrm{Spin}(V, Q).$$

Denote by P_B the bundle of frames of TB and consider the principal bundle

$P' = P'_0 \times_{\tilde{M}} \pi^* P_B$ over $\tilde{M}$, with structure group Spin $(V_0) \times GL(V_1)$. We have

$$\mathscr{V} = P' \times_{\mathrm{Spin}(V_0) \times GL(V_1)} (V_1 \oplus V_0 \oplus V_1^*)$$

$$\tilde{\mathscr{S}} = P' \times_{\mathrm{Spin}(V_0) \times GL(V_1)} \tilde{S}.$$

The covariant differentiation $\tilde{\nabla}$ on $\mathscr{V}$ gives rise (I.1.6) to an $\mathfrak{h}$-valued form:

1.11. $$\tilde{\theta} \in \mathscr{A}^1(P') \otimes \mathfrak{h}.$$

Consider $\tilde{\rho}(\tilde{\theta}) \in \mathscr{A}^1(P') \otimes \mathrm{End}\, \tilde{S}$.

This is a connection form for $\tilde{\mathscr{S}}$, thus defines a covariant differentiation on $\tilde{\mathscr{S}}$, still denoted by $\tilde{\nabla}$. It satisfies

1.12. $$\tilde{\nabla}_X \tilde{\rho}(Y) = \tilde{\rho}(Y) \tilde{\nabla}_X + \tilde{\rho}(\tilde{\nabla}_X . Y), \text{ for } X \in T\tilde{M},\ Y \in \Gamma(\mathscr{V}).$$

With the above preparations we can now define the Bismut superconnection. Consider the tensor product connection on $\tilde{\mathscr{S}} \otimes \mathscr{E}$ and denote it also by $\tilde{\nabla}$. Consider the following Dirac-looking operator on $\Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$. Let X_α, $1 \leqslant \alpha \leqslant n_1$, be a local basis of TB and let X_i, $n_1 + 1 \leqslant i \leqslant n_1 + n_0$, be a local oriented orthonormal basis of $V\tilde{M}$. Let X_i^*, $1 \leqslant i \leqslant n_1 + n_0$ be the dual basis in $V\tilde{M} \oplus \pi^* T^* B$. Thus $X_i^* = X_i$ for $n_1 \leqslant i \leqslant n_1 + n_0$, while $X_\alpha^* = f_\alpha$ is the dual basis of $\{X_\alpha\}$ in T^*B. Put

1.13. $$\mathbb{B} = \sum_{1 \leq i \leq n_1 + n_0} (\tilde{\rho}(X_i^*) \otimes 1) \tilde{\nabla}_{X_i}$$

$$= \sum_{1 \leq \alpha \leq n_1} f_\alpha \tilde{\nabla}_{X_\alpha} + \sum_{n_1 + 1 \leq i \leq n_1 + n_0} \rho_0(X_i) \tilde{\nabla}_{X_i}.$$

Let $\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-$ be the infinite dimensional supervector bundle over B such that $\mathscr{W}_x^\pm = \Gamma(M_x, \mathscr{S}_0^\pm \otimes \mathscr{E})$. We have an identification of $\Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$ with $\mathscr{A}(B) \underset{\mathscr{A}^0(B)}{\otimes} \Gamma(B, \mathscr{W})$. The even (respectively odd) part of $\Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$ is $\Gamma(\tilde{\mathscr{S}}^+ \otimes \mathscr{E})$ (resp. $\Gamma(\tilde{\mathscr{S}}^- \otimes \mathscr{E})$).

PROPOSITION 1.14. *Assume ∇^B is trace-free. Then*

(1) *The operator $\mathbb{B}$ is a superconnection, i.e. $\mathbb{B}$ maps $\Gamma(\tilde{\mathscr{S}}^+ \otimes \mathscr{E})$ into $\Gamma(\tilde{\mathscr{S}}^- \otimes \mathscr{E})$ and vice-versa, and satisfies:*

$$\mathbb{B}(\omega\phi) = (\mathrm{d}\omega)\phi + (-1)^{\deg \omega} \omega \mathbb{B}\phi$$

for $\omega \in \mathscr{A}(B)$, $\phi \in \Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$.

(2) *The term of exterior degree 0 of the superconnection $\mathbb{B}$ is given by the family $\mathbb{D} = (\mathrm{D}_x)$ of the Dirac operators in the fibers $\mathscr{W}_x$.*

(3) *The superconnection $\mathbb{B}$ does not depend on the choice of the torsion-free, trace-free connection ∇^B.*

Proof. (1) It is clear that $\mathbb{B}$ is an odd operator on $\Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$. For $b \in \mathfrak{gl}(V_1) \subset \mathfrak{so} \leqslant V, Q)$, we have on $\Lambda V_1^* \otimes S_0$,

1.15. $\tilde{\rho}(b) = b \otimes 1 + \frac{1}{2} tr\, b$, where we still denote by b the natural derivation of ΛV_1^*. Thus, we see that if ∇^B is trace-free, $\omega \in \mathscr{A}(B)$, $s \in \Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E})$, we have

$$\tilde{\nabla}_\xi(\omega s) = (\nabla_\xi^B \omega)s + (-1)^{\deg \omega} \omega \tilde{\nabla}_\xi s.$$

As ∇^B is torsion-free, $\mathrm{d}\omega = \sum_\alpha f_\alpha (\nabla_{X_\alpha}^B \omega)$ and we obtain (1).

(2) For $s \in \Gamma(\mathscr{S}_0 \otimes \mathscr{E})$, we have

$$\mathbb{B}^{[0]} \cdot s = \sum_{n_1+1 \leq i \leq n_1+n_0} \rho_0(X_i)(\tilde{\nabla}_{X_i} \cdot s)^{[0]}.$$

But clearly $(\tilde{\nabla}_{X_i} \cdot s)^{[0]} = \nabla^0_{X_i} \cdot s$, hence the result.

(3) Let $\mathbb{B}^1$, $\mathbb{B}^2$ be associated to ∇^{B_1}, ∇^{B_2}. Because of the superconnection property, it is enough to verify that $\mathbb{B}^1$ and $\mathbb{B}^2$ coincide on $\Gamma(\mathscr{S}_0 \otimes \mathscr{E})$. From formula 1.5 giving the connection $\tilde{\nabla}$ on $\mathscr{V}$, it is clear that the difference of connection matrices $\tilde{\theta}_1(X_i) - \tilde{\theta}_2(X_i)$ is valued in $\mathfrak{sl}(V_1) \subset \mathfrak{so}(V, Q)$. Therefore from (1.15) we see that $(\tilde{\nabla}^1_{X_i} - \tilde{\nabla}^2_{X_i}) . (1 \otimes s_0) = 0$.

Remark. When the fiber M is reduced to a point, Bismut superconnection $\mathbb{B}$ is just the usual covariant differentiation on $\mathscr{A}(B) \underset{\mathscr{A}^0(B)}{\otimes} \Gamma(\mathscr{E})$.

1.16. Let I_x $(x \in B)$ be the curvature of the Bismut superconnection $\mathbb{B}$ (I.1.3). Thus I_x is the operator on the fiber $\mathscr{W}_x$ defined by

$$(\mathbb{B}^2 \phi)(x) = I_x \cdot \phi(x) \quad \text{for} \quad \phi \in \Gamma(\tilde{\mathscr{S}} \otimes \mathscr{E}) = \mathscr{A}(B, \mathscr{W})$$

(i.e., $\phi(x) \in \Lambda T^*_x B \otimes \mathscr{W}_x = \Gamma(M_x, \mathscr{S} \otimes \mathscr{E})$).

We fix $x \in B$. The restriction of the connection $\tilde{\nabla}$ to M_x defines a connection on $\tilde{\mathscr{S}}|M_x$. Let $\tilde{\Delta}_x$ be the associated Laplacian. Let r be the scalar curvature of the Riemannian manifold M_x. Let $\Omega^{\mathscr{E}}$ be the curvature of the auxiliary bundle $\mathscr{E}$.

THEOREM 1.17. ([3], theorem 3.5)

$$I_x = \tilde{\Delta}_x + \frac{r}{4} + \sum_{1 \leq i < j \leq n_1+n_0} \tilde{\rho}(X_i^* X_j^*) \otimes \Omega^{\mathscr{E}}(X_i, X_j).$$

Proof. The proof is very similar to the computation for the usual Dirac operator. Using the commutation relation 1.12, the torsion property 1.4.3 of the connection $\tilde{\nabla}$ on $\mathscr{V}$ and the anticommutation relations in the Clifford algebra $C(V, Q)$, we obtain

$$\mathbb{B}^2 = \Delta + \tfrac{1}{2} \sum_{1 \leq i, j \leq n_1+n_0} \tilde{\rho}(X_i^* X_j^*)\, \Omega(X_i, X_j),$$

where Ω is the curvature tensor of $\tilde{\mathscr{S}} \otimes \mathscr{E}$. Since

$$\Omega(X_i, X_j) = \tilde{\rho}(\tilde{R}(X_i, X_j)) \otimes 1 + 1 \otimes \Omega^{\mathscr{E}}(X_i, X_j)$$

it remains to prove that

$$\sum_{1 \leq i, j \leq n_1+n_0} \tilde{\rho}(X_i^* X_j^*) \tilde{\rho}(\tilde{R}(X_i, X_j)) = \frac{r}{2}.$$

This is proven by a standard computation, using the symmetry formulas 1.7.1, 1.7.2 and the orthogonality relations in $TB \oplus V\tilde{M} \oplus T^*B$ with respect to the bilinear form g.

2. *The index density of the family* $(\not{D}_x)$.

2.1. Let $\not{D} = (\not{D}_x)$ be the family of Dirac operators on the fibers. Let $\mathbb{B}$ be the Bismut superconnection, $(I_x)_{x \in B}$ its curvature operator.

As in (I.1.4), we consider the automorphism δ_t of ΛT^*B given by $t^{-j/2}$ on $\Lambda^j T^*B$ and extend δ_t to a linear automorphism of $\tilde{\mathscr{S}}$. Then $\mathbb{B}_t = t^{1/2} \delta_t \circ \mathbb{B} \circ \delta_t^{-1}$ is again a superconnection on $\Gamma(\mathscr{S}\otimes\mathscr{E})=\Gamma(B, \mathscr{W})$ with curvature

$$I_{t,x}=\delta_t\circ(tI_x)\circ\delta_t^{-1}.$$

Let $x\in B$. By theorem I.17, I_x is a second order differential operator acting on $\Lambda T_x^*B\otimes\mathscr{W}_x$ with principal symbol the Riemannian metric of the manifold M_x. Therefore we can form the semi-group e^{-tI_x} $(t>0)$ acting on $\Lambda T_x^*B\otimes\mathscr{W}_x$. It is given by a C^∞ kernel $\langle y|e^{-tI_x}|y'\rangle\in\Lambda(T_x^*B)\otimes\mathrm{Hom}((\mathscr{S}_0\otimes\mathscr{E})_{y'}, (\mathscr{S}_0\otimes\mathscr{E})_y)$, $y, y'\in M_x$, where ΛT_x^*B acts on itself by exterior multiplication.

By theorem I.2.5, the cohomology class ch(Ind D) is represented, for every $t>0$, by the closed differential form on B given by

2.2. $$\delta_{2i\pi t}(\mathrm{str}(e^{-tI_x})).$$

We have

$$\delta_{2i\pi t}(\mathrm{str}(e^{-tI_x}))=\int_{M_x}\delta_{2i\pi t}(\mathrm{str}^0\langle y|e^{-tI_x}|y\rangle)dy,$$

where we have extended str^0: $\mathrm{End}(\mathscr{S}_0\otimes\mathscr{E})_y\to\mathbb{C}$ by $Id\otimes\mathrm{str}^0$ to $\Lambda(T_xB)^*\otimes\mathrm{End}(\mathscr{S}_0\otimes\mathscr{E})_y$.

We fix $x\in B$, denote I_x by I. We fix a basis X_x of T_xB identifying T_xB with $V_1=\mathbb{R}^{n_1}$. We denote by $\delta_{1,t}$ the dilatation on ΛV_1^* given by $\delta_{1,t}(\omega^{[i]})=t^{-i/2}\omega^{[i]}$. Consider the decomposition $T_y\tilde{M}=V_1\oplus T_yM_x$ in horizontal and vertical subspaces. Let T^0: $\Lambda(T_y\tilde{M})^*\to\Lambda V_1^*$ be the map given by taking the highest degree term in the vertical direction. Recall that we have a connection ∇^0 on $V\tilde{M}$ and a connection on $\mathscr{E}$. We use these connections to construct the Chern–Weil differential forms $\hat{\mathscr{A}}(V\tilde{M})$ and ch$(\mathscr{E})$. $\Big($Recall that $\hat{\mathscr{A}}$ is associated to the invariant function $\left(\det\frac{e^{X/2}-e^{-X/2}}{X}\right)^{-1/2}\Big)$. The aim of this section is to give a proof of the following theorem of Bismut ([3]).

THEOREM 2.3. *For $y\in M_x$, the index density $\delta_{1,2i\pi t}(\mathrm{str}\langle y|e^{-tI_x}|y\rangle$ has a limit when $t\to 0$ which is equal to $T^0(\hat{\mathscr{A}}(V\tilde{M})ch(\mathscr{E}))$.*

By integration over M_x, this theorem clearly implies the cohomological theorem of Atiyah–Singer [2]:

COROLLARY 2.4. *Let $D=(D_x)$ be the family of Dirac operators associated to a fibration π: $\tilde{M}\to B$ and an auxiliary bundle $\mathscr{E}$. Then*

$$\mathrm{ch}(\mathrm{ind}\ D)=\pi_*(\hat{\mathscr{A}}(V\tilde{M})\ \mathrm{ch}(\mathscr{E})).$$

Our proof of the theorem 2.3 is based (as in [5]) on the method of the heat equation on principal bundles developed in II. However in the situation where dim $B>0$ the Laplacian $\tilde{\Delta}_x$ is associated to a connection $\tilde{\nabla}$ on $\mathscr{S}|M_x$ which is valued in the non-compact Lie algebra $\mathfrak{h}_0$ so that II does not apply right away. This non compacity is due to the Grassmann multiplications which appear in the Bismut curvature operator I_x. Replacing Grassmann multiplications by Clifford ones, we will define an operator $I(\varepsilon)$ associated to a connection with compact holonomy group. Let us explain this construction.

2.5. Consider the vector space $\mathfrak{w}=\Lambda^2(V_0\oplus V_1^*)=\Lambda^2V_0\oplus(V_0\otimes V_1^*)\oplus\Lambda^2V_1^*$. It carries a Lie algebra structure isomorphic to $\mathfrak{h}_0$ given by the embedding $\mathfrak{w}\subset C(V_1\oplus V_0\oplus V_1^*, Q)$. We denote by $[x, y]_0$ this Lie algebra law. The subspace Λ^2V_0 is then a subalgebra $\mathfrak{g}_0$ isomorphic to $\mathfrak{so}(V_0)$ while $(V_0\otimes V_1^*)\oplus\Lambda^2V_1^*$ is a nilpotent ideal of $\mathfrak{h}_0$.

Consider the principal bundle $P'_x = P'_0|M_x$ with structure group $G_0 = \mathrm{Spin}\, V_0$. Let q: $P'_x \to M_x$ be the projection. We have

$$\mathscr{V}|M_x = P'_x \times_{G_0} (V_1 \oplus V_0 \oplus V_1^*)$$
$$T\tilde{M}|M_x = P'_x \times_{G_0} (V_1 \oplus V_0)$$
$$\tilde{\mathscr{S}}|M_x = P'_x \times_{G_0} (\Lambda V_1^* \otimes S_0).$$

From the formulae 1.7.2, 1.7.3, we see that the connection $\tilde{\nabla}$ on $\mathscr{V}|M_x$ is associated to a $\mathfrak{w}$ valued form $\tilde{\theta}$ on P'_x. We also consider the curvature $\tilde{R}$ of the connection $\tilde{\nabla}$. It is the 2-form

2.6. $$\tilde{R} = \mathrm{d}\tilde{\theta} + \tfrac{1}{2}[\tilde{\theta}, \tilde{\theta}]_0 \in \mathscr{A}^2(P'_x) \otimes \mathfrak{w}.$$

The connection $\tilde{\nabla}$ on $\tilde{\mathscr{S}}|M_x$ is associated to

2.7. $$\tilde{\rho}(\tilde{\theta}) \in \mathscr{A}^1(P'_x) \otimes \mathrm{End}(\Lambda V_1^* \otimes S_0).$$

Denote by $\tilde{\Delta} = \tilde{\Delta}_x$ the Laplacian on $\tilde{\mathscr{S}} \otimes \mathscr{E}|M_x$ associated to the tensor product connection. Consider the potential

$$\frac{r}{4} + \sum_{1 \le i < j \le n_1 + n_0} \tilde{\rho}(X_i^* X_j^*) \otimes \Omega^{\mathscr{E}}(X_i, X_j).$$

It is of the form $\tilde{\rho}(F)$, where $F = F^0 + F^1$,

2.8. $$F^0 \in \Gamma(M_x, \mathrm{End}\, \mathscr{E}),$$
$$F^1 \in (\Gamma(P'_x, q_* \mathrm{End}\, \mathscr{E}) \otimes \mathfrak{w})^{G_0},$$

with $$F^0 = \frac{r}{4}$$

$$F^1 = \sum_{1 \le i < j \le n_1 + n_0} \Omega^{\mathscr{E}}(X_i, X_j) \otimes (X_i^* \wedge X_j^*)$$

where X_i, $1 \leqslant i \leqslant n_1 + n_0$ is a basis of $V_1 \oplus V_0$. (Thus $(X_i)_u$ is a basis of $T_y\tilde{M}$, for $u \in P'_x$ and $y = q(u)$, and X_i^* is a basis of $V_1^* \oplus V_0$). We have:

2.9. $$I = \tilde{\Delta} + \tilde{\rho}(F).$$

However, as said before, we cannot apply the results of (II.2) to this operator, as the holonomy group of the connection $\tilde{\rho}(\tilde{\theta})$ is not compact.

We now introduce a deformation $I(\varepsilon)$ of I. Fix a positive inner product on V_1^* and consider the Clifford algebra $C(V_0 \oplus V_1^*)$ associated to the direct sum inner product on $V_0 \oplus V_1^*$. The embedding of the vector space $\mathfrak{w} \subset C(V_0 \oplus V_1^*)$ defines on $\mathfrak{w}$ another Lie algebra bracket which we denote by $[x, y]_1$. We denote by $\mathfrak{g}$ the Lie algebra $(\mathfrak{w}, [.]_1)$. We denote by G the spinor group $\mathrm{Spin}(V_0 \oplus V_1^*) = \mathrm{Spin}(n_0 + n_1)$ considered as a subset of $C(V_0 \oplus V_1^*)$. Then $\mathfrak{g}$ is the Lie algebra of G. It has an inner product (,). Consider the representation ρ_1 (II.3.12) of $C(V_1^* \oplus V_0)$ on $\Lambda V_1^* \otimes S_0$ (here V_1 is identified with V_1^*) given by

$$\rho_1(X) \cdot (\omega \otimes s) = (-1)^{\deg \omega} \omega \otimes \rho_0(X).s, \text{ for } X \in V_0,\ \omega \in \Lambda V_1^*,\ s \in S_0,$$
$$\rho_1(f) \cdot (\omega \otimes s) = (f \wedge \omega - i(f)\omega) \otimes s, \text{ for } f \in V_1^*.$$

Then ρ_1 induces a Lie algebra representation of $\mathfrak{g}$ and a representation of G. Denote by $\delta_{1,\varepsilon}$ the automorphism of ΛV_1^*, $\Lambda(V_0 \oplus V_1^*)$, $\Lambda V_1^* \otimes S_0$, etc. . . , which extend the dilatation

$\delta_{1,\varepsilon}(f)=\varepsilon^{-1/2}f$ on V_1^*. Consider the Lie bracket $[x,y]_\varepsilon=\delta_{1,\varepsilon}[(\delta_{1,\varepsilon})^{-1}x,(\delta_{1,\varepsilon})^{-1}y]$ on $\mathfrak{w}$. Then $\lim_{\varepsilon\to 0}[x,y]_\varepsilon=[x,y]_0$. Define $\rho_\varepsilon(x)=\delta_{1,\varepsilon}\circ\rho_1((\delta_{1,\varepsilon})^{-1}x)\circ(\delta_{1,\varepsilon})^{-1}$ for $x\in\mathfrak{w}$. Then ρ_ε is a Lie algebra representation of $(\mathfrak{w},[.]_\varepsilon)$. We have

$$\rho_\varepsilon(X)\cdot(\omega\otimes s)=(-1)^{\deg\omega}\omega\otimes\rho_0(X)s,\qquad \text{for } X\in V_0,$$

$$\rho_\varepsilon(f)\cdot(\omega\otimes s)=(f\wedge\omega-\varepsilon i(f)\omega)\otimes s,\qquad \text{for } f\in V_1^*,$$

so that $\lim_{\varepsilon\to 0}\rho_\varepsilon(x)=\tilde{\rho}(x)$ for $x\in\mathfrak{w}$.

Consider the form

$$\rho_\varepsilon(\tilde{\theta})\in\mathscr{A}^1(P'_x)\otimes\operatorname{End}(\Lambda V_1^*\otimes S_0).$$

Then it is easy to see that $\rho_\varepsilon(\tilde{\theta})$ is a connexion form (I.1.5), thus defines a connection $\tilde{\nabla}_\varepsilon$ on $\tilde{\mathscr{S}}|M_x$. Let $\tilde{\Delta}_\varepsilon$ be the Laplacian on $(\tilde{\mathscr{S}}\otimes\mathscr{E})|M_x$ associated to the tensor product connexion.

$$\text{Consider } I(\varepsilon)=\tilde{\Delta}_\varepsilon+\rho_\varepsilon(F).$$

Then $I(\varepsilon)$ is again a 2^d-order differential operator on $\Gamma(M_x,\tilde{\mathscr{S}})$ with principal symbol the metric of M_x. Recall that we have asymptotic expansions:

$$\langle y|e^{-tI}|y\rangle\sim t^{-n_0/2}\sum_{k\geq 0}t^kA_k(y),$$

$$\langle y|e^{-tI(\varepsilon)}|y\rangle\sim t^{-n_0/2}\sum_{k\geq 0}t^kA_{k,\varepsilon}(y),$$

where $A_k(y)$, $A_{k,\varepsilon}(y)$ are obtained by recurrence relations ([8]). It follows from the continuity of the operator $I(\varepsilon)$ that $\lim_{\varepsilon\to 0}A_{k,\varepsilon}(y)=A_k(y)$. We have, for $y\in M_x$,

$$A_{k,\varepsilon}(y)\in\operatorname{End}(\Lambda V_1^*)\otimes\operatorname{End}(\mathscr{S}_y^0)$$

$$A_k(y)\in\Lambda V_1^*\otimes\operatorname{End}(\mathscr{S}_y^0)\subset\operatorname{End}(\Lambda V_1^*)\otimes\operatorname{End}(\mathscr{S}_y^0),$$

where ΛV_1^* acts on ΛV_1^* by exterior multiplication. Thus, we have

$$\lim_{\varepsilon\to 0}\operatorname{str}^0(A_{k,\varepsilon}(y))\cdot 1=\operatorname{str}^0(A_k(y)).$$

We will see that, for every $\varepsilon\neq 0$, the asymptotic expansion of $\delta_{1,t}(\operatorname{str}^0\langle y|e^{-tI(\varepsilon)}|y\rangle\cdot 1)$ has no singular part. Put

2.10. $$\omega(\varepsilon,y)=\lim_{t\to 0}\delta_{1,t}(\operatorname{str}^0\langle y|e^{-tI(\varepsilon)}(y\rangle\cdot 1).$$

Thus, the asymptotic expansion of $\delta_{1,t}(\operatorname{str}^0\langle y|e^{-tI}|y\rangle)$ has no singular part, and

$$\lim_{t\to 0}\delta_{1,t}(\operatorname{str}^0\langle y|e^{-tI}|y\rangle)=\lim_{\varepsilon\to 0}\omega(\varepsilon,y).$$

It is easy to compute $\omega(\varepsilon,y)$, as we can apply to the operator $I(\varepsilon)$ the results of (II.2):

Consider the groups $G_0=\operatorname{Spin}(V_0)$, $G=\operatorname{Spin}(V_0\oplus V_1^*)$. Let $\mathfrak{g}_0\subset\mathfrak{g}$. For $\varepsilon>0$, consider the $\mathfrak{g}$-valued connection form and $\mathfrak{g}$-valued potentials given by

$$\tilde{\theta}_\varepsilon=\delta_{1,\varepsilon}^{-1}(\tilde{\theta})\in\mathscr{A}^1(P'_x)\otimes\mathfrak{g}$$

$$F_\varepsilon^1=\delta_{1,\varepsilon}^{-1}(F^1)\in\mathscr{A}^1(P'_x)\otimes\mathfrak{g}$$

$$F_\varepsilon^0 = F^0 = \frac{r}{4}$$

$$F_\varepsilon = F_\varepsilon^1 + F_\varepsilon^0.$$

Consider the representation $\delta_{1,\varepsilon} \circ \rho_1 \circ \delta_{1,\varepsilon}^{-1}$ of $\mathfrak{g}$ (or G). Then

$$\rho_\varepsilon(\tilde{\theta}) = \delta_{1,\varepsilon} \circ \rho_1(\tilde{\theta}_\varepsilon) \circ \delta_{1,\varepsilon}^{-1}$$

$$\rho_\varepsilon(F) = \delta_{1,\varepsilon} \circ \rho_1(F_\varepsilon) \circ \delta_{1,\varepsilon}^{-1}.$$

Thus, the operator $I(\varepsilon)$ is the operator H on $\Gamma(M_x, \mathscr{S}_x)$ associated to $(P'_x, \tilde{\theta}_\varepsilon, F_\varepsilon, \delta_{1,\varepsilon} \circ \rho_1 \circ \delta_{1,\varepsilon}^{-1})$ [II.2.19].

For $a \in \mathfrak{g}$, we have

2.11. $$(F_\varepsilon^1, \delta_{1,\varepsilon} a) = (F^1, a).$$

Let $\tilde{R}_\varepsilon$ be the curvature of $\tilde{\theta}_\varepsilon$. Then

$$\tilde{R}_\varepsilon = \delta_{1,\varepsilon}^{-1}(d\tilde{\theta}) + \tfrac{1}{2}[\delta_{1,\varepsilon}^{-1}\tilde{\theta}, \delta_{1,\varepsilon}^{-1}\tilde{\theta}]$$

and, for $a \in \mathfrak{g}$

2.12. $$\lim_{\varepsilon \to 0} (\tilde{R}_\varepsilon, \delta_{1,\varepsilon} a) = (\tilde{R}, a).$$

We now apply theorem II.2.20 to $I(\varepsilon)$. For $u \in P'_x$ above $y \in M_x$, there exist C^∞ functions ϕ_ε^j on $\mathfrak{g}$ with values in End $\mathscr{E}_y$ such that

$$\langle u | e^{-tI(\varepsilon)} | u \rangle \sim (4\pi t)^{-1/2(n_0 + \dim \mathfrak{g})} \oint_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}} \sum_{j=0}^{\infty} t^j \phi_\varepsilon^j(a) \otimes (\delta_{1,\varepsilon} \circ \rho_1(\exp a) \circ \delta_{1,\varepsilon}^{-1}) da$$

with

$$\phi_0^\varepsilon(a) = j_{\mathfrak{g}}^{1/2}(a)\, j_{V_0}^{-1/2}\,(\tfrac{1}{2}\tau_{V_0}(\tilde{R}_\varepsilon, a))\, e^{-1/2(F_\varepsilon^1, a)}.$$

From this, we obtain

$$\delta_{1,t}(\mathrm{str}^0 \langle u | e^{-tI(\varepsilon)} | u \rangle . 1) \sim$$

$$(4\pi t)^{-1/2(n_0 + \dim \mathfrak{g})} \delta_{1,\varepsilon} \left(\oint_{\mathfrak{g}} e^{-\frac{\|a\|^2}{4t}} \sum_{j=0}^{\infty} t^j tr\, \phi_\varepsilon^j(a) \delta_{1,t}(\mathrm{str}^0(\rho_1(\exp a)) \cdot 1) da \right).$$

We have $\mathrm{str}^0 \rho_1(\exp a) \cdot 1 = 2^{l_0} i^{-l_0} T^0(\exp_C a)$ (II.3.13). Thus by 3.14, the asymptotic expansion of $\delta_{1,t}((\mathrm{str}^0\langle u | e^{-tI(\varepsilon)} | u \rangle) \cdot 1)$ has no singular part and we obtain:

$$\omega_{\varepsilon, y} = (2i\pi)^{-l_0} \delta_{1,\varepsilon}(T^0(A(P_\varepsilon)))$$

with

$$P_\varepsilon(a) = j_{V_0}^{-1/2}(\tau_{V_0}(\tilde{R}_\varepsilon, a))\, tr(e^{-(F_\varepsilon^1, a)}).$$

By 2.11, 2.12, we have

$$\lim_{\varepsilon \to 0} \omega(\varepsilon, y) = (2i\pi)^{-l_0} P(\psi\psi^*)$$

with

2.13. $$P(a) = j_{V_0}^{-1/2}(\tau_{V_0}(\tilde{R}, a))\, tr(e^{-(F^1, a)}).$$

Now, as

$$(F^1, a) = \sum_{i<j} \Omega^{\mathscr{E}}(X_i, X_j) x_{ij},$$

we have

$$tr(e^{(F^1, \psi\psi^*)}) = \mathrm{ch}_Q(\mathscr{E}).$$

In the expression (2.13) $\tilde{R}$ is the curvature of the connection $\tilde{\nabla}$ of the bundle $\mathscr{V}|M_x$ and $a \in \Lambda^2(V_0 \oplus V_1^*)$, ie., $R \in \Lambda^2 V_0^* \otimes \Lambda^2(V_0 \ominus V_1^*)$ and $(R, a) = (1 \otimes i(a)) \cdot \tilde{R} \in \Lambda^2 V_0^* \simeq \mathfrak{so}(V_0)$.

Consider the curvature R^0 of the bundle $V\tilde{M}$ over $\tilde{M}$ for the Euclidean connection ∇^0, ie.

$$R^0 \in \Lambda^2(V_0^* \oplus V_1^*) \otimes \mathfrak{so}(V_0).$$

By the fundamental symmetry property of the curvature $\tilde{R}$ (1.8) we have

$$(1 \otimes i(a)) \cdot \tilde{R} = (i(a) \otimes 1) R^0.$$

(This equation corresponds to the theorem 4.14 of [3]). Thus

$$j_{V_0}^{-1/2}(\tau_{V_0}(\tilde{R}_0, \psi\psi^*)) = \hat{\mathscr{A}}_Q(V\tilde{M}),$$

where $\hat{\mathscr{A}}_Q$ is defined, as in [9], by omitting the factor $1/2i\pi$. This finishes the proof of Theorem 2.3.

REFERENCES

1. M. ANDLER, N. BERLINE, E. GETZLER and M. VERGNE: The index of Dirac operators. Heat equation methods (to appear).
2. M. ATIYAH and I. SINGER: The index of elliptic operators IV, *Ann. Math.* **87** (1968), 484–530.
3. J. M. BISMUT: The Atiyah–Singer index theorem for families of Dirac operators: Two heat equation proofs, *Invent. Math.* **83** (1986), 91–151.
4. M. BERGER, P. GAUDUCHON and E. MAZET: *Le spectre d'une variété riemanienne*, Lecture Notes in Mathematics, Springer–Verlag, Berlin, **194** (1971).
5. N. BERLINE and M. VERGNE: A computation of the equivariant index of the Dirac operator, *Bull. Soc. Math. France*, **113** (1985), 305–345.
6. E. GETZLER: Pseudo differential calculus on supermanifolds and the Atiyah–Singer Index theorem, *Commun. Math. Phys.* **92** (1983), 163–178.
7. E. GETZLER: A short proof of the Atiyah–Singer Index theorem, *Topology* **25** (1986), 111–117.
8. V. K. PATODI: An analytic proof of Riemann–Roch–Hirzebruch theorem for Kaehler manifolds, *J. Diff. Geometry* **5** (1971), 251–283.
9. D. QUILLEN: Superconnections and the Chern character, *Topology* **24** (1985), 89–95.

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