THE EQUIVARIANT INDEX AND KIRILLOV'S CHARACTER FORMULA

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Introduction. Let G be a compact Lie group acting by positive isometries on a compact oriented Riemannian manifold M of even dimension. Let $\mathcal{V}$ be a G equivariant Clifford module over M and let

$$D:\Gamma(\mathcal{V}^+) \to \Gamma(\mathcal{V}^-)$$

be the Dirac operator. The spaces Ker D and Coker D are finite dimensional representation spaces for G. The equivariant index of D is defined to be

$$(\text{index } D)(g) = \text{tr}(g, \text{Ker } D) - \text{tr}(g, \text{Coker } D).$$

Our main result, Theorem 3.18, is a formula which expresses (index D)($\exp X$) as an integral over M of a form μ_X which depends analytically on X near the origin in the Lie algebra $\mathfrak{g}$ of G. Of course, for $X = 0$, it coincides with Atiyah-Singer index formula. We also obtain a similar formula for (index D)($b \exp X$) for any element $b \in G$, with X in the Lie algebra of the centralizer of b, as an integral over the fixed point set of b.

Our main tool is a localization formula for differential forms (Theorem 2.8) generalizing results of R. Bott [11], [12]. Denote by X^* the vector field on M generated by $\exp tX$, denote by $c(X^*)$ the contraction with X^*, denote by d the exterior derivation; we consider the operator

$$d_X = d - 2i\pi c(X^*)$$

acting on differential forms. This operator was also introduced, independently, in [27], and related in [2] to equivariant cohomology. In [8], given a

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G-equivariant principal bundle with a G-invariant connection, we have defined a moment map and equivariant characteristic forms, which are d_X-closed forms analogous to Chern-Weil characteristic forms.

In our formula

$$(\text{index } D)(\exp X) = \int_M \mu_X$$

the form μ_X is defined in terms of such equivariant characteristic forms, associated to the geometric data.

Theorem 2.8 expresses the integral over M of a d_X-closed form as an integral over the zero-set M_0 of X^*. Using this, our formula for the equivariant index follows from the theorem of Atiyah and Segal [4] which expresses the equivariant index at a point $g \in G$ as an integral over the fixed point set of g.

When M is a regular admissible orbit $\mathcal{O}$ of the coadjoint representation of G, our formula gives Kirillov's formula for the character of the corresponding representation $T_{\mathcal{O}}$ (cf 3.21). As is well known, in this case the equivariant index theorem at a regular element $g \in G$ reduces to the Lefschetz-trace formula of [1] and is equivalent to Weyl's character formula. On the other hand, Kirillov's formula reads [19]

$$\operatorname{tr} T_{\mathcal{O}}(\exp X) = \int_{\mathcal{O}} e^{i\langle \cdot, X\rangle} e^{\omega/2\pi} \mathcal{J}^{-1/2}(\operatorname{ad} X)$$

where ω is the canonical 2-form on the orbit $\mathcal{O}$ and

$$\mathcal{J}(\operatorname{ad} X) = \det \frac{e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2}}{\operatorname{ad} X}.$$

The integrand in the right-hand side coincides with the form μ_X.

Consider now a noncompact semi-simple Lie group G with discrete series. The square-integrable representation $T_{\mathcal{O}}$ associated to a regular admissible elliptic orbit can be realized in the space of L^2-solutions of the twisted Dirac operator on $\mathcal{O}$ [22], [25]. Furthermore, the formula

$$\operatorname{tr} T_{\mathcal{O}}(\exp X) = \int_{\mathcal{O}} \mu_X$$

holds [24], when the trace is considered as a generalized function on a neighborhood of 0 in $\mathfrak{g}$. This expresses the L^2-equivariant index in terms of equivariant characteristic forms. In this situation no Lefschetz fixed point formula seems to hold over the whole noncompact group G. However the localization formula 2.8 remains true when $\exp tX$ is a relatively compact subgroup of G [7]. Thus on a compact torus T of G (but only there) can $\operatorname{tr} T_{\Theta}(g)$ be given alternatively by a fixed point formula, or by the integral formula above.

In the case of a homogeneous Riemannian manifold, Connes and Moscovici have obtained a formula for the L^2 index of the Dirac operator [14] involving the form μ_X, for $X = 0$. Similarly, we expect the range of validity of our equivariant index formula to extend to noncompact situations.

The results of this article have been announced in [9].

A particular case of the localization formula gives a formula of Duistermaat and Heckman [16], [17], for the moment map of a symplectic manifold with a Hamiltonian G-action (cf 2.10). We have used some ideas in [17] to simplify our original proof. The localization formula 2.8 has also been obtained in [2] using topological methods.

We thank Victor Kac for the proof of formula 2.9. Michel Duflo suggested us to model the formula for (index D)($b \exp X$) on Harish-Chandra description of the distribution $\theta_{\Lambda}^{(b)}$ [18].

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I. Moment map, group actions on vector bundles and equivariant characteristic forms.

1.1. Let M be a C^∞-manifold. Denote by $\mathcal{A}(M) = \bigoplus \mathcal{A}^r(M)$ the algebra of differential forms on M and by $\mathcal{A}^+(M)$ the commutative subalgebra of even forms. We denote by d the exterior differentiation. If ξ is a vector field on M, we denote by $c(\xi): \mathcal{A}(M) \to \mathcal{A}(M)$ the contraction, $\mathcal{L}(\xi)$ the Lie derivative. We have

1.2 $$\mathcal{L}(\xi) = d \cdot c(\xi) + c(\xi) \cdot d.$$

1.3. Let G be a Lie group acting on M. Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$, we denote by X_M^*, or simply X^*, the vector field on M defined by

$$(X^* \cdot f)(m) = \frac{d}{dt} f(\exp tX \cdot m)|_{t=0}.$$

We have: $[X^*, Y^*] = -[X, Y]^*$.

The operator $d_X = d - 2i\pi c(X^*)$ is an antiderivation of $\mathcal{A}(M)$, which respects the gradation in even and odd forms. By 1.2, $(d_X)^2 = -2i\pi\mathcal{L}(X^*)$. Let $\mathcal{A}_X$ be the subalgebra of forms $\mu \in \mathcal{A}(M)$ such that $\mathcal{L}(X^*)\mu = 0$. Then the square of the operator d_X is zero on $\mathcal{A}_X$. We define as in [8]:

$$Z(M, d_X) = \operatorname{Ker} d_X$$

$$B(M, d_X) = d_X(\mathcal{A}_X)$$

Thus $B(M, d_X) \subset Z(M, d_X) \subset \mathcal{A}_X$. We note by $H^*(M, d_X)$ the algebra $Z(M, d_X)/B(M, d_X)$. It is clear that, if $X_M^* = 0$, $H^*(M, d_X)$ is the usual De Rham cohomology ring $H^*(M)$ of M.

1.4. A map $X \to \mu_X$ from $\mathfrak{g}$ to $\mathcal{A}(M)$ will be called an equivariant form, if

$$\mu_X \in Z(M, d_X)$$

$$\mu_{g \cdot X} = g \cdot \mu_X \quad \text{for} \quad g \in G, X \in \mathfrak{g}.$$

We will study group actions on principal bundles with connections and see that such a situation leads naturally to the construction of equivariant forms on M.

Let H be a Lie group with Lie algebra $\mathfrak{h}$. Let $P \to M$ be a principal bundle with structure group H. If $U \in \mathfrak{h}$, we denote by $r(U)$ (or simply U) the vector field on P generated by the right action of H on P, i.e.

$$(U \cdot \varphi)(p) = \frac{d}{dt} \varphi(p \exp tU)|_{t=0}.$$

Let α be a connection form on P, D the covariant differentiation associated to α and Ω the curvature of α.

Suppose the action of G on M lifts to an action of G on P commuting with the action of H. For $X \in \mathfrak{g}$, let X_P^* the corresponding vector field on P. The function defined on P by $J_X = \alpha(X_P^*)$ satisfies $J_X(ph) = h^{-1} \cdot J_X(p)$.

1.5 LEMMA.

$$DJ_X + c(X_P^*)\Omega = \mathcal{L}(X_P^*)\cdot\alpha.$$

Proof. As $J_X(ph) = h^{-1}J_X(p)$, we have

$$DJ_X = dJ_X + [\alpha, J_X] = dc(X_P^*)\alpha + [\alpha, J_X].$$

Now $\Omega = d\alpha + (1/2)[\alpha, \alpha]$. Thus

$$c(X_P^*)\Omega = c(X_P^*)d\alpha + \frac{1}{2}[J_X, \alpha] - \frac{1}{2}[\alpha, J_X] = c(X_P^*)d\alpha + [J_X, \alpha],$$

$$DJ_X + c(X_P^*)\Omega = dc(X_P^*)\alpha + c(X_P^*)d\alpha = \mathcal{L}(X_P^*)\cdot\alpha.$$

1.6 *Definition.* If G acts on P and preserves the connection α, the map $J: P \to \mathfrak{g}^* \otimes \mathfrak{h}$ defined by $J_X = \alpha(X_P^*)$ is called the moment map of the action.

1.7 PROPOSITION. *The moment map J has the following properties*:

a) *J is $G \times H$ equivariant*,
b) $DJ_X + c(X_P^*)\Omega = 0$,
c) $[J_X, J_Y] - J_{[X,Y]} = \Omega(X_P^*, Y_P^*)$.

Proof. Let us prove c):

$$\Omega(X_P^*, Y_P^*) = d\alpha(X_P^*, Y_P^*) + \frac{1}{2}[\alpha, \alpha](X_P^* \wedge Y_P^*)$$

$$= X_P^*\cdot\alpha(Y_P^*) - Y_P^*\cdot\alpha(X_P^*) - \alpha([X_P^*, Y_P^*]) + [J_X, J_Y].$$

As α is invariant by X_P^*, Y_P^*, we have

$$(X_P^*\cdot\alpha)(Y_P^*) = \alpha[X_P^*, Y_P^*] = -\alpha([X, Y]_P^*) = J_{[X,Y]}$$

and we obtain c).

As Ω is a horizontal form, the form $c(X_P^*)\Omega$, as well as $\Omega(X_P^*, Y_P^*)$ depends only of the horizontal component of X_P^*, i.e. of X_M^* and α. We will reformulate the conditions a), b), c) in a slightly different form.

Let $\mathfrak{h}$ be the Lie algebra of H and consider the vector bundle $P(\mathfrak{h}) =$

$P \times \mathfrak{h}/H$, where H acts on $\mathfrak{h}$ by the adjoint action. We denote by $\Gamma(P(\mathfrak{h}))$ its space of sections. The map J_X can be considered as a section of $P(\mathfrak{h})$ and the curvature Ω of α as a section of $\Lambda^2 T^*M \otimes P(\mathfrak{h})$. As the bundle $P(\mathfrak{h})$ is a bundle of Lie algebras, given two sections φ and φ' of $P(\mathfrak{h})$, $[\varphi, \varphi']$ is again a section. Let us consider the covariant differentiation D on $P(\mathfrak{h})$ deduced from α. The conditions b) and c) read also as follows:

b′) $$DJ_X + c(X_M^*)\Omega = 0, \text{ as sections of } T^*M \otimes P(\mathfrak{h})$$

c′) $$[J_X, J_Y] - J_{[X,Y]} = \Omega(X_M^*, Y_M^*) \text{ as sections of } P(\mathfrak{h}).$$

Following Kostant [21], we now prove the converse.

1.8 THEOREM. *Let (P, α) be a principal bundle over M with structure group H and connection α. Let G be a simply connected group of automorphisms of M. Suppose there exists a linear map $J: \mathfrak{g} \to \Gamma(P(\mathfrak{h}))$ such that:*

1) $$DJ_X + c(X_M^*)\Omega = 0$$

2) $$\Omega(X_M^*, Y_M^*) = [J_X, J_Y] - J_{[X,Y]},$$

then the action of G on M lifts uniquely to an action of G on P commuting with H, leaving invariant the connection α, and such that $J_X = \alpha(X_P^*)$.

Proof. If ξ is a vector field on M, we denote by ξ^h its horizontal lift to P. Define $(X_P^*)_p = (X_M^*)^h + r(J_X(p))$. Thus $\alpha(X_P^*) = J_X$. As $J_X(ph) = h^{-1} \cdot J_X(p)$, the vector field X_P^* is invariant under the action of H, and it follows from 1.5 that X_P^* preserves the connection α.

Let us see that $[X_P^*, Y_P^*] = -[X, Y]_P^*$. As X_P^* projects to X_M^*, the difference between left and right hand side is a vertical vector. Thus, we need only to verify that

$$\alpha([X_P^*, Y_P^*]) = -\alpha([X, Y]_P^*) = -J_{[X,Y]}.$$

Consider $\Omega = d\alpha + (1/2)[\alpha, \alpha]$. Then:

$$\begin{aligned} \Omega(X_P^*, Y_P^*) &= (d\alpha)(X_P^*, Y_P^*) + [J_X, J_Y] \\ &= X_P^* \cdot \alpha(Y_P^*) - Y_P^* \cdot \alpha(X_P^*) - \alpha[X_P^*, Y_P^*] + [J_X, J_Y] \\ &= \alpha[X_P^*, Y_P^*] - \alpha[Y_P^*, X_P^*] - \alpha[X_P^*, Y_P^*] + [J_X, J_Y], \end{aligned}$$

as α is invariant under X_P^*,

$$= \alpha([X_P^*, Y_P^*]) + [J_X, J_Y].$$

The condition 2) implies then the desired equality.

It remains to see that the vector field X_P^* is complete. Let $x \in M$, p a point of P above x. Consider $x(t) = (\exp tX) \cdot x$. Consider $\gamma(t)$ the horizontal lift of $x(t)$ to P such that $\gamma(0) = p$. The equation $DJ_X + c(X_M^*)\Omega = 0$ implies in particular that $(X_M^*)^h \cdot J_X = 0$. Thus J_X is constant along the curve $\gamma(t)$.

Define $p(t) = \gamma(t) \exp tJ_X(p)$. Remark that

$$\begin{aligned} J_X(p(t)) &= e^{-tJ_X(p)} J_X(\gamma(t)) \\ &= e^{-tJ_X(p)} J_X(p) \\ &= J_X(p). \end{aligned}$$

As H leaves invariant horizontal subspaces, the tangent vector to the curve $\gamma(t + \epsilon)\exp tJ_X(p)$ is $(X_M^*)^h$. The tangent vector to $\gamma(t)\exp(t + \epsilon)J_X(p)$ is $r(J_X(p)) = r(J_X(p(t))$. Thus the tangent vector to $p(t)$ at the point t is $(X_P^*)_{p(t)}$, q.e.d.

1.9. Let (P, α) be a principal bundle over M with structure group H and connection α. Let ρ be a representation of H in a vector space V. We denote also by ρ the corresponding infinitesimal representation of $\mathfrak{h}$ in V. Let $\mathcal{V} = P \times V/H$ be the associated vector bundle and $\Gamma(\mathcal{V})$ its space of sections. Consider the linear bundle connection D on $\Gamma(\mathcal{V})$ defined by α.

Suppose G acts on (P, α). Let $J_X : P \to \mathfrak{h}$ be the associated moment map. The function $p \to \rho(J_X(p))$ can be identified with a section of the bundle $\Gamma(\mathrm{End}\, \mathcal{V})$. The group G acts on $\Gamma(\mathcal{V})$ by $(g \cdot s)(m) = g \cdot s(g^{-1}m)$. Let $\mathcal{L}(X)$ be the corresponding infinitesimal representation of $\mathfrak{g}$ in $\Gamma(\mathcal{V})$.

1.10 Lemma.

$$\mathcal{L}(X) = -D_{X_M^*} + \rho(J_X)$$

Proof. If we identify the space of sections of $\Gamma(\mathcal{V})$ to V-valued functions on P satisfying $\varphi(ph) = \rho(h)^{-1} \cdot \varphi(p)$, the action of G on $\Gamma(\mathcal{V})$ becomes $(g \cdot \varphi)(p) = \varphi(g^{-1}p)$. Thus

$$\mathcal{L}(X) \cdot \varphi = -X_P^* \cdot \varphi = -(X_M^*)^h \cdot \varphi - r(J_X(p)) \cdot \varphi.$$

But $r(J_X(p)) \cdot \varphi = -\rho(J_X(p)) \cdot \varphi$, and we obtain the lemma.

1.11. In particular, let $m \in M$ such that $(X_M^*)_m = 0$, then the infinitesimal action of X on $\mathcal{V}_m$ is given by $\rho(J_X)$.

1.12. Recall the context of symplectic geometry where the moment map was originally defined. Let (M, ω) be a symplectic manifold. Let φ be a function on M. The Hamiltonian vector field H_φ of φ is the vector field on M such that $d\varphi = c(H_\varphi) \cdot \omega$. Define the Poisson bracket of the functions φ, φ' by

$$\{\varphi, \varphi'\} = \omega(H_{\varphi'}, H_\varphi)$$

1.13. Let G be a group of symplectic transformations of M. We say that the action of G on M is Hamiltonian if there exists a G-equivariant linear map f from $\mathfrak{g}$ to functions on M such that

a) $$df_X + c(X_M^*)\omega = 0.$$

The G-equivariance of f and the condition a) implies:

b) $$\{f_X, f_Y\} = f_{[X,Y]}.$$

Reciprocally, if the group G is connected, the conditions a) and b) implies the G-equivariance.

The map $f\colon \mathfrak{g} \to \mathcal{A}^0(M)$ was then defined to be the moment map of the Hamiltonian action.

Suppose we have an Hermitian line bundle (L, α) over (M, ω) with curvature form $K = -i\omega$. Such a line bundle exists if and only if $\omega/2\pi$ is integral. Consider the associated principal bundle with structure group the one-dimensional torus T. As T is commutative, the conditions 1), 2) of Theorem 1.8 can be simply rewritten as

1) $$dJ_X + c(X_M^*)K = 0$$

2) $$K(X_M^*, Y_M^*) = -J_{[X,Y]}.$$

If G lifts to a group of Hamiltonian transformations of M, the conditions of Theorem 1.8 are satisfied with $J_X = -if_X$. Thus the action of G on M lifts uniquely to an action of G on L preserving the connection α and such that $\alpha(X_L^*) = -if_X$. Remark that if $(X_M^*)_m = 0$, the action of $\exp tX$ on M leaves stable the point m and acts on the fiber by $e^{-itf_X(m)}$.

1.14. An important example of Hamiltonian action arises as follows [20], [21]. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit of the coadjoint representation. For $\ell \in \mathcal{O}$, let $G(\ell)$ be the stabilizer of ℓ. The group $G(\ell)$ has Lie algebra

$$\mathfrak{g}(\ell) = \{X \in \mathfrak{g};\ \ell[X, Y] = 0, \text{ for all } Y \in \mathfrak{g}\}.$$

The tangent space $T_\ell(\mathcal{O})$ to $\mathcal{O}$ at ℓ is $\mathfrak{g} \cdot \ell = \{X \cdot \ell\} \subset \mathfrak{g}^*$. Define the 2-form ω on $\mathcal{O}$ by $\omega_\ell(X \cdot \ell, Y \cdot \ell) = \ell([X, Y])$. The manifold $(\mathcal{O}, \omega)$ is then a symplectic manifold and G acts by symplectic transformations on $\mathcal{O}$.

If $X \in \mathfrak{g}$, the vector field $X_{\mathcal{O}}^*$ on $\mathcal{O}$ is given by $(X_{\mathcal{O}}^*)_\ell = X \cdot \ell$. Let $f_X(\xi) = -(\xi, X)$ be the restriction to $\mathcal{O}$ of the linear function $-X$ on $\mathfrak{g}^*$. It is immediate to verify that

a) $$df_X + c(X^*)\omega = 0$$

b) $$\{f_X, f_Y\} = f_{[X,Y]}.$$

Thus if $\omega/2\pi$ is integral, there exists a line bundle (L, α) over $\mathcal{O}$ and an action of G on (L, α), such that if $X \in \mathfrak{g}(\ell)$ the action of $(\exp tX)$ in the fiber L_ℓ of L above ℓ is given by multiplication by $e^{it(\ell,X)}$. In particular, there exists a character χ_ℓ of $G(\ell)$ of differential $i\ell$. This is Kostant's integrality criterion for $\omega/2\pi$ to be integral [21].

1.15. Let us consider a Hamiltonian action of G on (M, ω). Define $\nu_X = f_X - \omega/2i\pi$. As $df_X + c(X^*)\omega = 0$, $d\omega = 0$, we see that $d_X(\nu_X) = 0$. The map $X \to \nu_X$ is thus an equivariant form on M.

1.16. We return now to the general situation of a manifold M and a G-equivariant principal bundle (P, α) with structure group H and connection form α. Let D be the covariant differentiation on P, Ω the curvature of α, J_X the moment of X. Then $J_X - \Omega/2i\pi$ is a $\mathfrak{h}$-valued form (not homogeneous, but even) on P. From Bianchi's identity $D\Omega = 0$, it follows that $(D - 2i\pi c(X^*))(J_X - \Omega/2i\pi) = 0$. This relation is the analogue of the equality $(d - 2i\pi c(X^*))(f_X - \omega/2i\pi) = 0$ relating the moment map of a Hamiltonian action to the symplectic form ω. Let φ be a H-invariant polynomial function on $\mathfrak{h}$. We extend φ by multilinearity to a polynomial function on $\mathfrak{A}^+(P)\otimes\mathfrak{h}$ with values in $\mathfrak{A}^+(P)$. It is easy to see that the form $\varphi(J_X - \Omega/2i\pi)$ projects on a form on M denoted $\varphi(X, \alpha)$. Recall

1.17 THEOREM. [8]

1) $\varphi(X, \alpha) \in Z(M, d_X)$.

2) *The class* $\varphi(X, P)$ *of* $\varphi(X, \omega)$ *in* $H^*(M, d_X)$ *is independent of the G-invariant connection* α.

In particular suppose $\mathcal{V}$ is a G-equivariant vector-bundle with a G-invariant linear connection α. Let P be the associated principal bundle, with structure group $H = GL(n, \mathbf{C})$. Denote by c_k the coefficients of the characteristic polynomial

$$\det(1 + tA) = \sum_{k=0}^{n} t^k c_k(A).$$

The corresponding form $c_k(J_X - \Omega/2i\pi)$ coincides for $X = 0$ with the k^{th}-Chern form. The last one, $\det(J_X - \Omega/2i\pi)$, will be denoted by $\chi(X, \mathcal{V})$. It will occur in the localization formula of Section II.

In Section III, we will express the equivariant index theorem in terms of equivariant characteristic forms.

II. A localization formula. The structure of $H^*(M, d_X)$ is particularly simple to describe when the one parameter subgroup $(\exp tX)$ is relatively compact in G and M is a compact manifold. In this section, we will thus assume that T is a compact torus with Lie algebra $\mathfrak{t}$ acting on a compact manifold M. (These hypotheses may be relaxed in applications.) We fix $X \in \mathfrak{t}$. The zeros of X_M^* form a submanifold of M denoted by M_0.

2.1 .PROPOSITION. *The restriction map* $i^*: H^*(M, d_X) \to H^*(M_0)$ *is an isomorphism.*

Remark. As the operator d_X was shown to be related to equivariant cohomology in [2], this result follows from Quillen [23]. Nevertheless we give a proof, as it is very simple in our differential forms setting. This proof will follow from the next lemmas:

2.2 LEMMA. *Let Y be a manifold with a G-action. Let* $X \in \mathfrak{g}$. *Suppose there exists a* 1*-form* α *on Y such that*

a) $\alpha(X^*) = 1$
b) $\mathcal{L}(X^*)\alpha = 0$.

Then, if $\mu \in Z(X)$, $\mu = d_X(\alpha(d\alpha - 2i\pi)^{-1}\mu)$.

Proof. This is clear, as $d_X\alpha = d\alpha - 2i\pi$ and $(d\alpha - 2i\pi)$ is an invertible element of $Z(M, d_X)$.

2.3. If Y is a manifold with a G-invariant Riemannian structure g, such a 1-form α can be constructed on the complement of the zeros of X^* by setting

$$\alpha(\xi) = \frac{g(X^*, \xi)}{g(X^*, X^*)}.$$

In particular, we obtain:

$$H^*(M - M_0, d_X) = 0.$$

Remark. If $T = \exp tX$ is a torus, α is a connection form for the bundle $M - M_0 \to M - M_0/T$.

2.4 LEMMA. *Let U be a T-invariant open subset of M containing M_0. The restriction map $i^*: H^*(M, d_X) \to H^*(U, d_X)$ is an isomorphism.*

Proof. Choose a T-invariant Riemannian metric g on M. Let ψ be a T-invariant function on M, identically equal to 1 on a neighborhood of M_0 and whose support is contained in U. Let α be the 1-form on $M - M_0$ constructed as in 2.3. If $\mu \in Z(U, d_X)$, the form $\mu - d_X((1 - \psi)\alpha(d\alpha - 2i\pi)^{-1}\mu)$ represents the same element of $H^*(U, d_X)$, and is compactly supported in U, thus can be considered as a form on M. Therefore the restriction map is surjective. Let now μ be a form in $Z(M, d_X)$ such that $\mu = d_X\beta$ on U. The form $\mu' = \mu - d_X(\psi\beta)$ is a form on M, which is identically 0 in a neighborhood of M_0. Thus $\mu' = d_X(\alpha(d\alpha - 2i\pi)^{-1}\mu')$.

2.5. Let N be the normal bundle to M_0 in M. Identify, via the metric g, N_x to a subspace of $T_x(M)$ and let $N_\epsilon = \{(x, v); x \in M_0, v \in N_x; \|v\| < \epsilon\}$. For ϵ sufficiently small, the exponential map $E(x, v) = \exp_x v$ is defined and is a diffeomorphism of N_ϵ onto an open tubular neighborhood of M_0 in M. Consider the homothety $H(t)$ along the fibers of N given by $H(t)(x, v) = (x, e^t v)$. Denote by H the corresponding vector field. Let $i: M_0$

$\to N$ be the embedding of M_0 in N as the zero section, p the projection map $(x, v) \to x$ of N to M_0. If $t < 0$, $H(t)$ sends N_ϵ to N_ϵ and if $t \to -\infty$, the map $H(t)$ tends to the map $i \cdot p$.

2.6. Define $h : \mathfrak{A}(N_\epsilon) \to \mathfrak{A}(N_\epsilon)$ by

$$h\omega = \int_0^\infty H(-t)^*(c(H) \cdot \omega) dt.$$

We have

$$(dh + hd)\omega = \omega - p^* i^* \omega$$

as

$$\begin{aligned}(dh + hd)\omega &= \int_0^\infty H(-t)^*(dc(H) + c(H)d) \cdot \omega dt \\ &= \int_0^\infty H(-t)^*(\mathcal{L}(H) \cdot \omega) dt \\ &= -\int_0^\infty \frac{d}{dt}(H(-t)^*\omega) dt \\ &= \omega - p^* i^* \omega.\end{aligned}$$

Let us now prove the Proposition 2.1. From 2.4, it is sufficient to prove that the restriction map $i^* : H^*(N_\epsilon, d_X) \to H^*(M_0)$ is an isomorphism. If ω is a closed form on M_0, $p^*\omega$ is a closed form on N_ϵ. As X^* is a vertical vector field, $c(X^*)p^*\omega = 0$. Thus $p^*\omega$ is an element of $Z(N_\epsilon, d_X)$ restricting to ω.

Now let ω be a form on N_ϵ such that $i^*\omega = 0$ on M_0. Consider $h\omega$; as X^* and H commute, $c(X^*)h = -hc(X^*)$. Thus

$$\begin{aligned}d_X(h\omega) &= d(h\omega) - 2i\pi c(X^*)h\omega \\ &= dh\omega + 2i\pi hc(X^*)\omega \\ &= (dh + hd)\omega \\ &= \omega.\end{aligned}$$

2.7. We will now compute the integral over M of an element $\mu \in Z(M, d_X)$ in terms of its restriction to M_0.

Let $\mu = \Sigma\mu^{[r]}$ be a form on M. If Y is a connected oriented compact submanifold of M, we write $\int_Y \mu$ for $\int_Y \mu^{[\dim Y]}$. If $Y = \cup Y_i$ is a finite disjoint union of compact oriented submanifolds (of possibly different dimensions), we write $\int_Y \mu$ for $\Sigma_i \int_{Y_i} \mu$. If Y is invariant under $\exp tX$ and μ is d_X-closed, $\int_Y \mu$ depends only on the class of μ in $H^*(M, d_X)$.

The normal bundle $N \to M_0$ over M_0 can be provided with a T-invariant complex structure. This may be seen as follows. The infinitesimal transformation J_X acting on $N_x = T_x(M)/T_x(M_0)$ ($x \in M_0$) is invertible. As T is compact, the eigenvalues of J_X on N_x are purely imaginary. For $\lambda \in \mathbf{R}$, define

$$N_x^{i\lambda} = \{v \in N_x \otimes_{\mathbf{R}} \mathbf{C}; J_X v = i\lambda v\}$$

and define

$$N_x^+ = \bigoplus_{\lambda > 0} N_x^{i\lambda}.$$

If a complex structure N^+ on N is chosen, the orientation of M determines an orientation M_0^+ of M_0.

We now state:

2.8 THEOREM. *If* $\mu \in H^*(M, d_X)$

$$\int_M \mu = \int_{M_0^+} (\mu | M_0) \chi(X, N^+)^{-1}.$$

Remark. Recall that the term $\chi(X, N^+)$ is constructed as follows: Choose a T-invariant linear connection D on N^+. Let Ω be its curvature, considered as a section of $\Lambda^2 T^* M_0 \otimes \operatorname{End} N^+$. Let $J_X \in \Gamma(\operatorname{End} N^+)$ be the infinitesimal action of X in N^+. Then $\chi(X, N^+)$ is the class in $H^*(M_0)$ of the form $\det(J_X - \Omega/2i\pi)$. As T is compact, the eigenvalues of J_X on $N_x \otimes_{\mathbf{R}} \mathbf{C}$ are locally constant functions of x. The form $\det(J_X - \Omega/2i\pi)$ is then a closed form on M_0, whose term of degree 0 is the nonvanishing locally constant function $\det J_X$. Thus the class $\chi(X, N^+)$ is invertible in $H^*(M_0)$.

The following proof is a simplification of our original proof, using some ideas in [17].

Consider N_ϵ and identify it to the tubular neighborhood $E(N_\epsilon)$ of M_0 in M. Suppose α_0 is a 1-form on $N - M_0$ invariant by $\exp tX$ and such that $c(X^*)\alpha_0 = 1$. Using a T-invariant partition of unity (φ_0, φ) for the covering $(N_\epsilon, M - M_0)$ and the form α on $M - M_0$ of 2.3, the form $\alpha' = \varphi_0\alpha_0 + \varphi\alpha$ still satisfies the conditions $\mathcal{L}(X^*)\alpha' = 0$, $c(X^*)\alpha' = 1$ and coincides with α_0 in a neighborhood of M_0.

We will construct α_0 as follows: Choose a T-invariant complex structure on N. Choose a T-invariant Hermitian structure on the complex vector bundle $N \to M$. Let D be a T-invariant linear connection on N preserving the Hermitian structure h and let Ω be the curvature of D. If ξ is a tangent vector on N, denote by $\xi^\perp$ its vertical component determined by D. The vector field X_N^* is the vertical vector field given by $(X_N^*)_{(x,v)} = J_X(x) \cdot v$. Let

$$\alpha_0(\xi) = \frac{h(\xi^\perp, X_N^*)}{h(X_N^*, X_N^*)}.$$

Let $\mu \in Z(M, d_X)$. Recall that, by Lemma 2.2, on $M - M_0$,

$$\mu = (d - 2i\pi c(X^*))(\alpha'(d\alpha' - 2i\pi)^{-1}\mu).$$

Thus

$$\mu^{[\dim M]} = d(\alpha'(d\alpha' - 2i\pi)^{-1}\mu)^{[(\dim M)-1]}.$$

Let $\partial(N_\epsilon) = \{(x, v) \in N; \|v\| = \epsilon\}$. By Stokes' theorem:

$$\int_M \mu = \lim_{\epsilon \to 0} \int_{M - N_\epsilon} \mu = -\lim_{\epsilon \to 0} \int_{\partial(N_\epsilon)} \alpha'(d\alpha' - 2i\pi)^{-1}\mu.$$

Let $\mu_0 = \mu | M_0$, $p: N \to M_0$ the projection, then $\mu - p^*(\mu_0)$ is an element of $B(N_\epsilon, d_X)$ (2.1). We therefore have:

$$\int_M \mu = -\lim_{\epsilon \to 0} \int_{\partial(N_\epsilon)} \alpha'(d\alpha' - 2i\pi)^{-1} p^*(\mu_0).$$

For ϵ sufficiently small, α' coincides with α_0 on N_ϵ. Remark that α_0 is invariant by homotheties along the fibers. Let $N^1 = \{(x, v); \|v\| = 1\}$.

The proper map $p^1: N^1 \to M_0$ of oriented manifolds determines a push-forward p_*^1 of forms by integration over the fibers. Theorem 2.8 follows from the equality

$$p_*^1(\alpha_0(2i\pi - d\alpha_0)^{-1}) = \det\left(J_X - \frac{\Omega}{2i\pi}\right)^{-1}.$$

We prove this equality in local coordinates.

Let x_0 be a point of M_0. Consider the Hermitian form h on N. We may construct for x in a small neighborhood U of x_0 in M_0, an orthonormal frame $(e_1^x, e_2^x, \ldots, e_n^x)$ of N_x by parallel transport from an orthonormal basis of N_{x_0}. Let $\langle z, w\rangle$ be the canonical Hermitian form on $\mathbf{C}^n$. Identify locally N with $U \times \mathbf{C}^n$ by $(x, z) \to (x, \Sigma z_i e_i^x)$. The form h becomes $h_x(z, w) = \langle z, w\rangle$ and J_X is identified with a fixed anti-Hermitian matrix J. If θ is the $\mathfrak{su}(n)$-valued 1-form on U corresponding to the connection D in these coordinates, and if (ξ, v) is a tangent vector to N at (x, z), its vertical component is $v + \theta(\xi)(x) \cdot z$. Remark that our frame is chosen such that $\theta_{x_0} = 0$.

Let f be the 1-form on N given by $f(\xi) = h(\xi^\perp, X_N^*)$, Q the function $h(X_N^*, X_N^*)$.

In these coordinates:

$$f = \langle dz + \theta z, J \cdot z\rangle$$

$$Q = \langle Jz, Jz\rangle$$

$$df = -\langle dz, Jdz\rangle + \langle d\theta z, Jz\rangle - \langle \theta dz, Jz\rangle - \langle \theta z, Jdz\rangle.$$

We write, as $\alpha_0 = Q^{-1} f$

$$\alpha_0(2i\pi - d\alpha_0)^{-1} = Q^{-1} f(2i\pi - Q^{-1} df)^{-1}, \text{ as } f^2 = 0$$

$$= f(2i\pi Q - df)^{-1}.$$

At the point x_0, $\theta_{x_0} = 0$ and $\Omega_{x_0} = (d\theta)_{x_0}$. Consider $f(2i\pi Q - df)_{x_0}^{-1}$ as a form on $S^1 = \{(x_0, z); \|z\| = 1\}$ with values in $\Lambda T_{x_0}^* M_0$.

Consider $A = (J - \Omega/2i\pi)_{x_0}$ as a matrix with entries in the commutative ring $\Lambda^{\text{even}} T_{x_0}^* M_0$. We have:

$$(2i\pi Q - df)_{x_0} = \langle 2i\pi Az, Jz\rangle + \langle dz, Jdz\rangle$$

and it remains to prove:

2.9 $$\int_{S^1} \langle dz, Jz\rangle(\langle 2i\pi Az, Jz\rangle + \langle dz, Jdz\rangle)^{-1} = (\det A)^{-1}.$$

The term of degree $\dim S^1 = 2n - 1$ (as a form on S^1) of this expression is

$$(-1)^{n-1}\langle 2i\pi Az, Jz\rangle^{-n}\langle dz, Jz\rangle\langle dz, Jdz\rangle^{n-1}.$$

Now $\langle dz, Jz\rangle\langle dz, Jdz\rangle^{n-1} = \overline{(\det J)}\langle dz, z\rangle\langle dz, dz\rangle^{(n-1)}$ for any Hermitian matrix J as it may be seen by diagonalizing J by a unitary transformation. Setting $B = \bar{J}A$, we need to prove

$$\int_{S^1} (-1)^{(n-1)}\langle 2i\pi Bz, z\rangle^{-n}\langle dz, z\rangle\langle dz, dz\rangle^{n-1} = (\det B)^{-1}.$$

Considering this as an algebraic identity with respect to the entries of the matrix B, it is sufficient to prove this equality when B is a positive definite Hermitian matrix with complex entries.

Let μ be the volume form of the sphere S^1. We have:

$$(-1)^{n-1}\langle dz, z\rangle\langle dz, dz\rangle^{n-1} = \frac{1}{2}(-1)^n(n-1)!(-2i)^n\mu.$$

Thus the preceding equality is true, for $B = 1$, as $\operatorname{vol} S^1 = 2\pi^n/(n-1)!$.

By the change of variables $z \to B^{-1/2}z$, the integral

$$\int_{S^1} (-1)^{(n-1)}\langle 2i\pi Bz, z\rangle^{-n}\langle dz, z\rangle\langle dz, dz\rangle^{(n-1)}$$

is transformed to

$$(\det B)^{-1}\int_{S_B} (-1)^{n-1}\langle 2i\pi z, z\rangle^{-n}\langle dz, z\rangle\langle dz, dz\rangle^{(n-1)}$$

where $S_B = \{z;\ \langle Bz, z\rangle = 1\}$.

Remark that the $(2n-1)$-form $\langle z, z\rangle^{-n}\langle dz, z\rangle\langle dz, dz\rangle^{(n-1)}$ is d-closed. Thus its integral on S_B is equal to its integral on S^1, q.e.d.

Remark. When M_0 consists of isolated points, this proof was given in [8].

2.10. Application to the moment map of a symplectic manifold with a Hamiltonian group action.

Recall the notations of 1.12 to 1.15. Assume that the symplectic manifold M is compact and that the one-parameter group $\exp tX$ is relatively compact. Let M be of dimension $2n$. By 1.15 we can apply Theorem 2.8 to the form

$$\exp\left(f_X - \frac{\omega}{2i\pi}\right) = 1 + \left(f_X - \frac{\omega}{2i\pi}\right) + \frac{1}{2!}\left(f_X - \frac{\omega}{2i\pi}\right)^2 + \cdots$$

the component of degree $2n$ of which is

$$i^n e^{f_X}\frac{\omega^n}{(2\pi)^n n!}.$$

We get the Duistermaat-Heckman formula [16], [17].

$$\int_M e^{f_X}\frac{\omega^n}{(2\pi)^n n!} = \int_{M_0^+}\exp\left(f_X - \frac{\omega}{2i\pi}\right)\chi(X, N^+)^{-1}$$

2.11. When M is a coadjoint orbit $\mathcal{O}$ of a compact Lie group G, the formula above gives Harish-Chandra formula for the Fourier transform of the invariant measure on $\mathcal{O}$. The proof of Theorem 2.8 can be adapted so as to give an analogous formula for a closed orbit of a noncompact semisimple group on the set of elliptic elements of $\mathfrak{g}$ [7]. For a regular elliptic orbit, this formula was obtained by Rossman [24].

III. The equivariant index for the Dirac operator. In this section we present a formula for the equivariant index of the Dirac operator as an integral over M of an equivariant form.

We formulate the result (Theorem 3.18) in the context of a G-equivariant Clifford module, in order to avoid assuming that M admits a spin structure.

In particular, a coadjoint orbit, which is admissible in the sense of Duflo [15], carries always a canonical G-equivariant Clifford module, while not always a spin structure. In this case, our equivariant index formula gives Kirillov's universal formula for the character of the representation of G associated to the orbit (3.21).

Let M be a compact manifold with a G-action. Recall that if (P, α) is a G-equivariant principal bundle with connection α and structure group H, we have associated to any H-invariant polynomial function on $\mathfrak{h}$ a characteristic form $\varphi(X, \alpha)$, whose class $\varphi(X, P)$ does not depend on the choice of the G-equivariant connection α. Denote by $I(\mathfrak{h}^*)$ the algebra of H-invariant polynomial functions on $\mathfrak{h}$. The application $\varphi \to \varphi(X, P)$ is a homomorphism from $I(\mathfrak{h}^*)$ to $H^*(M, d_X)$.

Let $\hat{I}(\mathfrak{h}^*)$ be the algebra of germs of H-invariant analytic functions on $\mathfrak{h}$. If φ is entire, $\varphi(X, \alpha)$ is a form on M, whose coefficients depend analytically on X. If φ has a finite radius of convergence, we can define $\varphi(X, \alpha)$ on any relatively compact open set of M, for X sufficiently small.

Suppose $\mathfrak{h} = \mathfrak{gl}(V)$. We denote by $ch(X, P)$ the class in $H^*(M, d_X)$ associated to the function $A \to \operatorname{tr}(e^A)$.

Let $\mathcal{V} \to M$ be a G-equivariant vector bundle over M with typical fiber V and G-invariant linear connection D. If $R(\mathcal{V})$ is its associated $GL(n)$-principal bundle of frames, we simply denote by $ch(X, \mathcal{V})$ the characteristic class $ch(X, R(\mathcal{V}))$.

3.1 Lemma.

$$ch(X, \mathcal{V}_1 \oplus \mathcal{V}_2) = ch(X, \mathcal{V}_1) + ch(X, \mathcal{V}_2)$$

$$ch(X, \mathcal{V}_1 \otimes \mathcal{V}_2) = ch(X, \mathcal{V}_1)ch(X, \mathcal{V}_2)$$

Let $Q = \Sigma\, a_n z^n$ be an analytic function of one variable z. The function $A = \det Q(A)$ is an analytic function on $\mathfrak{gl}(V)$. Let $(\mathcal{V}_i, \alpha_i)$ $(i = 1, 2)$ be two G-equivariant vector bundles over M with connections. It is clear that $\det Q(X, \mathcal{V}_1 \oplus \mathcal{V}_2) = \det Q(X, \mathcal{V}_1)\det Q(X, \mathcal{V}_2)$.

The function $j(z) = (e^{z/2} - e^{-z/2})/z$ has an analytic square root $j^{1/2}$ in a neighborhood of 0, such that $j^{1/2}(0) = 1$. We define

$$\mathcal{J}^{1/2}(X, \mathcal{V}) = \det j^{1/2}(X, \mathcal{V})$$

$$\mathcal{J}^{-1/2}(X, \mathcal{V}) = \det j^{-1/2}(X, \mathcal{V}).$$

Let us remark now the:

3.2 LEMMA. a) *Let B be a nondegenerate symplectic form on V. Let $Sp(B)$ be the group of symplectic transformations of (V, B) and $\mathfrak{sp}(B)$ its Lie algebra. The function* $\det j(A) = \det((e^{A/2} - e^{-A/2})/A)$ *admits a $Sp(B)$-invariant entire square root $\mathfrak{J}^{1/2}$ on $\mathfrak{sp}(B)$.*

b) *Let Q be a nondegenerate symmetric form on V. Let $O(Q)$ be the group of orthogonal transformations of (V, Q) and $\mathfrak{so}(Q)$ its Lie algebra. The function* $\det j(A) = \det((e^{A/2} - e^{-A/2})/A)$ *admits a $O(Q)$-invariant entire square root $\mathfrak{J}^{1/2}$ on $\mathfrak{so}(Q)$.*

Proof. a) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sp}(B)$. There exists a basis $e_1, e_2, \ldots, e_{2n}$ of V such that

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} a_1 & & & & & & & \\ & a_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & a_n & & & & \\ & & & & -a_1 & & & \\ & 0 & & & & -a_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -a_n \end{pmatrix} ; a_i \in \mathbf{C} \right\}.$$

The Weyl group W of $\mathfrak{h}$ in $Sp(B)$ is generated by the permutations of the a_i and the changes of signs. We have

$$\det j(H) = \left(\prod_{i=1}^{n} \frac{e^{a_i/2} - e^{-a_i/2}}{a_i} \right)^2 .$$

The function

$$\varphi(H) = \prod_{i=1}^{n} \frac{e^{a_i/2} - e^{-a_i/2}}{a_i}$$

is an entire function on $\mathfrak{h}$, invariant by W. Thus, by Chevalley's theorem, φ is the restriction to $\mathfrak{h}$ of an entire function φ on $\mathfrak{sp}(B)$, which is $Sp(B)$-invariant. Clearly φ is a square root of $\det j$, proving a). The proof of b) is entirely similar.

3.3. Suppose that P is the trivial bundle $M \times H$ with G-action $g \cdot (m, h) = (gm, \gamma(g)h)$, for a homomorphism $\gamma : G \to H$. Denote also by γ the corresponding infinitesimal map from $\mathfrak{g}$ to $\mathfrak{h}$. Let α be the flat connection on P, reciprocal image of the Maurer-Cartan form on H. Then α is G-invariant and $\varphi(X, \alpha)$ is the constant function on M equal to $\varphi(\gamma(X))$.

Let $\rho : G \to O(V)$ be an orthogonal representation of G in a real vector space V. Let M be a closed G-invariant submanifold of V. Let $N(M)$ be the normal bundle to M in V, $T(M)$ its tangent bundle. Then $N(M) \oplus T(M) = M \times V$. Consequently:

3.4 $$\mathcal{J}^{1/2}(X, N(M))\mathcal{J}^{1/2}(X, T(M)) = \mathcal{J}^{1/2}(\rho(X)).$$

3.5. If Z is a compact group with Lie algebra $\mathfrak{z}$ acting trivially on M, then for $X \in \mathfrak{z}$, the ring $H^*(M, d_X)$ is the usual cohomology ring of M. Recall the description of the ring $K_Z(M)$ of equivariant K-theory of M [26]. Let $\mathcal{E} \to M$ be a vector bundle over M with trivial action of Z. Let (ρ, V) be a finite dimensional representation of Z. Consider the trivial bundle $\mathcal{V}_\rho = M \times V$ with action $t \cdot (m, v) = (m, \rho(t)v)$. Then the map $\mathcal{E} \otimes \rho \to \mathcal{E} \otimes \mathcal{V}_\rho$ determines an isomorphism of $K(M) \otimes R(Z)$ with $K_Z(M)$. Denote by $u \to u(g)$ the evaluation map from $K_Z(M)$ to $K(M) \otimes_{\mathbf{Z}} \mathbf{C}$ defined by $(\mathcal{E} \otimes \mathcal{V}_\rho)(g) = (\operatorname{tr} \rho(g))\mathcal{E}$. If $X \in \mathfrak{z}$, by Lemma 3.1, $\mathcal{V} \to ch(X, \mathcal{V})$ extends to a character on $K_Z(M)$. By 3.3 we obtain:

3.6 LEMMA. *If* $u \in K_Z(M)$, $X \in \mathfrak{z}$, $ch(X, u) = ch(u(\exp X))$.

3.7. Let V be a complex vector space. Denote $GL(V)/\pm id$ by $DL(V)$. The Lie algebra of $DL(V)$ is $\mathfrak{gl}(V)$. If $g \in GL(V)$, its class $(g, -g)$ in $DL(V)$ is denoted by $\dot{g}$. We denote $GL(n; \mathbf{C})/(\pm id)$ by $DL(n)$. We refer to a principal $DL(n)$-bundle $\mathcal{W}$ as a pseudo vector bundle.

If $\mathcal{V} \to M$ is a vector bundle, its frame bundle $R(\mathcal{V})$ is a principal $GL(n)$-bundle, thus it determines a principal $DL(n)$-bundle $\dot{\mathcal{V}}$. The condition that a pseudo vector bundle $\mathcal{W}$ is obtained from a vector bundle $\mathcal{V}$ is expressed by the vanishing of a class $\epsilon_{\mathcal{W}} \in H^2(M, \mathbf{Z}/2\mathbf{Z})$ described as follows in Cech-cohomology. Let $\{U_i\}$ be a contractible covering of M, $c_{ij}(x)$ continuous lifts to $GL(W)$ of the transition functions $\dot{c}_{ij}(x) \in DL(W)$ of $\mathcal{W}$. Then $\epsilon_{i,j,k}(x)Id = c_{ij}(x)c_{jk}(x)c_{ki}(x)$ is a Cech-cochain representing $\epsilon_{\mathcal{W}}$.

If W_1 and W_2 are vector spaces, the map $(g_1, g_2) \to g_1 \otimes g_2$ gives a homomorphism of $DL(W_1) \times DL(W_2)$ in $DL(W_1 \otimes W_2)$. Thus if $\mathcal{W}_1$, $\mathcal{W}_2$ are two pseudo vector bundles, $\mathcal{W}_1 \otimes \mathcal{W}_2$ defined by this homomorphism, is a pseudo vector bundle, and we have $\epsilon_{\mathcal{W}_1 \otimes \mathcal{W}_2} = \epsilon_{\mathcal{W}_1} \cdot \epsilon_{\mathcal{W}_2}$. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are two pseudo vector bundles with the same class $\epsilon_{\mathcal{W}_1} = \epsilon_{\mathcal{W}_2}$, the pseudo vector bundle $\mathcal{W}_1 \oplus \mathcal{W}_2$ is well defined, and $\epsilon_{(\mathcal{W}_1 + \mathcal{W}_2)} = \epsilon_{\mathcal{W}_1} = \epsilon_{\mathcal{W}_2}$.

3.8. We set notations for the half spin representations.

Let E be a real vector space of even dimension $n = 2\ell$ with a positive definite form Q. We denote by $C(Q)$ or $C(E)$ the Clifford algebra of Q. $C(Q)$ is the quotient of the tensor algebra $T(E \otimes_{\mathbf{R}} \mathbf{C})$ by the ideal generated by the elements $\{x \otimes y + y \otimes x + 2Q(x, y)\}$. Recall that $C(Q)$ has a unique irreducible representation c_0 in a complex vector space S. The space S is called the spinor space and the map $(v, s) \to c_0(v)s$ the Clifford multiplication. Thus, if c is any representation of $C(Q)$ in a vector space V, there exists a vector space W and an isomorphism I between V and $S \otimes W$, transporting the representation c to $c_0 \otimes id$.

Suppose E is oriented and let $e_1, e_2, \ldots, e_{2\ell}$ be an oriented orthonormal basis of E. The element $\alpha = e_1 e_2 \cdots e_{2\ell-1} e_{2\ell}$ of $C(Q)$ is such that $\alpha^2 = (-1)^\ell$, $\alpha e_i = -e_i \alpha$. We define then:

$$S^+ = \{s \in S, c_0(\alpha)s = i^{-\ell} s\}$$

$$S^- = \{s \in S, c_0(\alpha)s = -i^{-\ell} s\}$$

Then:

$$S = S^+ \oplus S^-$$

$$c_0(E)S^+ \subset S^-$$

$$c_0(E)S^- \subset S^+.$$

Let $E = E_1 \oplus E_2$ be an orthogonal sum of oriented euclidean spaces of even dimensions, S, S_1, S_2 the corresponding spinor spaces. Then:

$$S \simeq S_1 \otimes S_2, \quad \text{as} \quad C(E_1) \otimes C(E_2) \text{ modules}$$

$$S^+ \simeq S_1^+ \otimes S_2^+ + S_1^- \otimes S_2^-$$

$$S^- \simeq S_1^- \otimes S_2^+ + S_1^+ \otimes S_2^-.$$

Consider the universal covering group $\mathrm{Spin}(E) \xrightarrow{p} SO(E)$ of $SO(E)$. Let $p^{-1}(e) = (e, \epsilon)$. There exists a representation ρ of $\mathrm{Spin}(E)$ in S such that $\rho(\tilde{g})c_0(x)\rho(\tilde{g})^{-1} = c_0(g \cdot x)$ for all $x \in E$, $\tilde{g}$ in $\mathrm{Spin}(E)$ above g in $SO(E)$. We have $\rho(\epsilon) = -Id$. The spaces S^+ and S^- are invariant under ρ and are called respectively the space of even and odd spinors. We denote by ρ_+, ρ_- the restriction of ρ to S^+, S^-.

As $\rho(\epsilon) = -Id$, the homomorphism $\rho: \mathrm{Spin}(E) \to GL(S)$ defines homomorphisms

$$\dot{\rho}: SO(E) \to DL(S)$$

$$\dot{\rho}_+: SO(E) \to DL(S^+)$$

$$\dot{\rho}_-: SO(E) \to DL(S^-).$$

Consider the $SO(E)$-invariant function on $\mathfrak{so}(E)$ defined by

$$D(X) = \mathrm{tr}\, \rho_+(\exp X) - \mathrm{tr}\, \rho_-(\exp X).$$

We have:

3.9 $$D(X)^2 = (-1)^\ell \det(1 - e^X; E).$$

Suppose that E has an Hermitian structure and that E is oriented accordingly to the complex structure, then if X is an infinitesimally unitary matrix, we have

3.10 $$D(X) = (-1)^\ell \det{}_{\mathbf{C}}(X) \cdot j^{1/2}(X).$$

If $E = \mathbf{R}^n$ with its canonical form, we denote by $SO(n)$, $\mathrm{Spin}(n)$, C_n the corresponding orthogonal, spin group and Clifford algebra.

3.11. Let (M, g) be a Riemannian oriented manifold of dimension $n = 2\ell$. The frame bundle $\mathbf{F}(M) = \mathbf{F}$ of orthogonal oriented frames is a principal $SO(n)$-bundle. We denote by $\mathcal{S}$, $\mathcal{S}^+$, $\mathcal{S}^-$ the corresponding pseudo-bundles $\mathbf{F} \times_{SO(n)} DL(S)$, $\mathbf{F} \times_{SO(n)} DL(S^+)$, $\mathbf{F} \times_{SO(n)} DL(S^-)$, deduced from $\mathbf{F}$ and the homomorphisms $\dot{\rho}$, $\dot{\rho}_+$, $\dot{\rho}_-$. We call $\mathcal{S}$, $\mathcal{S}^+$, $\mathcal{S}^-$ the spin pseudo-bundles. The existence of a spin-bundle $\tilde{\mathcal{S}}$ lifting the spin pseudo-bundles $\mathcal{S}$ (or $\mathcal{S}_+$, $\mathcal{S}_-$) is equivalent to the vanishing of the second

Whitney class $w_2(M)$ of M. Explicitly w_2 is described as follows: Let (r_α) be local sections of the frame bundle $\mathbf{F}$ over a contractible covering U_α. Let $c_{\beta\alpha}(x)$ be the $SO(n)$-valued transitions functions over $U_\alpha \cap U_\beta$ determined by $r_\alpha(x) = r_\beta(x)c_{\beta\alpha}(x)$. Choose a continuous lift $\tilde{c}_{\beta\alpha}(x)$ of $c_{\beta\alpha}(x)$ in Spin(n). Then $w_{\alpha,\beta,\gamma}(x) = \tilde{c}_{\alpha\beta}(x)\tilde{c}_{\beta\gamma}(x)\tilde{c}_{\gamma\alpha}(x)$ takes values in the group $(e, \epsilon) \simeq \mathbf{Z}/2\mathbf{Z}$ and represents w_2. In particular $\epsilon_{\mathcal{S}} = \epsilon_{\mathcal{S}_+} = \epsilon_{\mathcal{S}_-} = w_2(M)$ and $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$.

3.12. Let G be a group acting on M by orientation preserving isometries. The group G acts on $\mathbf{F}$, thus acts on $\mathcal{S}$, $\mathcal{S}^+$, $\mathcal{S}^-$.

3.13. Let $\mathcal{C}(M) \to M$ be the bundle of Clifford algebras $C_x(M) = C(T_xM, g_x)$ over M. A vector bundle $\mathcal{V} \to M$ is said to be a Clifford module, if there exists a bundle map $c: \mathcal{C}(M) \otimes \mathcal{V} \to \mathcal{V}$ such that at each x the map $c_x: C_x(M) \otimes V_x \to V_x$ is a representation of the Clifford algebra $C_x(M)$.

Let G be a group acting on M by orientation preserving isometries. The bundle $\mathcal{C}(M)$ is G-equivariant. Let $\mathcal{V}$ be a G-equivariant vector bundle over M. If there exists a G-equivariant multiplication $c: \mathcal{C}(M) \otimes \mathcal{V} \to \mathcal{V}$, $\mathcal{V}$ will be called a G-equivariant Clifford module.

If $w_2(M) = 0$, there exists a principal bundle $\tilde{\mathbf{F}}$ covering $\mathbf{F}$ with structure group Spin(n). Let $\tilde{\mathcal{S}}$ be the vector bundle associated to the representation ρ of Spin(n) in S. The map $(\xi, v) \to c_0(\xi)v$ from $C_n \otimes S$ to S determines a bundle map $c_0: \mathcal{C}(M) \otimes \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$ which gives to $\tilde{\mathcal{S}}$ the structure of a Clifford module. If $\mathcal{W}$ is any vector bundle on M, consider $\tilde{\mathcal{S}} \otimes \mathcal{W}$ and define a Clifford structure on $\tilde{\mathcal{S}} \otimes \mathcal{W}$ by $c_0 \otimes id: \mathcal{C}(M) \otimes \tilde{\mathcal{S}} \otimes \mathcal{W} \to \tilde{\mathcal{S}} \otimes \mathcal{W}$. It is easy to see that every Clifford module over M is obtained by this construction [3].

3.14. In the general case, where $w_2(M)$ is not necessarily equal to 0, we will see that we can still associate to a Clifford module $\mathcal{V}$ a pseudo vector bundle $\mathcal{W}$, such that $\epsilon_2(\mathcal{W}) = w_2(M)$ and such that $\dot{\mathcal{V}} = \mathcal{W} \otimes \mathcal{S}$. If $\mathcal{V}$ is G-equivariant, so will be $\mathcal{W}$.

We describe $\mathcal{W}$ as follows:

Choose a system of local sections r^i of the frame bundle $\mathbf{F}$, over a contractible covering U_i. These define isomorphisms

$$r_x^i: T_xM \xrightarrow{\sim} \mathbf{R}^n$$

and

$$C_x(M) \rightrightarrows C_n .$$

Thus we obtain a representation c_x^i of C_n in V_x. By uniqueness of the Clifford representation, there exists an integer m and trivializations

$$R_x^i : \mathcal{V}_x \to \mathbf{C}^m \otimes S$$

taking c_x^i to the representation $1 \otimes c_0$. Trivializations r^i and R^i as above will be called compatible trivializations of TM and $\mathcal{V}$.

Let $c_{ji}(x) \in SO(n)$ be the corresponding transition function $r_x^j \circ (r_x^i)^{-1}$ of the tangent bundle. Lift $c_{ji}(x)$ to elements $\tilde{c}_{ji}(x)$ of $\mathrm{Spin}(n)$. The map $R_x^j \circ (R_x^i)^{-1} \circ (1 \otimes \rho(\tilde{c}_{ji}(x)))^{-1}$ commutes with $1 \otimes c_0$. As c_0 is irreducible, this implies that

$$R_x^j \circ (R_x^i)^{-1} = h_{ji}(x) \otimes \rho(\tilde{c}_{ji}(x)).$$

The $GL(m)$-valued maps $h_{ji}(x)$ satisfies the relation:

$$h_{ij}(x) \circ h_{jk}(x) \circ h_{ki}(x) = \epsilon_{ijk}(x) Id = \pm Id.$$

Thus the $DL(m)$-valued maps $\dot{h}_{\sigma\iota}(x)$ define a pseudo vector bundle $\mathcal{W}$ such that $\epsilon_2(\mathcal{W}) = w_2(M)$.

Let us analyze the action of G in $\mathcal{V}$. Consider the action of G on TM. Let $g \in G$ and let $m_{ji}(g, x)$ the $SO(n)$-valued functions defined on the sets $\{(g, x); x \in U_i, g \cdot x \in U_j\}$ by the relation

$$r_{gx}^j \circ g \circ (r_x^i)^{-1} = m_{ji}(g, x).$$

Let $\tilde{m}_{ji}(g, x)$ be a lift of $m_{ji}(g, x)$ to $\mathrm{Spin}(n)$. Using similarly the commutation relation with $1 \otimes c_0$, we see that necessarily the action of g on $\mathcal{V}$ is given by

$$R_{gx}^j \circ g \circ (R_x^i)^{-1} = n_{ji}(g, x) \otimes \rho(\tilde{m}_{ji}(g, x)).$$

The maps $\dot{n}_{ji}(g, x)$ provide then an action of G in the pseudo vector bundle $\mathcal{W}$.

3.15. If $\mathcal{V}$ is a G-equivariant Clifford module, we denote by $s_{-1}(\mathcal{V})$ the G-equivariant pseudo-bundle $\mathcal{W}$ defined by compatible trivializations of $\mathcal{V}$ and TM.

3.16. If $\mathcal{V}$ is a Clifford module over M, we define $\mathcal{V}^+$ and $\mathcal{V}^-$ as follows: Choose an oriented orthogonal basis $e_1, e_2, \ldots, e_{2\ell}$ of T_xM, and consider the transformation

$$\alpha_x = c_x(e_1 \cdot e_2 \cdot \cdots \cdot e_{2\ell}).$$

Define

$$\mathcal{V}^+ = \{v;\ \alpha_x v = i^{-\ell} v\}$$

$$\mathcal{V}^- = \{v;\ \alpha_x v = -i^{-\ell} v\}.$$

As G acts on M by orientation preserving isometries, the bundle $\mathcal{V}^+$ and $\mathcal{V}^-$ are G-equivariant.

Recall how an equivariant Clifford module $\mathcal{V}$ defines an element of equivariant K-theory of TM. Consider the map $\pi: TM \to M$. The map $((x, \xi), v) \to ((x, \xi), c(\xi) \cdot v)$ defines a bundle map from $\pi^*(\mathcal{V}^+)$ to $\pi^*(\mathcal{V}^-)$ which is an isomorphism on $TM - M$. We denote by $d(\mathcal{V})$ the corresponding element of G-equivariant K-theory.

Let $\mathcal{W}$ be the G-equivariant $DL(m)$-principal bundle $s_{-1}(\mathcal{V})$ determined by $\mathcal{V}$. Let $\psi: H^*(M) \to H^*_{cpt}(TM)$ be the Thom isomorphism. Then [6].

3.17 Lemma. $\psi^{-1} ch(d(\mathcal{V})) = (-1)^\ell ch \mathcal{W} \cdot \mathcal{J}^{1/2}(TM)$. We will now prove:

3.18 Theorem. *For X in a neighborhood of* 0 *in* $\mathfrak{g}$

$$(\text{index } d(\mathcal{V}))(\exp X) = \int_M ch(X, \mathcal{W}) \mathcal{J}^{-1/2}(X, TM).$$

Proof. We will deduce this theorem from the localization formula of Atiyah-Segal [4] and from our Theorem (2.8). We need some lemmas.

Suppose Y is a compact Riemannian manifold of dimension $2m$. Let Z be a compact group with Lie algebra $\mathfrak{z}$ acting trivially on Y and let $\mathcal{V}$ be a Z-equivariant Clifford module over Y. Let $\mathcal{W}$ be the pseudo-bundle over Y determined by $\mathcal{V}$.

3.19 LEMMA. *For* $X \in \mathfrak{z}$

$$\psi^{-1} ch(d(\mathcal{V})(\exp X)) = (-1)^m ch(X, \mathcal{W}) \mathcal{J}^{1/2}(TY).$$

Proof. If $\mathcal{V} = \oplus \mathcal{V}(\lambda)$ is the decomposition of $\mathcal{V}$ with respect to the locally constant eigenvalues of $g \in Z$, each of the $\mathcal{V}(\lambda)$ is a Clifford submodule of $\mathcal{V}$. We have

$$ch(d(\mathcal{V}))(g) = \sum_\lambda \lambda(g) ch(d(\mathcal{V}(\lambda)))$$

$$\psi^{-1} ch(d(\mathcal{V}))(g) = \sum_\lambda \lambda(g)(-1)^m ch(\mathcal{W}(\lambda)) \mathcal{J}^{1/2}(TY),$$

if $\mathcal{W}(\lambda)$ are the pseudo bundles $s_{-1}(\mathcal{V}(\lambda))$. But it is clear that $\mathcal{W} \simeq \Sigma$ $\mathcal{W}(\lambda)$ as a sum of pseudo vector bundles and that

$$ch(X, \mathcal{W}) = \Sigma\, \lambda(\exp X) ch(\mathcal{W}(\lambda)).$$

Let us come back to the proof of the Theorem 3.18. Let $g \in G$, T be the closure in G of the group generated by g, M_g be the submanifold of fixed points of g in M. Let $M_g = \cup_\alpha M_g^\alpha$ be the decomposition of M_g in connected components. Suppose that M_g^α is oriented and of even dimension $2\ell_g^\alpha$. Let N be the normal bundle to M_g in M, $\lambda_{-1}N$ the element of $K(M_g)$ defined by $\lambda_{-1}N = \Sigma (-1)^i \Lambda^i N$. Let $i: TM_g \to TM$ be the inclusion. Then we have, by [4], [6]

3.20 index $d(\mathcal{V})(g)$

$$= \sum_\alpha (-1)^{\ell_g^\alpha} \int_{M_g^\alpha} \psi^{-1} ch((i^* d(\mathcal{V}))(g))(ch(\lambda_{-1}N)(g))^{-1} \mathcal{J}^{-1}(TM_g^\alpha).$$

Let us analyze $i^*d(\mathcal{V})$ over a connected component of M_g. Consider the orthogonal decomposition $TM_x = T_x M_g \oplus N_x$. Consider on N_x the orientation determined by the orientation of M and M_g. Let $\dim N_x = 2q$. Let $\alpha_N = c_x(f_1 f_2 \cdots f_{2q-1} f_{2q})$ be the endomorphism of $\mathcal{V}_x$ determined by the choice of an oriented orthonormal basis of N_x. Let

$$(\mathcal{V}_1)_x = \{v \in \mathcal{V}_x;\ \alpha_N \cdot v = i^{-q} v\}$$

$$(\mathcal{V}_2)_x = \{v \in \mathcal{V}_x;\ \alpha_N \cdot v = -i^{-q} v\}.$$

Then $(\mathcal{V}_1)_x$ and $(\mathcal{V}_2)_x$ are stable under the action of $C_x(M_g)$. Thus we have:

$$i^*(\mathcal{V}) = \mathcal{V}_1 \oplus \mathcal{V}_2$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are Clifford modules over M_g. Let $\mathcal{V}_i^{\pm}$ be the sub-bundles of $\mathcal{V}_1$, $\mathcal{V}_2$ determined by the orientation of M_g. Then

$$\mathcal{V}^+ = \mathcal{V}_1^+ \oplus \mathcal{V}_2^-$$

$$\mathcal{V}^- = \mathcal{V}_1^- \oplus \mathcal{V}_2^+$$

and

$$i^*d(\mathcal{V}) = d(\mathcal{V}_1) - d(\mathcal{V}_2) \quad \text{in} \quad K(TM_g).$$

Consider the normal bundle N to M_g; its oriented frame bundle is a principal $SO(2q)$-bundle. Let S_N be the spinor representation of Spin$(2q)$. Let $\mathcal{S}_N^+$, $\mathcal{S}_N^-$ be the pseudo-bundles over M_g determined by the homomorphisms ρ_N^+, ρ_N^- of $SO(2q)$ in $DL(S_N^+)$, $DL(S_N^-)$. It is then easy to see that

$$s_{-1}(\mathcal{V}_1) \simeq \mathcal{W} \otimes \mathcal{S}_N^+$$

$$s_{-1}(\mathcal{V}_2) \simeq \mathcal{W} \otimes \mathcal{S}_N^-.$$

If $g = \exp X$, then for X sufficiently small, the manifold M_g is the manifold M_0 of zeros of X^*.

Let $D(X) = \operatorname{tr} \rho_N^+(\exp X) - \operatorname{tr} \rho_N^-(\exp X)$.

Then, by 3.6

$$\psi^{-1} ch(i^*d(\mathcal{V}))(\exp X) = (-1)^{\ell_g} ch(X, \mathcal{W}) D(X, N) \mathcal{J}^{1/2}(TM_g)$$

$$ch(\lambda_{-1}N)(\exp X) = (-1)^q D(X, N)^2.$$

Choose a T-invariant complex structure on N as in (2.8) then $D(X, N) = (-1)^q \chi(X, N^+) \mathcal{J}^{1/2}(X, N)$. From 3.20 we obtain

$$\operatorname{index} d(\mathcal{V})(\exp X) = \int_{M_g} ch(X, \mathcal{W}) \mathcal{J}^{-1/2}(X, N) \mathcal{J}^{-1/2}(TM_g) \chi(X, N^+)^{-1}.$$

As T acts trivially on M_g, the class $\mathcal{J}^{1/2}(X, N)\mathcal{J}^{1/2}(TM_g)$ is the restriction to M_g of the class of the element $\mathcal{J}^{1/2}(X, TM)$ of $H^*(M, d_X)$. Similarly $ch(X, \mathcal{W})$ is the restriction to M_g of the element $ch(X, \mathcal{W})$ of $H^*(M, d_X)$. Thus the theorem is deduced from the localization formula 2.8.

3.21. Application to the Kirillov character formula. Let $\mathcal{V} \to M$ be a G-equivariant Clifford module, ∇ a G-invariant connection on $\mathcal{V}$. We may then consider the Dirac operator $D = \Sigma\, c(e_i)\nabla_{e_i}$, where $e_1, e_2, \ldots, e_n$ is an orthonormal basis of TM. We denote by $D^{\pm}$ the restriction of D to $\Gamma(\mathcal{V}^+)$, $\Gamma(\mathcal{V}^-)$. The difference $\operatorname{Ker} D^+ - \operatorname{Ker} D^-$ is then a virtual representation of G and the Atiyah-Singer index theorem [5] asserts that

$$\operatorname{tr}_{\operatorname{Ker} D^+}(g) - \operatorname{tr}_{\operatorname{Ker} D^-}(g) = \operatorname{index} d(\mathcal{V})(g).$$

Every irreducible representation of a connected compact Lie group G is obtained by the following construction [13]: Let $\mathcal{O}_\Lambda = G \cdot \Lambda$ be an orbit of the coadjoint representation of G in $\mathfrak{g}^*$. Suppose Λ is admissible and regular, then there exists a canonical Clifford module $\mathcal{V}_\Lambda$ over $\mathcal{O}_\Lambda$ such that the virtual representation $\operatorname{Ker} D^+ - \operatorname{Ker} D^-$ is the irreducible representation T_Λ of G with character

$$\frac{\sum_{w} \epsilon(w) e^{w \cdot \Lambda}}{\prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Suppose M is a submanifold of an orthogonal representation space (ρ, V) of G. Let N_M be the normal bundle to M in V, then by (3.4) we may rewrite 3.18, as

$$\operatorname{index} d(\mathcal{V})(\exp X)\mathcal{J}^{1/2}(\rho(X)) = \int_M ch(X, \mathcal{W})\mathcal{J}^{1/2}(X, N_M)$$

as an equality of entire functions on $\mathfrak{g}$. As the normal bundle to $\mathcal{O}_\Lambda$ in $\mathfrak{g}^*$ is trivial, the formula above reads in this case [10]

$$\operatorname{tr} T_\Lambda(\exp X)\mathcal{J}^{1/2}(\operatorname{ad} X) = \int_{\mathcal{O}_\Lambda} e^{i\langle \xi, X\rangle} d\beta_\Lambda(\xi)$$

which is the Kirillov integral formula for the character of the representation T_Λ.

3.22. The Theorem 3.18 gives us the expression of index $d(\mathcal{V})(g)$ as an analytic function of g near the identity element of G. We will now give a similar formula at every point b of G as an analytic function on the centralizer of b in G.

We introduce characteristic classes adapted to this situation.

Let W be a complex vector space. Consider a semi-simple transformation $B \in GL(W)$. Let $\dot{B}$ be the corresponding element of $DL(W)$. Let D_B be the centralizer of $\dot{B}$ in $DL(W)$. The Lie algebra $\mathfrak{g}_B$ of D_B is the subalgebra of $\mathfrak{gl}(W)$ of matrices X commuting with B.

The functions

$$ch_B(X) = \mathrm{tr}(Be^X)$$

$$\nu_B(X) = \det(1 - Be^X)$$

are D_B-invariant functions on $\mathfrak{g}_B$. Remark that $ch_B(X)$ is determined by $\dot{B}$ up to sign.

Let Z be a compact Lie group with Lie algebra $\mathfrak{z}$ acting on a manifold Y. If $\mathcal{W}$ is a Z-equivariant D_B-principal bundle over Y, we can thus define the equivariant characteristic form $X \to ch_B(X, \mathcal{W})$, on the Lie algebra $\mathfrak{z}$ of Z.

Similarly, let (E, Q) be an oriented Euclidean space of dimension $2q$. Let $A \in SO(E)$ and $\tilde{A} \in \mathrm{Spin}(E)$ be an element above A. Let $SO_A(E)$ be the centralizer of A in $SO(E)$, $\mathfrak{so}_A$ its Lie algebra. Consider the SO_A invariant functions on $\mathfrak{so}_A(E)$ defined by

$$d_{\tilde{A}}(X) = \mathrm{tr}\, \rho_+(\tilde{A} \exp X) - \mathrm{tr}\, \rho_-(\tilde{A} \exp X).$$

The function $d_{\tilde{A}}$ is determined by A up to sign and

$$d_{\tilde{A}}(X)^2 = (-1)^q \det(1 - Ae^X; E).$$

If $\mathcal{W}$ is a Z-equivariant principal bundle over Y with structure group SO_A, we denote by $X \to d_{\tilde{A}}(X, \mathcal{W})$ the corresponding equivariant form.

Such principal bundles arise in our situation as follows: Consider our compact group G acting on (M, g). Let $b \in G$ and let M_b be the submanifold of fixed points of b. Let Z be the connected component of the centralizer of b in G. Consider a pseudo vector bundle $\mathcal{W}$ on M with structure group $DL(W)$. Let M_b^α be a connected component of M_b. The restriction of $\mathcal{W}$ to M_b^α is a pseudo vector bundle over M_b^α which we denote by $\mathcal{W}^\alpha$. Choose a point $p_0 \in \mathcal{W}^\alpha$. As b acts trivially on M_b, there exists an element $B \in GL(W)$ with image $\dot{B} \in DL(W)$ such that $b \cdot p_0 = p_0 \dot{B}$. As the group G is compact, the action of b in $\mathcal{W}^\alpha$ is the same in each fiber, thus the set

$$\mathcal{W}_B^\alpha = \{p \in \mathcal{W}^\alpha,\ b \cdot p = p\dot{B}\}$$

is a D_B-principal bundle over M_b^α which is still Z-equivariant. It gives rise to an equivariant class $X \to ch_B(X, \mathcal{W}_B^\alpha)$ on M_b^α. This class is determined up to sign by the action of b in $\mathcal{W}^\alpha$. To simplify the notations we will drop the subscript α which identifies a connected component of M_b.

Let N be the normal bundle to M_b in M. Suppose that M_b is oriented and of codimension $2q$. The bundle $\mathbf{F}(N)$ of oriented orthonormal frames over M_b is a $SO(2q)$-principal bundle. Let $s_0 \in \mathbf{F}(N)$ and let $A \in SO(2q)$ be such that $b_0 s_0 = s_0 A$. Consider the bundle $\mathbf{F}_A(N) = \{s \in \mathbf{F}(N),\ b \cdot s = sA\}$. This is a SO_A-principal bundle over M_b, which is Z-equivariant. Choose an element $\tilde{A}$ in Spin$(2q)$ above A and consider $d_{\tilde{A}}(X)$. We denote by $X \to d_{\tilde{A}}(X, N)$ the corresponding equivariant characteristic form on M_b. This class is determined by the action of b on N up to sign.

Let $\mathcal{V}$ be a G-equivariant Clifford module over M. Recall that compatible trivializations of the tangent bundle TM and of the bundle $\mathcal{V}$ define a pseudo bundle $\mathcal{W} = s_{-1}(\mathcal{V})$. We will now make a particular choice of the elements $B \in GL(W)$ and $\tilde{A} \in$ Spin$(2q)$ employed in the preceding paragraphs. We suppose that the trivialization $r_x^i: T_x M \to \mathbf{R}^{2\ell}$ is an isomorphism of the direct sum of oriented Euclidean spaces

$$T_x M = N_x \oplus T_x M_b \to \mathbf{R}^{2q} \oplus R^{2(\ell - q)}$$

for $x \in M_b$, and that the action of b on N_x corresponds through r_x^i to the action of A (b acts trivially on $T_x M_b$). Let S be the spinor space over $\mathbf{R}^{2\ell}$, S_N the spinor space over $\mathbf{R}^{2q}$ and S_b the spinor space over $\mathbf{R}^{2(\ell-q)}$. Then $S = S_N \otimes S_b$ as $C(2q) \times C(2(\ell - q))$-modules. Choose a trivialization $R_x: V_x \to W \otimes S$ such that the action of $C(2\ell)$ in V_x becomes $1 \otimes c$. Let $\tilde{A}$ be an element in Spin$(2q)$ above A. The action of b in $V_x \simeq W \otimes S_N \otimes S_b$ is thus given by $B \otimes \rho(\tilde{A}) \otimes 1$ for some $B \in GL(W)$ determined by the choice of $\tilde{A}$. If $\tilde{A}$ is changed to $-\tilde{A}$, B is changed to $-B$, thus the product $ch_B(X, \mathcal{W}) d_{\tilde{A}}(X, N)^{-1}$ is well defined and depends only on the action of b in $\mathcal{V}$. Abusing notations, we denote it by

$$ch_b(X, \mathcal{W})d_b(X, N)^{-1}.$$

The tangent bundle TM_b over M_b is a Z-equivariant vector bundle. Thus the characteristic class $X \to \mathcal{J}^{-1/2}(X, TM_b)$ is well defined on $\mathfrak{z}$. Using these notations, we then formulate the:

3.23 THEOREM. *Let G be a compact group acting on a compact oriented Riemannian manifold M of even dimension by orientation preserving isometries. Let* $\mathcal{V} \to M$ *be a G-equivariant Clifford module over M,* $\mathcal{W} = s_{-1}(\mathcal{V})$ *the G-equivariant pseudo vector bundle determined by* $\mathcal{V}$. *Let* $b \in G$, *Z be the centralizer of b in G,* $\mathfrak{z}$ *its Lie algebra. Suppose the manifold* M_b *of fixed points of b is a oriented submanifold of M. Let N be the normal bundle to* M_b *in M. Then for* $X \in \mathfrak{z}$ *in a small neighborhood of* 0

$$(\text{index } d(\mathcal{V}))(b \exp X) = \int_{M_b} ch_b(X, \mathcal{W}) d_b(X, N)^{-1} \mathcal{J}^{-1/2}(X, TM_b).$$

Proof. It is entirely similar to the proof of the Theorem 3.18. We remark that for $X \in \mathfrak{z}$ small and $g = b \exp X$, the manifold M_g of fixed points of g in M coincides with the set $(M_b)_0$ of zeros of X^* in M_b. The Atiyah-Segal formula for index $d(\mathcal{V})(g)$ as an integral over $M_g = (M_b)_0$ can be transformed to an integral over M_b of elements in $H^*(M_b, d_X)$ by using the Theorem 2.8.

Note added in Proof. It has been called to our attention that the operator d_x was introduced and related to equivariant cohomology in H. Cartau, [Colloque de Topologie, Bruxelles, 1950, Centre Belge de Recherches Mathematiques Georges Thone, Liège].

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