Restrictions and Expansions of Holomorphic Representations

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We compute tensor products of representations of the holomorphic discrete series of a Lie group $G$, or restrictions to some subgroup $G'$. A detailed study is done for the case of the conformal group $O(4,2)$.

**INTRODUCTION**

Let $D_n$ be a homogeneous bounded domain in $\mathbb{C}^n$ containing the origin 0. Let $G_n$ be the group of holomorphic transformations of $D_n$ and let us consider a representation $T_n$ of $G$ inside a Hilbert space $H_n$ of holomorphic functions on $D_n$.

As a simple example, suppose that $D_{n-1}' = \{D_n \cap z_n = 0\}$ is a homogeneous bounded domain in $\mathbb{C}^{n-1}$ for a subgroup $G_{n-1}$ of $G_n$. We consider the restriction of the representation $T_n$ to $G_{n-1}$. Clearly, as the Hilbert space $H_n$ consists of holomorphic functions, the restriction map $R_0 : f(z_1, z_2, \ldots, z_{n-1}, z_n) \mapsto f(z_1, z_2, \ldots, z_{n-1}, 0)$ intertwines the representation $T_n$ with a representation $T_{n-1}$ of $G_{n-1}$ inside a space of holomorphic functions on $D_{n-1}$. The kernel of $R_0$ is the subspace of holomorphic functions in $H_n$ which vanish where $z_n = 0$.

Similarly, the maps $R^p = ((\partial \partial z_n)^p \cdot f) |_{z_n = 0}$ are well defined, therefore it is natural, in order to calculate $T_n |_{G_{n-1}}$, to expand the functions $f$ in $H_n$ in Taylor series with respect to the variable $z_n$.

Let $D = G/K$ be a Hermitian symmetric space. We will consider a representation $T$ of $G$ of the holomorphic discrete series. The preceding simple idea of taking normal derivatives gives us the decomposition of the restriction of $T$ to any

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subgroup $G'$ of $G$ for which $G'/K'$ is a complex submanifold of $G/K$ (Proposition 2.5).

The tensor product of two representations of the holomorphic discrete series is included in this study, as it corresponds to the diagonal submanifold $D' = (z, z) \subset D \times D$ (Corollary 2.6). The idea of considering filtrations according to the order of vanishing along submanifolds is due to Martens [14], who used it to compute the characters of the holomorphic discrete series. We will refer to it as the Martens method.

Sections 3 and 4 deal with matters relevant to theoretical physics. The problems arose in connection with the extension to microphysics of the macrophysical theory of Segal [11]. We have benefited from discussions with B. Orsted, B. Speh, and I. Segal concerning these problems. The physical significance of the restriction question for representations of $O(4, 2)$ to $O(3, 2)$ is indicated in part in our joint note, (Jacobsen et al. [6]) and in part by Segal [11]. The scalar case has been treated by another method in particularly explicit form by Orsted [9], who also considered the analogous question for the action of $O(p, q)$ on $S^{p-1} \times S^{q-1}$. We believe that Martens method shed some light on these matters, for the particular case of the group $O(2, n)$.

In Section 3, we study the modules of $K$-finite functions on the Minkowski-space $\mathbb{R}^{1+n}$ obtained by taking boundary values of holomorphic functions on the associated homogeneous tube domain $\mathbb{R}^{1+n} + iC^+$, where $C^+$ is the solid light cone. These modules are the "positive-energy" subspaces of some of the classical representation of the group $O(2, n + 1)$ acting by conformal transformations on the Minkowski space. We give the decomposition of the positive energy subspace under the Lie algebra of $O(2, n)$. In particular, it is multiplicity free. We also obtain information about the Fourier transforms of the modules.

In Section 4, we study the decomposition of tensor products of unitary representations of $SU(2, 2)$ in reproducing kernel Hilbert spaces of holomorphic functions. Our main interest is in the case in which at least one factor in the tensor product lives in a solution space to a "Mass-zero" equation. Some of the results apply equally well to $SU(n, n)$ and $Mp(n, \mathbb{R})$; however, to complete the analysis some concrete computations are needed, and we only do these for $SU(2, 2)$. We finish this paragraph by observing that certain differential equations appear naturally in the decomposition of such tensor products.

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1. The General Case; Modules

Let $G$ be a Lie group with Lie algebra $g$. Let $G_1$ and $G_2$ be connected subgroups of $G$ with Lie algebras $g_1$ and $g_2$. Let $\tau_1$ be a representation of $g_1$ in a finite-
dimensional vector space $V_{\tau_1}$, let $\mathcal{U}(g)$ be the enveloping algebra of $g$ and let

$$M = M(g, g_1, V_{\tau_1}) = \mathcal{U}(g) \otimes_{\mathcal{U}(g_1)} V_{\tau_1}$$

be the induced module.

We consider $M$ as a $g_2$-module and let $\mathcal{U}_0(g) \subseteq \cdots \subseteq \mathcal{U}_n(g) \subseteq \cdots$ be the canonical filtration of $\mathcal{U}(g)$. Then the modules

$$M_n = \mathcal{U}(g_2) \mathcal{U}_n(g)(1 \otimes V_{\tau_1})$$

clearly form an increasing sequence of $g_2$-modules.

For a vector space $V$ we denote by $S(V) = \bigoplus^n S^n(V)$ the symmetric algebra of $V$ with its canonical gradation. We consider the natural action of $g_1 \cap g_2$ on $g/g_1 + g_2$; with this $S^n(g/g_1 + g_2)$ is a $g_1 \cap g_2$ module.

It is easy to check that the $(\mathcal{U}(g_2)$ module $M_n/M_{n-1}$ is isomorphic to

$$\mathcal{U}(g_2) \otimes_{\mathcal{U}(g_1 \cap g_2)} (S^n(g/g_1 + g_2) \otimes V_{\tau_1})$$

We assume that we are given a representation $\tau_1$ of $G_1$ in the dual space $V'_{\tau_1}$ of $V_{\tau_1}$, whose differential, also denoted by $\tau_1'$, equals the contragredient representation of $\tau_1$. The space $\mathcal{A}(G, G_1, V'_{\tau_1})$ of analytic functions $\varphi: G \to V'_{\tau_1}$ for which, for all $(g, g_1) \in G \times G_1$,

$$\varphi(g g_1) = \tau_1'(g_1)^{-1} \varphi(g)$$

(1.3)

can be identified with the space of analytic sections of the bundle

$$G \times_{G_1} V'_{\tau_1} \to G/G_1.$$  

(1.4)

$G$ acts on $\mathcal{A}(G, G_1, V'_{\tau_1})$ by left translation.

For $x \in g$ we denote by $r(x)$ the differential operator which acts on $C^\infty$-functions $\varphi$ from $G$ to $V'_{\tau_1}$ by

$$(r(x)\varphi)(g) = \frac{d}{dt} \varphi(g \exp tx) \big|_{t=0},$$

(1.5)

and extend $r$ to $\mathcal{U}(g^C)$. Clearly, if $\varphi \in \mathcal{A}(G, G_1, V'_{\tau_1})$ and if $x \in g_1$,

$$(r(x)\varphi)(g) = -\tau_1'(x) \varphi(g).$$

(1.6)

For $u \in \mathcal{U}(g^C)$, $v \in V_{\tau_1}$, and $\varphi \in \mathcal{A}(G, G_1, V'_{\tau_1})$ we now define

$$(u \otimes v, \varphi)(g) = \langle v, (r(u)\varphi)(g) \rangle,$$

(1.7)
and observe that this only depends upon \( u \) and \( v \) through the image \( \xi \) of \( u \otimes v \) in \( M(g, g_1, V_{\tau_1}) \). The corresponding function is denoted by \( (\xi, \varphi)(g) \), and we define

\[
\langle \xi, \varphi \rangle = (\xi, \varphi)(e).
\] (1.8)

If \( G \) is connected and if, for some \( \varphi \) in \( \mathcal{A}(G, G_1, V'_{\tau_1}) \), \( \langle \xi, \varphi \rangle = 0 \) for all \( \xi \) in \( M(g, g_1, V_{\tau_1}) \), then \( \varphi \) is zero, since it is assumed to be analytic.

The group \( G_2 \) acts on \( G/G_1 \), and since

\[
G_2 \cdot e \cong G_2/G_1 \cap G_2,
\] (1.9)

at the point \( e \), the tangent space to the submanifold \( G_2 \cdot e \) of \( G/G_1 \) is \( g_1 + g_2/g_1 \), and hence the complexified normal space is \( (g_1 + g_2)^c = g_1^c + g_2^c \).

The restriction map \( R_0 : \mathcal{A}(G, G_1, V'_{\tau_1}) \to \mathcal{A}(G_2, G_1 \cap G_2, V'_{\tau_1}) \) defined by

\[
(R_0\varphi)(g_2) = \varphi(g_2)
\] (1.10)

clearly commutes with the action of \( G_2 \). Moreover, if \( G_2 \) is connected, it is easy to see that

\[
R_0\varphi = 0 \iff \forall \xi \in M_0 = \mathcal{A}(g_2)(1 \otimes V_{\tau_1}) : \langle \xi, \varphi \rangle = 0.
\] (1.11)

Let \( \mathcal{A}_0 \) be the kernel of \( R_0 \):

\[
\mathcal{A}_0 = \{ \varphi \in \mathcal{A}(G, G_1, V'_{\tau_1}) \mid R_0\varphi = 0 \}.
\] (1.12)

We can then define a map \( R_1 \) from \( \mathcal{A}_0 \) to \( \mathcal{A}(G_2, G_1 \cap G_2, (V_{\tau_2} \otimes (g_1^c + g_2^c)))' \) by

\[
\langle (R_1\varphi)(g_2), v \otimes x \rangle = \langle (r(x)\varphi)(g_2), v \rangle.
\] (1.13)

Here, \( v \in V'_{\tau_2}, x \in (g_1^c + g_2^c), \) and \( x \in g^c \) is chosen to be such that the equivalence class of \( x \) modulo \( g_1^c + g_2^c \) is \( \bar{x} \).

That \( R_1 \) is a well-defined map with values in \( \mathcal{A}(G_2, G_1 \cap G_2, (V_{\tau_2} \otimes (g_1^c + g_2^c)))' \) follows by noting that if \( R_0\varphi \) is zero, then so is \( R_0\varphi(y)\varphi \) for any \( y \) in \( g_1^c + g_2^c \). Also, if \( g_0 \in G_1 \cap G_2 \), then \( \text{Ad}_h(g_0) \) leaves \( g_1^c + g_2^c \) invariant.

\( R_1 \) can be interpreted as the map which associates to each function \( \varphi \) which vanishes on \( G_2 \cdot e \) the derivatives of \( \varphi \) along the "normal directions" to \( G_2 \cdot e \).

This procedure can of course be continued. We let

\[
\mathcal{A}_{n-1} = \{ \varphi \mid \forall \xi \in M_{n-1} : R_0(\xi, \varphi) = 0 \}
\] (1.14)

and define

\[
R_n : \mathcal{A}_{n-1} \to \mathcal{A}(G_2, G_1 \cap G_2, (V_{\tau_1} \otimes S^n(g_1^c + g_2^c)))'
\] (1.15)
by
\[ \langle (R_\nu)(g_2), \psi \otimes b \rangle = \langle (r(b)\nu)(g_2), \psi \rangle \] (1.16)
where \( b \in \text{Hom}(g \to n) \) is such that its image under the map \( \text{Hom}(g \to n) \to S^n(\text{Hom}(g \to n)) \) is \( b \).

We will apply this line of thought to the study of certain modules of holomorphic functions on Hermitian symmetric spaces.

2. The Case of a Hermitian Symmetric Space

Let \( g \) be a semisimple Lie algebra and let \( g = k \oplus p \) be a Cartan decomposition of \( g \). We assume that \( k \) has a nontrivial center and choose a Cartan subalgebra \( k \) contained in \( k \), and a system of positive roots \( \Delta^+ \) of \( (g, k) \) such that
\[ g^c = k^c \oplus p^+ \oplus p^- \] (2.1)
where \( p^+ \subseteq \sum_{a \in \Delta^+} g^a, p^- \subseteq \sum_{a \in \Delta^+} g^a \), the spaces \( p^+ \) and \( p^- \) are abelian subalgebras of \( g^c \), and
\[ [k^c, p^+] \subseteq p^+; \quad [k^c, p^-] \subseteq p^- \] (2.2)

We also define
\[ \Delta_{k^+} = \{ \alpha \in \Delta^+ | g^a \subseteq k^c \} \] (2.3)
\[ \Delta_{p^+} = \{ \alpha \in \Delta^+ | g^a \subseteq p^+ \} \]

The center \( \zeta \) of \( k \) is one dimensional. We choose \( z \in \zeta \) such that
\[ \forall x \in p^+: [z, x] = ix. \] (2.4)

Let \( G^c \) be the simply connected Lie group with Lie algebra \( g^c \), and let \( K^c, G, \) and \( K \) be the connected subgroups corresponding to \( k^c, g, \) and \( k \). The space \( D = G/K \) then has the structure of a Hermitian symmetric space, and the space of holomorphic functions on \( D \) can be identified with the set of analytic functions \( \varphi \) on \( G/K \) for which \( r(x)\varphi = 0 \) for all \( x \) in \( p^- \). We finally recall that any element \( g \in G \) uniquely can be written as
\[ g = \exp X_+(g) \cdot k(g) \cdot \exp X_-(g) \] (2.5)
where \( X_+(g) \in p^+, X_-(g) \in p^-, \) and \( k(g) \in K^c \), and that the map \( g \to X_+(g) \) is an isomorphism of \( D \) onto a bounded domain \( D(p^+) \) contained in \( p^+ \) [3, 4].
We shall use this version of $D$, and observe that the action of $k \in K$ on $D(p^+)$ is given by $\text{Ad}(k)$. In particular, the group $\exp tz$ acts by

$$\exp tz \xi = e^{it} \cdot \xi \quad \text{for} \quad \xi \in D = D(p^+). \quad (2.6)$$

Let $\tau$ be a finite-dimensional irreducible unitary representation of $K$ in $V$. We let

$$\mathcal{O}(G, K, V_\tau) = \{ \text{Analytic functions } \varphi: G \to V_\tau \mid \forall k \in K: \varphi(gk) = \tau(k)^{-1} \varphi(g),$$

and $\forall x \in p^-: r(x)\varphi = 0 \}. \quad (2.7)$$

The group $G$ acts on $\mathcal{O}(G, K, V_\tau)$ by left translation. We let $\mathfrak{h}^c$ act on $V_\tau$ by the extension of the contragredient representation of $\mathfrak{h}$, and let $\mathfrak{p}^-$ act trivially. As before we can then form the module

$$M(g^c, \mathfrak{h}^c \oplus \mathfrak{p}^-, V_\tau) = \mathcal{U}(g^c) \otimes_{\mathcal{U}(\mathfrak{h}^c \oplus \mathfrak{p}^-)} V_\tau \quad (2.8)$$

and we can define, for any $\varphi \in \mathcal{O}(G, K, V_\tau)$ and any $\xi \in M(g^c, \mathfrak{h}^c \oplus \mathfrak{p}^-, V_\tau)$, the function $(\xi, \varphi)(g)$.

If $\varphi \in \mathcal{O}(G, K, V_\tau)$, the function $(P\varphi)(g) = \tau(k(g)) \varphi(g)$ is invariant under right translation by $K$ and hence the map $\varphi \to P\varphi$ is an isomorphism between $\mathcal{O}(G, K, V_\tau)$ and $\mathcal{O}(D, V_\tau)$, the space of holomorphic functions from $D$ to $V_\tau$. The action of $K$ on $\mathcal{O}(D, V_\tau)$ becomes

$$(k \cdot \psi)(\xi) = \tau(k) \cdot \psi(k^{-1} \xi) \quad (2.9)$$

and it is then clear from (2.6) that any finite-dimensional subspace of $\mathcal{O}(D, V_\tau)$, that transforms according to an irreducible representation of $K$, consists of restrictions of polynomials on $p^+$ of a certain homogeneity to $D$. In other words: If $\mathcal{O}'(G, K, V_\tau)$ denotes the subspace of $\mathcal{O}(G, K, V_\tau)$ spanned by the $K$-finite vectors, and if we identify the polynomials on $p^+$ with $S(p^-)$ via the Killing form, then we have that as $K$-modules $S(p^-) \otimes V_\tau$.

Similar to (1.5) we define $l(x)$ for $x \in g$, by

$$l(x)\varphi(g) = \frac{d}{dt} \varphi(\exp - txg) \mid_{t=0}, \quad (2.10)$$

and extend $l$ to $\mathcal{U}(g^c)$.

If we identify $\mathcal{O}'(G, K, V_\tau)$ with the space of polynomials on $p^+$ with values in $V_\tau$, and if, for any $C^\infty$-function $f$ on $p^+$,

$$\left(\hat{\delta}(\nu_0)f\right)(\nu) = \frac{d}{ds} f(\nu + s\nu_0) \mid_{s=0} \quad (2.11)$$
for \( v_0 \) and \( v \) in \( p^+ \), the action of \( l(x) \) for \( x \in g^c \) is given as follows:

If \( x \in p^+ \): 
\[
(l(x) \ p)(v) = -(\delta(x) \ p)(v),
\]
if \( x \in k^c \): 
\[
(l(x) \ p)(v) = d_\tau(x) \ p(v) - (\delta([x, v]) \ p)(v),
\]
and if \( x \in p^- \): 
\[
(l(x) \ p)(v) = d_\tau([x, v]) \ p(v) - \frac{1}{2}(\delta([x, v], v)) \ p(v).
\]

(2.12)

For any \( v \) in \( V_\tau \), the function
\[
(\psi_{\tau}^v)(g) = \tau(k(g))^{-1} \cdot v
\]
belongs to \( \mathcal{O}(G, K, V_\tau) \) and satisfies \( l(\ p^+) \psi_{\tau}^v = 0 \). \( \psi_{\tau}^v \) is defined on the set \( \exp p^+ K^c \exp p^- \). In particular, \( P_{\psi_{\tau}^v} \) is the constant function \( v \rightarrow v \) on \( D \).

We now define a module map \( \Pi : \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes V_\tau \rightarrow \mathcal{O}(G, K, V_\tau) \) by
\[
\Pi(u \otimes v) = l(u) \psi_{\tau}^v,
\]
and let \( W(\tau) \) denote the image of \( \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes V_\tau \) under \( \Pi \). It is easy to see that every \( g \)-submodule of \( \mathcal{O}(G, K, V_\tau) \) contains \( W(\tau) \). In particular, \( W(\tau) \) is irreducible (and hence is the irreducible quotient of \( \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes V_\tau \)). Since \( \mathcal{O}(G, K, V_\tau) \) has the same decomposition under \( K \) as \( \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes V_\tau \), the preceding remark also shows that \( \mathcal{O}(G, K, V_\tau) \) is isomorphic to \( \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes V_\tau \) if and only if the latter is irreducible.

Consider the following subsets of \( \hat{K} \):
\[
I = \left\{ \tau \in \hat{K} \mid \mathcal{U}(g^c) \otimes \mathcal{U}(k^c \oplus p^+) \otimes \mathcal{U}(g^c) \otimes V_\tau \text{ is irreducible} \right\},
\]
and
\[
P = \left\{ \tau \in \hat{K} \mid W(\tau) \text{ is unitarizable} \right\}.
\]

We shall need the following result:

PROPOSITION 2.1 [3]. Let \( \tau \in P \). Then there exists a reproducing kernel Hilbert space \( H(\tau) \) such that \( W(\tau) \subseteq H(\tau) \subseteq \mathcal{O}(G, K, V_\tau) \) and \( H(\tau) \) is the completion of \( W(\tau) \). Moreover, if \( H \) is a Hilbert space contained in \( \mathcal{O}(G, K, V_\tau) \) on which \( G \) acts unitarily, and if the evaluation map \( \psi \rightarrow \psi(\tau) \) is a continuous map from \( H \) to \( \mathbb{C} \), then \( H = H(\tau) \). To be precise: As sets, \( H \) and \( H(\tau) \) are equal, and the Hilbert space structures are proportional.

If \( \tau \in P \) we let \( T_\tau \) denote the corresponding unitary representation of \( G \) in \( H(\tau) \). Finally we observe that the set \( P \) is not known apart from some special cases [8, 10, 13].

Let \( g_1 \) be a semisimple subalgebra of \( g \) such that \( g_1 = g_1 \cap k \oplus g_1 \cap p \) is a Cartan decomposition of \( g_1 \). Assume that \( p_1^c = p_1^c \cap p^+ \oplus p_1^c \cap p^- \). We let
$G_1$ and $K_1$ be the connected subgroups of $G$ corresponding to $g_1$ and $k_1 = g_1 \cap k$, respectively. By the above assumptions $D_1 = G_1/K_1$ is a Hermitian symmetric space, and in the Harish-Chandra realization $D_1 \subset D \subset \mathfrak{p}^+$, and $D_1 = D \cap \mathfrak{p}_1^+$. As before we identify $\mathcal{O}(G, K, V_r)$ with the space of all polynomials on $\mathfrak{p}^+$.

Since $\mathfrak{p}_1^+ \subseteq \mathfrak{p}^+$ we have that $\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}^{'+}$ where $\mathfrak{p}^{'+}$ is the complement of $\mathfrak{p}_1^+$ in $\mathfrak{p}^+$. Corresponding to this we let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ be a set of coordinates on $\mathfrak{p}^+$, with $x \in \mathfrak{p}_1^+$, and $y \in \mathfrak{p}^{'+}$.

Intuitively speaking we shall decompose $G$-modules of holomorphic functions on $D$ under $G_1$ by expressing the corresponding functions $\varphi(x, y)$ by Taylor series;

$$\varphi(x, y) = \varphi(x, 0) + \sum_a y^a \varphi_a(x).$$

In this spirit we define subspaces $\mathcal{C}_r$ of $\mathcal{O}(G, K, V_r)$ for $r = 0, 1, 2, \ldots$, by

$$\mathcal{C}_r = \text{span}\{P_a(x_1, \ldots, x_p, y_1, \ldots, y_q) \mid P_a \text{ is a polynomial on } \mathfrak{p}^+ \text{ with values in } V_r, \text{ and } |\alpha| = \alpha_1 + \cdots + \alpha_q \geq r\}.$$  

These subspaces $\mathcal{C}_r$ are in fact $\mathcal{U}(g_1)$-modules as can be seen from (3.12): If $x \in \mathfrak{p}_1^-$ then $\delta(x)$ only differentiates the polynomials through the $x$-variables, and if $x \in k_1^C$ $d\tau(x)$ is a linear operator that leaves both $\mathfrak{p}_1^+$ and $\mathfrak{p}^{'+}$ invariant whereas $\delta(x, v)$ splits into two terms, one of which is as above, and one which is a differential operator with first order polynomials in the $y$-variables as coefficients. Finally, if $x \in \mathfrak{p}_1^-$, one can similarly see that $\ell(x)$ leaves $\mathcal{C}_r$ invariant.

We have $g_1^C = k_1^C \oplus \mathfrak{p}_1^+ \oplus \mathfrak{p}_1^-$ with $k_1^C = k^C \cap g_1^C$, $\mathfrak{p}_1^+ = \mathfrak{p}^+ \cap g_1^C$, and $\mathfrak{p}_1^- = \mathfrak{p}^- \cap g_1^C$. Hence, with $b^C = b^C \oplus \mathfrak{p}^-$,

$$g_1^C + b^C = k_1^C \oplus \mathfrak{p}_1^+ \oplus \mathfrak{p}_1^-, \quad \text{and} \quad g_1^C \cap b^C = k_1^C \oplus \mathfrak{p}_1^-.$$  

Thus, $g_1^C/g_1^C \overset{\sim}{=} \mathfrak{p}^+/[\mathfrak{p}_1^+]$ as representation spaces for $k_1^C \oplus \mathfrak{p}_1^-$ where $\mathfrak{p}_1^-$ acts trivially, and $k_1^C$ by the natural representation. $\mathfrak{p}^+/[\mathfrak{p}_1^+]$ can be viewed as the space of holomorphic vector fields normal to the domain $D_1 \subset D$ at the point $\varepsilon$.

The preceding analysis then leads to

**Proposition 2.2.** $\mathcal{O}(G, K, V_r)$ has a filtration by $\mathcal{U}(g_1)$-submodules $\mathcal{C}_r$ ($r = 0, 1, 2, \ldots$). The quotients $\mathcal{C}_r/\mathcal{C}_{r+1}$ are canonically isomorphic to $\mathcal{O}(g_1, K_1, \tau \otimes (S^r(\mathfrak{p}^+/[\mathfrak{p}_1^+]))$.

**Corollary 2.3.** Suppose that the representations $\mu$ of $K_1$ occurring in the decomposition of $\tau \otimes (S^r(\mathfrak{p}^+/[\mathfrak{p}_1^+]))$, for every $r$, all are in the set $I$, for $K_1$ (cf. (2.15)). Then the $\mathcal{U}(g_1)$-module $\mathcal{O}(G, K, V_r)$ splits into a direct sum.
Proof. The functions $w = y_1^{a_1} \cdots y_n^{a_n} \cdot v (v \in V_\tau)$ all satisfy: $\ell(p_+^+)w = 0$. Hence we can send $\mathcal{H}(g_1 c) \otimes \mathcal{H}(g_2 c \otimes p_+^+) (V_\tau \otimes (S^n(p_+^+/p_+^+)''))$ into $\mathcal{O}_\tau$. If the hypothesis of the corollary is satisfied this map is an isomorphism into a complement of $\mathcal{O}_{\tau+1}$.

We now turn to the case in which $G$ acts unitarily in a Hilbert space $H(\tau) \subseteq \mathcal{O}(G, K, V_\tau)$ in which the point-evaluation maps are continuous, and consider the decomposition of $H(\tau)$ under $G_1$. We proceed as in Section 1 and let

$$H = H(\tau),$$

and

$$H_0 = \{ \varphi \in H(\tau) \mid \varphi|_{\zeta_1} = 0 \}. \quad (2.18)$$

$H_0$ is the kernel of the map $R_0: H \rightarrow \mathcal{O}(G, K, V_\tau)$ defined by $(R_0\varphi)(g_1) = \varphi(g_1)$. (We shall continue to use the notation $R_n$ for the maps corresponding to (1.15), even though the maps here in general are defined on different spaces.) Since $H_0$ clearly is a closed $G_1$-invariant subspace of $H$, $H/H_0$, and hence $R_0(H)$, has a canonical Hilbert space structure. It is easy to see that $R_0(H) = H(G_1, K_1, V_\tau)$ is a space of functions on $G_1$ in which the point-evaluation maps are continuous.

Let $\tau|_{K_1} = \bigoplus_{\mu_j} \mu_j$ be the decomposition of $\tau|_{K_1}$ into irreducible pieces. Then canonically $\mathcal{O}(G_1, K_1, V_\tau) = \bigoplus_{\mu_j} \mathcal{O}(G_1, K_1, \mu_j)$. Since $\psi_i^\sigma(e) = v$ for $v \in V_\tau$, we see that $R_0(H)$ intersects each $\mathcal{O}(G_1, K_1, \mu_j)$ nontrivially, and it then follows that

$$R_0(H) = H(G_1, K_1, V_\tau) = \bigoplus_{\mu_j} H(G_1, K_1, V_\mu_j). \quad (2.19)$$

The next step is to consider the map $R_1$ (cf. (1.13)) from $H_0$ to $\mathcal{O}(G_1, K_1, V_\tau \otimes (p_+^+/p_+^+)')$. In the present setting, the kernel of $R_1$ is seen to be

$$H_1 = \{ \varphi \in H_0 \mid x \in g^c: R_0\varphi(x)\varphi = 0 \}. \quad (2.20)$$

We recall: If a sequence of holomorphic functions $\{f_n\}$ converges to a function $f$ in some region $\Omega$, uniformly on compact sets, then $f$ is holomorphic in $\Omega$, and $\{f'_n\}$ converges to $f'$, uniformly on compact sets (Weierstrass). From this it is easy to see that $H_1$ is a closed subspace of $H_0$, that the point-evaluation maps are continuous on $R_1(H_0)$, equipped with the Hilbert space structure from $H_0/H_1$, and that similar facts are true for all the following steps.

It is then clear that $H_0/H_{n+1}$ can be identified with a subspace of $H(G_1, K_1, V_\tau \otimes (S^n(p_+^+/p_+^+)''))$. This subspace will in general be proper (see Sections 3 and 4). However, we do have

**Proposition 2.5.** Let $\tau \in I \cap P$. Then $H(\tau)$ under the action of $G_1$ breaks into

$$H(\tau)|_{G_1} = \bigoplus_{n=0}^\infty H(G_1, K_1, V_\tau \otimes (S^n(p_+^+/p_+^+)'')).$$
Furthermore, the modules on the right-hand side are all finite sums of \(H(\mu_i)\)'s where \(\mu_i \in I_1 \cap P_1 \subset K_1\).

Proof. \(H_n = \{ \varphi \in H(\tau) \mid R_0 \varphi = 0 \text{ and } R_0^r(x^r) \varphi = 0 \text{ for all } r(x^r) = r(x_1) \cdots r(x_n) \text{ with } x_i \in g^\alpha, \text{ } i = 1, \ldots, n \alpha \leq n \}.\) We must prove that the map \(R_{n+1}: H_n \to H(G_1, K_1, V_\tau \otimes (S^{n+1}(p^+/p_1^+))')\) is surjective. For this, it is sufficient to prove that the map from \(H_n\) to \(V_\tau \otimes (S^{n+1}(p^+/p_1^+))'\),

\[
\varphi \mapsto (R_{n+1}\varphi)(e) \to V_\tau \otimes (S^{n+1}(p^+/p_1^+))',
\]

is surjective. That is: for any \(v \in V_\tau\) and any set \(\{x_1, \ldots, x_{n+1}\} \subset p^+\) there must exist a \(\varphi \in H_n\) such that

\[
(r(x_1) \cdots r(x_{n+1})\varphi)(e) = v.
\]

In the given case we know that the space of \(K\)-finite vectors in \(H(\tau)\) equals the space of polynomials on \(D\). Hence functions of the form \(\varphi(g) = \tau(k(g))^{-1} P(X_+(g))\), where \(P\) is any polynomial on \(D\), are all in \(H(\tau)\), and we have

\[
(r(x_1) \cdots r(x_{n+1})\varphi)(e) = \frac{d}{ds_1} \cdots \frac{d}{ds_{n+1}} \varphi(\exp s_1 x_1 \cdots \exp s_{n+1} x_{n+1}) \big|_{s_1 = \cdots = s_{n+1} = 0} = \frac{d}{ds_1} \cdots \frac{d}{ds_{n+1}} P(s_1 x_1 + \cdots + s_{n+1} x_{n+1}) \big|_{s_1 = \cdots = s_{n+1} = 0}.
\]

This completes the proof since evidently \(P\) can be chosen such that the last expression equals \(v\).

Corollary 2.6. Let \(G\) be as before, and let \(\tau_i \in I \cap P\) for \(i = 1, 2\). Then

\[
H(\tau_1) \otimes H(\tau_2) = \bigoplus_{n=0}^\infty H(\tau_1 \otimes \tau_2 \otimes S^n(p^-)).
\]

Proof. Imbed \(G\) in \(G \times G\) by the diagonal map and apply Proposition 2.5 with \(G \equiv G \times G\) and \(G_1 \equiv G\). (Comment: \(\tau_1 \otimes \tau_2 \in I \cap P \subset K_{G \times G}\).)

We shall now leave the general theory and turn to some special cases particularly important to physics. For these we shall also consider elements \(\tau\) of \(K\) that do not belong to \(P\). Typically, in this case \(H(\tau)\) will consist of solutions to certain differential equations, and the restriction to \(G_1\) will then correspond to fixing Cauchy data on \(D_1\). It is then clear that some of the spaces \(H_n\) may be rather small, if not zero, and that \(H(\tau)\) in some cases will equal the direct sum of only finitely many irreducible representations of \(G_1\).
3. EXAMPLE: THE MASS ZERO EQUATIONS AND $O(2, n + 1)$

Consider $\mathbb{R}^{2n+1}$ with basis $(e_{-1}, e_0, e_1, \ldots, e_{n+1})$ and let $q(x, x)$ denote the quadratic form which, in the given basis, is given by

$$q(x, x) = x_{-1}^2 + x_0^2 - \sum_{i=1}^{n+1} x_i^2.$$  \hfill (3.1)

Let $O(2, n + 1)$ be the group of linear transformations of $\mathbb{R}^{2n+1}$ leaving $q$ invariant, $G_{n+1}$ its connected component containing the identity, and let $g_{n+1}$ denote the Lie algebra of $G_{n+1}$. The maximal compact subgroup $K_{n+1}$ of $G_{n+1}$ is isomorphic to $SO(2) \times SO(n + 1)$, and any irreducible representation $\tau$ of $K_{n+1}$ is then of the form $\tau = (\alpha, \mu)$, where the integer $\alpha$ denotes the representation $\exp(-i\alpha \theta \mathbf{1})$, and $\mu$ is an irreducible representation of $SO(n + 1)$ in $V_{\tau}$. We let $z$ denote the element of $g_{n+1}$ which is the generator of the subgroup $\exp(-i\alpha \theta \mathbf{1})$, and consider, as above, the representation $T_{\tau} = T_{\alpha, \mu}$ of $G_{n+1}$ in $\mathfrak{g}(G_{n+1}, K_{n+1}, V_{\tau})$. Let us denote this module by $\mathcal{O}(\tau, \alpha, n + 1)$.

The group $G_n$ is imbedded in $G_{n+1}$ by extending its action on $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n+1}$ in the obvious way. The preceding analysis then shows that the space of $K$-finite vectors $\mathcal{O}(\alpha, \mu, n + 1)$ as a $\mathfrak{g}(G_{n+1})$-module has a filtration for which the set of composition factors is in a one-to-one onto correspondence by isomorphisms to the set

$$\{\mathcal{O}(\alpha + i, \mu, n + 1) \mid i = 0, 1, 2, \ldots \text{ and } r = 1, 2, \ldots, l\},$$

where $\mu_1 \oplus \cdots \oplus \mu_l$ is the decomposition of $\mu$'s restriction to $SO(n)$ into irreducible representations. As is well known this decomposition is multiplicity free, and it follows that no pair of composition factors are isomorphic as $g_n$-modules.

We obtain in particular the following as a corollary.

**COROLLARY 3.1.** Let $\tau \in P$. The restriction of the representation $T_{\tau}$ of $G_{n+1}$ in $H(\tau)$ to $G_n$ is a direct sum of representations, and is multiplicity free.

We shall analyze the restriction of $T_{\tau}$ to $G_n$ further for particular representations.

Let $M = \mathbb{R}^{1+n}$ denote Minkowski space for $n + 1$-dimensional space-time. Let $(e_0, e_1, \ldots, e_n)$ be a basis and let

$$x \cdot x = x_0^2 - x_1^2 - \cdots - x_n^2.$$  \hfill (3.2)

The group $G_{n+1}$ acts on $M$ by (locally defined) conformal transformations. We recall that a space of solutions to the wave equation can be equipped with a
Hilbert space structure in which $G_{n+1}$ acts unitarily [7, 10]. Let us recall the precise construction involved in the above statement.

Consider the quadratic form $q'$ on $\mathbb{R}^{2+(n+1)}$ given by

$$q'(t) = t_0^2 - t_1^2 - \cdots - t_n^2 + t_{-1}t_{n+1}.$$ (3.3)

This form has signature $(2, n+1)$ and the linear group that leaves $q'$ invariant is then $O(2, n+1)$. We extend $q'$ to $\mathbb{C}^{2+(n+1)}$ by linearity and consider in the complex projective space the open submanifold

$$D_{n+1} = \{ \mathbf{C} \cdot \mathbf{v} | \mathbf{v} \in \mathbb{C}^{2+(n+1)}, q' (\mathbf{v}, \mathbf{v}) = 0, \text{ and } q'(\mathbf{v}, \overline{\mathbf{v}}) > 0 \}. \quad (3.4)$$

If $\mathbf{v} = z_0e_{-1} + z_0\overline{e}_0 + \cdots + z_{n+1}e_{n+1}$ then $\mathbf{C} \cdot \mathbf{v}$ is in $D_{n+1}$ if and only if

$$z_{-1}z_{n+1} + z \cdot z = 0,$$ (3.5a)

$$\frac{1}{2}(z_{-1}\overline{z}_{n+1} + \overline{z}_{-1}z_{n+1}) + |z_0|^2 - |z_1|^2 - \cdots - |z_n|^2 > 0 \quad (3.5b)$$

where $z \cdot z = z_0^2 - z_1^2 - \cdots - z_n^2 = z^2$.

These equations imply that $z_{-1}$ (as well as $z_{n+1}$) is different from zero, and we can then normalize $\mathbf{v}$ such that $z_{-1} = 1$. If $\mathbf{v}$ is normalized we let

$$T(\mathbf{C} \cdot \mathbf{v}) = T(\mathbf{v}) = (x_0, x_1, \ldots, x_n)$$ (3.6)

and can then parametrize $D$ by the subset $\Omega_{n+1} = T(D_{n+1}) \subseteq \mathbb{C}^{n+1}$. Expressed on $\Omega_{n+1}$ Eq. (3.5b) becomes

$$(\text{Im } z_0)^2 > (\text{Im } x_1)^2 + \cdots + (\text{Im } z_n)^2. \quad (3.7)$$

$\Omega_{n+1}$ is a domain with two connected components $\Omega_{n+1}^+$ and $\Omega_{n+1}^-$ (corresponding to $\text{Im } z_0 \leq 0$). We consider the domain $\Omega_{n+1}^+$. Its Shilov boundary is the set

$$\{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} | \text{Im } z_i = 0, \ i = 0, 1, \ldots, n \} = \mathbb{R}^{n+1} = M.$$ Finally, if $(x_0, x_1, \ldots, x_n)$ is in $\Omega_{n+1}$, we let

$$v(z_0, z_1, \ldots, z_n) = e_{-1} + z_0e_0 + \cdots + z_ne_n - (z \cdot z)e_{n+1}.$$ (3.8)

The group $G_{n+1}$ acts on $\mathbb{C}^{2+(n+1)}$ by $\mathbf{v} \rightarrow g \cdot \mathbf{v}$. If $v \rightarrow \overline{v}$ denotes the natural map from $\mathbb{C}^{2+(n+1)}$ onto the projective space $\mathbb{P}^{2+(n+1)}$ then $g \cdot \overline{\mathbf{v}} = \overline{g \cdot \mathbf{v}}$ defines an action of $G_{n+1}$ on this space which clearly preserves $D_{n+1}$, and which in fact, since $G_{n+1}$ is connected, leaves both connected components, $D_{n+1}^-$ and $D_{n+1}^+$, of $D_{n+1}$ invariant.
We now define a function $j$ on $\mathbb{C}^{2(n+1)}$ by, if $v = x_{-1}e_{-1} + \cdots + x_{n+1}e_{n}$,

$$j(v) = x_{-1}. \tag{3.9}$$

The function

$$j(g, \bar{v}) = j(g \cdot v)j(v)^{-1} \tag{3.10}$$

is then well defined on $G_{n+1} \times D_{n+1}$ and there satisfies

$$j(g_1g_2, \bar{v}) = j(g_1, g_2\bar{v})j(g_2, \bar{v}). \tag{3.11}$$

This means that we can define a family of representations $T_\alpha$ of $G_{n+1}$ on $\mathcal{O}(D_{n+1})$ (or $\mathcal{O}(D_{n+1}^+)$) by

$$(T_\alpha(g)f)(\bar{v}) = j(g^{-1}, \bar{v})^{-1}f(g^{-1}\bar{v}) \tag{3.12}$$

for $\alpha \in \mathbb{Z}$.

As in Section 2, we denote by $\mathcal{O}'(\alpha; n+1)$ the space of $K$-finite vectors of the representation $T_\alpha$ of the group $G_{n+1}$ acting on $\mathcal{O}(D_{n+1})$, and by $W(\alpha; n+1)$ the minimal submodule of $\mathcal{O}'(\alpha; n+1)$. We shall below realize these last modules as modules of functions on $\Omega_{n+1}^+$, the unbounded realization of $G/K$.

We consider the function $R$ on $\Omega_{n+1}^+ \times \Omega_{n+1}^+$,

$$R(z, z') = g'(v(z), \overline{v(z')}) = -\frac{1}{2}(z - z')^2, \tag{3.13}$$

and define

$$\widetilde{j}(g, z) = j(g, \overline{v(z)}). \tag{3.14}$$

Since by definition $g \cdot v(z) = \overline{v(gz)}$, $g \cdot v(z) = j(g \cdot v(z)) v(gz)$. Hence

$$R(gz, gz') = j(g, z)^{-1}R(z, z')j(g, z')^{-1}. \tag{3.15}$$

The stabilizer $K$ of $i = (i, 0, 0, \ldots, 0) \in \Omega_{n+1}$ is isomorphic to $SO(2) \times SO(n+1)$. Hence under $T_\alpha$'s restriction to $K$ the function $z \mapsto K(z, i)^{-\alpha}$ on $\Omega_{n+1}^+$ transforms by a character of $K$, and we obtain the irreducible module $W(\alpha; n+1)$ by letting $\mathcal{W}(g_{n+1})$ act on it.

If $\alpha \geq (n - 1)/2$, it is known that $\alpha \in P$ and if $\alpha > (n - 1)/2, \alpha \in I \cap P$. We denote in this case the representation of $G_{n+1}$ in the space $H(\alpha; n+1)$ of holomorphic functions on $\Omega_{n+1}^+$ by $T(\alpha; n+1)$.

We are interested in a study of the value $\alpha_0 = (n - 1)/2$; in this case we have the following inclusion: $W(\alpha_0; n+1) \subset \mathcal{O}'(\alpha_0; n+1)$. We will study the decomposition under $g_n$ of both of these modules.
Let us consider the wave operator

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$  \hspace{1cm} (3.16)$$
on O_{n+1}^{+}$. Since (with $z^a$ as in (3.5))

$$\Box(z^a)^{-\alpha} = -2\alpha(n - 1 - 2\alpha)(z^a)^{-\alpha - 1};$$  \hspace{1cm} (3.17)$$

$(z^a)^{-\alpha}$, for $\alpha = (n - 1)/2$, is a solution to the wave equation. Moreover, the module spanned by the function $k(z, i)^{-(n-1)/2} = (-h(z + i)^2)^{-(n-1)/2}$ under the action of $\mathcal{U}(g_{n+1})$ consists of all the $K$-finite functions $\varphi$ on $\mathbf{O}_{n+1}^{+}$ that are solutions to $\Box \varphi = 0$. We let $H(\frac{n-1}{2})$ denote the corresponding Hilbert space of holomorphic functions on $\mathbf{O}_{n+1}^{+}$, and let $U_{n-1/2}$ denote the representation of $G_{n+1}$ in $H(\frac{n-1}{2})$.

**Proposition 3.2.** The restriction of $U_{(n-1)/2}$ to $G_{n}$ is the direct sum of the representations $T(\frac{n-1}{2}, n)$ and $T(\frac{n+1}{2}, n)$.

**Proof.** Consider the filtration of $H(\frac{n-1}{2})$ by the closed subspaces

$$H\left(\frac{n-1}{2}, r\right) = \left\{ \varphi \in H\left(\frac{n-1}{2}\right) \middle| \left(\frac{\partial}{\partial z_n}\right)^{\beta} \varphi \right|_{z_n=0} = 0 \text{ for } 0 \leq \beta \leq r \right\}.$$  

These spaces are invariant under $G_{n}$, and clearly, for $r \geq 1$, $H(\frac{n-1}{2})$, is zero since if $\varphi$, as well as $\partial \varphi/\partial z_n$, is zero on the subset $z_n = 0$ and if $\Box \varphi = 0$, then $\varphi$ is zero everywhere. It thus only remains to be seen that there exists a $K$-finite function in $H(\frac{n-1}{2})$ which vanishes on the set $z_n = 0$, but this follows since the operator $\partial/\partial z_n$ on holomorphic functions equals $\partial/\partial x_n$ where $z_n = x_n + iy_n$, and the latter comes from the subgroup of translations on Minkowski space. Thus $(\partial/\partial z_n)((z + i)^2)^{-(n-1)/2}$ belongs to $H(\frac{n-1}{2})$ and is clearly proportional to $z_n((z + i)^2)^{-(n+1)/2}$.

The space of restrictions to Minkowski space of the span of all $K$-finite vectors of the representations $T_{\alpha}$, $\alpha \geq \frac{n-1}{2}$, is a space of analytic functions. We denote the restriction map by $RM$. The subspace $RM(\mathcal{U}(\frac{n-1}{2}); n + 1)$ is the positive energy subspace consider in [6] for $n = 3$.

The preceding analysis leads to

**Proposition 3.3.** $\mathcal{U}(\frac{n-1}{2}, n + 1)$ on $\mathbf{O}_{n+1}^{+}$ splits under $\mathcal{U}(g_{n})$ into a direct sum of the modules $W(\frac{n-1}{2} + \lambda, n)$, $\lambda = 0, 1, 2, \ldots$. Each of these submodules is unitarizable as a $g_{n}$-module. The highest weight vector in $W(\frac{n-1}{2} + \lambda, n)$ under the action of $g_{n}$ is the function $z_n((z + i)^2)^{-(\lambda + (n-1)/2)}$.

**Proof.** We have seen that $\mathcal{U}(\frac{n-1}{2}, n + 1)$ on $\mathbf{O}_{n+1}^{+}$ has a composition series given by the modules $\mathcal{U}(\frac{n-1}{2} + \lambda, n)$ with respect to $g_{n}$. But if $\alpha = \frac{n-1}{2} + \lambda$ with
\( \lambda \geq 0 \), then clearly \( \alpha > \frac{(n-1)-1}{2} \), where \( \frac{(n-1)-1}{2} \) is the parameter for the wave equation for \( G_n \). Hence \( \mathcal{O}'(\frac{n-1}{2} + \lambda, n) = \mathcal{O}(\frac{n-1}{2} + \lambda, n) \), and the conclusion now follows from Corollary 2.4.

**Corollary 3.4.** Let \( \varphi \in \text{RM}(\mathcal{O}'(\frac{n-1}{2}, n + 1)) \). Then, considered as a distribution, the Fourier transform \( \varphi \) is supported by the cone \( C^+ = \{ k \in \mathbb{R}^{n+1}; k_0^2 \geq k_1^2 + \cdots + k_n^2; k \geq 0 \} \).

**Proof.** The decomposition of \( \mathcal{O}'(\frac{n-1}{2}, n + 1) \) under \( g_n \) is \( \bigoplus \lambda \mathcal{O}(\frac{n-1}{2} + \lambda, n) \). We have \( \mathcal{O}(\frac{n-1}{2} + \lambda, n) = \mathcal{U}(g_n \cdot (x_n^\lambda((x + i)^2))^{\lambda+(n-1)/2} \). Compute the Fourier transform of the function \( x_n^\lambda((x + i)^2))^{\lambda+(n-1)/2} \). If \( x \in \Omega_{n+1}^+ \), \( (x^2) \neq 0 \) and \( (x^2)^{-(s+(n-1)/2)} = m(s) \int_{C^+} e^{i\langle k, x \rangle/(k^2)^{s-1}} \, dk \) whenever \( s > 0 \). The function \( m(s) \) has a simple zero for \( s = 0 \).

Let us consider \( b(C^+) = \{ k; k_0^2 = k_1^2 + \cdots + k_n^2; k_0 \geq 0 \} \), the boundary of the cone \( C^+ \), equipped with the surface measure \( dm(k) \). If we consider the distribution \( f^s \) defined by the function \( (k^2)^{s-1} \) restricted to \( C^+ \), for \( s > 0 \), this distribution has a meromorphic continuation in \( s \) and \( s^m(f^s) \) tends to \( \int_{b(C^+)} \varphi(k) \, dk(k) \) when \( s \to 0 \). From

\[
\left( \frac{\partial}{\partial k_m} \right)^p (k^2)^{s-1} = (-2)^p(s - 1) \cdots (s - p) k_n^p(k^2)^{s-1-p} + \sum_{i<p} p_i(s \cdot)(k^2)^{s-1-i}
\]

where \( p_i(s, k_n) \) is a polynomial in \( s \) and \( k_n \) of degree \( i \), we see that for \( p < s \), the Fourier transform of \( x_n^\lambda((x + i)^2))^{\lambda+(n-1)/2} \) is supported in the interior of the cone \( C^+ \). Letting \( s \to p \) we see that the Fourier transform of our function is a sum of a function supported by \( b(C^+) \) and a function supported in the interior of \( C^+ \).

Now consider the subgroup \( P'_{n+1} \) of \( O(2, n + 1) \) of affine transformations of the domain \( \Omega_{n+1}^+ \), and denote by \( \mathfrak{l}_{n+1} \) its Lie algebra. We have \( g_n = g_{n+1} + k_n \) by the Iwasawa decomposition, hence since the function \( x_n^\lambda((x + u)^2))^{\lambda+(n-1)/2} \) transforms by a character of \( k_n \sim SO(2) \times SO(n) \), we have \( W(\lambda + \frac{n-1}{2}, n) = \mathcal{U}(g_{n+1} \cdot (x_n^\lambda((x + i)^2))^{\lambda+(n-1)/2} \).

As the action of \( \mathcal{U}(g_{n+1}) \) is given by differential operators with polynomial coefficients, the corollary follows.

4. **Example: Tensor Products of Analytic Continuations of the Holomorphic Discrete Series for SU(2, 2)**

In this example we analyze tensor products of highly singular type for \( SU(2, 2) \). Some of the material applies equally well to \( SU(n, n) \) and \( SP(n, \mathbb{R}) \), and indicates a direction of attack which may work for other singular restriction problems. In the end, however, it turns out that to get even some of the simplest cases,
a detailed knowledge of the representations of holomorphic type of $SU(2,2)$
and some concrete computations are needed.

For this case it is easiest to work with the unbounded realization of the
associated Hermitian symmetric space. Specifically, we take the domain to be
$D \times D$, where $D = \{ z \in gl(2, \mathbb{C}) \mid (z - z^*)/2i > 0 \}$, and we restrict to the
diagonal in $D \times D$, which we identify with $D$. We recall from [5] that if $(z_1, z_2) \in
D \times D$, $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SU(2,2)$, and if $y = (z_1 - z_2)/2$, then for $g \cdot z =
(az + b)(cz + d)^{-1}$

$$\left( \frac{z_1 - z_2}{2} \right) = (zc^* + d^*)^{-1}y(cz_1 + d)^{-1}. \quad (4.1)$$

From now on we let $G = SU(2,2)$, and let $K$ be the maximal compact subgroup.

To begin with, we need some observations:

If $\tau$ is a finite-dimensional unitary representation of $K$ in $V_\tau$, we identify
$\mathcal{O}(G, K, V_\tau)$ with the space $\mathcal{O}(D, V_\tau)$ of holomorphic functions from $D$ to $V_\tau$ and
let, as before, $T_\tau$ denote the corresponding action of $G$ on $\mathcal{O}(D, V_\tau)$.

Let us be explicit in this case: We have $K = \{(a, b) \mid (a + ib), (a - ib) \in
U(2) \times U(2)$ and $\det(a + ib)(a - ib) = 1 \}$. Let $u_1 = a + ib$, and $u_2 = a - ib$.
Then $\tau$ is of the form

$$\tau = \bigoplus_{i=1}^n \left( \det u_1^\otimes_i \left( \otimes_s u_1 \right) \otimes \left( \otimes_s u_2 \right) \right), \quad (4.3)$$

where $\otimes_s^\alpha$ denotes the $\alpha$th fold symmetrized tensor product. Then with

$$J_\tau(g, z) = \bigoplus_{i=1}^n \left( \det(cz + d)^{\otimes_i} \left( \otimes_s (cz + d) \right) \otimes \left( \otimes_s (zc^* + d^*)^{-1} \right) \right), \quad (4.4)$$

for $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G$, the representation $T_\tau$ on $\mathcal{O}(D, V_\tau)$ is given by

$$(T_\tau(g), f)(z) = J_\tau(g^{-1}, z)^{-1}f(g^{-1}z). \quad (4.5)$$

**Lemma 4.1.** Let $\tau_1$ and $\tau_2$ be given finite-dimensional unitary representations
of $K$, and let $\tau_1 \otimes \tau_2 = \bigotimes_{i,j} H_j$ be the decomposition of $\tau_1 \otimes \tau_2$, as a representation of $K$, into irreducible subrepresentations. Suppose that there exists a reproducing
kernel Hilbert space $H$ of holomorphic functions from $D$ to $V_{\tau_1} \otimes V_{\tau_2}$ such that
$T_{\tau_1 \otimes \tau_2}$ is unitary in $H$, and let $H = \bigoplus_{j \in J} H_j$ be the decomposition of $H$ corresponding
to $T_{\tau_1 \otimes \tau_2} = \bigoplus_{j \in J} T_{\tau_j}$. Then the Hilbert space structures on the $H_j$'s can be scaled
in such a way that the reproducing kernel $K : D \times D \to \text{Aut}(V_{\tau_1} \otimes V_{\tau_2})$ is the
restriction of a function $K : D \times D \times D \to \text{Aut}(V_{\tau_1} \otimes V_{\tau_2})$, holomorphic in
the two first variables, antiholomorphic in the last, in the sense that

$$\forall \psi \in V_{\tau_1} \otimes V_{\tau_2}, \quad \forall \omega \in D : K(\omega, \omega, \omega) = (R_{\omega}K(\cdot, \cdot, \omega) \psi)(\omega) \quad (4.6)$$
and such that moreover
\[ \forall \psi \in V_{\tau_1} \otimes V_{\tau_2}, \quad \forall \omega \in D: \]
\[ ((T_{\tau_1} \otimes T_{\tau_2})(g) K_{\otimes}(\cdot, \cdot, \omega)\psi)(z_1, z_2) \]
\[ = K_{\otimes}(z_1, z_2, \omega) J_{\tau_1 \otimes \tau_2}(g, \omega)^{-1} \psi. \quad (4.7) \]

Proof. Let \( K \) be a reproducing kernel on \( H \). Then we have [5], for \( g \in G \), and all \( z, w \) in \( D \)
\[ K(gz, gw) = J_{\tau_1 \otimes \tau_2}(g, z) K(z, w) J_{\tau_1 \otimes \tau_2}(g, w)^*. \quad (4.8) \]
In particular, for all \( k \) in \( K \)
\[ K(i, i) = ((\tau_1 \otimes \tau_2)(k)) K(i, i)((\tau_1 \otimes \tau_2)(k))^{-1}. \quad (4.9) \]
\( K(i, i) \) is a positive (self-adjoint) operator, and we see from (4.9) that each eigenspace of \( K(i, i) \) is invariant under \( \tau_1 \otimes \tau_2 \). Hence \( K(i, i) \) commutes with the extension of \( \tau_1 \otimes \tau_2 \) to \( GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \) (cf. (4.3)). \( K(z, w) \) is completely determined by \( k(z, z) \), and it follows from (4.8), (4.4), and the above that
\[ k(z, z) = J_{\tau_1 \otimes \tau_2}(g, i) J_{\tau_1 \otimes \tau_2}(g, i)^* k(i, i) \quad (4.10) \]
where \( g \cdot i = z \). It is now clear that \( k(i, i) \) can be chosen to be the identity operator on \( V_{\tau_1} \otimes V_{\tau_2} \). Thus we shall assume that
\[ k(z, z) = J_{\tau_1 \otimes \tau_2}(g, i) J_{\tau_1 \otimes \tau_2}(g, i)^*, \quad (4.11) \]
where \( g \cdot i = z \). Since \( J_{\tau_1 \otimes \tau_2}(g, z) = J_{\tau_1}(g, z) \otimes J_{\tau_2}(g, z) \) we can now define a function \( K'(z_1, z_2, w_1, w_2) \) on \( D \times D \times D \times D \); holomorphic in \( z_1, z_2 \), and antiholomorphic in \( w_1, w_2 \) by its value on the diagonal
\[ K'(z_1, z_2, z_1, z_2) = J_{\tau_1}(g_1, i) J_{\tau_1}(g_1, i)^* \otimes J_{\tau_2}(g_2, i) J_{\tau_2}(g_2, i)^* \quad (4.12) \]
where \( g_1 \cdot i = z_1 \) and \( g_2 \cdot i = z_2 \). This is clearly well defined and \( K_{\otimes}(z_1, z_2, w) = K'(z_1, z_2, w, w) \) is then the desired function.

Finally, the following will be useful:

**Lemma 4.2.** Let \( T \) denote a representation of \( G \) in the space of holomorphic functions from \( D \times D \) to some finite-dimensional vector space \( V \) and assume that there are reproducing kernel Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), both contained in \( \mathcal{O}_{\mathbb{C}}(\mathbb{C}^n) \) (1.14), to which the restriction of \( T \) is unitary and irreducible. Assume moreover that there exists an irreducible unitary representation \( T_n \) of \( G \) in a reproducing
kernel Hilbert space \( \mathcal{K}_1 \) of holomorphic functions from \( D \) to \( V \) such that \( R_n(\mathcal{K}_1) = R_n(\mathcal{K}_2) = \mathcal{K}_n \) (cf. 1.15), and

\[
R_n(T \mid_{\mathcal{K}_1}) = R_n(T \mid_{\mathcal{K}_2}) = T_n R_n.
\]

Then \( \mathcal{K}_1 = \mathcal{K}_2 \).

**Remark.** This means that they coincide as sets and that the Hilbert space structures are proportional.

**Proof.** If \( \mathcal{K}_1 \cap \mathcal{K}_2 \neq \{0\} \) it is obvious by irreducibility, so we assume that \( \mathcal{K}_1 \cap \mathcal{K}_2 = \{0\} \). Then the map \( S \) from \( \mathcal{K}_1 \oplus \mathcal{K}_2 \) to the space of holomorphic functions on \( D \times D \) with values in \( V \) given by

\[
S(f, g) = f + g
\]

is injective. The range \( S(\mathcal{K}_1 \oplus \mathcal{K}_2) \) can therefore be equipped with a Hilbert space structure. We let \( \mathcal{K} \) denote this space. It is easy to see that point evaluation is continuous and hence that \( \mathcal{K} \) is a reproducing kernel Hilbert space which accordingly can be decomposed by differentiation and restriction. We know that we must pick up \( T_n \oplus T_n \) by this procedure, but in the step where we use \( R_n \) (which is the first nontrivial step) we clearly only get \( T_n \). This means that the other \( T_n \) must be picked up in a later step, which, however, is impossible, since the \( K \)-types change under differentiation.

The following is then obvious:

**Corollary 4.3.** In Lemma 4.2 the group \( G \) can be replaced by \( K \) provided one assumes that \( \dim \mathcal{K}_1 = \dim \mathcal{K}_2 < \infty \). (And maintain the other hypotheses.)

**Remark.** This does not imply that there are no multiplicities in the \( K \)-types.

We are now ready to turn to the topic of this section: Let \( T_r \) and \( T_r \) be two irreducible unitary representations of \( G \) in reproducing kernel Hilbert spaces \( H(\tau_1) \) and \( H(\tau_2) \) consisting of holomorphic functions from \( D \) to \( V_{\tau_1} \) and \( V_{\tau_2} \), respectively, and consider \( T_r \otimes T_r \) acting unitarily in the reproducing kernel Hilbert space \( H(\tau_1) \otimes H(\tau_2) \) of holomorphic functions from \( D \times D \) to \( V_{\tau_1} \otimes V_{\tau_2} \).

As in [S] we introduce variables \( z \) and \( y \) on \( D \times D \) where, for \( (z_1, z_2) \in D \times D \)

\[
z = \frac{z_1 + z_2}{2} \quad \text{and} \quad y = \frac{z_1 - z_2}{2}.
\]

We let

\[
H_r = \left\{ \varphi \in H_{\tau_1} \otimes H_{\tau_2} \mid R_0 \varphi = 0 \text{ and } R_0 \left( \frac{\partial^{\alpha_1}}{\partial y_1} \cdots \frac{\partial^{\alpha_4}}{\partial y_4} \right) \varphi = 0 \right\}
\]

for all \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq r \).
and define maps $R_r$ analogous to the preceding cases. If it is also convenient to introduce the notation $Q_{r+1}$ for the orthogonal complement of $H_{r+1}$ in $H_r$. Finally, let $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ denote the module of all holomorphic functions from $D \times D$ to $V_{\tau_1} \otimes V_{\tau_2}$, let $\mathcal{O}_{r+1}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ be the analog of $H_{r+1}$, and let $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ denote the space of $K$-finite vectors in $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$.

Let $N(r)$ denote the dimension of the space of homogeneous polynomials in four variables of degree $r$ ($N(r) = (r+3)^4$). We identify this space with $\otimes^r_{g} \mathfrak{g}l(2, \mathbb{C})$ and let $T^r(\tau_1, \tau_2)$ denote the action of $G$ on the space of functions from $D \times D$ to $\otimes^r_{g} \mathfrak{g}l(2, \mathbb{C}) \otimes V_{\tau_1} \otimes V_{\tau_2}$, which on functions of the form $f_M \otimes h$, where $f_M(z_1, z_2) = M(z_1, z_2) \otimes \cdots \otimes M(z_1, z_2)$ is a function from $D \times D$ to $\otimes^r_{g} \mathfrak{g}l(2, \mathbb{C})$ and $h$ is a function from $D \times D$ to $V_{\tau_1} \otimes V_{\tau_2}$, is given by

\begin{equation}
(T^r(\tau_1, \tau_2)(g)f_M \otimes h)(z_1, z_2) = \otimes_{s}((cz_1 + d)^{-1}M(g^{-1}z_1, g^{-1}z_2)(cz_1 + d)^{-1})
\end{equation}

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Let us assume that there is a linear subspace $S_{0^r}$ of $\mathcal{O}(D \times D, \otimes^r_{g} \mathfrak{g}l(2, \mathbb{C}) \otimes V_{\tau_1} \otimes V_{\tau_2})$ such that:

- If $f \in S_{0^r}$ and $R_0 f = 0$ then $f = 0$. (4.16a)
- $S_{0^r}$ is invariant under $T^r(\tau_1, \tau_2)(g)$ for all $g$ in $G$. (4.16b)

As a basis of $\otimes^r_{g} \mathfrak{g}l(2, \mathbb{C})$ we choose

\begin{equation}
e_i = e_{i_1, i_2, i_3, i_4} = P_{s}(M_1 \otimes \cdots \otimes M_4)_{i_1} \otimes \cdots \otimes M_4_{i_4}
\end{equation}

where $M_1, \ldots, M_4$ is a basis of $\mathfrak{g}l(2, \mathbb{C})$, $i_1 + i_2 + i_3 + i_4 = r$, and $P_s$ denotes the symmetrization map (projection).

Any function $F$ in $\mathcal{O}(D \times D, \otimes^r_{g} \mathfrak{g}l(2, \mathbb{C}) \otimes (V_{\tau_1} \otimes V_{\tau_2}))$ can uniquely be written as

\begin{equation}
F(z_1, z_2) = \sum_{i=1}^{N(r)} e_i \otimes f_i(z_1, z_2)
\end{equation}

where $f_i$ is a holomorphic function for $i = 1, 2, \ldots, N(r)$. Hence we can define a map $P^r(\tau_1, \tau_2)$ from $S_{0^r}$ to $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ by

\begin{equation}
(P^r(\tau_1, \tau_2)F)(z_1, z_2) = \sum_{i=1}^{N(r)} (\text{tr } y M_1)^{i_1}(\text{tr } y M_2)^{i_2}(\text{tr } y M_3)^{i_3}(\text{tr } y M_4)^{i_4}f_i(z_1, z_2).
\end{equation}
Lemma 4.4. (a) \( P'(\tau_1, \tau_2) T'(\tau_1, \tau_2) = T_{\tau_1} \otimes T_{\tau_2} P'(\tau_1, \tau_2) \)

(b) \( P'(\tau_1, \tau_2) \) is injective.

Proof. (a) is essentially contained in (4.1).

To prove (b) assume that \( P'(\tau_1, \tau_2)F = 0 \). Then clearly \( R_\tau P'(\tau_1, \tau_2)F = 0 \) and since this equals \( R_\tau F, F \) is zero by assumption.

One application of this lemma is the following: Consider \( T'(\tau_1, \tau_2) \) and let \( \tau_1^* \) and \( \tau_2^* \) be the representations of \( K \) given by

\[
\tau_1^*(u_1, u_2) = (u_1 \otimes \cdots \otimes u_1) \otimes \tau_1(u_1, u_2),
\]

and

\[
\tau_2^*(u_1, u_2) = (det u_2)^{-\tau}(u_2 \otimes \cdots \otimes u_2) \otimes \tau_2(u_1, u_2).
\] (4.20)

Then \( T'(\tau_1, \tau_2) \) is in a natural way contained in \( T_{\tau_1} \otimes T_{\tau_2}^* \). Of course, \( \tau_1^* \) and \( \tau_2^* \) are not irreducible representations, but writing them as finite sums of elements of \( \hat{K} \), we see that \( T'(\tau_1, \tau_2) \) is contained in a direct sum of representations \( T_{\mu_1} \otimes T_{\mu_2}^* \), where \( \mu_1 \) as well as \( \mu_2^* \), is contained in \( \hat{K} \).

Corollary 4.5. Let \( T_{u_1}^* \) and \( T_{u_2}^* \) be as above. If there are reproducing kernel Hilbert spaces \( H(\mu_1^*) \) and \( H(\mu_2^*) \) of holomorphic functions in which \( T_{u_1}^* \) and \( T_{u_2}^* \), respectively, are unitary, and if \( T_{u_1}^* \otimes T_{u_2}^* \) is contained in \( T'(\tau_1, \tau_2) \), then \( T_{\tau_1} \otimes T_{\tau_2} \) is unitary in a reproducing kernel Hilbert space of holomorphic functions.

Restricted to this space, it is unitarily equivalent to \( T_{u_1}^* \otimes u_2^* \).

Proof. The orthogonal complement in \( H(\mu_1^*) \otimes H(\mu_2^*) \) of the space of functions that vanish on the diagonal is a reproducing kernel Hilbert space. If we apply \( P'(\tau_1, \tau_2) \) the resulting space is one in which point-evaluation is continuous.

If we only demand that the space \( S_0^* \) (4.16) should be invariant under \( K \) then \( P'(\tau_1, \tau_2) \) can be used, by means of Lemma 4.1 and Corollary 4.3, to compare minimal \( K \)-types. This is used in an example below.

We shall say that a representation \( T(\tau) \) in \( \mathcal{C}(D, V_\tau) \) is supported by the forward light cone \( C^+ = \{ k \in H(2) \mid \det k > 0, \operatorname{tr} k > 0 \} \) if it is unitary in a space \( H(\tau) \subset \mathcal{C}(D, V_\tau) \) and if the elements of \( H(\tau) \) are Fourier–Laplace transforms of functions from \( C^+ \) to \( V_\tau \).

If the above space \( H(\tau) \) is a reproducing kernel Hilbert space, it is easy to see that there exists a continuous function \( F_\tau : C^+ \rightarrow \text{Aut}(V_\tau) \) such that the kernel \( K_\tau \), is given by

\[
K_\tau(z, w) = \int_{C^+} F_\tau(k) e^{i\tau(z-w^*)k} \, dk.
\] (4.21)

If, for all \( k \in C^+ \), \( F_\tau(k) \) is nonsingular we say that \( T(\tau) \) is strongly supported.
by $C^+$. By restricting to the point $i \in D$ the $K$-types in $H(\tau)$ of a representation $T(\tau)$ which is strongly supported by $C^+$ can be found by arguments analogous to those in the proof of theorem 3.2 in [5].

**Proposition 4.6.** If $\tau \in K$ and if $T(\tau)$ is strongly supported by $C^+$, then $\Theta'(D, V_{\tau}) \subseteq H(\tau)$.

We shall now turn to some concrete example of the above. We recall from [1, 2, 7] that if $\tau$ denotes the defining representation of $GL(2, \mathbb{C})$; $\tau(g) = g$, if $\tau_n$ denotes the $n$th fold symmetrized tensor product of $\tau$, and if $\tau_0 = 1$, then one can define two series $T_1(n, \alpha)$ and $T_2(m, \beta)$ of representations of $G$ by

$$(T_1(n, \alpha)(g)f)(z) = \tau_n(cz + d)^{-1} \det(cz + d)^{-\alpha} f(g^{-1}z), \quad (4.22)$$

and

$$(T_2(m, \beta)(g)f)(z) = \tau_m(zc^* + d^*) \det(zc^* + d^*)^{-\beta} f(g^{-1}z),$$

for $\alpha, \beta \in \mathbb{Z}$, and $n, m \geq 0$. For $\alpha > -1$ and $\beta > -1$ the representations $T_1(n, \alpha)$ and $T_2(m, \beta)$ are unitary, irreducible, and strongly supported by $C^+$ in reproducing kernel Hilbert spaces $H_1(n, \alpha)$ and $H_2(m, \beta)$, respectively. $T_1(n, -1)$ and $T_2(m, -1)$ are unitary and irreducible in reproducing kernel Hilbert spaces $H_1(n, -1)$ and $H_2(m, -1)$, respectively, where $H_1(n, -1)$, as well as $H_2(m, -1)$, consists of holomorphic solutions to certain wave equations.

Finally, we define

$$(T_3(n, m, \gamma)(g)f)(z) = \tau_n(\gamma) \otimes \tau_m(zc^* + d^*) \det(cz + d)^{-\gamma} f(g^{-1}z). \quad (4.23)$$

**Proposition 4.7** [2]. For $\gamma \geq m + 2$, $T_3(n, m, \gamma)$ is unitary and irreducible in a reproducing kernel Hilbert space $H_3(n, m, \gamma)$.

**Proof.** For such $\gamma$ the representation $T_3(n, m, \gamma)$ can be obtained as the restriction to the diagonal of $T_1(n, \alpha) \otimes T_2(m, \beta) \alpha, \beta > -1$, since $\det(zc^* + d^*) = \det(cz + d)$.

**Lemma 4.8** [2]. For $\gamma \geq m + 2$, $T_3(n, m, \gamma)$ is supported by $C^+$.

**Proof.** The reproducing kernel is given by

$$\tau_n \left( \frac{x - w^*}{2i} \right)^{-\gamma} \otimes \tau_m \left( \frac{x^* - w}{2i} \right) \det \left( \frac{x^* - w}{2i} \right)^{-\gamma} \quad (4.24)$$

Let $\epsilon = \gamma - m - 2$. Then the kernel is given by $M_{n,m}(\frac{x - w^*}{2i}) \det(\frac{x - w^*}{2i})^{-\epsilon - m - n}$ where $M_{n,m}(\frac{z^* - w}{2i})$ is a matrix whose entries are polynomials of degree $n + m$ in
the entries of \((\frac{z-w^*}{2i})\). We know [7] that there are constants \(c_\rho\) such that for \(\rho \geq 0\)

\[
\det \left( \frac{z-w^*}{2i} \right)^{-\rho - 2} = c_\rho \int_{C^+} e^{i\text{tr}(z-w^*)k} \det k^\rho \, dk \quad (4.25)
\]

and the proof is completed by observing that the kernel can be obtained by differentiation of \(\det(\frac{z-w^*}{2i})^{-\rho - \epsilon}\) modulo some correction terms obtainable by differentiation of functions of the form \(\det(\frac{z-w^*}{2i})^{-\rho - \epsilon - i}\) for \(i \geq 1\).

**Remark.** It is proved in [2] that for \(\gamma > m + 2\), \(T_\gamma(n, m, \gamma)\) is strongly supported by \(C^+\), whereas for \(\gamma = m + 2\) it is not.

Let \(T_\tau_1\) and \(T_\tau_2\) be two limit points of the series (4.22). It then remains to be seen how \(H(T_\tau_1) \otimes H(T_\tau_2)\) decomposes. We give a typical example of how the above approach leads to the answer:

**Example 4.1.** \(T_1(n, -1) \otimes T_1(m, -1)\). The reproducing kernel \(K_p\) for \(T_1(p, -1)\) is given by [7, p. 75]

\[
K_p(z, w) = \int_{\mathbb{C}^+} \tau_n(k) e^{i\text{tr}(z-w^*)k} \, dm(k). \quad (4.26)
\]

Consider \(T_{\tau'}(\tau_1, \tau_2)\) and let \(T_{\tau'}(\tau_1, \tau_2)\) be the representation defined by \(R_0 T_{\tau'}(\tau_1, \tau_2) = T_0 T_{\tau'}(\tau_1, \tau_2) R_0\). It is straightforward to see that the corresponding kernel \(K_{\tau'}(z_1, z_2, w)\) (Lemma 4.1) is the operator which maps \((\otimes_s M) \otimes v_1 \otimes v_2\) into

\[
\otimes_s \left( \left( \frac{z_1 - w^*}{2i} \right)^{-1} M \left( \frac{z_2 - w^*}{2i} \right)^{-1} \right) \otimes K_{\tau'}(z_1, w) v_1 \otimes K_{\tau'}(z_2, w) v_2 \quad (4.27)
\]

for \(M \in g(l(2, \mathbb{C}) \otimes (\otimes_s \mathbb{C}^2) \otimes (\otimes_s \mathbb{C}^2)\). The map \(P_{\tau'}\) can be chosen in such a way that the expression in (4.27) is mapped into

\[
\left( \text{tr} \left( \frac{z_1 - z_2}{2i} \right) \left( \frac{z_1 - w^*}{2i} \right)^{-1} M \left( \frac{z_2 - w^*}{2i} \right)^{-1} \right) \otimes K_{\tau'}(z_1, w) v_1 \otimes K_{\tau'}(z_2, w) v_2 \quad (4.28)
\]

and the question becomes: For which linear combinations of elements of \((\otimes_s g(l(2, \mathbb{C})) \otimes (\otimes_s \mathbb{C}^2) \otimes (\otimes_s \mathbb{C}^2)\) do the corresponding linear combinations of the functions (4.28) belong to \(H_1(n, -1) \otimes H_1(m, -1)\)? This may seem, especially for large \(n\) and \(m\)'s, to be a hopeless case. However, there are some obvious combinations that do work: We write \((z_1 - z_2)(z_1 - w^*) - (z_2 - w^*)\) and obtain:

\[
\text{tr}((z_1 - z_2)(z_1 - w^*)^{-1} M(z_2 - w^*)^{-1})
\]

\[
= \frac{\text{tr} M(z_2 - w^*)}{\det(z_2 - w^*)} - \frac{\text{tr} M(z_1 - w^*)}{\det(z_1 - w^*)}, \quad (4.29)
\]
where, for $M = (m_1, m_2, m_3) = (m_{21}, m_{22}, m_{23})$, and $M = (m_4, -m_3)$. We parametrize the elements $k$ in $H(2)$ by

$$k = \begin{pmatrix} k_0 + k_1 & k_2 - ik_3 \\ k_2 + ik_3 & k_0 - k_1 \end{pmatrix},$$

and we let

$$\frac{\partial}{\partial k} = \begin{pmatrix} \frac{\partial}{\partial k_0} + \frac{\partial}{\partial k_1} & \frac{\partial}{\partial k_2} - i \frac{\partial}{\partial k_3} \\ \frac{\partial}{\partial k_2} + i \frac{\partial}{\partial k_3} & \frac{\partial}{\partial k_0} - \frac{\partial}{\partial k_1} \end{pmatrix}. 
$$

Then the operator $\left( \text{tr} \bar{M}(\partial/\partial k) \right)$ applied to $k$ gives $2\bar{M}$; applied to $\det k$ it gives $2 \text{tr} M k$, and applied to $\text{tr} M k$ it gives $2 \frac{\text{tr} \bar{M} M}{\det M}$.

Let us now look at (4.28). According to (4.29) this breaks down into a sum of terms of the form

$$\frac{\text{det}(z_1 - w^*)^s}{\text{det}(z_1 - w^*)^s} K_n'(z_1, w) v_1 \otimes \frac{(\text{det}(z_2 - w^*)^{r-s})}{\text{det}(z_2 - w^*)^{r-s}} K_m'(z_2, w) v_2. \quad (4.22)$$

In this expression, for $s \geq 1$, and up to a constant,

$$\frac{1}{\text{det}(z_1 - w^*)^s} K_n'(z_1, w) v_1 = \int_{\mathbb{C}^1} \tau_n(k) \det k^{s-1} e^{\text{tr}(z_1 - w^*) k} \, dk v_1, \quad (4.23)$$

and $\text{tr} \bar{M}(z_1 - w^*) e^{\text{tr}(z_1 - w^*) k}$ is proportional to $(\text{tr} \bar{M}(\partial/\partial k)) e^{\text{tr}(z_1 - w^*) k}$.

From this and the preceding remarks it follows that if $e_1 = (1) \in \mathbb{C}^1$, if $M = (0, 1) = (0, 1, 0)\), if $v_1 = e_1 \otimes \cdots \otimes e_1$, and if $v_2 = e_1 \otimes \cdots \otimes e_1$, then the corresponding function (4.28) does indeed belong to $H_s(n, -1) \otimes H_t(m, -1)$. Thus, for $r \geq 1$, the representation $T_s(n + m + r, r, r + 2)$ is contained in the tensor product. To complete the analysis one could now either examine functions of the form (4.28) corresponding to the other highest weight vectors, or compare the $K$-types. We mention that the $K$-types for the mass-zero representations can be found by combining [7, pp. 100-104] with [12]. These possibilities are mentioned because the tensor product of a mass-zero representation with a representation supported by $C^+$ can be treated among these lines. However, for the given case, the intuitively clear result that besides what we pick up for $r = 0$, the above representations are the only representations that appear, follows readily from the remarks following Proposition 4.9 below. We mention that the case $T_s(n, -1) \otimes T_s(m, -1)$ and $T_s(n, -1) \otimes T_s(m, -1)$ are similar, and that the results also can be obtained from [8].
Proposition 4.9.

\[ T_1(n, -1) \otimes T_1(m, -1) \]
\[ = \bigoplus_{q=0}^{\min(n,m)} T_1(n + m - 2q, q) \oplus \bigoplus_{r=1}^{\infty} T_3(n + m + r, r, r + 2); \]

\[ T_1(n, -1) \otimes T_2(m, -1) \]
\[ = \bigoplus_{q=0}^{\min(n,m)} T_2(n + m - 2q, q) \oplus \bigoplus_{r=1}^{\infty} T_3(r, n + m + r, n + m + r + 2); \]

\[ T_1(n, -1) \otimes T_1(m, -1) \]
\[ = \bigoplus_{r=1}^{\infty} T(n + r, m + r, m + r + 2). \]

We conclude this article by observing that several interesting phenomena occur at the decomposition of the tensor product of a mass-zero representation with a mass-zero representation. One such is that most of the representations that appear in the decomposition themselves live in solution spaces to differential equations. The simplest example of this is the tensor product of \( U_{-1} = T_1(0, -1) = T_2(0, -1) \) with itself. Since \( U_{-1} \) is unitary in a space of solutions to \( \Box \psi = 0 \), the Hilbert space in which \( U_{-1} \otimes U_{-1} \) acts consists of functions that are solutions to \( \Box_{x_i} f = \Box_{x_i} f = 0 \). Under the change of variables (4.13) these equations can be combined into

\[ (\Box_z + \Box_y) f = 0, \]

and

\[ \left( \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} - \frac{\partial}{\partial x^1} \frac{\partial}{\partial y^1} - \frac{\partial}{\partial x^2} \frac{\partial}{\partial y^2} - \frac{\partial}{\partial x^3} \frac{\partial}{\partial y^3} \right) f = 0, \]

where we have used the Pauli matrices as a basis of \( gl(2, \mathbb{C}) \).

Let us assume for simplicity that \( f \) is of the form \( y_0 f_0 + y_1 f_1 + y_2 f_2 + y_3 f_3 \). Then \( f \) under our method of decomposing tensor products is mapped into a function \( h = (h_1, h_2, h_3, h_4) \) with \( h_i(z) = f_i(z, z) \) for \( i = 0, 1, 2, 3 \), and hence, by (4.34), \( h \) satisfies the equation

\[ \frac{\partial}{\partial x_0} h_0 - \frac{\partial}{\partial x_1} h_1 - \frac{\partial}{\partial x_2} h_2 - \frac{\partial}{\partial x_3} h_3 = 0. \]

If \( f \) is of the form \( \sum_i p_i(y) f_i \) where the \( p_i \)'s are homogeneous polynomials of degree, \( r \), the resulting equation of course is more complicated.

The Eq. (4.35) is in nature a "gauge-condition." However, it may also indicate a "conserved current."
A similar phenomenon takes place for the tensor product of any two limits of the series (4.22) and this leads to the conclusion that with the possible exception of what is picked up for $r = 0$, the representations occurring in the decomposition cannot be strongly supported by $C^+$. 

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