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The Campbell-Hausdorff Formula and Invariant Hyperfunctions

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Introduction

Let **G** be a Lie group and $\mathfrak{g}$ its Lie algebra. We denote by V the underlying vector space of $\mathfrak{g}$.

There is a canonical isomorphism between the ring $\mathbf{Z}(\mathfrak{g})$ of the biinvariant differential operators on **G** and the ring $\mathbf{I}(\mathfrak{g})$ of the constant coefficient operators on V which are invariant by the adjoint action of **G**. When $\mathfrak{g}$ is semi-simple, this is the "Harish-Chandra isomorphism"; for a general Lie algebra, this was established by Duflo [4].

We shall prove here, that when **G** is solvable the Duflo isomorphism extends to an isomorphism Φ of the algebra of "local" invariant hyperfunctions under the group convolution and the algebra of invariant hyperfunctions on V under additive convolution (the exact result will be stated below). This gives a partial answer to a conjecture of Rais [12].

The existence of such an isomorphism Φ is of importance for the harmonic analysis on **G**, and for the study of the solvability of biinvariant operators on **G** (see [7]). It reflects and explains the "orbit method" ([8, 9]), i.e. the correspondence between orbits of **G** in V^*, the dual vector space of V, and unitary irreducible representations of **G**: let T be an irreducible representation of **G**, then the infinitesimal character of T is a character of the ring $\mathbf{Z}(\mathfrak{g})$. Let $\mathcal{O}$ be an orbit in V^*, the map $\rho_{\mathcal{O}}(P)=P(f)$ $(f\in\mathcal{O})$ is a character of the ring $\mathbf{I}(\mathfrak{g})$ ($\mathbf{I}(\mathfrak{g})$ being identified with the ring of invariant polynomials on V^*). The principle of the orbit method is to assign to a (good) orbit $\mathcal{O}$ a representation $T_{\mathcal{O}}$ of **G** (or $\mathfrak{g}$), whose infinitesimal character corresponds to $\rho_{\mathcal{O}}$ via the isomorphism Φ. This is the technique used by M. Duflo to construct the ring isomorphism Φ.

Furthermore let $t_{\mathcal{O}}$ be (when defined) the distribution on V which is the Fourier transform of the canonical measure on the orbit $\mathcal{O}$, then $t_{\mathcal{O}}$ is clearly an invariant positive eigendistribution of every operator P in $\mathbf{I}(\mathfrak{g})$ of eigenvalue $\rho_{\mathcal{O}}(P)$. Kirillov conjectured that the global character of the representation $T_{\mathcal{O}}$ (when

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defined) should be intimately connected with the "orbit distribution" $\Phi^{-1}(t_\varnothing)$, as proven in numerous cases. It is an essential result of Duflo [4] that these "orbit distributions" are indeed eigenfunctions for every biinvariant operator P in $\mathbf{Z}(\mathfrak{g})$; as in Rais [11], this implies the local solvability of P [4].

We will here derived the existence of Φ from a property of the Campbell-Hausdorff formula, that we conjecture and can prove in the solvable case. It is then a natural corollary of our conjecture, that biinvariant operators are locally solvable and that "orbit distributions" are eigendistributions for $\mathbf{Z}(\mathfrak{g})$. Hence the correspondence between orbits and representations is already engraved in the structure of the multiplication law.

Let us describe with some details our technique and results: We denote by $\mathfrak{g}_t$ the Lie algebra whose underlying vector space is V itself and in which the bracket $[*,*]_t$ is given by $[X,Y]_t=t[X,Y]$. Then $\mathfrak{g}_t$ gives a deformation between $\mathfrak{g}$ and the abelian Lie algebra, in which the fact is trivial.

In the course of the proof we encounter the following problem: Let $\mathbf{L}$ be the free Lie algebra generated by two indeterminates x and y and $\hat{\mathbf{L}}$ its completion. Since $x+y-\log e^y e^x$ belongs to $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$, by Campbell-Hausdorff formula, we can write it in $x+y-\log e^y e^x=(1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G$ for F and G in $\hat{\mathbf{L}}$. F and G are not uniquely determined by this property.

Conjecture. *For any Lie algebra* $\mathfrak{g}$ *of finite dimension, we can find F and G such that they satisfy*

a) $x+y-\log e^y e^x=(1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G$.

b) *F and G give $\mathfrak{g}$-valued convergent power series on* $(x,y)\in\mathfrak{g}\times\mathfrak{g}$.

c) $\operatorname{tr}((\operatorname{ad} x)(\partial_x F);\mathfrak{g})+\operatorname{tr}((\operatorname{ad} y)(\partial_y G);\mathfrak{g})$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}-\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1;\mathfrak{g}\right).$$

Here $z=\log e^x e^y$ *and* $\partial_x F$ (resp. $\partial_y G$) *is the* $\mathbf{End}(\mathfrak{g})$*-valued real analytic function defined by*

$$\mathfrak{g}\ni a\mapsto\frac{d}{dt}F(x+ta,y)|_{t=0}\quad\left(\text{resp. }\mathfrak{g}\ni a\mapsto\frac{d}{dt}G(x,y+ta)|_{t=0}\right),$$

and tr *denotes the trace of an endomorphism of* $\mathfrak{g}$.

When $\mathfrak{g}$ is nilpotent, this conjecture is easily verified because $(\operatorname{ad} x)(\partial_x F)$, $1-(\operatorname{ad} x/(e^{\operatorname{ad} x}-1))$ etc. are nilpotent endomorphisms of $\mathfrak{g}$ so that their traces vanish. However, we get the following fact.

Proposition 0. *If* $\mathfrak{g}$ *is solvable, then* Conjecture *is true.*

Let K be a non-empty closed cone in $\mathfrak{g}$. Let $\mathscr{I}(K)$ (resp. $\tilde{\mathscr{I}}(K)$) be the vector space of the germs at the unit element $e\in\mathbf{G}$ (resp. the origon $0\in\mathfrak{g}$) of the functions (i.e. either distributions, or hyperfunctions or micro-functions) $u(g)$ (resp. $\tilde{u}(x)$) such that $\operatorname{supp} u\subset\exp K$ (resp. $\operatorname{supp}\tilde{u}\subset K$) infinitesimally (see § 2) and that $u(ghg^{-1})=|\det(\operatorname{Ad}(g);\mathfrak{g})|^{-1}u(h)$ (resp. $\tilde{u}(\operatorname{Ad}(g)x)=|\det(\operatorname{Ad}(g);\mathfrak{g})|^{-1}\tilde{u}(x)$). We shall set $j(x)=\det((1-e^{-\operatorname{ad} x})/\operatorname{ad} x;\mathfrak{g})$ for $x\in\mathfrak{g}$ sufficiently near the origin. We define the isomorphism Φ: $\mathscr{I}(K)\to\tilde{\mathscr{I}}(K)$ by $(\Phi u)(x)=j(x)^{1/2}u(e^x)$ for $u\in\mathscr{I}(K)$. If two closed cones K_1 and K_2 satisfy $K_1\cap(-K_2)=\{0\}$, then we can define the product $\mathscr{I}(K_1)\times\mathscr{I}(K_2)\to\mathscr{I}(K_1+K_2)$ (resp. $\tilde{\mathscr{I}}(K_1)\times\tilde{\mathscr{I}}(K_2)\to\tilde{\mathscr{I}}(K_1+K_2)$) by the convolution $*$, i.e.

$$(u*v)(g)=\int_{\mathbf{G}} u(h)\,v(h^{-1}g)\,dh \quad\text{and}\quad (\tilde{u}*\tilde{v})(x)=\int_{\mathfrak{g}}\tilde{u}(y)\,\tilde{v}(-y+x)\,dy.$$

The exact statement which we shall prove is the following:

Theorem. *If* Conjecture *is true for the group* **G**, *then we have*

$$(\Phi u)*(\Phi v)=\Phi(u*v)$$

for $u\in\mathscr{I}(K_1)$ *and* $v\in\mathscr{I}(K_2)$.

If we apply this theorem when v is supported at the origin, then we obtain the following corollary:

Corollary 0. *Suppose that* Conjecture *is true for* **G**, *then with any biinvariant differential operator* P *on* **G** *we can associate a constant coefficient differential operator* $\tilde{P}$ *on* $\mathfrak{g}$ *so that* $\tilde{P}\Phi(u)=\Phi(Pu)$ *holds for any* $u\in\mathscr{I}(\mathfrak{g})$.

In paragraph 4, we will prove directly this particular case of our theorem. In fact, applying the same technique, we can prove a more precise result, giving a partial answer to a conjecture of Dixmier.

Let $\gamma(P)=\beta(D(j^{1/2})P)$ the Duflo isomorphism from $\mathbf{I}(\mathfrak{g})$ to $\mathbf{Z}(\mathfrak{g})$, where β is the symmetrization map and $D(j^{1/2})$ the "differential" operator (of infinite order) defined by $j^{1/2}$, let us look at the operator $\gamma(P)$ as a biinvariant differential operator on **G**; we denote by $(\exp)^*(\gamma(P))$ the differential operator on $\mathfrak{g}$ with analytic coefficients, which is the inverse image of $\gamma(P)$ by the exponential mapping. Let D be the ring of the germs at 0 of differential operators with analytic coefficients. We consider the left ideal $\mathscr{L}$ of D generated by the elements $\langle[A,x],\partial_x\rangle+\mathrm{tr}(\mathrm{ad}A;\mathfrak{g})$, $A\in\mathfrak{g}$ (here $\langle[A,x],\partial_x\rangle$ is the adjoint vector field given by $\frac{d}{d\varepsilon}\varphi(\exp\varepsilon A\cdot x)|_{\varepsilon=0}$). Every invariant distribution on $\mathfrak{g}$ is annihilated by $\mathscr{L}$. So Corollary 0 is implied by:

Corollary 1. *Suppose that* Conjecture *is true for* **G**, *then*

$$(\exp)^*(\gamma(P))-j(x)^{-\frac{1}{2}}Pj(x)^{\frac{1}{2}}\in\mathscr{L}.$$

Since Conjecture is solved in the solvable case the above theorem and its corollaries are true for a solvable group **G**. Recall that the result stated in Corollary 0 holds for $\mathfrak{g}$ semi-simple as proved by Harish-Chandra [6]. Howe [16] says that he proved Theorem for a nilpotent group **G** and a restricted class of functions u, v.

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§1

For The Theory of Microfunctions, we Refer to [1, 10, 15]. Let $\mathbf{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and exp: $\mathfrak{g} \to G$ the exponential map. Let $\mathbf{M}$ be a real analytic manifold on which $\mathbf{G}$ acts real analytically. A hyperfunction $u(x)$ on $\mathbf{M}$ is called a *relative invariant* with respect to a character χ of G if $u(gx) = \chi(g)u(x)$ holds on $G \times \mathbf{M}$. Here $u(gx)$ is the pull-back of u by the map r: $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ defined by $(g, x) \mapsto gx$, and $\chi(g)u(x)$ is the product of a real analytic function $\chi(g)$ on $\mathbf{G} \times \mathbf{M}$ and the pull-back of u by the projection from $\mathbf{G} \times \mathbf{M}$ onto $\mathbf{M}$. More generally, let A be a subset of M, $\mathbf{G}_A = \{g \in \mathbf{G};\ gA = A\}$. A hyperfunction $u(x)$ defined in a neighborhood U of A is called *relative invariant* locally at A if there is a neighborhood W of $\mathbf{G}_A \times A$ such that $\mathrm{r}(W) \subset U$ and that $u(gx) = \chi(g)u(x)$ on W.

For any $X \in \mathfrak{g}$, we denote by D_X the vector field defined by $(D_X u)(x) = \frac{d}{dt} u(\exp(-tX)x)|_{t=0}$, and by $\delta\chi$ the derivative of χ $\left(\text{i.e. } \delta\chi(X) = \frac{d}{dt}\chi(\exp tX)|_{t=0}\right)$.

Lemma 1.1. *If u is a relative invariant locally on A hyperfunction then $(D_X + \delta\chi(X))u = 0$ in a neighborhood of A for any $X \in \mathfrak{g}$.*

Proof. We define the map φ: $\mathbb{R} \times \mathbf{M} \to \mathbf{G} \times \mathbf{M}$ by $(t, x) \mapsto (\exp(-tX), x)$. Then the pull-back of $u(gx)$ is the pull-back $(r\varphi)^* u$ of u by the map $r \circ \varphi$, and the pull-back of $\chi(g)u(x)$ is $\chi(e^{-tX})u(x)$. Since $r \circ \varphi$ has maximal rank, this is justified. Thus $(r \circ \varphi)^* u = \chi(e^{-tX})u(x)$. If we differentiate the both-sides with respect to t, and restrict them at the variety $t = 0$ in $\mathbb{R} \times \mathbf{M}$, we obtain $D_X u$ from the left hand side and $-\delta\chi(X)u$ from the right hand side. Q.E.D.

§2

Let $\mathbf{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and exp: $\mathfrak{g} \to \mathbf{G}$ the exponential map. We denote by $\mathbf{d}g$ the left invariant Haar measure and by $\mathbf{d}x$ the Euclidean measure on $\mathfrak{g}$. After the normalization, $\mathbf{d}g$ and $\mathbf{d}x$ are related under the exponential map by the formula: $d(e^x) = j(x)\,dx$, where $j(x) = \det((1 - e^{-\mathrm{ad}\,x})/\mathrm{ad}\,x; \mathfrak{g})$ in a neighborhood of $x = 0$, because the derivative of $\exp x$ at x is given by $(1 - e^{-\mathrm{ad}\,x})/\mathrm{ad}\,x$ when we identify $\mathbf{T}\mathbf{G}$ with $\mathfrak{g} \times \mathbf{G}$ by the left translation. We define the character $\chi_0(g)$ of G by $|\det(\mathrm{Ad}(g); \mathfrak{g})|$, we denote by $d\chi_0$ the corresponding character of $\mathfrak{g}$, i.e. $d\chi_0(x) = \mathrm{tr}(\mathrm{ad}\,x; \mathfrak{g})$.

Let A and B be subsets of a C^1-manifold $\mathbf{M}$, x a point in $\mathbf{M}$. Take a local coordinate system $(x_1, \ldots, x_l)$ of $\mathbf{M}$. The set of limits of the sequence $a_n(y_n - z_n)$ where $a_n > 0$, $y_n \in A$, $z_n \in B$ and y_n, z_n converge to x when $n \to \infty$, is denoted by $\mathbf{C}_x(A; B)$ regarded as a closed subset of the tangent space $\mathbf{T}_x\mathbf{M}$ of $\mathbf{M}$ at x. $\mathbf{C}_x(A; \{x\})$ is simply denoted by $\mathbf{C}_x(A)$. If f is a differential map from $\mathbf{M}$ to a C^1 manifold N, then we have $(df)_x(\mathbf{C}_x(A; B)) \subset \mathbf{C}_{f(x)}(fA; fB)$. If $\mathbf{C}_x(A; B) \cap \operatorname{Ker} df(x) \subset \{0\}$, then there is a neighborhood U of x such that

$$(df)_x \mathbf{C}_x(A; B) = \mathbf{C}_{f(x)}(f(A \cap U); f(B \cap U)).$$

If $\mathbf{C}_x(A;B)=\{0\}$, then x is an isolated point of $\bar{A}$ and $\bar{B}$. $\mathbf{C}_x(A;B)=\emptyset$ if and only if $\bar{A}\cap\bar{B}\not\ni x$.

Let K be a closed cone of $\mathfrak{g}$. We shall denote by $\mathscr{I}(K)$ (resp. $\tilde{\mathscr{I}}(K)$) the space of the germs of function $u(g)$ (resp. $\tilde{u}(x)$) on $\mathbf{G}$ (resp. on $\mathfrak{g}$) at $e\in\mathbf{G}$ (resp. $0\in\mathfrak{g}$) satisfying

(2.1) $\mathbf{C}_e(\operatorname{supp} u)\subset K\subset\mathfrak{g}=\mathbf{T}_e\mathbf{G}$ (resp. $C_0(\operatorname{supp}\tilde{u})\subset K\subset\mathfrak{g}=\mathbf{T}_0\,\mathfrak{g}$)

and

(2.2) u is a relative invariant locally at e with respect to the character $\chi_0(g)^{-1}$.

Let K_1 and K_2 be two closed cones in $\mathfrak{g}$ such that $K_1\cap(-K_2)=\{0\}$. If $u\in\mathscr{I}(K_1)$ and $v\in\mathscr{I}(K_2)$, then $(\operatorname{supp} u)\cap(\operatorname{supp} v)^{-1}$ is contained in $\{e\}$ locally. Suppose that u and v are defined on a neighborhood U_0 of e. For any open neighborhood $U\subset U_0$ of e, we can find neighborhoods W and V of e such that $W\subset U$, $W^{-1}\subset U$, $\{h\in W; h\in\operatorname{supp} u, h^{-1}\in\operatorname{supp} v\}\subset\{e\}$ and that the map $(g,h)\mapsto g$ from $\{(g,h)\in V\times W; h^{-1}g\in\operatorname{supp} v, h\in\operatorname{supp} u\}$ to V is a proper map. Hence we can define $(u*v)(g)$ by

$$\int_W u(h)\,v(h^{-1}g)\,dh \quad \text{on } g\in V.$$

This gives the bilinear homomorphism $\mathscr{I}(K_1)\times\mathscr{I}(K_2)\to\mathscr{I}(K_1+K_2)$ because $\mathbf{C}_e((\operatorname{supp} u)\cdot(\operatorname{supp} v))\subset K_1+K_2$. In the same way, we can define the convolution

$$(\tilde{u}*\tilde{v})(x)=\int_{\mathfrak{g}}\tilde{u}(y)\,\tilde{v}(-y+x)\,dy$$

which gives the homomorphism $\tilde{\mathscr{I}}(K_1)\times\tilde{\mathscr{I}}(K_2)\to\tilde{\mathscr{I}}(K_1+K_2)$.

Note that if u belongs to $\mathscr{I}(\mathfrak{g})$, then we have $\chi_0(g)\,u(g)=u(g)$. In fact, if we restrict the identity $u(g_1 g g_1^{-1})=\chi_0(g_1)^{-1}u(g)$ on the submanifold $\{(g_1,g)\in\mathbf{G}\times\mathbf{G};\ g_1=g^{-1}\}$, then we obtain the above identity. Hence we have $u(g)=\chi_0(g)^{\lambda}u(g)$ for any $\lambda\in\mathbb{C}$. We shall define the isomorphism Φ: $\mathscr{I}(K)\to\tilde{\mathscr{I}}(K)$ by $(\Phi u)(x)=j(x)^{\frac{1}{2}}u(e^x)$. The above remark shows us $(\Phi u)(x)=\chi_0(e^x)^{\lambda}j(x)^{\frac{1}{2}}u(e^x)$ for any λ.

For any $\tilde{u}(x)$ in $\tilde{\mathscr{I}}(\mathfrak{g})$, we have $d\chi_0(x)\,\tilde{u}(x)=0$. In fact, by Lemma 1.1, we have $\langle[A,x],\partial_x\rangle\,\tilde{u}(x)=-d\chi_0(A)\,\tilde{u}(x)$ for any $A\in\mathfrak{g}$. Here, for any $\mathfrak{g}$-valued real analytic function $E(x)$ on $\mathfrak{g}$, $\langle E(x),\partial_x\rangle$ is the vector field defined by $\langle E(x),\partial_x\rangle\,u(x)=\frac{d}{dt}u(x+tE(x))|_{t=0}$. Thus, we have the identity $\langle[A,x],\partial_x\rangle\,\tilde{u}(x)=-d\chi_0(A)\,\tilde{u}(x)$ on $(x,A)\in\mathfrak{g}\times\mathfrak{g}$. If we restrict this on the submanifold $A=x$, we obtain $d\chi_0(x)\,\tilde{u}(x)=0$. These observations also show the following:

Let us denote by $\mathbf{G}_0$ the kernel of χ_0 and $\mathfrak{g}_0$ its Lie algebra. Then, $\mathbf{G}_0$ is a unimodular group. For any $u\in\mathscr{I}(\mathfrak{g})$, we can find an absolute invariant v on $\mathbf{G}_0$ such that $u=v\,\delta(\chi_0)$. Similarly, for any $\tilde{u}\in\tilde{\mathscr{I}}(\mathfrak{g})$, we can find an absolute invariant $\tilde{v}$ on $\mathfrak{g}_0$ such that $\tilde{u}=\tilde{v}\,\delta(d\chi_0)$. Thus we can reduce the study of $\mathscr{I}(\mathfrak{g})$ and $\tilde{\mathscr{I}}(\mathfrak{g})$ into the case where the group is unimodular, although we will not employ this fact.

§3. We Shall Prove Theorem

Take two closed cones K_1 and K_2 of $\mathfrak{g}$ such that $K_1 \cap (-K_2) = \{0\}$ and two functions u in $\mathscr{I}(K_1)$ and v in $\mathscr{I}(K_2)$. Set $w(g) = \int_G u(h)\, v(h^{-1} g)\, dh$, and $\tilde{u} = \Phi u$, $\tilde{v} = \Phi v$, $\tilde{w} = \Phi w$.

In order to prove Theorem we shall compute $\tilde{w}$.

$$\begin{aligned}\tilde{w}(z) &= j(z)^{\frac{1}{2}} \int_G u(h)\, v(h^{-1} e^z)\, dh \\ &= j(z)^{\frac{1}{2}} \int_{\mathfrak{g}} u(e^x)\, v(e^{-x} e^z)\, j(x)\, dx \\ &= j(z)^{\frac{1}{2}} \int_{\mathfrak{g}} dx \int_{\mathfrak{g}} dy\, u(e^x)\, v(e^y)\, j(x)\, \delta(y - \log e^{-x} e^z).\end{aligned}$$

Lemma 3.1. $\delta(y - \log e^{-x} e^z) = j(y)\, j(z)^{-1}\, \delta(z - \log e^x e^y)$.

Proof. We have $\delta(y - f(z)) = |Jf|^{-1}\, \delta(z - f^{-1}(y))$ where Jf is the Jacobian of f. Setting $f(z) = \log e^{-x} e^z$, we shall apply this. We have, for $a \in \mathfrak{g}$

$$f(z + \varepsilon a) = \log e^{-x} e^{z + \varepsilon a}$$

which equals $\log e^{-x} e^z \exp(\varepsilon(1 - e^{-\operatorname{ad} z})/\operatorname{ad} z)\, a$ modulo ε^2. As we can set $y = \log e^{-x} e^z$, this is equal to

$$\log e^y \exp(\varepsilon(1 - e^{-\operatorname{ad} z})/\operatorname{ad} z)\, a = y + \varepsilon \frac{\operatorname{ad} y}{1 - e^{-\operatorname{ad} y}} \frac{1 - e^{-\operatorname{ad} z}}{\operatorname{ad} z} a \quad \text{modulo } \varepsilon^2.$$

Thus we obtain $Jf = \det \dfrac{\operatorname{ad} y}{1 - e^{-\operatorname{ad} y}} \dfrac{1 - e^{-\operatorname{ad} z}}{\operatorname{ad} z}$, which implies the desired result. Q.E.D.

By this lemma, we have

$$\begin{aligned}(3.1) \quad \tilde{w}(z) &= \iint u(e^x)\, v(e^y)\, j(x)\, j(y)\, j(z)^{-\frac{1}{2}}\, \delta(z - \log e^x e^y)\, dx\, dy \\ &= \iint \left(\frac{j(x)\, j(y)}{j(z)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - \log e^x e^y)\, dx\, dy.\end{aligned}$$

We want to prove that this integral equals

$$(\tilde{u} * \tilde{v})(z) = \int \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - x - y)\, dx\, dy.$$

Given a vector space V and two functions $\tilde{u}$ and $\tilde{v}$ on V, given a structure μ of Lie algebra on V, we want to prove for the Lie algebra $\mathfrak{g} = (V, \mu)$ the equality:

$$\int \left(\frac{j(x)\, j(y)}{j(z)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - \log e^x e^y)\, dx\, dy = \int \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - x - y)\, dx\, dy.$$

If we consider the Lie algebra $\mathfrak{g}_t = (V, t\mu)$ i.e. $[x, y]_t = t[x, y]$, the first member of the equality becomes

$$(3.2) \quad \varphi_t(z) = \int \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta\left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) dx\, dy,$$

and this must be equal to the second member which is the value of φ_t for $t=0$. Therefore it is enough to show that φ_t does not depend on t, or equivalently $\frac{\partial}{\partial t}\varphi_t=0$. Let us calculate this derivative.

Lemma 3.2. *Let $F(x,y)$ and $G(x,y)$ be two $\mathfrak{g}$-valued real analytic functions on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$ defined in a neighborhood of the origin. Suppose that $F(0,0)=G(0,0)=0$ and that*

$$x+y-\log e^y e^x=(1-e^{-\operatorname{ad}x})F(x,y)+(e^{\operatorname{ad}y}-1)G(x,y).$$

Then, we have

$$\frac{\partial}{\partial t}\frac{1}{t}\log e^{tx}e^{ty}=\left(\left\langle\left[x,\frac{1}{t}F(tx,ty)\right],\partial_x\right\rangle+\left\langle\left[y,\frac{1}{t}G(tx,ty)\right],\partial_y\right\rangle\right)\frac{1}{t}\log e^{tx}e^{ty}. \tag{3.3}$$

Here $\langle A(x),\partial_x\rangle$ is the derivation defined by

$$(\langle A(x),\partial_x\rangle u)(x)=\frac{d}{d\varepsilon}u(x+\varepsilon A(x))|_{\varepsilon=0}.$$

Proof. Set $F_t=t^{-1}F(tx,ty)$ and $G_t=t^{-1}G(tx,ty)$. Then, the right hand side of (3.3) is the value of

$$t^{-1}\frac{d}{d\varepsilon}\log\exp(tx+\varepsilon[tx,F_t])\exp(ty+\varepsilon[ty,G_t])$$

at $\varepsilon=0$. We shall calculate

$$A=\exp(tx+\varepsilon[tx,F_t])\exp(ty+\varepsilon[ty,G_t])$$

modulo ε^2. We have

$$\begin{aligned}\exp(tx+\varepsilon[tx,F_t])&=e^{tx}\exp\varepsilon\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}(tx)}[tx,F_t]\\&=e^{tx}\exp\varepsilon(1-e^{-\operatorname{ad}tx})F_t\quad\text{modulo }\varepsilon^2,\end{aligned}$$

and similarly $\exp(ty+\varepsilon[ty,G_t])=\exp\varepsilon(e^{\operatorname{ad}ty}-1)G_t\exp ty$ modulo ε^2. Thus, we have

$$\begin{aligned}A&=e^{tx}\exp\varepsilon((1-e^{-\operatorname{ad}tx})F_t+(e^{\operatorname{ad}ty}-1)G_t)e^{ty}\\&=e^{tx}\exp\varepsilon\left(x+y-\frac{1}{t}\log e^{ty}e^{tx}\right)e^{ty}\\&=e^{(t+\varepsilon)x}\exp\varepsilon\left(y-\frac{1}{t}\log e^{ty}e^{tx}\right)e^{ty}\\&=e^{(t+\varepsilon)x}e^{ty}\exp\varepsilon\left(y-\frac{1}{t}\log e^{tx}e^{ty}\right)\\&=e^{(t+\varepsilon)x}e^{(t+\varepsilon)y}\exp-\varepsilon\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\quad\text{modulo }\varepsilon^2.\end{aligned}$$

We have therefore

$$\log A = \log e^{(t+\varepsilon)x} e^{(t+\varepsilon)y} - \frac{\varepsilon}{t} \log e^{tx} e^{ty}$$

$$= \frac{t}{t+\varepsilon} \log e^{(t+\varepsilon)x} e^{(t+\varepsilon)y}.$$

This implies Lemma 3.2. Q.E.D.

This lemma shows in particular

$$(3.4) \quad \frac{\partial}{\partial t} \delta \left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) = (\langle [x, F_t], \partial_x \rangle + \langle [y, G_t], \partial_y \rangle)\, \delta \left(z - \frac{1}{t} \log e^{tx} e^{ty}\right).$$

Therefore, integrating by parts, we have the equality

$$(3.5) \quad p_1 = \int \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y) \frac{\partial}{\partial t} \delta \left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) dx\, dy$$

$$= -\int \Big\{ (\langle [x, F_t], \partial_x \rangle + \langle [y, G_t], \partial_y \rangle + \mathrm{div}_x [x, F_t]$$

$$+ \mathrm{div}_y [y, G_t]) \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y) \Big\} \;\; \delta \left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) dx\, dy.$$

Here div_x (resp. div_y) signifies the divergent with respect to the variable x (resp. y), i.e. the function $\mathrm{div}_x E(x)$ is the sum of the vector field $\langle E(x), \partial_x \rangle$ and its formal adjoint.

If a function $\varphi(x)$ satisfies $\varphi(\mathrm{Ad}(g)x) = \chi(g)\, \varphi(x)$ with a character $\chi(g)$, then we have

$$\langle [A, x], \partial_x \rangle \varphi = (\delta \chi)(A)\, \varphi(x) \qquad \text{for } A \in \mathfrak{g}.$$

Here, $\delta \chi$ is the derivative of χ. Hence, if φ is an absolute invariant, φ and $\langle [A, x], \partial_x \rangle$ commute. Since $(j(x)\, j(y)/j(z))^{\frac{1}{2}}$ is an absolute invariant

$$\langle [x, F_t], \partial_x \rangle + \langle [y, G_t], \partial_y \rangle + \mathrm{div}_x [x, F_t] + \mathrm{div}_y [y, G_t]$$

commutes with this function. Since $\tilde{u}(x)$ is a relative invariant with respect to the character $|\det(\mathrm{Ad}(g); \mathfrak{g})^{-1}|$, we have

$$\langle [A, x], \partial_x \rangle\, \tilde{u}(x) = -\mathrm{tr}(\mathrm{ad}\, A)\, \tilde{u}(x).$$

Thus, we obtain

$$(3.6) \quad p_1 = -\int (\mathrm{tr}(\mathrm{ad}(F_t + G_t), \mathfrak{g}) + \mathrm{div}_x [x, F_t]$$
$$+ \mathrm{div}_y [y, G_t])(j(tx)\, j(ty)/j(tz))^{\frac{1}{2}}\, \tilde{u}(x)\, \tilde{v}(y)\; \delta \left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) dx\, dy.$$

Lemma 3.3. $\frac{\partial}{\partial t} \log j(tx) = \mathrm{tr} \left(\frac{\mathrm{ad}\, x}{e^{t\, \mathrm{ad}\, x} - 1} - \frac{1}{t}\right).$

Proof.

$$\frac{\partial}{\partial t}\log\det\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}(tx)}=\operatorname{tr}\frac{\operatorname{ad}tx}{1-e^{-\operatorname{ad}tx}}\frac{\partial}{\partial t}\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}tx}$$
$$=\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}-\frac{1}{t}\right).$$

By this lemma we have

$$\frac{\partial}{\partial t}\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}.$$

We obtain finally

$$\frac{\partial}{\partial t}\varphi_t=-\int\Big\{\operatorname{div}_x[x,F_t]+\operatorname{div}_y[y,G_t]+\operatorname{tr}\operatorname{ad}(F_t+G_t)$$
$$-\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)\Big\}$$
$$\cdot\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}\tilde{u}(x)\,\tilde{v}(y)\,\delta\left(z-\frac{1}{t}\log e^{tx}e^{ty}\right)dx\,dy.$$

In order to see that $\partial\varphi_t/\partial t$ vanishes, it is enough to show

$$\text{(3.7)}\quad \operatorname{div}_x[x,F_t]+\operatorname{div}_y[y,G_t]+\operatorname{tr}\operatorname{ad}(F_t+G_t)$$
$$-\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)=0$$

when $z=\frac{1}{t}\log e^{tx}e^{ty}$. Since the left hand side of this formula is homogeneous of degree 1 when we assign degree -1 to t and degree 1 to x and y, it is enough to show (3.7) when $t=1$.

For a $\mathfrak{g}$-valued function $A(x)$, let us denote by $\partial_x A$ the endomorphism of $\mathfrak{g}$ defined by $\mathfrak{g}\ni a\mapsto\frac{d}{dt}A(x+ta)|_{t=0}$. Then $\operatorname{div}_x A(x)=\operatorname{tr}\partial_x A(x)$.

Since $\partial_x[x,A(x)]=(\operatorname{ad}x)\,\partial_x A-\operatorname{ad}A$, the formula (3.7) is equivalent to

$$\text{(3.8)}\quad \operatorname{tr}(\operatorname{ad}x)(\partial_x F)+\operatorname{tr}(\operatorname{ad}y)(\partial_y G)=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right)$$

with $z=\log e^x e^y$. This completes the proof of Theorem.

§4. Biinvariant Differential Operators

We consider the algebra $\mathbf{I}(\mathfrak{g})$ of the $\mathbf{G}$-invariant elements of $\mathbf{S}(\mathfrak{g})$. We identify $\mathbf{S}(\mathfrak{g})$ with the algebra of constant coefficient differential operators on $\mathfrak{g}$, hence $\mathbf{I}(\mathfrak{g})$ is identified with the ring of constant coefficient differential operators on $\mathfrak{g}$

invariant by the action of $\mathbf{G}$. We consider the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ and its center $\mathbf{Z}(\mathfrak{g})$. We identify $\mathbf{U}(\mathfrak{g})$ with the algebra of the left invariant differential operators, hence $\mathbf{Z}(\mathfrak{g})$ will be identified with the ring of biinvariant differential operators on $\mathbf{G}$.

We denote by δ the Dirac function on $\mathbf{G}$ supported at the unit e, then $u * \delta = \delta * u = u$. On the other hand, we have $P(u * v) = u * Pv$ for $P \in \mathbf{U}(\mathfrak{g})$. This shows that $Pu = u * P\delta$. We shall denote by the same letter δ the Dirac function on $\mathfrak{g}$ supported at the origin. Similarly if $P \in \mathbf{S}(\mathfrak{g})$, $Pu = u * P\delta = P\delta * u$. We shall denote by $(\exp)^*$ (resp. $(\exp)_*$) the pull-back of functions or differential operators on $\mathbf{G}$ to those on $\mathfrak{g}$ (resp. the inverse of $(\exp)^*$), by the exponential map.

We shall denote by β the linear mapping from $\mathbf{S}(\mathfrak{g})$ onto $\mathbf{U}(\mathfrak{g})$ obtained by symmetrization. We have $(\beta(P)\varphi)(e) = (P\tilde{\varphi})(0)$ with $\tilde{\varphi}(x) = \varphi(e^x)$, hence $(\beta(P)\delta)(e^x) = j(-x)^{-1}(P\delta)(x)$.

For a real analytic function $f(x)$ on $\mathfrak{g}$ defined on a neighborhood of the origin, and $P \in \mathbf{S}(\mathfrak{g})$, we define

$$D(f)P \in \mathbf{S}(\mathfrak{g}) \quad \text{by} \quad ((D(f)P)\delta)(x) = f(-x)P\delta(x),$$

or

$$((D(f)P)\varphi)(0) = P(x \mapsto f(x)\varphi(x))(0).$$

We shall denote by γ the map from $\mathbf{I}(\mathfrak{g})$ onto $\mathbf{Z}(\mathfrak{g})$ defined by $P \mapsto \beta(D(j^{\frac{1}{2}})P)$.

Duflo [4] has proved that for any Lie algebra $\mathfrak{g}$, γ is an isomorphism of the rings $\mathbf{I}(\mathfrak{g})$ and $\mathbf{Z}(\mathfrak{g})$.

We have seen that for any $P \in \mathbf{I}(\mathfrak{g})$,

$$\chi_0(e^x)(P\delta)(x) = P\delta(x),$$

and hence $\chi_0(e^x)$ and P commute. In fact,

$$\chi_0(e^{x-y})(P\delta)(x-y) = (P\delta)(x-y)$$

and this implies

$$\chi_0(e^x)(P\delta)(x-y) = \chi_0(e^y)(P\delta)(x-y).$$

Let us denote by $\mathfrak{g}_0 = \{A \in \mathfrak{g}; \operatorname{tr} \operatorname{ad} A = 0\}$, this implies that $P \in \mathbf{S}(\mathfrak{g}_0)$ (see also [3, 13]). In particular, we have $j(x)^{\frac{1}{2}}(P\delta)(x) = j(-x)^{\frac{1}{2}}(P\delta)(x)$, as $j(x) = (\det e^{-\operatorname{ad} x})\, j(-x)$. So we have $\Phi(\gamma(P)\delta) = P\delta$. If we take $v = \gamma(P)\delta$ then we can get from Theorem the following proposition.

Proposition 4.1. *If* Conjecture *is true for* $\mathfrak{g}$, *then for every* $\tilde{u} \in \tilde{\mathscr{S}}(\mathfrak{g})$ *and* $P \in \mathbf{I}(\mathfrak{g})$

$$((\exp)^* \gamma(P))\tilde{u} = (j(x)^{-\frac{1}{2}} P j(x)^{\frac{1}{2}})\tilde{u}.$$

(*In particular* γ *is an isomorphism of the ring* $\mathbf{I}(\mathfrak{g})$ *and* $\mathbf{Z}(\mathfrak{g})$.)

However, we can get a more precise result applying the same method as in the preceeding paragraphs. Let us denote by $\mathbf{D}$ the ring of the germs of the differential operators at the origin.

Proposition 4.2. *Suppose that* Conjecture *is true for* $\mathfrak{g}$, *then for any* $P \in \mathbf{I}(\mathfrak{g})$

$$j(x)^{\frac{1}{2}}((\exp)^* \gamma(P))\, j(x)^{-\frac{1}{2}} - P$$

is contained in the left ideal of $\mathbf{D}$ *generated by the* $(\langle [A,x], \partial_x\rangle + \operatorname{tr}\operatorname{ad} A)$'s $(A \in \mathfrak{g})$.

(As we have $(\langle [A,x], \partial_x\rangle + \operatorname{tr}\operatorname{ad} A)\,\tilde{u}(x) = 0$ for every $\tilde{u} \in \tilde{\mathscr{I}}(\mathfrak{g})$, this implies Proposition 4.1.)

Proof. Remark that for $P \in \mathbf{S}(\mathfrak{g})$, $\exp^*(\beta(P))$ is the differential operator defined by

$$((\exp)^* \beta(P) u)(x) = P_y(u(\log e^x e^y))|_{y=0},$$

where P_y means that P operates on the y variable. Hence

$$Q = j(x)^{\frac{1}{2}}((\exp)^* \gamma(P))\, j(x)^{-\frac{1}{2}}$$

is the operator:

$$(Q u)(x) = P_y \left(\frac{j(x)^{\frac{1}{2}} j(y)^{\frac{1}{2}}}{j(\log e^x e^y)^{\frac{1}{2}}} u(\log e^x e^y) \right)\Bigg|_{y=0}.$$

As before we introduce the Lie algebra $\mathfrak{g}_t$ and the corresponding operator Q_t, then

$$(Q_t u)(x) = P_y \left(\frac{j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{tx} e^{ty})^{\frac{1}{2}}} u\left(\frac{1}{t} \log e^{tx} e^{ty}\right) \right)\Bigg|_{y=0}.$$

Let us remark that if we define the left ideal $\mathscr{L}_t$ of $\mathbf{D}$ generated by the element $\langle [x, A]_t, \partial_x\rangle + \operatorname{tr}(\operatorname{ad}_t A; \mathfrak{g}_t)$ then for $t \neq 0$ $\mathscr{L}_t = \mathscr{L}$. Hence we have to prove that: $Q_t - P \in \mathscr{L}$. As $Q_0 = P$ it is sufficient to prove that $\frac{\partial}{\partial t} Q_t \in \mathscr{L}$, where

$$\left(\left(\frac{\partial}{\partial t} Q_t \right) u \right)(x) = \frac{\partial}{\partial t} (Q_t u)(x)$$
$$= P_y \left(\frac{\partial}{\partial t} \frac{j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{tx} e^{ty})^{\frac{1}{2}}} u\left(\frac{1}{t} \log e^{tx} e^{ty}\right) \right)\Bigg|_{y=0}.$$

Let F and G be as in Lemma 3.2,

$$F_t(x, y) = \frac{F(t x, t y)}{t}, \qquad G_t(x, y) = \frac{G(t x, t y)}{t},$$

and

$$d(x, y, t) = \tfrac{1}{2} \operatorname{tr} \left(\frac{\operatorname{ad} x}{e^{t \operatorname{ad} x} - 1} + \frac{\operatorname{ad} y}{e^{t \operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{t \operatorname{ad} z} - 1} - \frac{1}{t} \right)$$
$$- \operatorname{tr}((\operatorname{ad} x)\, \partial_x F_t + (\operatorname{ad} y)\, \partial_y G_t)$$

where $z = \dfrac{\log e^{tx} e^{ty}}{t}$. Then we prove:

$$(4.1)\quad \frac{\partial}{\partial t}\left(j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx} e^{ty}\right)\right)$$
$$= d(x,y,t)\, j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx} e^{ty}\right)$$
$$+\sum_{i=1}^{n} \alpha_i(x,y,t)((\langle [e_i,z],\partial_z\rangle + \operatorname{tr}\operatorname{ad} e_i)\cdot u)\left(\frac{1}{t}\log e^{tx} e^{ty}\right)$$
$$+\sum_{i=1}^{n} \langle [y,e_j],\partial_y\rangle \cdot \beta_i(x,y,t)\, u\left(\frac{1}{t}\log e^{tx} e^{ty}\right).$$

Here, $e_i\,(i=1,2,\ldots,n)$ is a basis of the Lie algebra $\mathfrak{g}$, $\langle [e_i,z],\partial_z\rangle$ denotes the adjoint field corresponding to e_i, and $\alpha_i(x,y,t)$, $\beta_i(x,y,t)$ are analytic functions defined near the origin.

To prove (4.1), we compute as in Lemma 3.2

$$\frac{1}{t+\varepsilon}\log e^{(t+\varepsilon)x} e^{(t+\varepsilon)y} \qquad \text{modulo } \varepsilon^2$$
$$=\frac{1}{t}\log e^{tx} e^{\varepsilon x} e^{ty} e^{\varepsilon y} e^{-\varepsilon\frac{\log e^{tx}e^{ty}}{t}}$$
$$=\frac{1}{t}\log e^{tx} e^{ty} e^{\varepsilon\left(e^{-t\operatorname{ad} y}\left(x+y-\frac{\log e^{ty}e^{tx}}{t}\right)\right)}$$
$$=\frac{1}{t}\log e^{tx} e^{ty} e^{\varepsilon(e^{-t\operatorname{ad} y}((1-e^{-t\operatorname{ad} x})F_t+(e^{t\operatorname{ad} y}-1)G_t))}.$$

We write

$$e^{-t\operatorname{ad} y}((1-e^{-t\operatorname{ad} x})F_t+(e^{t\operatorname{ad} y}-1)G_t)$$
$$=(1-e^{-\operatorname{ad}(\log e^{tx}e^{ty})})F_t+(1-e^{-t\operatorname{ad} y})(G_t-F_t).$$

So we have

$$\frac{d}{dt}\left(\frac{1}{t}\log e^{tx} e^{ty}\right)=\langle [z,F_t],\partial_z\rangle\left(\frac{1}{t}\log e^{tx} e^{ty}\right)+\langle [y,G_t-F_t],\partial_y\rangle\left(\frac{1}{t}\log e^{tx} e^{ty}\right)$$

(if $F_t(x,y)=\sum f_i(x,y,t)\,e_i$, and $\mathbf{I}(x)=x$

$$\langle [F_t,z],\partial_z\rangle\left(\frac{1}{t}\log e^{tx} e^{ty}\right)=\sum f_i(x,y)(\langle [e_i,z],\partial_z\rangle\cdot \mathbf{I})\left(\frac{1}{t}\log e^{tx} e^{ty}\right)\Big).$$

We write

$$\frac{\partial}{\partial t} j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx} e^{ty}\right)\Bigg|_{t=t_0}$$
$$=\frac{\partial}{\partial t} j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j\left(t\frac{\log e^{tx} e^{ty}}{t}\right)^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx} e^{ty}\right)\Bigg|_{t=t_0}$$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)$$

$$\cdot j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}\frac{\partial}{\partial t}\left(j\left(t_0\frac{\log e^{tx}e^{ty}}{t}\right)^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\right)\bigg|_{t=t_0}$$

by Lemma 3.3.

Now if $(G_t-F_t)(x,y)=\sum\lambda_i(x,y,t)\,e_i$ we have

$$\langle[y,G_t-F_t],\partial_y\rangle=\sum_{i=1}^{n}\langle[y,e_i],\partial_y\rangle\,\lambda_i(x,y,t)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t).$$

As j is an absolute invariant, j commutes with the adjoint fields.

Hence from the preceeding calculation, we obtain that the left hand side of (4.1) is equal to

$$\left(\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t)\right)$$

$$\cdot j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}(\langle[z,F_t],\partial_z\rangle\cdot u)\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+\sum_{i=1}^{n}\langle[y,e_i],\partial_y\rangle\cdot\left(\beta_i(x,y,t)\,u\left(\frac{1}{t}\log(e^{tx}e^{ty})\right)\right).$$

But, we have

$$\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t)$$
$$=d(x,y,t)+\operatorname{tr}(\operatorname{ad}y\,\partial_y F_t+\operatorname{ad}x\,\partial_x F_t).$$

Let us remark here that if E is in $\hat{\mathbf{L}}$, we have $g\cdot E(x,y)=E(gx,gy)$ for every $g\in\mathbf{G}$. The operator $(\partial_x E)\operatorname{ad}x+(\partial_y E)\operatorname{ad}y$ is the linear operator

$$c\mapsto\frac{d}{d\varepsilon}E(x+\varepsilon[x,c],y+\varepsilon[y,c])|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}E(\exp\varepsilon c\cdot x,\exp\varepsilon c\cdot y)|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\exp\varepsilon c\cdot E(x,y)|_{\varepsilon=0}$$

$$=-[E(x,y),c]$$

hence is the operator $-\operatorname{ad}E$.

We then obtain that the left side of (4.1) is equal to

$$d(x,y,t)\,j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx}e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$
$$-j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx}e^{ty})^{-\frac{1}{2}}((\langle[F_t,z],\partial_z\rangle+\operatorname{tr}\operatorname{ad}F_t)\cdot u)\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$
$$+\sum_{i=1}^{n}\langle[y,e_i],\partial_y\rangle\,\beta_i(x,y,t)\,u\left(\frac{1}{t}\log(e^{tx}e^{ty})\right),$$

which is of the required form.

Now if our conjecture is true for $\mathfrak{g}$, then we can find F and G such that $d(x,y,t)=0$. Now we remark that if $P\in\mathbf{I}(\mathfrak{g})$,

$$P_y\langle[y,e_i],\partial_y\rangle=\langle[y,e_i],\partial_y\rangle P_y$$

hence $(P_y\langle[y,e_i],\partial_y\rangle\,\psi(y))|_{y=0}=0$. Let $R_i(t)$ denote the differential operator

$$(R_i(t)\varphi)(x)=P_y\left(\alpha_i(x,y,t)\,\varphi\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\right)\bigg|_{y=0}.$$

We obtain from (4.1)

$$\frac{\partial}{\partial t}Q_t=\sum_{i=1}^{n}R_i(t)(\langle[e_i,x],\partial_x\rangle+\operatorname{tr}\operatorname{ad}e_i),$$

i.e. $\frac{\partial}{\partial t}Q_t\in\mathscr{L}$. Q.E.D.

Remark. The same proof shows the corresponding fact for biinvariant integral operators.

Remark. We will see in the next section that our conjecture is true for $\mathbf{G}$ solvable; we can easily deduce from Proposition 4.1, the fact that every biinvariant operator on $\mathbf{G}$ is locally solvable, which was already obtained by Rouvière [14] and Duflo-Raïs [5]. In fact P being invariant by the action of $\mathbf{G}$ we can find a fundamental solution for P, which is invariant by $\mathbf{G}$. It follows that $(\exp)^*\,\gamma(P)$ has a local fundamental solution. If $\mathbf{G}$ is exponential solvable, the maps F and G can be constructed in the whole space $\mathfrak{g}$ hence the Propositions 4.1 and 4.2 hold on the whole space $\mathfrak{g}$. So $\exp^*(j(P))$ has a fundamental solution on the space G, (Weita Chang [2] has proven recently that every biinvariant operator on an simply connected solvable group is globally solvable). We recall that M. Duflo has shown that every biinvariant differential operator on a Lie group $\mathbf{G}$ is locally solvable [4].

§ 5. Proof of Proposition 0

First we shall translate our conjecture into another form. Let us write for an $A\in\hat{\mathbf{L}}$

$$2(x+y-\log e^y e^x)=((x+y-\log e^y e^x)+A)+(x+y-\log e^y e^x)-A.$$

Hence we will consider $A \in \hat{\mathbf{L}}$ such that $(x+y-\log e^y e^x)+A$ is divisible by x (i.e. in $[x, \hat{\mathbf{L}}]$) and $(x+y-\log e^y e^x)-A$ is divisible by y (i.e. in $[y, \hat{\mathbf{L}}]$). As $x+y-\log e^y e^x \equiv \frac{1}{2}[x, y] \bmod [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], \hat{\mathbf{L}}]$ and $[x, y]$ is divisible by x and y, we may take A in $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], \hat{\mathbf{L}}]$. We will write $x+y-\log e^y e^x+A=[x, P]$, $A-(x+y-\log e^y e^x)=[y, Q]$, choose $F=\frac{1}{2}\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}P$, $G=-\frac{1}{2}\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}Q$ and translate our conjecture in terms of A.

We shall first give two preliminary lemmata.

Lemma 5.1.

i) $\partial_x \log e^x e^y = \dfrac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}\,\dfrac{e^{\operatorname{ad} x}-1}{\operatorname{ad} x}$

and

ii) $\partial_y \log e^x e^y = \dfrac{\operatorname{ad} z}{1-e^{-\operatorname{ad} z}}\,\dfrac{1-e^{-\operatorname{ad} y}}{\operatorname{ad} y}$.

Here $z=\log e^x e^y$.

Proof. We have, modulo ε^2,

$$\log e^{(x+\varepsilon a)} e^y = \log e^{\varepsilon \frac{e^{\operatorname{ad} x}-1}{\operatorname{ad} x} a} e^x e^y = \log e^{\varepsilon \frac{e^{\operatorname{ad} x}-1}{\operatorname{ad} x} a} e^z$$
$$= z+\varepsilon \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}\,\frac{e^{\operatorname{ad} x}-1}{\operatorname{ad} x} a.$$

The formula ii) is shown in the same way. Q.E.D.

Lemma 5.2. *Let* $a \in \mathfrak{g}$, $f(\lambda)$ *and* $g(\lambda)$ *two power series on* λ. *Then*

$$\operatorname{tr}(f(\operatorname{ad} x)\,\partial_x(g(\operatorname{ad} x)a)) = \operatorname{tr}\left(f(\operatorname{ad} x)\frac{g(0)-g(\operatorname{ad} x)}{\operatorname{ad} x}\operatorname{ad} a; \mathfrak{g}\right).$$

Proof. By linearity, we may assume $g(\lambda)=\lambda^n$. If $n=0$, the lemma is evident. Suppose $n \geqq 1$. Then we have

$$\operatorname{ad}(x+\varepsilon c)^n a-(\operatorname{ad} x)^n a = \varepsilon \sum_{k=0}^{n-1} (\operatorname{ad} x)^{n-1-k}(\operatorname{ad} c)(\operatorname{ad} x)^k a$$
$$= -\varepsilon \sum_{k=0}^{n-1} (\operatorname{ad} x)^{n-1-k} \operatorname{ad}((\operatorname{ad} x)^k a) c \quad \text{modulo } \varepsilon^2.$$

Thus we have

$$\partial_x(g(\operatorname{ad} x)a) = -\sum_{k=0}^{n-1} (\operatorname{ad} x)^{n-1-k} \operatorname{ad}((\operatorname{ad} x)^k a).$$

If $k>0$, $\operatorname{tr} f(\operatorname{ad} x)(\operatorname{ad} x)^{n-1-k} \operatorname{ad}((\operatorname{ad} x)^k a)$ vanishes. In fact, if we set $b=(\operatorname{ad} x)^{k-1} a$ and $\varphi(\lambda)=\lambda^{n-1-k} f(\lambda)$, then

$$\operatorname{tr} \varphi(\operatorname{ad} x) \operatorname{ad}((\operatorname{ad} x) b) = \operatorname{tr} \varphi(\operatorname{ad} x)(\operatorname{ad} x \operatorname{ad} b-\operatorname{ad} b \operatorname{ad} x)=0.$$

Therefore, we obtain

$$\operatorname{tr} f(\operatorname{ad} x)\,\partial_x g(\operatorname{ad} x) a = -\operatorname{tr} f(\operatorname{ad} x)(\operatorname{ad} x)^{n-1}(\operatorname{ad} a). \quad \text{Q.E.D.}$$

Proposition 5.3. Conjecture *is implied from the following: For any Lie algebra* $\mathfrak{g}$, *we can find* A *in* $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], \hat{\mathbf{L}}]$ *satisfying the conditions* i), ii) *and* iii):

i) *There is* P *in* $\hat{\mathbf{L}}$ *such that* $A + x + y - \log e^y e^x = [x, P]$ *and that* P *gives a convergent power series on* $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

ii) *There is* Q *in* $\hat{\mathbf{L}}$ *such that* $A - (x + y - \log e^y e^x) = [y, Q]$ *and that* Q *gives a convergent power series on* $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

iii) $\operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x A - \operatorname{tr} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y A = \operatorname{tr} \left(\frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} - 1 + \frac{1}{2} \operatorname{ad} z \right)$,
where $z = \log e^x e^y$.

Proof. We have $x + y - \log e^y e^x = \frac{1}{2}[x, P] - \frac{1}{2}[y, Q]$. Let $F = \frac{1}{2} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} P$ and $G = -\frac{1}{2} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} Q$. Then we have

$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x}) F + (e^{\operatorname{ad} y} - 1) G.$$

We have $[x, P] = 2(1 - e^{-\operatorname{ad} x}) F$. Therefore, by Lemma 5.2, we have

$$\begin{aligned} \operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x [x, P] &= 2 \operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \frac{0 - (1 - e^{-\operatorname{ad} x})}{\operatorname{ad} x} \operatorname{ad} F + 2 \operatorname{tr} (\operatorname{ad} x) \partial_x F \\ &= 2 \operatorname{tr} (\operatorname{ad} x) \partial_x F - 2 \operatorname{tr} \operatorname{ad} F. \end{aligned}$$

Similarly, we have $-\operatorname{tr} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y [y, Q] = 2 \operatorname{tr} (\operatorname{ad} y) \partial_y G - 2 \operatorname{tr} \operatorname{ad} G$. Set $\tilde{z} = \log e^y e^x$, we have, by Lemma 5.1

$$\partial_x [x, P] = \partial_x (x + y - \log e^y e^x + A) = 1 - \frac{\operatorname{ad} \tilde{z}}{1 - e^{-\operatorname{ad} \tilde{z}}} \frac{1 - e^{-\operatorname{ad} x}}{\operatorname{ad} x} + \partial_x A.$$

Hence, we obtain

$$\begin{aligned} \operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x [x, P] &= \operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x A + \operatorname{tr} \left(\frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} - \frac{\operatorname{ad} \tilde{z}}{1 - e^{-\operatorname{ad} \tilde{z}}} \right) \\ &= \operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x A + \operatorname{tr} \left(\frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} - \frac{\operatorname{ad} z}{1 - e^{-\operatorname{ad} z}} \right). \end{aligned}$$

In the same way, we have

$$-\operatorname{tr} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y [y, Q] = -\operatorname{tr} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y A + \operatorname{tr} \left(\frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} \right).$$

Thus, we obtained

$$\begin{aligned} &\operatorname{tr} (\operatorname{ad} x)(\partial_x F) + \operatorname{tr} (\operatorname{ad} y)(\partial_y G) \\ &\quad = \operatorname{tr} (\operatorname{ad} F) + \operatorname{tr} (\operatorname{ad} G) + \tfrac{1}{2} \operatorname{tr} \left(\frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x A - \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y A \right) \\ &\quad\quad + \tfrac{1}{2} \operatorname{tr} \left(\frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} - \frac{\operatorname{ad} z}{1 - e^{-\operatorname{ad} z}} \right) \end{aligned}$$

$$=\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}-1+\tfrac{1}{2}\operatorname{ad}z\right).$$

Since $\lambda/(1-e^{-\lambda})=\lambda/(e^{\lambda}-1)+\lambda$ and $\operatorname{tr}\operatorname{ad}z=\operatorname{tr}(\operatorname{ad}x+\operatorname{ad}y)$, this equals

$$\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)+\tfrac{1}{4}\operatorname{tr}(\operatorname{ad}x-\operatorname{ad}y)+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right).$$

Hence, it is enough to show that

(5.1) $\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)=\tfrac{1}{4}\operatorname{tr}(\operatorname{ad}y-\operatorname{ad}x).$

However, adding a constant multiple of x (resp. y) to P (resp. Q), we may assume that P (resp. Q) is equal to αy (resp. βx) modulo $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$. However $x+y-\log e^y e^x\equiv-\tfrac{1}{2}[x,y]$ modulo $[[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$ and hence $P\equiv\tfrac{1}{2}y$ (resp. $Q\equiv\tfrac{1}{2}x$). Thus, we have $F\equiv\tfrac{1}{4}y$ (resp. $G=-\tfrac{1}{4}x$) modulo $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$. Since $\operatorname{tr}\operatorname{ad}[\hat{\mathbf{L}},\hat{\mathbf{L}}]=0$, (5.1) is satisfied. Q.E.D.

Let A satisfy i), ii), iii), of the Proposition 4.3. We may remark that $A'(x,y)=\tfrac{1}{4}(A(x,y)-A(y,x)-A(-x,-y)+A(-y,-x))$ satisfies also 1), 2), and 3). This follows from the following observations:

a) if $m(x,y)=x+y-\log e^y e^x$, then $m(x,y)=-m(-y,-x)$;

$$\begin{aligned}m(x,y)-m(y,x)&=\log e^x e^y-\log e^y e^x\\&=(e^{\operatorname{ad}x}-1)\log e^y e^x\\&=(1-e^{\operatorname{ad}y})\log e^x e^y\end{aligned}$$

hence is divisible by x and y.

b) if $t(x,y)=\operatorname{tr}\left(\dfrac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1+\dfrac{\operatorname{ad}z}{2}\right)$ then $t(x,y)=t(y,x)=t(-x,-y)$.

c) for any $E\in[\hat{\mathbf{L}},\hat{\mathbf{L}}]$,

$$\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x E-\operatorname{tr}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y E=\operatorname{tr}\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}\partial_x E-\operatorname{tr}\frac{\operatorname{ad}y}{1-e^{-\operatorname{ad}y}}\partial_y E.$$

In fact the difference is

$$\begin{aligned}\operatorname{tr}(\operatorname{ad}x\,\partial_x E+\operatorname{ad}y\,\partial_y E)&=\operatorname{tr}(\partial_x E\operatorname{ad}x+\partial_y E\operatorname{ad}y)\\&=-\operatorname{tr}\operatorname{ad}E(x,y)\quad\text{(see 4.2)}\\&=0\quad\text{as } E\in[\hat{\mathbf{L}},\hat{\mathbf{L}}].\end{aligned}$$

We will now construct A in $[[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$ such that

$$A(x,y)=-A(y,x)=-A(-x,-y)$$

and i) $x+y-\log e^y e^x+A(x,y)=[x,P]$ and P gives a convergent power series on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$. ((ii) follows then). If $\mathfrak{g}$ is solvable we will be able to prove that A satisfies also the condition iii).

We consider now the condition i):

$$x+y-\log e^y e^x+A(x,y)=[x,P(x,y)].$$

Then for every t, we will have

$$tx+ty-\log e^{ty}e^{tx}+A(tx,ty)=t[x,P(tx,ty)].$$

Hence $\frac{\partial}{\partial t}(tx+ty-\log e^{ty}e^{tx})+\frac{\partial}{\partial t}A(tx,ty)\in[x,\hat{\mathbf{L}}]$ and $\frac{\partial}{\partial t}A(tx,ty)$ satisfies the same antisymmetry relation as A.

Let θ be the vector field $\langle x,\partial_x\rangle+\langle y,\partial_y\rangle$ (or the derivation of $\hat{\mathbf{L}}$ defined by $\theta|\mathbf{L}_n=n\,\mathrm{id}\,\mathbf{L}_n$ where $\mathbf{L}_n$ is the space of elements of $\mathbf{L}$ of degree n) then $t\frac{\partial}{\partial t}B(tx,ty)|_{t=1}=\theta B$, for $B\in\hat{\mathbf{L}}$. We compute

$$\theta(x+y-\log e^y e^x)=x+y-\frac{\mathrm{ad}\,\tilde{z}}{e^{\mathrm{ad}\,\tilde{z}}-1}\cdot y-\frac{\mathrm{ad}\,\tilde{z}}{1-e^{-\mathrm{ad}\,\tilde{z}}}\cdot x$$

with $\tilde{z}=\log e^y e^x$ and we will write $\theta(x+y-\log e^y e^x)$ as an antisymmetric element mod $[x,\hat{\mathbf{L}}]$.

For any real analytic function $g(\lambda)$, we have $g(\mathrm{ad}\,z)=e^{\mathrm{ad}\,x}g(\mathrm{ad}\,\tilde{z})e^{-\mathrm{ad}\,x}$, in particular $g(\mathrm{ad}\,\tilde{z})\cdot x\equiv g(\mathrm{ad}\,z)\cdot x$ modulo $[x,\hat{L}]$ and

$$g(\mathrm{ad}\,z)\cdot y\equiv g(\mathrm{ad}\,\tilde{z})e^{-\mathrm{ad}\,x}y\equiv g(\mathrm{ad}\,\tilde{z})e^{-\mathrm{ad}\,\tilde{z}}\cdot y \quad \text{modulo } [x,\hat{L}].$$

Hence we write modulo $[x,\hat{L}]$

$$\begin{aligned}\theta(x+y-\log e^y e^x)&=\left(1-\frac{\mathrm{ad}\,\tilde{z}}{1-e^{-\mathrm{ad}\,\tilde{z}}}\right)\cdot x+y-\frac{\mathrm{ad}\,\tilde{z}}{1-e^{-\mathrm{ad}\,\tilde{z}}}e^{-\mathrm{ad}\,\tilde{z}}\cdot y\\ &\equiv f(\mathrm{ad}\,\tilde{z})\cdot x+f(\mathrm{ad}\,z)\cdot y,\quad\text{where}\quad f(\lambda)=\left(1-\frac{\lambda}{1-e^{-\lambda}}\right)\\ &\equiv f(\mathrm{ad}\,\tilde{z})\cdot x-f(\mathrm{ad}\,z)\cdot y+2f(\mathrm{ad}\,z)\cdot y.\end{aligned}$$

We write, as $f(0)=0$,

$$\begin{aligned}f(\mathrm{ad}\,z)\cdot y&=\frac{f(\mathrm{ad}\,z)}{e^{\mathrm{ad}\,z}-1}(e^{\mathrm{ad}\,z}-1)y\\ &=\frac{f(\mathrm{ad}\,z)}{e^{\mathrm{ad}\,z}-1}(e^{\mathrm{ad}\,x}-1)y,\end{aligned}$$

therefore $f(\mathrm{ad}\,z)\cdot y\equiv\left(\frac{f(\mathrm{ad}\,\tilde{z})}{e^{\mathrm{ad}\,\tilde{z}}-1}-\frac{f(\mathrm{ad}\,z)}{e^{\mathrm{ad}\,z}-1}\right)\cdot y.$

As $\left(\frac{f(\mathrm{ad}\,\tilde{z})}{e^{\mathrm{ad}\,\tilde{z}}-1}-\frac{f(\mathrm{ad}\,z)}{e^{\mathrm{ad}\,z}-1}\right)\cdot x\equiv 0$ we obtain that

$$\theta(x+y-\log e^y e^x)\equiv f(\mathrm{ad}\,z)\cdot x-f(\mathrm{ad}\,z)\cdot y+2\left(\frac{f(\mathrm{ad}\,\tilde{z})}{e^{\mathrm{ad}\,\tilde{z}}-1}-\frac{f(\mathrm{ad}\,z)}{e^{\mathrm{ad}\,z}-1}\right)\cdot(x+y).$$

Let us denote by $\alpha(x, y)$ the second member of this equality. We have obviously $\alpha(x, y) = -\alpha(y, x)$, hence if we define $\beta(x, y) = \frac{1}{2}(\alpha(x, y) + \alpha(-y, -x))$, β will satisfy the relation $\beta(x, y) = -\beta(y, x) = -\beta(-x, -y)$ and $\theta(x + y - \log e^y e^x) \equiv \beta(x, y)$ mod $[x, \hat{L}]$. We remark that the function $h(\lambda) = \left(1 - \frac{\lambda}{1 - e^{-\lambda}}\right) \frac{1}{e^\lambda - 1}$ verifies $h(\lambda) = -h(-\lambda) - 1$ as $\frac{1}{1 - e^{-\lambda}} = \frac{1}{e^\lambda - 1} + 1$, hence

$$\beta(x, y) = 2\left(\frac{f(\operatorname{ad}\tilde{z})}{e^{\operatorname{ad}\tilde{z}} - 1} - \frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z} - 1}\right) \cdot (x + y) + \tfrac{1}{2}(f(\operatorname{ad}\tilde{z}) + f(-\operatorname{ad}z)) \cdot x - \tfrac{1}{2}(f(\operatorname{ad}z) + f(-\operatorname{ad}\tilde{z})) \cdot y.$$

We can therefore define $A(x, y)$ by the differential equation:

$$\begin{aligned}(5.2)\quad \theta A = {} & 2\left(1 - \frac{\operatorname{ad}z}{1 - e^{-\operatorname{ad}z}}\right) \frac{1}{e^{\operatorname{ad}z} - 1}(x + y) - 2\left(1 - \frac{\operatorname{ad}\tilde{z}}{1 - e^{-\operatorname{ad}\tilde{z}}}\right) \frac{1}{e^{\operatorname{ad}\tilde{z}} - 1} \cdot (x + y) \\ & + \tfrac{1}{2}\left(\frac{\operatorname{ad}z}{e^{\operatorname{ad}z} - 1} - 1\right) \cdot x + \tfrac{1}{2}\left(\frac{\operatorname{ad}\tilde{z}}{1 - e^{-\operatorname{ad}\tilde{z}}} - 1\right) \cdot x - \tfrac{1}{2}\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}} - 1} - 1\right) \cdot y \\ & - \tfrac{1}{2}\left(\frac{\operatorname{ad}z}{1 - e^{-\operatorname{ad}z}} - 1\right) \cdot y,\end{aligned}$$

with the initial condition $A(0, 0) = 0$ ($\tilde{z} = \log e^y e^x$, $z = \log e^x e^y$). As the second member is a convergent power series at the origin, so is $A(x, y)$.

The preceding calculation implies now 1) and 2) of the:

Lemma 5.4.

1) $A(x, y) = -A(y, x) = -A(-x, -y)$,
2) $x + y - \log e^y e^x + A \in [x, \hat{L}]$,
3) $A \in [\hat{L}, [\hat{L}, \hat{L}]]$.

For 3) we remark that $A \in [\hat{L}, \hat{L}]$, and the properties $A(x, y) = -A(-x, -y)$ implies that $A \in [\hat{L}, [\hat{L}, \hat{L}]]$. The lemma is proven. Q.E.D.

Let q be a power series of the two non commutative variables x and y, i.e. q is in the completion of the tensor algebra $\hat{T}(x, y)$ of the vector space $\mathbb{C}x + \mathbb{C}y$. We denote by $c(q)$ the image of q under the map $\hat{T}(x, y) \to \hat{S}(x, y) = C[[x, y]]$, i.e. $c(q)$ is a power series in the commutative variables x and y.

Lemma 5.5. *If* $\mathfrak{g}$ *is solvable,* $\operatorname{tr}(q(\operatorname{ad}x, \operatorname{ad}y))$ *depends only on* $c(q)$.

Proof. There is a basis of $\mathfrak{g}^{\mathbb{C}}$ where the operators $\operatorname{ad}x, \operatorname{ad}y$ are lower triangular, then $\operatorname{ad}[x, y] = \operatorname{ad}x\operatorname{ad}y - \operatorname{ad}y\operatorname{ad}x$ have zeros on the diagonal, and the lemma follows.

Let us write $A = p(\operatorname{ad}x, \operatorname{ad}y) \cdot [x, y]$, where p is a convergent power series in the non commutative variables x and y.

Lemma 5.6. *Let* $\mathfrak{g}$ *be solvable, then*

$$\operatorname{tr}\frac{\operatorname{ad}x}{1 - e^{-\operatorname{ad}x}}\,\partial_x A - \operatorname{tr}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y} - 1}\,\partial_y A$$

$$= -\operatorname{tr}\left((e^{\operatorname{ad} z}-1)\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}\right)\left(\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}\right)p(\operatorname{ad} x, \operatorname{ad} y)\right).$$

Proof. Let us consider the endomorphism

$$\mathfrak{g} \ni c \mapsto \frac{d}{d\varepsilon} p(\operatorname{ad} x + \varepsilon \operatorname{ad} c, \operatorname{ad} y)\cdot[x, y]|_{\varepsilon=0};$$

this is a sum of terms of the form

$$\begin{aligned} &p_1(\operatorname{ad} x, \operatorname{ad} y)\operatorname{ad} c\, p_2(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y] \\ &\quad = -p_1(\operatorname{ad} x, \operatorname{ad} y)\operatorname{ad}(p_2(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y])\cdot c. \end{aligned}$$

The trace of the endomorphism $\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} p_1(\operatorname{ad} x, \operatorname{ad} y)\operatorname{ad}(p_2(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y])$ vanishes by the preceding lemma. So the only term appearing in $\operatorname{tr}\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}\partial_x A$ will come from the trace of the endomorphism

$$c \mapsto \frac{d}{d\varepsilon}\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} p(\operatorname{ad} x, \operatorname{ad} y)[x+\varepsilon c, y]|_{\varepsilon=0}.$$

We obtain that the left side of the equality is:

$$\begin{aligned} &-\operatorname{tr}\left(\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}\operatorname{ad} y + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}\operatorname{ad} x\right)p(\operatorname{ad} x, \operatorname{ad} y) \\ &\quad = -\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}\right)\left(\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}\right)(e^{\operatorname{ad} z}-1)\,p(\operatorname{ad} x, \operatorname{ad} y). \end{aligned}$$

If we restrict our attention when $\mathfrak{g}$ is solvable, we have to prove:

$$-\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}\,\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}(e^{\operatorname{ad} z}-1)\,p(\operatorname{ad} x, \operatorname{ad} y)\right) = \operatorname{tr}\left(\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1 + \tfrac{1}{2}\operatorname{ad} z\right).$$

Hence, considering the commutative ring $\mathbb{C}[[x, y]]$ we need only to prove:

$$\mathbf{c}(p)(x, y) = \left(1 - \frac{x+y}{2} - \frac{x+y}{e^{x+y}-1}\right)\frac{1}{e^{x+y}-1}\,\frac{e^x-1}{x}\,\frac{e^y-1}{y}.$$

We denote by $q(x, y)$ the right hand side.

Let us consider the homomorphism h: $[\hat{\mathbf{L}}, \hat{\mathbf{L}}] \to [\hat{\mathbf{L}}, \hat{\mathbf{L}}]/[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$ and let us write for $m \in [\hat{\mathbf{L}}, \hat{\mathbf{L}}]$, $m = \varphi(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y]$ then clearly $\mathbf{h}(m)$ depends only on $\mathbf{c}(\varphi)$. Therefore, for $f(x, y) \in \mathbb{C}[[x, y]]$, we shall write $f(\operatorname{ad} x, \operatorname{ad} y)[x, y]$ for the element $\varphi(\operatorname{ad} x, \operatorname{ad} y)[x, y]$ modulo $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$ with $f = \mathbf{c}(\varphi)$.

Remark 5.7. If $f(x, y) \in \mathbb{C}[[x, y]]$ is such that $f(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y] \equiv 0$ modulo $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$, then $f(x, y) = 0$. In fact if $\varphi(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y] \in [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$, with $f = c(\varphi)$ then $\operatorname{tr}(\partial_x \varphi(\operatorname{ad} x, \operatorname{ad} y)\cdot[x, y]; \mathfrak{g}) = 0$ for any solvable Lie algebra $\mathfrak{g}$.

On the other hand the same calculation as in Lemma 5.6 shows that

$$\operatorname{tr}(\partial_x(\varphi(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]); \mathfrak{g}) = -\operatorname{tr}(\varphi(\operatorname{ad}x, \operatorname{ad}y)\operatorname{ad}y; \mathfrak{g}).$$

Considering the 2 dimension Lie algebra $\mathfrak{g}$ with basis H, A and relation $[H, A] = A$, we have for $x = x_1 H + x_2 A$, $y = y_1 H + y_2 A$,

$$\operatorname{tr}(\varphi(\operatorname{ad}x, \operatorname{ad}y)\operatorname{ad}y; \mathfrak{g}) = f(x_1, y_1)y_1,$$

hence $f(x_1, y_1)y_1 = 0$, and so is f.

Proposition 0 will result from the following lemma.

Lemma 5.8. *Let*

$$\alpha = \left(1 - \frac{\operatorname{ad}\tilde z}{e^{\operatorname{ad}\tilde z} - 1}\right)\frac{1}{\operatorname{ad}\tilde z}\cdot(x+y-\tilde z) + \tfrac{1}{2}\tilde z - \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}\cdot(x+y-z) - \tfrac{1}{2}z$$

then

1) $\mathbf{h}(\alpha) = q(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]$,
2) $\mathbf{h}(\alpha) = \mathbf{h}(A)$.

Proof. 1) We have as $(x+y-\tilde z) \in [\hat{\mathbf{L}}, \hat{\mathbf{L}}]$,

$$\begin{aligned}\alpha &\equiv \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}\cdot(x+y-\tilde z) + \tfrac{1}{2}\tilde z \\ &\quad - \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}(x+y-z) - \tfrac{1}{2}z \quad \text{modulo } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]] \\ &\equiv \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}(z-\tilde z) - \tfrac{1}{2}(z-\tilde z)\end{aligned}$$

and 1) will result from the following formula:

$$(5.3)\quad (z-\tilde z) \equiv \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\,\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}\,\frac{e^{\operatorname{ad}y}-1}{\operatorname{ad}y}\cdot[x,y] \quad \text{modulo } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]].$$

Proof of (5.3). Let

$$\varphi_1(x,y) = (e^x - e^{-y})^{-1}\left(\frac{e^x-1}{x} - \frac{1-e^{-y}}{y}\right)$$

$$\varphi_2(x,y) = (e^y - e^{-x})^{-1}\left(\frac{1-e^{-x}}{x} - \frac{e^y-1}{y}\,.\right)$$

then φ_1 and φ_2 are analytic functions at the origin. We have

a) $x+y-\tilde z \equiv \varphi_1(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]$,
b) $(x+y-z) \equiv \varphi_2(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y] \quad \text{mod } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$.

For a) we consider

$$e^{\operatorname{ad}x} - e^{-\operatorname{ad}y})(x+y-\tilde z) = (e^{\operatorname{ad}x} - e^{-\operatorname{ad}y})(x+y)$$

(as $(e^{\operatorname{ad}x}-e^{-\operatorname{ad}y})(\tilde{z})=e^{-\operatorname{ad}y}(e^{\operatorname{ad}\tilde{z}}-1)\cdot\tilde{z}=0$) so

$$(e^{\operatorname{ad}x}-e^{-\operatorname{ad}y})(x+y-\tilde{z})=(e^{\operatorname{ad}x}-1)+(1-e^{-\operatorname{ad}y})\cdot(x+y)$$
$$=\left(\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}-\frac{1-e^{-\operatorname{ad}y}}{\operatorname{ad}y}\right)\cdot[x,y]$$

and we obtain the equality a) by Remark 5.7. Now

$$z-\tilde{z}\equiv(\varphi_1-\varphi_2)(\operatorname{ad}x,\operatorname{ad}y)\cdot[x,y]$$

but

$$\varphi_1=(e^z-1)^{-1}\left(\frac{e^z-e^y}{x}-\frac{e^y-1}{y}\right),\quad \varphi_2=(e^z-1)^{-1}\left(\frac{e^x-1}{x}-\frac{e^z-e^x}{y}\right),$$

with $z=x+y$, and

$$\varphi_1-\varphi_2=(e^z-1)^{-1}\left(\frac{(e^x-1)(e^y-1)}{x}+\frac{(e^x-1)(e^y-1)}{y}\right)$$
$$=(e^z-1)^{-1}\frac{z}{xy}(e^x-1)(e^y-1)$$

and this proves Formula (5.3).

Let us prove 2) in Lemma 5.8. We let

$$\zeta(x,y)=\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}(x+y-\tilde{z})+\tfrac{1}{2}\tilde{z},$$

then $\alpha(x,y)=\zeta(x,y)-\zeta(y,x)$. As $x+y-\tilde{z}\in[\hat{\mathbf{L}},\hat{\mathbf{L}}]$, we have

$$\zeta(tx,ty)\equiv\left(1-\frac{t\operatorname{ad}\tilde{z}}{e^{t\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{t\operatorname{ad}\tilde{z}}(tx+ty-\tilde{z}(tx,ty))+\tfrac{1}{2}\tilde{z}(tx,ty)$$
$$\equiv\left(1-\frac{t\operatorname{ad}\tilde{z}}{e^{t\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\left(x+y-\frac{1}{t}\tilde{z}(tx,ty)\right)+\tfrac{1}{2}\tilde{z}(tx,ty)$$
$$\text{modulo } [[\hat{\mathbf{L}},\hat{\mathbf{L}}],[\hat{\mathbf{L}},\hat{\mathbf{L}}]].$$

Here $\tilde{z}$ still denotes $\log e^y e^x$ and $\tilde{z}(tx,ty)=\log e^{ty}e^{tx}$. We have

$$\frac{\partial}{\partial t}\left(\left(1-\frac{tz}{e^{tz}-1}\right)\frac{1}{z}\right)_{t=1}=\frac{1}{e^z-1}\left(\frac{z}{1-e^{-z}}-1\right).$$

So

$$(\theta\zeta)(x,y)\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y-\tilde{z})+\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}(\tilde{z}-\theta\tilde{z})+\tfrac{1}{2}\theta\tilde{z}$$
$$\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)-\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\cdot\theta\tilde{z}+\tfrac{1}{2}\theta\tilde{z}$$

as

$$\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot\tilde{z}=\tfrac{1}{2}\tilde{z}=\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\cdot\tilde{z}.$$

Recalling that

$$\theta\tilde{z}=\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x=\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\operatorname{ad}\tilde{z}\cdot x,$$

we obtain:

$$\begin{aligned}\theta\zeta(x,y)&\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}-1\right)\cdot x+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x\\&\equiv 2\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}x-x+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x\\&\equiv 2\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x-x-\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y.\end{aligned}$$

After antisymmetrization, we obtain

$$\begin{aligned}\theta\alpha(x,y)&\equiv 2\left(1-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}\right)\frac{1}{e^{\operatorname{ad}z}-1}\cdot(x+y)-2\left(1-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\left(\tfrac{1}{2}\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\cdot x-\left(\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}+\tfrac{1}{2}\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}-1\right)\cdot y\\&\equiv\theta A.\quad \text{c.q.f.d.}\end{aligned}$$

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