Lattice points in simple polytopes

Michel Brion and Michèle Vergne

May 1995

1 Introduction

Consider a convex $n$-dimensional polytope $P$ in $\mathbb{R}^n$ with all vertices in the lattice $\mathbb{Z}^n$. In this article, we give a formula for the number of lattice points in $P$, in the case where $P$ is simple, that is if there are exactly $n$ edges through each vertex of $P$. More generally, for any polynomial function $\phi$ on $\mathbb{R}^n$, we express

$$\sum_{m \in \mathbb{Z}^n \cap P} \phi(m)$$

in function of $\int_{P(h)} \phi(x)dx$ where the polytope $P(h)$ is obtained from $P$ by independent parallel motions of all facets. This extends to simple lattice polytopes the Euler-MacLaurin summation formula of Khovanskii-Pukhlikov [8] (valid for lattice polytopes such that the primitive vectors on edges through each vertex of $P$ form a basis of the lattice). As a corollary, we recover results of Pommersheim [9] and Kantor-Khovanski [6] on the coefficients of the Ehrhart polynomial of $P$. Our proof is elementary. In a subsequent article, we will show how to adapt it to compute the equivariant Todd class of any complete toric variety with quotient singularities.

The Euler-MacLaurin summation formula for simple lattice polytopes has been obtained independently by Ginzburg-Guillemin-Karshon [4]. They used the dictionary between convex polytopes and projective toric varieties with an ample divisor class, in combination with the Riemann-Roch-Kawasaki formula ([1], [7]) for complex manifolds with quotient singularities. A counting formula for lattice points in lattice simplices has been announced by Cappell and Shaneson [2], as a consequence of their computation of the Todd class of toric varieties with quotient singularities.
2 Euler-Mac Laurin formula for polytopes

Let $V$ be a real vector space of dimension $n$. Let $M$ be a lattice in $V$. Points of $M$ will be called integral points. The vector space $V$ has a canonical Lebesgue measure $dx$ giving volume 1 to a fundamental domain for $M$. More precisely,

$e_1, e_2, ..., e^n$ be a basis of $V$ such that $M = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e^n$. If $x = x_1e_1 + x_2e_2 + ... + x_ne^n$ is a point in $V$, then $dx = dx_1dx_2...dx_n$.

We denote by $V^*$ the dual vector space to $V$. If $L$ is a lattice in a vector space $W$, we denote by $L^*$ its dual lattice in the dual vector space $W^*$:

$$L^* = \{y \in W^*, (x,y) \in \mathbb{Z}, \text{ for all } x \in L\}.$$ 

We will denote $M^*$ by $N$. Then $N$ is a lattice in $V^*$.

Let $P$ be a convex polytope contained in $V$ with non-empty interior $P^0$. We denote by $\text{vol}(P)$ the volume of $P$ with respect to the measure $dx$ on $V$.

We denote by $\mathcal{F}$ the set of closed faces of $P$. We have

$$\mathcal{F} = \bigcup_{k=0}^n \mathcal{F}(k)$$

where $\mathcal{F}(k)$ is the set of faces of dimension $k$. We have $\mathcal{F}(n) = \{P\}$. By definition, the set $\mathcal{F}(0)$ of extremal points of $P$ is the set of vertices of $P$. The set $\mathcal{F}(1)$ is the set of edges of $P$. A face of codimension 1 is called a facet. A facet $F \in \mathcal{F}(n-1)$ is the intersection of $P$ with an affine hyperplane $\{y; (u_F, y) + \lambda_F = 0\}$. We choose the normal vector $u_F \in V^*$ to the facet $F$ such that $P$ is contained in $\{y, (u_F, y) + \lambda_F \geq 0\}$. In other words, we choose the inward-pointing normal vector $u_F$. This normal vector is determined modulo multiplication by an element of $\mathbb{R}^+$.

If $f$ is a face of $P$, we define

$$(1) \quad \mathcal{F}^f = \{F \in \mathcal{F}(n-1); f \subset F\}.$$ 

We denote by $<f>$ the vector space generated by elements $\{p - q | p \in f, q \in f\}$. If $f$ is in $\mathcal{F}(k)$, then $<f>$ is of dimension $k$.

Define $C_f$ to be the convex cone generated by elements $p - q$ with $p \in P$ and $q \in f$. We say that $C_f$ is the barrier cone to $P$ at its face $f$. The cone $C_f$ contains $<f>$ as its largest linear subspace.

Define $\sigma_f$ to be the polar cone to $C_f$:

$$\sigma_f = \{y \in V^* | (x,y) \geq 0 \text{ for all } x \in C_f\}.$$ 

2
We have
\[ \sigma_f = \sum_{F \in \mathcal{F}} \mathbb{R}^+ u_F. \]

If \( f = \{P\} \), then \( \sigma_f = \{0\} \).

**Definition 1** A convex polytope \( P \) is said to be simple if there are exactly \( n \) edges through each vertex.

For example, in \( \mathbb{R}^3 \), a cube, a pyramid with triangular basis, a dodecahedron are simple.

**Definition 2** A convex polytope \( P \) is a lattice polytope if all vertices of \( P \) are in the lattice \( M \).

Consider a convex lattice polytope \( P \). We can then choose for each facet \( F \) the normal vector \( u_F \) in the dual lattice \( N \). We normalize \( u_F \) in order that \( u_F \) is a primitive element of \( N \), that is if \( tu_F \in N \) then \( t \) is an integer.

Let us number facets of \( P \) as \( F_1, F_2, \ldots, F_d \). We denote by \( u_i \in N \) the normalized normal vector \( u_{F_i} \) to \( F_i \). Let \( \lambda_i = \lambda_{F_i} \). Thus \( P \) is the intersection of \( d \) half-spaces:
\[ P = \{ x \in V, (u_i, x) + \lambda_i \geq 0, 1 \leq i \leq d \}. \]

Consider \( h \in \mathbb{R}^d \), \( h = (h_1, h_2, \ldots, h_d) \). For \( h \in \mathbb{R}^d \), define
\[ P(h) = \{ x \in V, (u_i, x) + \lambda_i + h_i \geq 0, 1 \leq i \leq d \}. \]

Then \( P(h) \) is a convex polytope. Moreover, for small \( h \), \( P(h) \) and \( P \) have the same directions of faces. In particular, \( P(h) \) is simple if \( P \) is simple and \( h \) is small enough.

Let \( C \) be a closed convex cone in a vector space \( W \) with a lattice \( L \). We denote by \( \langle C \rangle \) the vector space spanned by \( C \). The dimension of \( C \) is defined to be equal to the dimension of the vector space \( \langle C \rangle \). We say that \( C \) is acute (or pointed) if \( C \) does not contain any non-zero linear subspace. A cone \( C \) is said to be polyhedral (respectively rational polyhedral) if \( C \) is generated by a finite number of elements of \( W \) (respectively of \( L \)). An acute polyhedral cone \( C \) of dimension \( k \) is said to be simplicial if \( C \) has exactly \( k \)
edges. If $P$ is a convex lattice polytope, then cones $C_f$ associated to faces $f$ of $P$, and their polar cones $\sigma_f$ are rational and polyhedral. The cone $C_f$ is acute if and only if $f$ is a vertex of $P$. The cones $\sigma_f$ are acute for all $f \in \mathcal{F}$.

A finite collection $\Sigma$ of rational polyhedral acute cones in $V^*$ is called a fan if

1) if $\tau$ is a face of an element $\sigma \in \Sigma$, then $\tau \in \Sigma$.
2) if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau \in \Sigma$.

The fan is complete if $\cup_{\sigma \in \Sigma} \sigma = V^*$.

We denote by $\Sigma(k)$ the set of cones in the fan $\Sigma$ of dimension $k$.

If $P$ is a convex lattice polytope, the collection $\Sigma_P = \{\sigma_f, f \in \mathcal{F}\}$ is a complete fan, called the normal fan of $P$ ([11]). If $f \in \mathcal{F}(n-k)$ is a face of codimension $k$ of $P$, then $\sigma_f$ has dimension $k$. The fan $\Sigma_P$ depends only on the directions of the faces of $P$. In particular, the homothetic polytope $qP$ ($q$ positive integer) has same fan than $P$.

**Definition 3** A fan $\Sigma$ is said to be simplicial if each cone $\sigma \in \Sigma$ is simplicial.

A lattice polytope is simple if and only if its fan is simplicial.

Let $\Sigma$ be a simplicial fan. Let $d$ be the cardinal of $\Sigma(1)$. We denote elements of $\Sigma(1)$ as $\ell_1, \ell_2, ..., \ell_d$. Let $u_i, 1 \leq i \leq d$ be the primitive integral vector (with respect to $N$) on the half line $\ell_i$.

**Definition 4** Let $\sigma \in \Sigma$. We denote by $E(\sigma)$ the subset of $\{1, 2, ..., d\}$ consisting of those $i$ such that the half-line $\ell_i$ is an edge of $\sigma$.

The elements $\{u_i, i \in E(\sigma)\}$ are linearly independent. Let

$$U(\sigma) = \oplus_{i \in E(\sigma)} \mathbb{Z}u_i$$

and

$$T(\sigma) = \langle \sigma \rangle / U(\sigma).$$

Let $k$ be the dimension of $\sigma$. Then $U(\sigma) = \mathbb{Z}^k$ is a lattice in $\langle \sigma \rangle = \mathbb{R}^k$ and $T(\sigma) = \mathbb{R}^k / \mathbb{Z}^k$ is a $k$-dimensional torus.

Consider the lattice $N(\sigma) = N \cap \langle \sigma \rangle$ of $\langle \sigma \rangle$. Then $U(\sigma)$ is a sublattice of $N(\sigma)$ which is usually different from $N(\sigma)$. Define

$$G(\sigma) = N(\sigma) / U(\sigma).$$

4
Then $G(\sigma)$ is a finite subgroup of $T(\sigma)$. The order of $G(\sigma)$ is called the multiplicity of $\sigma$ in toric geometry.

If $\tau$ is a face of $\sigma$, we have $U(\tau) = <\tau > \cap U(\sigma)$ as $E(\tau) \subset E(\sigma)$ and elements $u_i, i \in E(\sigma)$, are linearly independent. Thus we have a natural inclusion $T(\tau) \subset T(\sigma)$. This induces a natural inclusion of the finite group $G(\tau)$ in $G(\sigma)$. By definition of $u_i$ as primitive vector, the group $G(\sigma)$ is trivial if $\sigma \in \Sigma(1)$.

**Definition 5** Let $\Sigma$ be a simplicial fan. We denote by $T_\Sigma$ the set obtained from the disjoint union of the tori $T(\sigma)$ ($\sigma \in \Sigma$) by identifying the subsets $T(\sigma \cap \tau)$ of $T(\sigma)$ and $T(\tau)$ for all $(\sigma, \tau) \in \Sigma \times \Sigma$.

We write $T_\Sigma = \cup_{\sigma \in \Sigma} T(\sigma)$. In $T_\Sigma$, we have $T(\sigma) \cap T(\tau) = T(\sigma \cap \tau)$. A visual way to represent the set $T_\Sigma$ associated to a rational fan $\Sigma$ is the following.

We denote by $Q(\sigma)$ the subset
\[
Q(\sigma) = \sum_{i \in E(\sigma)} [0, 1] u_i
\]
of $<\sigma>$. It is clear that the map $Q(\sigma) \mapsto T(\sigma)$ (restriction of the quotient map $<\sigma> \mapsto <\sigma>/U(\sigma)$) is an isomorphism. Furthermore $Q(\sigma) \cap Q(\tau) = Q(\sigma \cap \tau)$. Consider the subset $Q_\Sigma$ of $V^*$ defined by
\[
(5) \quad Q_\Sigma = \cup_{\sigma \in \Sigma} Q(\sigma).
\]
Then the set $Q_\Sigma$ is isomorphic to the set $T_\Sigma$.

Consider the finite subgroup $G(\sigma) \subset T(\sigma)$.

**Definition 6** The subset $\Gamma_\Sigma$ of $T_\Sigma$ is defined to be
\[
\Gamma_\Sigma = \cup_{\sigma \in \Sigma} G(\sigma).
\]

Thus we can think of $\Gamma_\Sigma$ as the union of all finite groups $G(\sigma)$ ($\sigma \in \Sigma$) with equivalence relations given by $G(\sigma \cap \tau) = G(\sigma) \cap G(\tau) \subset T_\Sigma$. In particular the neutral elements of all the groups $G(\sigma)$ are identified to a unique element of $\Gamma_\Sigma$, denoted by $1$. In the identification of $T_\Sigma$ with the subset $Q_\Sigma$ of $V^*$, the subset $\Gamma_\Sigma$ of $T_\Sigma$ is identified with $Q_\Sigma \cap N$.

**Example:** Let $a, b, c$ be pairwise coprime integers and let $P(a, b, c)$ be the simplex in $\mathbb{R}^3$ with vertices
\[
O = (0, 0, 0) \quad A = (a, 0, 0) \quad B = (0, b, 0) \quad C = (0, 0, c).
\]
The rational fan $\Sigma_P$ associated to $P$ has edges

$$\ell_1 = \mathbb{R}^+ e_1 \quad \ell_2 = \mathbb{R}^+ e_2 \quad \ell_3 = \mathbb{R}^+ e_3 \quad \ell_0 = \mathbb{R}^+ (-bce_1 - cae_2 - abe_3)$$

where $(e_1, e_2, e_3)$ is the canonical basis of $(\mathbb{R}^3)^*$. Let us list the non trivial abelian groups $G(\sigma)$ for $\sigma \in \Sigma_P$. Denote by $G(j, \ldots, k)$ the group associated to a cone in $\Sigma_P$ generated by $(\ell_j, \ldots, \ell_k)$. We have

$$G(0, 2, 3) = \mathbb{Z}/bc\mathbb{Z}, \quad G(0, 3, 1) = \mathbb{Z}/ca\mathbb{Z}, \quad G(0, 1, 2) = \mathbb{Z}/ab\mathbb{Z}$$

and

$$G(0, 1) = \mathbb{Z}/a\mathbb{Z}, \quad G(0, 2) = \mathbb{Z}/b\mathbb{Z}, \quad G(0, 3) = \mathbb{Z}/c\mathbb{Z}.$$ 

Our set $\Gamma_\Sigma$ is equal to

$$\Gamma = (\mathbb{Z}/bc\mathbb{Z}) \cup (\mathbb{Z}/ac\mathbb{Z}) \cup (\mathbb{Z}/ab\mathbb{Z})$$

where we identify the common subsets $\mathbb{Z}/a\mathbb{Z}$, $\mathbb{Z}/b\mathbb{Z}$, $\mathbb{Z}/c\mathbb{Z}$.

**Definition 7** A simple lattice polytope $P$ is called a Delzant polytope if each cone $\sigma \in \Sigma_P$ is spanned by a part of a basis of $\mathbb{N}$, i.e. if $G(\sigma) = \{1\}$ for each element $\sigma \in \Sigma_P$.

Equivalently, $P$ is a Delzant polytope if the set $\Gamma_\Sigma$ constructed from the complete fan $\Sigma_P$ of $P$ is reduced to $\{1\}$. This is a very strong hypothesis. As shown by the example above, many lattice simplices are not Delzant. As another example, consider the lattice simplex $P(a)$ in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,a)$ where $a \geq 2$ is an integer. Then $P(a)$ is not Delzant, and the only lattice points in $P(a)$ are its vertices. It follows that $P(a)$ is not a union of Delzant polytopes.

**Remark 2.1** In the dictionary (that we do not use here) between convex polytopes and toric varieties, to a simple lattice polytope $P$ is associated a toric variety with quotient singularities. This toric variety is non singular if and only if the polytope $P$ is a Delzant polytope (see [3]).

We now define for $k \in \{1, 2, \ldots, d\}$ functions $a^k$ on $T_{\Sigma}$ associated to the $d$ elements $\ell_k$ of $\Sigma(1)$. The torus $T(\sigma) = \langle \sigma \rangle / U(\sigma)$ comes equipped with a basis of its lattice of characters: for each $k \in \mathcal{E}(\sigma)$, we define $\chi^k_{\sigma}(g) = e^{2\pi i y^k_{\sigma}}$ if $y = \sum_{j \in \mathcal{E}(\sigma)} y^k_{\sigma} u_j$ is an element of $\langle \sigma \rangle$ representing $g$.

The following lemma is obvious.
Lemma 8 For any $k \in \{1, 2, \ldots, d\}$, there exists a unique function $a^k : T_\Sigma \to \mathbb{C}^*$ such that

1) if $k \notin E(\sigma)$, then $a^k(g) = 1$ for all $g \in T(\sigma) \subset T_\Sigma$.
2) if $k \in E(\sigma)$, then $a^k(g) = \chi^k(\sigma)$ if $g \in T(\sigma) \subset T_\Sigma$.

Observe that there exists a unique continuous function $\xi^k$ on $V^*$ which is linear on each cone of $\Sigma$, and such that $\xi^k(u_k) = 1$ and $\xi^k(u_j) = 0$ for all $j \neq k$. Let us identify $T_\Sigma$ with the subset $Q_\Sigma$ of $V^*$. Then if $g \in T_\Sigma$ is represented by the element $y \in Q_\Sigma$, we have $a^k(g) = e^{2\pi i \xi^k(y)}$.

We can characterize the subset $G(\sigma)$ of $\Gamma_\Sigma$ as follows.

Lemma 9 Let $\sigma \in \Sigma$. We have

$$G(\sigma) = \{ \gamma \in \Gamma_\Sigma, a^k(\gamma) = 1 \text{ for all } k \notin E(\sigma) \}.$$ 

Now we turn to the definition of Todd operators. Consider the analytic function

$$\text{Todd}(z) = \frac{z}{1 - \exp(-z)} = 1 + \frac{1}{2} z + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k}$$

where $B_k$ are the Bernoulli numbers.

Let $a$ be a complex number. Consider more generally the function

$$\text{Todd}(a, z) = \frac{z}{1 - a \exp(-z)}.$$ 

This function is analytic in a neighborhood of 0. Consider its Taylor expansion

$$\text{Todd}(a, z) = \sum_{k=0}^{\infty} c(a, k) z^k$$

for $z$ small.

Let $h$ be a real variable. For any $a \in \mathbb{C}$, consider the operator

$$\text{Todd}(a, \partial/\partial h) = \sum_{k=0}^{\infty} c(a, k)(\partial/\partial h)^k.$$ 

We have

(6) \hspace{1cm} \text{Todd}(1, \partial/\partial h) = 1 + \frac{1}{2} \partial/\partial h + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} (\partial/\partial h)^{2k} \hspace{1cm} 7
while for $a \neq 1$, we have

$$
(7) \quad \text{Todd}(a, \partial/\partial h) = (1 - a)^{-1}\partial/\partial h + \sum_{k=2}^{\infty} c(a, k)(\partial/\partial h)^k.
$$

We denote Todd$(1, \partial/\partial h)$ simply by Todd$(\partial/\partial h)$. If $\phi$ is a polynomial function of $h$, then Todd$(a, \partial/\partial h)\phi(h)$ is well defined, as $(\partial/\partial h)^k \phi = 0$ for large $k$.

**Definition 10** Let $\Sigma$ be a complete simplicial fan. For $g \in T_\Sigma$, define

$$
\text{Todd}(g, \partial/\partial h) = \prod_{k=1}^{d} \text{Todd}(a^k(g), \partial/\partial h_k).
$$

Define

$$
\text{Todd}(\Sigma, \partial/\partial h) = \sum_{\gamma \in \Gamma_\Sigma} \text{Todd}(\gamma, \partial/\partial h).
$$

Recall a version of the Euler-MacLaurin formula. If $\phi(x)$ is a polynomial function on $\mathbb{R}$ and $s \leq t$ are integers, then

$$
(8) \quad \sum_{k=s}^{t} \phi(k) = \text{Todd}(\partial/\partial h_1)\text{Todd}(\partial/\partial h_2)(\int_{s-h_1}^{t+h_2} \phi(x) dx)|_{h_1=0}\, h_2=0.
$$

We will generalize this formula to simple lattice polytopes.

Let $P$ be a simple lattice polytope with $d$ facets. Let $|M \cap P|$ be the number of lattice points in $P$ and $|M \cap P^0|$ be the number of lattice points in the interior $P^0$ of $P$. Let $P(h)$ be the deformed polytope obtained from $P$ after $d$ independent parallel motions of its facets (formula (2)). The main theorem of this article is

**Theorem 11** Let $V$ be a vector space with a lattice $M$. Let $P \subset V$ be a simple lattice polytope and let $\Sigma$ be its associated fan. Then, for small $h$, the volume $\text{vol}(P(h))$ of the deformed polytope $P(h)$ is a polynomial function of $h$, and we have

$$
|M \cap P| = \text{Todd}(\Sigma, \partial/\partial h) \text{vol}(P(h))|_{h=0}
$$

while

$$
|M \cap P^0| = \text{Todd}(\Sigma, -\partial/\partial h) \text{vol}(P(h))|_{h=0}.
$$
More generally, if $\phi$ is a polynomial function on $V$, then $I(\phi)(h) = \int_{P(h)} \phi(x) dx$ is a polynomial function of $h$, for small $h$, and
\[
\sum_{m \in M \cap P} \phi(m) = \text{Todd}(\Sigma, \partial/\partial h) I(\phi)(h)|_{h=0}
\]
while
\[
\sum_{m \in M \cap P^0} \phi(m) = \text{Todd}(\Sigma, -\partial/\partial h) I(\phi)(h)|_{h=0}.
\]

Remark 2.2 If moreover $P$ is a Delzant polytope, then the corresponding set $\Gamma_\Sigma$ is reduced to $\{1\}$, Todd($\Sigma, \partial/\partial h$) is the usual Todd operator considered by Khovanskii and Pukhlikov [8] and Theorem 11 is due to them in this case.

We will prove Theorem 11 in the next section.

2.1 Integral formulas

As an example of our method, let us first prove identity (8). It will be convenient to extend the action of Todd operators to exponential functions $h \mapsto e^{hz}$, for $z$ a small complex number. Indeed, for $z$ small, the series
\[
\text{Todd}(\partial/\partial h)e^{hz} = e^{hz}(1 + \frac{1}{2}z + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k})
\]
is convergent and equal to Todd($z$)e^{hz}.

Let $[s, t]$ be an interval. Then we have
\[
\int_s^t e^{zx} dx = \frac{e^{tx}}{z} - \frac{e^{sx}}{z}.
\]
Assume $t$ and $s$ are integers, then
\[
\sum_{k=s}^{t} e^{kz} = e^{sz}(1 + e^z + \cdots + e^{(t-s)z}) = e^{sz} \frac{1 - e^{(t-s+1)z}}{1 - e^z}
\]
that is
\[
\sum_{k=s}^{t} e^{kz} = \frac{e^{tz}}{1 - e^{-z}} + \frac{e^{sz}}{1 - e^z}.
\]
On the other hand,

\[(11) \int_{s-h_1}^{t+h_2} e^{zx} \, dx = e^{t+z} z - e^{-h_1 z} e^{s z} z.\]

Therefore

\[
\text{Todd}(\partial/\partial h_1)\text{Todd}(\partial/\partial h_2)\left( \int_{s-h_1}^{t+h_2} e^{zx} \, dx \right)_{|h_1=h_2=0} = \text{Todd}(z) \frac{e^{t z}}{z} - \text{Todd}(-z) \frac{e^{s z}}{z}.
\]

Comparing with formula (10), we obtain

\[
\text{Todd}(\partial/\partial h_1)\text{Todd}(\partial/\partial h_2)\left( \int_{s-h_1}^{t+h_2} e^{zx} \, dx \right)_{|h_1=h_2=0} = \sum_{k=s}^{t} e^{k z}.
\]

If we take the Taylor expansion at the origin of this identity in \(z\), we obtain formula (8).

Our proof of Theorem 11 for a \(n\)-dimensional lattice polytope \(P\) will be based on the same approach. Let \(y \in V^*_C\) and let \(P \subset V\) be a polytope (not necessarily a lattice polytope). Define

\[
E(P)(y) = \int_P e^{(x,y)} \, dx.
\]

Then the volume of \(P\) is the value of \(E(P)\) at \(y = 0\).

If moreover \(P\) is a lattice polytope, define

\[
D(P)(y) = \sum_{m \in M \cap P} e^{(m,y)}
\]

and

\[
D(P^0)(y) = \sum_{m \in M \cap P^0} e^{(m,y)}.
\]

Then the number \(|M \cap P|\) of lattice points in \(P\) is the value of \(D(P)\) at \(y = 0\). Although \(E(P)(y)\), \(D(P)(y)\) and \(D(P^0)(y)\) are analytic functions of \(y\), "simple" expressions (similar to formulae (9) and (10)) for \(E(P)(y)\) \(D(P)(y)\) and \(D(P^0)(y)\) will be given only when \(P\) is simple and \(y\) "generic". On this formula for \(E(P(h))(y)\), it will be easy to analyze the action of the Todd operator \(\text{Todd}(\Sigma, \partial/\partial h)\) and to compare it with \(D(P)(y)\).
Recall that $C_f$ denotes the barrier cone to $P$ at its face $f$. Choose $v_0 \in f$. Set
\[ C_P^+(f) = v_0 + C_f. \]
As $C_f$ is invariant by translation by vectors in $< f >$, the affine cone $C_P^+(f)$ does not depend of the choice of $v_0 \in f$. We call it the inward pointing affine cone tangent to $P$ at $f$. Thus $C_P^+(f)$ contains $P$ and $P = \cap_{f \in F} C_P^+(f)$.

Let
\[ C_P^-(f) = v_0 - C_f \]
be the outward pointing affine cone at $f$.

If $E$ is a subset of $V$, we denote by $\chi_E$ its characteristic function.

**Proposition 12** Let $P$ be a convex polytope with non empty interior $P^0$. Then we have the identities
\[
\begin{align*}
(1) \quad \chi_P &= \sum_{f \in F} (-1)^{\dim f} \chi_{C_P^+(f)} \\
(2) \quad (-1)^n \chi_{P^0} &= \sum_{f \in F} (-1)^{\dim f} \chi_{C_P^-(f)} \\
(3) \quad \chi_{\{0\}} &= \sum_{f \in F} (-1)^{\dim f} \chi_{C_f}.
\end{align*}
\]

**Proof.** A version of these identities can be found in [5]. We give another proof, based on Euler's identities
\[
\sum_{f \in F} (-1)^{\dim f} = 1
\]
and, for any point $m$ in the boundary of $P$:
\[
\sum_{f \in F, m \in f} (-1)^{\dim f} = 0.
\]

Let $m$ be an arbitrary point of $V$. We have to prove the relations
(1) If \( m \in P \), then
\[
\sum_{f \in \mathcal{F}, m \in C_\mathcal{P}(f)} (-1)^{\dim f} = 1
\]
(2) If \( m \notin P \), then
\[
\sum_{f \in \mathcal{F}, m \in C_\mathcal{P}(f)} (-1)^{\dim f} = 0.
\]
(3) If \( m \in P^0 \), then
\[
\sum_{f \in \mathcal{F}, m \in C_{\mathcal{P}^0}(f)} (-1)^{\dim f} = (-1)^n.
\]
(4) If \( m \notin P^0 \), then
\[
\sum_{f \in \mathcal{F}, m \in C_{\mathcal{P}^0}(f)} (-1)^{\dim f} = 0.
\]
(5)
\[
\sum_{f \in \mathcal{F}, 0 \in C_f} (-1)^{\dim f} = 1.
\]
(6) If \( m \neq 0 \), then
\[
\sum_{f \in \mathcal{F}, m \in C_f} (-1)^{\dim f} = 0.
\]

First observe that \( m \in P \) if and only if \( m \in C_\mathcal{P}(f) \) for all \( f \in \mathcal{F} \). So assertion (1) is just Euler’s identity (12), and the same holds for (5). For (3), if \( m \in P^0 \), then the unique face \( f \) such that \( m \in C_{\mathcal{P}^0}(f) \) is \( f = P \).

Let us prove (4). Let \( m \notin P^0 \). Consider the convex hull \( H \) of \( P \) and \( m \). Let \( \mathcal{F}(H) \) be the set of faces of \( H \). Let \( \mathcal{F}_m(H) \) be the set of faces of \( H \) containing \( m \). Write \( \mathcal{F}(H) \) as the disjoint union \( \mathcal{F}_m(H) \cup \mathcal{F}_n(H) \). Using relations (12, 13) for the polytope \( H \) and its boundary point \( m \), we obtain
\[
\sum_{g \in \mathcal{F}_n(H)} (-1)^{\dim g} = 1.
\] It is easy to see that faces \( g \) in \( \mathcal{F}_n(H) \) are faces of \( P \) and that these are all the faces \( f \) of \( P \) such that \( m \notin C_{\mathcal{P}}(f) \). Thus we have
\[
\sum_{f \in \mathcal{F}, m \notin C_{\mathcal{P}}(f)} (-1)^{\dim f} = 1.
\] Subtracting Euler identity for \( P \), we obtain (4).

Let us prove (2). Let \( m \notin P \). Consider the convex hull \( H \) of \( P \) and \( m \). Let \( R \) be the closure of \( H \setminus P \). The set \( R \) is not convex in general, however
it can be contracted to \( m \). Therefore the Euler identities holds for \( R \). Let \( \mathcal{F}(R) \) be the set of faces of \( R \). Let \( \mathcal{F}_m(R) \) be the set of faces of \( R \) containing \( m \). Write \( \mathcal{F}(R) \) as the disjoint union \( \mathcal{F}_m(R) \cup \mathcal{F}_n(R) \). Using relations (12, 13) for the polytope \( R \) and its vertex \( m \), we obtain \( \sum_{g \in \mathcal{F}_n(R)} (-1)^{\dim g} = 1 \). It is easy to see that faces \( g \) in \( \mathcal{F}_n(R) \) are faces of \( P \) and that these are all the faces \( f \) of \( P \) such that \( m \not\in C_P(f) \). Thus we have \( \sum_{f \in \mathcal{F}_m, m \not\in C_P(f)} (-1)^{\dim f} = 1 \). Subtracting Euler’s identity for \( P \), we obtain relation (2).

Finally, let us prove (6). We may assume that 0 is an interior point of \( P \). Let \( m \neq 0 \). Choose a small positive number \( t \). Recall that \( C_{tp}(tf) \) denotes the inward pointing affine cone for the face \( tf \) of \( tp \), where \( t \) is a small positive number. Then there exists \( t \) sufficiently small such that \( m \in C_{tf} \) if and only if \( m \in C_{tp}(tf) \). Thus the last relation is deduced from relation (2) by considering the polytope \( tp \) for \( t \) sufficiently small. •

To a point \( m \) of \( V \), we associate its \( \delta \)-measure \( \delta(m) \), defined as follows. For any continuous function \( \phi \) on \( V \), we have \( (\delta(m), \phi) = \phi(m) \). If \( S \) is a discrete subset of \( V \), we denote by \( \delta(S) = \sum_{s \in S} \delta(s) \) its \( \delta \)-measure.

The following proposition follows immediately from Proposition 12.

**Proposition 13** Let \( P \) be a convex lattice polytope. We have the equalities

\[
\delta(M \cap P) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \delta(M \cap C^+_P(f)),
\]

\[
(-1)^n \delta(M \cap P^0) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \delta(M \cap C_P(f)),
\]

\[
\delta(\{0\}) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \delta(M \cap C_f).
\]

We will consider Fourier transforms of the measures \( \delta(M \cap C_f) \). They make sense in the framework of generalized functions. We will use the function notation \( \Theta(y) \) for a generalized function \( \Theta \) on \( V^* \), although the value of \( \Theta \) at a particular point \( y \) may not have a meaning. We denote by \( \int_{V^*} \Theta(y)\phi(y)dy \) the value of \( \Theta \) on a test density \( \phi(y)dy \). We will say that \( \Theta \) is smooth on an open subset \( U \) of \( V^* \) if there exists a smooth function \( \theta(y) \) on \( U \) such that \( \int_{V^*} \Theta(y)\phi(y)dy = \int_{V^*} \theta(y)\phi(y)dy \) for all test functions \( \phi \) with compact support contained in \( U \). Then the value at \( y \in U \) of \( \Theta \) is defined to
be $\theta(y)$. If there exists two smooth functions $f, g$ on $V^*$, with $g$ not identically 0, such that the equation $g(y)\theta(y) = f(y)$ holds in the space of generalized functions on $V^*$, then $\theta$ is smooth on the open set $U = \{y, g(y) \neq 0\}$ and $\theta(y) = f(y)/g(y)$ on $U$.

Consider for example $V = \mathbb{R}$. Consider the discrete measure $\delta(Z) = \sum_{n \in \mathbb{Z}} \delta(n)$. We denote its Fourier transform by $\Theta(y) = \sum_{k \in \mathbb{Z}} e^{iky}$. This means that the generalized function $\Theta(y)$ is the limit in the space of generalized functions of the smooth functions $\sum_{|k| \leq K} e^{iky}$. We have thus for a smooth test function $\phi$ on $\mathbb{R}$

$$\int_{\mathbb{R}} \Theta(y)\phi(y)dy = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{iky}\phi(y)dy.$$ 

Clearly $(1 - e^{iy})(\sum_{k \in \mathbb{Z}} e^{iky}) = 0$ so that $\Theta(y)$ is supported on $2\pi\mathbb{Z}$. In fact, Poisson summation formula is

$$(2\pi)^{-1} \int_{\mathbb{R}} \Theta(y)\phi(y)dy = \sum_{k \in \mathbb{Z}} \phi(2\pi k).$$

Let

$$Z^+ = \{0, 1, 2, 3, \ldots\}.$$ 

Let $a$ be a complex number of modulus 1. Consider the discrete measure $h(a) = \sum_{n \in \mathbb{Z}^+} a^n \delta(n)$ and its Fourier transform $\Theta^+_a(y) = \sum_{n \in \mathbb{Z}^+} a^n e^{iny}$. We have the equality

$$(14) \quad (1 - ae^{iy})\Theta^+_a(y) = 1. $$

Thus the generalized function $\Theta^+_a(y)$ is smooth outside the set $i\text{Log}a + 2\pi\mathbb{Z}$ and for $y \notin i\text{Log}a + 2\pi\mathbb{Z}$,

$$(15) \quad \Theta^+_a(y) = \frac{1}{1 - ae^{iy}}.$$ 

In particular, $\Theta^+_a(y)$ is a rational function of $e^{iy}$. We will generalize this formula to higher dimensions.

Call a meromorphic function $\Phi(y)$ on $V_\mathbb{C}^*$ a rational function of $e^y$ if for some basis $(e_1, \ldots, e_n)$ of $N$, writing $y = \sum_{i=1}^n y_i^* e_i$, $\Phi(y_1^*, \ldots, y_n^*)$ is a rational function of $e^{y_1^*}, \ldots, e^{y_n^*}$. This does not depend on the choice of the basis of $N$. 

14
Definition 14 Let $C$ be a rational polyhedral convex cone in $V$. Denote by $\Theta(C)(y)$ the Fourier transform of $\delta(M \cap C)$:

$$\Theta(C)(y) = \sum_{m \in M \cap C} e^{i(m, y)}.$$

Proposition 15 Let $C$ be a rational polyhedral convex cone in $V$; let $W$ be the largest linear subspace contained in $C$.

(1) The generalized function $\Theta(C)$ is supported on a discrete union of translates of $W^\perp$.

(2) If $C$ is acute (i.e. $W = 0$), then there exists a meromorphic function $\phi$ on $V^*_\mathbb{C}$ such that for $y$ outside a union of a discrete set of affine hyperplanes

$$\Theta(C)(y) = \phi(iy).$$

The order at 0 of the function $\phi$ is at least $-n$. Moreover, $\phi(y)$ is a rational function of $e^y$.

Proof.

Observe that $W$ is a rational subspace of $V$. Moreover, for any $m_0 \in M \cap W$, $(1 - e^{i(m_0, y)}) \Theta(C)(y) = 0$ as $M \cap C$ is invariant by translation by elements of $M \cap W$. Hence the support of $\Theta$ is contained in the set

$$\{ y \in V^* \mid (y, m_0) \in 2\pi \mathbb{Z} \text{ for all } m_0 \in M \cap W \}$$

and this set is a discrete union of translates of $W^\perp$.

It is enough to prove (2) when $C$ is simplicial. Indeed we can always subdivide $C$ by simplicial cones $C_j$ and then $\Theta(C)$ is a sum (with signs) of the generalized functions $\Theta(C_j)$. Now consider a simplicial cone $C \subset V$ intersection of $n$ distinct hyperplanes $(w_k, x) \geq 0$ with $w_k$ in the lattice $N$ of $V^*$. Let $w^k \in V$ be the dual basis of $w_k$. An element $x \in V$ is written as $x = \sum \eta_k w^k$, with $\eta_k = (w_k, x)$. Thus

$$C = \bigoplus_k \mathbb{R}^+ w^k.$$

Let $L = \bigoplus_{k=1}^n \mathbb{Z} w^k$ be the lattice of $V$ spanned by $w^k$. We have

$$L \cap C = \bigoplus_k \mathbb{Z}^+ w^k.$$

Let $\chi$ be a multiplicative character of $L$. Consider the discrete measure

$$h(C, \chi, L) = \sum_{m \in L \cap C} \chi(m) \delta(m)$$

15
and its Fourier transform
\[ \Theta(C, \chi, L)(y) = \sum_{m \in \mathbb{L} \cap C} \chi(m) e^{i(m,y)}. \]

Formula (14) gives
\[ \prod_{k=1}^{n} (1 - \chi(w^k)e^{i(w^k,y)}) \Theta(C, \chi, L)(y) = 1. \]

This equation implies that \( \Theta(C, \chi, L) \) is smooth outside the zeroes of the analytic function \( g(y) = \prod_{k=1}^{n} (1 - \chi(w^k)e^{i(w^k,y)}) \). This zero set is a union of a discrete set of hyperplanes. Thus we obtain

Lemma 16 For \( y \) outside a union of a discrete set of hyperplanes,
\[ \Theta(C, \chi, L)(y) = \frac{1}{\prod_{k=1}^{n} (1 - \chi(w^k)e^{i(w^k,y)})}. \]

With the notation as above, a basis of the dual lattice \( L^* \) to \( L \) consists in \( w_1, \ldots, w_n \). Let \( T = V^*/L^* = V^*/(\oplus_k \mathbb{Z}w_k) \). Then \( T \) is a \( n \)-dimensional torus. Characters of \( L \) are parametrized by \( T \): an element \( g \in T \) gives a character \( \chi_g \) by writing \( \chi_g(x) = e^{2\pi i(x,y)} \) if \( x \in L \) and if \( y \in V^* \) represents \( g \in V^*/L^* \).

Consider the finite subgroup \( G = N/(\oplus_k \mathbb{Z}w_k) \subset T \). Recall that \( N \) is the dual lattice to \( M \). Thus, if \( x \in L \):
\[ \sum_{g \in G} \chi_g(x) = 0, \text{ if } x \notin M \]
\[ \sum_{g \in G} \chi_g(x) = |G|, \text{ if } x \in M. \]

We obtain
\[ \delta(M \cap C) = |G|^{-1} \sum_{g \in G} h(C, \chi_g, L). \]

From Lemma 16, we obtain

Lemma 17 Let \( C \subset V \) be a rational simplicial cone. Then, for \( y \) outside an union of a discrete set of hyperplanes, we have
\[ \Theta(C)(y) = |G|^{-1} \prod_{g \in G} \frac{1}{\prod_{k=1}^{n} (1 - \chi_g(w^k)e^{i(w^k,y)})}. \]
This explicit formula for simplicial cones implies Proposition 15. To check that \( \phi(y) \) is a rational function of \( e^y \), let us state another formula for \( \Theta(C)(y) \). Let \( \alpha_1, \ldots, \alpha_n \) be the primitive vectors of \( M \) on edges \( \mathbb{R}^+w_1, \ldots, \mathbb{R}^+w_n \) of \( C \). Then the set
\[
S(C) = M \cap \{ \sum_{k=1}^n t_k \alpha_k \mid 0 \leq t_k < 1 \}
\]
is finite and
\[
\Theta(C)(y) \cdot \prod_{k=1}^n (1 - e^{i(\alpha_k, y)}) = \sum_{m \in S(C)} e^{i(m, y)}.
\]

In the sequel, we will say that a property holds for generic \( y \in V^* \) if there exists some non-zero analytic function \( g \) such that the property holds for all \( y \) with \( g(y) \neq 0 \).

Let \( P \) be a lattice polytope. Consider the Fourier transform of identities (1), (2), (3) of Proposition 13. By definition, the Fourier transform of \( \delta(M \cap P) \) is the function \( y \mapsto D(P)(iy) \). We choose on each face \( f \) of \( P \) an integral element \( v_0 \in M \). Then \( M \cap C^*_P(f) = v_0 + (M \cap C_f) \) while \( M \cap C^-_P(f) = v_0 - (M \cap C_f) \). We obtain the following lemma.

**Proposition 18** We have the equality of generalized functions:
\[
(1) \quad D(P)(iy) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} e^{i(v_0, y)} \Theta(C_f)(y),
\]
\[
(2) \quad (-1)^n D(P^0)(iy) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} e^{i(v_0, y)} \Theta(C_f)(-y),
\]
\[
(3) \quad 1 = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \Theta(C_f)(y).
\]

For each vertex \( s \), consider the acute cone \( C_s \). Proposition 15 shows that there exists a meromorphic function \( \phi(C_s) \) on \( V^*_C \) (in fact, a rational function of \( e^y \)) such that \( \Theta(C_s)(y) = \phi(C_s)(iy) \) for \( y \) generic.

**Proposition 19** We have the equalities of meromorphic functions on \( V^*_C \):

\[
D(P)(y) = \sum_{s \in \mathcal{F}(0)} e^{(s,y)} \phi(C_s)(y),
\]

\[
D(P^0)(y) = (-1)^n \sum_{s \in \mathcal{F}(0)} e^{(s,y)} \phi(C_s)(-y).
\]

Furthermore
\[
\sum_{s \in \mathcal{F}(0)} \phi(C_s)(y) = 1.
\]

**Proof.** Consider formulae (1) and (2) of Proposition 18. When \( f \in \mathcal{F} \) is a face of strictly positive dimension, the cone \( C_f \) contains the non zero linear vector space \(< f >\) and \( \Theta(C_f) \) is supported on an union of affine spaces of dimension strictly less than \( n \). Thus we obtain the identities above for \( y \in iV^* \) generic. The last identity is obtained from relation (3).

Consider a simple lattice polytope \( P \) with associated fan \( \Sigma \). In this case, we have explicit expressions for the meromorphic functions \( \phi(C_s) \). Indeed \( C_s \) is the intersection of the \( n \) half-spaces \( (u_F, y) \geq 0, F \in \mathcal{F}^* \). The elements \( u_F \) belong to \( N \). Let \( \sigma_s \in \Sigma \) be the polar cone to \( C_s \). Recall the definitions of \( U(\sigma) \) and of \( G(\sigma) \) by formulae (3) and (4). The lattice \( U(\sigma_s) \) is the lattice with \( \mathbb{Z} \)-basis \( (u_F, F \in \mathcal{F}^*) \) and the group \( G(\sigma_s) \) is the group \( N/U(\sigma_s) \). Let \( (m^F_s, F \in \mathcal{F}^*) \) be the dual basis to \( (u_F, F \in \mathcal{F}^*) \). Applying Lemma 17, we obtain
\[
\phi(C_s)(y) = |G(\sigma_s)|^{-1} \sum_{g \in G(\sigma_s)} 1 / \prod_{F \in \mathcal{F}^*} (1 - \chi_g(m^F_s) e^{(m^F_s,y)}).
\]

This leads to the following explicit formulae for \( D(P) \) and \( D(P^0) \) as a sum of meromorphic functions attached to each vertex \( s \) of \( P \). These formulae are the generalization of formula (10) in the 1-dimensional case.

**Proposition 20** For \( y \in V_C^* \) generic, we have
\[
\sum_{m \in M \cap P} e^{(m,y)} = \sum_{s \in \mathcal{F}(0)} e^{(s,y)} |G(\sigma_s)|^{-1} \sum_{g \in G(\sigma_s)} 1 / \prod_{F \in \mathcal{F}^*} (1 - \chi_g(m^F_s) e^{(m^F_s,y)}).
\]
\[
\sum_{m \in M \cap P^0} e^{(m,y)} = (-1)^n \sum_{s \in \mathcal{F}(0)} e^{(s,y)|G(\sigma_s)|^{-1}} \sum_{g \in G(\sigma_s)} \frac{1}{\prod_{F \in \mathcal{F}_s}(1 - \chi_g(m^F_s)e^{-(m^F_s,y)})}.
\]

Let \(P\) be a simple polytope in \(V\). We now analyze the continuous version
\[
E(P)(y) = \int_P e^{(x,y)}dx
\]
of \(D(P)\). We do not assume that \(P\) is a lattice polytope, as we will have to consider deformed polytopes \(P(h)\). Let \(s\) be a vertex of \(P\). We choose inward pointing normal vectors \((u_F, F \in \mathcal{F}^s)\) and dual elements \((m^F_s, F \in \mathcal{F}^s)\). Then the volume of the parallelepiped constructed on \((m^F_s, F \in \mathcal{F}^s)\) is equal to \(|\det(m^F_s)|_{F \in \mathcal{F}^s}\).

The following formula expresses the analytic function \(E(P)\) as a sum over all vertices of meromorphic functions attached to each vertex \(s\) of \(P\). This is the \(n\)-dimensional analogue of formula (9).

**Proposition 21** Let \(P\) be a simple polytope. Let \(y \in V^*_C\) be such that \((m^F_s,y) \neq 0\) for all vertices \(s\) and all \(F \in \mathcal{F}^s\). Then
\[
\int_P e^{(x,y)}dx = (-1)^n \sum_{s \in \mathcal{F}(0)} e^{(s,y)}|\det(m^F_s)|_{F \in \mathcal{F}^s} \frac{1}{\prod_{F \in \mathcal{F}_s}(m^F_s,y)}.
\]

**Proof.** It is possible to give a direct argument for this proposition using Proposition 12 and explicit formulas for Fourier transforms of characteristic functions of simplicial cones. However, we can also deduce the value of \(E(P)\) from the value of \(D(P)\) by a limit argument using Riemann sums to evaluate an integral. Indeed, it is sufficient to prove this formula for lattice polytopes (choosing lattice \(M\) with smaller and smaller fundamental domain). We have
\[
E(P)(y) = \lim_{q \to \infty} \sum_{m \in (M/q) \cap P} e^{(m,y)}
\]
when \(q\) becomes a large integer.

We replace \(M\) by \(M/q\) in the formula of Proposition 20. We obtain
\[
q^{-n} \sum_{m \in (M/q) \cap P} e^{(m,y)} = \sum_{s \in \mathcal{F}(0)} e^{(s,y)|G(\sigma_s)|^{-1}} \sum_{g \in G(\sigma_s)} \frac{1}{\prod_{F \in \mathcal{F}_s}(1 - \chi_g(m^F_s)e^{-(m^F_s,y/q)})}.
\]

We see that only the trivial term \(g = 1\) in each group \(G(\sigma_s)\) will contribute to the limit at \(q = \infty\) and we obtain our proposition, as we observe that \(|G(\sigma_s)|^{-1}\) is the absolute value of \(\det(m^F_s)_{F \in \mathcal{F}^s}\).
In particular, we have \( \text{vol}(P) = \lim_{t \to 0} E(P)(ty) \) and we obtain that for any \( y \) generic

\[
\text{vol}(P) = \frac{(-1)^n}{n!} \sum_{s \in \mathcal{F}(0)} (|\det(m^F_s)|_{F \in \mathcal{F}^*}) \frac{(s, y)^n}{\prod_{F \in \mathcal{F}^*}(m^F_s, y)}.
\]

Let \( P \) be a simple lattice polytope with \( d \) facets \( F_1, F_2, \ldots, F_d \). Let \( h = (h_1, \ldots, h_d) \) be a small parameter of deformation.

Lemma 22 Let \( \phi \) be a polynomial function on \( V \). For \( h \in \mathbb{R}^d \) small, the function \( I(\phi)(h) = \int_{P(h)} \phi(x)dx \) is polynomial in \( h \).

Proof. Consider

\[
E(y)(h) = E(P(h))(y) = \int_{P(h)} e^{(x,y)}dx.
\]

We compute \( E(y)(h) \) using Proposition 21. Let \( s \) be a vertex of \( P \). Let \( \sigma_s \) be the polar cone to \( C_s \). We have

\[
\sigma_s = \sum_{j, F_j \in \mathcal{F}^*} \mathbb{R}^+ u_j.
\]

The subset of \( \{1, 2, \ldots, d\} \) consisting of those \( j \) with \( F_j \in \mathcal{F}^* \) is the set \( \mathcal{E}(\sigma_s) \) of Definition 4. We denote by \( (m^j_s, j \in \mathcal{E}(\sigma_s)) \) the dual basis to \( (u_j, j \in \mathcal{E}(\sigma_s)) \).

When \( h \) is small, the point \( s(h) \) given by

\[
s(h) = s - \sum_{j \in \mathcal{E}(\sigma_s)} h_j m^j_s
\]

is a vertex of \( P(h) \). Thus, for \( y \) generic,

\[
E(y)(h) = \sum_{s \in \mathcal{F}(0)} E(s, y)(h)
\]

where

\[
E(s, y)(h) = (-1)^n |G(\sigma_s)|^{-1} e^{(s - \sum_{j \in \mathcal{E}(\sigma_s)} h_j m^j_s, y)} \prod_{j \in \mathcal{E}(\sigma_s)} (m^j_s, y)^{-1}.
\]
The function $E(y)(h)$ is analytic in $y$. Considering the Taylor expansion of $t \mapsto E(ty)(h)$ at $t = 0$, we obtain for every $k$ and $y$ generic

\begin{equation}
\frac{1}{k!} \int_{P(h)} (x, y)^k dx = \frac{(-1)^n}{(n + k)!} \sum_{s \in \mathcal{F}(0)} \left| G(\sigma_s) \right|^{-1} \frac{\left( (s, y) - \sum_{j \in \mathcal{E}(\sigma_s)} h_j (m^j_s, y) \right)^{n+k}}{\prod_{j \in \mathcal{E}(\sigma_s)} (m^j_s, y)}.
\end{equation}

The polynomial behaviour in $h$ of $\int_{P(h)} (x, y)^k dx$ is apparent on this formula. As this result holds for any $y$ generic and any $k$, we obtain our lemma.

Remark that the Todd operator $\text{Todd}(a, \partial/\partial h)$ is well defined on functions $h \mapsto e^{hz}$ provided $z$ is sufficiently small. We rewrite also formula (17) as

\begin{equation}
E(y)(h) = \sum_{s \in \mathcal{F}(0)} E(s, y)(h)
\end{equation}

with

$$E(s, y)(h) = (-1)^n \left( \prod_{j \in \mathcal{E}(\sigma_s)} e^{-h_j (m^j_s, y)} \right) e^{(s, y)} |G(\sigma_s)|^{-1} \prod_{j \in \mathcal{E}(\sigma_s)} \frac{1}{(m^j_s, y)}.$$ 

This shows that $\text{Todd}(\Sigma, \partial/\partial h)$ is well defined on $E(y)(h) = \int_{P(h)} e^{(x, y)} dx$ provided that $y$ is sufficiently small.

**Theorem 23** Let $P$ be a simple lattice polytope and let $\Sigma$ be the associated fan. If $y \in V^*_C$ is small, then

$$\text{Todd}(\Sigma, \partial/\partial h)(\int_{P(h)} e^{(x, y)} dx)|_{h=0} = \sum_{m \in M \cap P} e^{(m, y)}.$$ 

$$\text{Todd}(\Sigma, -\partial/\partial h)(\int_{P(h)} e^{(x, y)} dx)|_{h=0} = \sum_{m \in M \cap P^0} e^{(m, y)}.$$ 

Consider the Taylor expansion of both members of the first equality above at $y = 0$. We obtain

$$\text{Todd}(\Sigma, \partial/\partial h)(\int_{P(h)} (x, y)^k dx)|_{h=0} = \sum_{m \in M \cap P} (m, y)^k$$

for all $y \in V^*_C$ and $k \in \mathbb{N}$. Thus Theorem 23 implies Theorem 11.
Proof.

Consider formula (19) for the function $E(y)(h)$. For $s$ a vertex of $P$, the function $E(s, y)(h)$ depends only of the variables $h_j$ such that $j \in \mathcal{E}(\sigma_s)$. Let $k$ be such that $k \notin \mathcal{E}(\sigma_s)$. From formula (7), we see that $\text{Todd}(a^k, \partial/\partial h_k)E(s, y)(h) = 0$ if $a^k \neq 1$, while if $a^k = 1$, we have $\text{Todd}(1, \partial/\partial h_k)E(s, y)(h) = E(s, y)(h)$. By Lemma 9, if $\gamma \in \Gamma_{\Sigma}$ is not in $G(\sigma_s)$, then there is $k \notin \mathcal{E}(\sigma_s)$ such that $a^k(\gamma)$ is not 1. Thus $\text{Todd}(\gamma, \partial/\partial h)E(s, y)(h) = 0$ if $\gamma \notin G(\sigma_s)$. We obtain

$$\text{Todd}(\Sigma, \partial/\partial h)E(s, y)(h) = \sum_{\gamma \in G(\sigma_s)} \text{Todd}(\gamma, \partial/\partial h)E(s, y)(h)$$

and for $\gamma \in G(\sigma_s)$

$$\text{Todd}(\gamma, \partial/\partial h)E(s, y)(h) = (\prod_{j \in \mathcal{E}(\sigma_s)} \text{Todd}(a^j(\gamma), \partial/\partial h_j))E(s, y)(h).$$

We have

$$\text{Todd}(a, \partial/\partial h)e^{uh}|_{h=0} = \text{Todd}(a, u) = \frac{u}{1 - ae^{-a}}.$$  

We obtain, for $\gamma \in G(\sigma_s)$,

$$\text{Todd}(\gamma, \partial/\partial h)E(s, y)(h)|_{h=0}$$

$$= (-1)^n \prod_{j \in \mathcal{E}(\sigma_s)} \frac{-(m^j, y)}{(1 - a^j(\gamma)e^{m^j, y})}e^{(s, y)|G(\sigma_s)|^{-1}} \prod_{j \in \mathcal{E}(\sigma_s)} \frac{1}{(m^j, y)}$$

$$= e^{(s, y)|G(\sigma_s)|^{-1}} \prod_{j \in \mathcal{E}(\sigma_s)} (1 - a^j(\gamma)e^{m^j, y}).$$

By definition of $a^j$, $a^j(\gamma) = \chi_\gamma(m^j)$. Comparing with the first formula of Proposition 20, we obtain the first formula of our theorem. By a similar proof, we obtain the second formula. •

3 The coefficients of the Ehrhart polynomial

Let $P$ be a convex lattice polytope in $V$ with non empty interior $P^0$. Consider, for $q$ a positive integer, the polytope $qP$. Let $\phi$ be a function on $V$. Let

$$i(\phi, P)(q) = \sum_{m \in M \cap (qP)} \phi(m)$$

22
and
\[ i(\phi, P^0)(q) = \sum_{m \in M \cap (qP^0)} \phi(m). \]

As a consequence of Proposition 19, let us prove the following generalization of a well-known theorem of Ehrhart.

**Proposition 24.** If \( \phi \) is a homogeneous polynomial function of degree \( k \), then the functions \( q \mapsto i(\phi, P)(q) \) and \( q \mapsto i(\phi, P^0)(q) \) are polynomial of degree \( n + k \). Moreover, we have

\[ i(\phi, P^0)(q) = (-1)^{n+k}i(\phi, P)(-q) \]

and
\[ i(\phi, P)(0) = \phi(0) \]

**Proof.** Observe that the vertices of \( qP \) are the \( qs \) (\( s \) a vertex of \( P \)) and that the tangent cone at \( qs \) to \( qP \) is \( C_s \). Therefore, using Proposition 19, we obtain for generic \( y \):

\[ \sum_{m \in M \cap (qP)} e(m,y) = \sum_{s \in F(0)} e(qs,y) \phi(C_s)(y) \]

and
\[ \sum_{m \in M \cap (qP^0)} e(m,y) = (-1)^n \sum_{s \in F(0)} e(qs,y) \phi(C_s)(-y). \]

Now replace \( y \) by \( ty \) for small non-zero \( t \) and consider the expansion into Laurent series in \( t \). As \( \phi(C_s)(y) \) is of order at least \(-n\), we have \( \phi(C_s)(ty) = \sum_{j \geq -n} t^j a^s_j(y) \) where \( a^s_j(y) \) are homogeneous rational functions of degree \( j \). We thus see that

\[ \frac{1}{k!} \sum_{m \in M \cap (qP)} (m, y)^k = \sum_{s \in F(0)} \sum_{j=0}^{k+n} \frac{1}{j!} q^j(s, y)^j a^s_{k-j}(y) \]

and that
\[ \frac{1}{k!} \sum_{m \in M \cap (qP^0)} (m, y)^k = (-1)^n \sum_{s \in F(0)} \sum_{j=0}^{k+n} \frac{1}{j!} q^j(s, y)^j (-1)^{k-j} a^s_{k-j}(y). \]
We thus obtain the polynomial behaviour in $q$ of $i(\phi, P)(q)$ and $i(\phi, P^0)(q)$ for the polynomial function $\phi(x) = (x, y)^k$ and the first identity as well. As this result holds for all $y$ and $k$, it holds for all polynomial functions on $V$. We also obtain, for $q = 0$ and for $\phi(x) = (x, y)^k$, that

$$\sum \frac{q^k}{k!} i(\phi, P)(0)$$

is equal to the Laurent series expansion of $\sum_{s \in \mathcal{F}(0)} \phi(C_s)(ty)$. Thus we obtain the second identity from formula (3) of Proposition 19 as we have identically $\sum_{s \in \mathcal{F}(0)} \phi(C_s) = 1$.

For $\phi = 1$, the polynomial $i(\phi, P)(q)$ is called the Ehrhart polynomial. We denote it simply by $i(P)$. We write

$$i(P)(q) = |M \cap (qP)| = \sum_{k=0}^{n} q^k a_k(P).$$

It follows from Proposition 24 that the term $a_0(P)$ is equal to 1.

Let us give for example the values of $a_k(P)$ for the simplex $P(a, b, c)$ considered in Example 22.

Let $p$ and $q \geq 1$ be two coprime integers. Let $s(p, q)$ be the Dedekind sum, defined by

$$s(p, q) = \sum_{i=1}^{q} \left( \frac{(i)}{q} \right) \left( \frac{pi}{q} \right)$$

where $((x)) = 0$ if $x$ is integral and $((x)) = x - [x] - \frac{1}{2}$ otherwise.

We have

$$a_3(P) = abc/6, \quad a_2(P) = (ab + bc + ca + 1)/4, \quad a_0(P) = 1$$

while

$$a_1(P) = \frac{1}{12} \left( \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + \frac{1}{ab} \right) + \frac{1}{4} - s(bc, a) - s(ca, b) - s(ab, c) + \frac{3}{4}.$$

This formula, originally due to Mordell, has been generalized recently by Pommersheim [9] and Kantor-Khovanskii [6]: more generally, they computed the coefficient $a_{n-2}(P)$ of the Ehrhart polynomial. Let us show how to deduce
their results from Theorem 11, which gives (in principle) an explicit formula for the Ehrhart polynomial of any simple polytope.

Let $P$ be a simple lattice polytope and let $\Sigma$ be its fan. Consider the Todd operator $\text{Todd}(\Sigma, \partial/\partial h)$. We write it as the sum of its homogeneous components

$$\text{Todd}(\Sigma, \partial/\partial h) = \sum_{k=0}^{\infty} A_k(\partial/\partial h).$$

Lemma 25 We have

$$a_{n-k}(P) = A_k(\partial/\partial h) \text{vol}(P(h)) |_{h=0}.$$

Proof. Let $q$ be a positive integer. We have $|M \cap (qP)| = T(\Sigma, \partial/\partial h) \text{vol}((qP)(h)) |_{h=0}$. Formula (16) shows us that for any $y$ generic

$$\text{vol}((qP)(h)) = \frac{(-1)^n}{n!} \sum_{s \in \mathcal{F}(0)} |G(\sigma_s)|^{-1} \left( (qs, y) - \sum_{j \in \mathcal{E}(\sigma_s)} h_j(m^j_s, y) \right)^n \prod_{j \in \mathcal{E}(\sigma_s)} (m^j_s, y).$$

Thus the lemma follows.

Let us write

$$\text{Todd}(\Sigma, \partial/\partial h) = \text{Todd}(\partial/\partial h) + R(\partial/\partial h)$$

with

$$\text{Todd}(\partial/\partial h) = \prod_{j=1}^{d} \text{Todd}(\partial/\partial h_j)$$

and

$$R(\partial/\partial h) = \sum_{\gamma \in \Gamma_{\Sigma}, \gamma \neq \gamma} \text{Todd}(\gamma, \partial/\partial h).$$

We write

$$\text{Todd}(\partial/\partial h) = \sum_{k=0}^{\infty} T_k(\partial/\partial h)$$

and

$$R(\partial/\partial h) = \sum_{k=0}^{\infty} R_k(\partial/\partial h)$$

where $T_k, R_k$ are homogeneous polynomials of degree $k$. 

25
We have thus
\[ a_{n-k}(P) = m_{n-k}(P) + r_{d-k}(P) \]
with
\[ m_{n-k}(P) = T_k(\partial/\partial h) \cdot \text{vol}(\Delta(h))|_{h=0} \]
and
\[ r_{n-k}(P) = R_k(\partial/\partial h) \cdot \text{vol}(\Delta(h))|_{h=0}. \]

**Lemma 26** We have
\[ T_0(\partial/\partial h) = I, \quad T_1(\partial/\partial h) = \frac{1}{2} \sum_{j=1}^d \partial/\partial h_j \]
while
\[ R_0(\partial/\partial h) = 0, \quad R_1(\partial/\partial h) = 0. \]

**Proof.** The first two equalities follow readily from formula (6).

The groups \( G(\sigma) \) are trivial for \( \sigma \in \Sigma(1) \). Thus there is no element \( \gamma \) of \( \Gamma_{\Sigma} \) with \( a_k(\gamma) = 1 \) for all \( k \) but one, and the last equalities follow from formula (7).

More generally, by the same argument, we obtain the following lemma

**Lemma 27** Assume \( G(\sigma) = \{1\} \) for all cones \( \sigma \in \Sigma \) of dimension at most \( K \). Then \( R_k(\partial/\partial h) = 0 \) and hence \( a_{n-k}(P) = m_{n-k}(P) \) for all \( k \leq K \).

Let \( f \in \mathcal{F} \) be a face of \( P \). Consider the vector space \( < f > \) and its lattice \( M \cap < f > \). We denote by \( \text{vol}(f) \) the volume of the face \( f \) with respect to the Lebesgue measure on \( < f > \) determined by this lattice.

Let \( f \) be a face of codimension 2. Then \( f \) is the intersection of two facets. To simplify notations, we assume that \( f = F_1 \cap F_2 \). Then \( \sigma_f \) (the polar cone of \( C_f \)) is generated by the two normal vectors \( u_1 \) and \( u_2 \) to \( F_1 \) and \( F_2 \). The elements \( u_1, u_2 \) generates the sublattice \( U(\sigma_f) \) of \( N \cap < \sigma_f > \). We have
\[ G(\sigma_f) = (N \cap < \sigma_f >)/U(\sigma_f). \]
As \( u_1 \) is primitive, we can always a \( \mathbb{Z} \)-basis \( n_1, n_2 \) of \( N \cap < \sigma_f > \) such that \( u_1 = n_1 \) and such that \( u_2 = pn_1 + qn_2 \) with \( 1 \leq p \leq q \). The integers \( (p, q) \) are coprime. We have \( q = |G(\sigma)|. \) Recall formula (20) for \( s(p, q) \).
Definition 28 Let $f$ be a face of codimension 2. Using notations above, define
$$
\mu(f) = \frac{1}{4} - \frac{1}{4q} + s(p, q).
$$

Proposition 29 [9], [6] We have
$$
m_n(P) = \text{vol}(P), \quad r_n(P) = 0,
$$
$$
m_{n-1}(P) = \frac{1}{2} \sum_{F \in F(n-1)} \text{vol} F, \quad r_{n-1}(P) = 0,
$$
$$
r_{n-2}(P) = \sum_{f \in F(n-2)} \mu(f) \text{vol}(f).
$$

Proof. Let $\sigma$ be an element of the fan $\Sigma$. Define
$$
e(\sigma, \partial/\partial h) = \prod_{j \in E(\sigma)} \partial/\partial h_j.
$$

From formula (21), we obtain readily

Lemma 30 Let $f$ be a face of $P$ and let $\sigma_f \in \Sigma$ the corresponding cone. Then we have
$$
e(\sigma_f, \partial/\partial h) \text{vol}(P(h))|_{h=0} = |G(\sigma_f)|^{-1} \text{vol}(f).
$$

The values of $a_n(P)$, $a_{n-1}(P)$ are well known and easily obtained from Lemmas 26 and 30. It remains to obtain the value of $r_{n-2}(P)$. Consider the subset $\Gamma_2$ of $\Gamma_\Sigma$ defined by
$$
\Gamma_2 = \cup_{\sigma \in \Sigma(2)} G(\sigma).
$$
Let $\Gamma'_2 = \Gamma_2 - \{1\}$. We have
$$
R_2(\partial/\partial h) = \sum_{\gamma \in \Gamma'_2} \text{odd}(\gamma, \partial/\partial h).
$$

For $\sigma \in \Sigma(2)$, let $G(\sigma)' = G(\sigma) - \{1\}$. Then $\Gamma'_2$ is the disjoint union of the sets $G(\sigma)'$ when $\sigma$ varies in $\Sigma(2)$. We study for $\sigma \in \Sigma(2)$
$$
R_2(\sigma, \partial/\partial h) = \sum_{\gamma \in G(\sigma)'} \text{odd}(\gamma, \partial/\partial h).
$$
To simplify notations, we write $\sigma = \mathbb{R}^+ u_1 + \mathbb{R}^+ u_2$ and as before we choose a $\mathbb{Z}$-basis $n_1, n_2$ of $< \sigma > \cap N$ such that $u_1 = n_1, u_2 = pm_1 + qn_2$. Elements of $G(\sigma) = ( < \sigma > \cap N ) / U(\sigma)$ are represented by elements $jn_2$ with $0 \leq j < q$. If $\gamma = jn_2$ we write $\gamma = (jp/q)u_1 + (j/q)u_2$. By definition $a^k(\gamma) = 1$ except for $k = 1, 2$. We have $a^1(\gamma) = e^{-2\pi ij p/q}$ while $a^2(\gamma) = e^{2\pi i j/q}$. Thus by formula (7)

$$R_2(\sigma, \partial/\partial h) = (\sum_{j=1}^{q-1} (1 - e^{-2\pi ij p/q})^{-1}(1 - e^{2\pi i j/q})^{-1})(\partial/\partial h_1)(\partial/\partial h_2).$$

If $f$ is the face of $P$ such that $\sigma = \sigma_f$, we have by Lemma 30,

$$(\partial/\partial h_1)(\partial/\partial h_2) \text{vol}(P(h))|_{h=0} = q^{-1} \text{vol}(f).$$

By formula (18 a) of [10], we have

$$q^{-1} \sum_{j=1}^{q} (1 - e^{2\pi i j p/q})^{-1}(1 - e^{-2\pi i j/q})^{-1} = -s(-p,q) + \frac{q-1}{4q} = s(p,q) - \frac{1}{4q} + \frac{1}{4}.$$ 

Thus we obtain

$$R_2(\sigma_f) \text{vol}(P(h))|_{h=0} = \mu(f) \text{vol}(f).$$

Summing over all faces of codimension 2, we obtain the stated formula for $r_{n-2}(P)$.

References


Michel Brion:
Ecole Normale Supérieure de Lyon
46 allée d’Italie
69364 LYON Cedex 07. France
E-mail: mbrion@fourier.grenet.fr

Michèle Vergne: E.N.S. et UA 762 du CNRS.
DMI, Ecole Normale Supérieure.
45 rue d’Ulm, 75005 Paris. France
E-mail: vergne@DMI.ENS.FR