Convergence of Krylov Solvers and Choice of Basis and Weighting Set of Functions in the Moment Method Solution of Electrical Field Integral Equation

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Abstract — In Computational Electromagnetics, iterative techniques for solving algebraic linear system of equations are of fundamental importance, since actual problems give rise to linear systems too large to be practically solved by direct methods. In this work we investigate as performances of the major Krylov subspace iterative solver (i.e., GMRES), is affected by different choice of these set of functions. Specifically, we consider the algebraic linear system of equations obtained by reducing the electrical field integral equation (EFIE) from the TM subscattering of a plane wave by a metallic strip. It can be observed that exists a critical threshold \( \Delta_0 \) such that, whenever either the basis or the weight pulses are given with an amplitude greater than \( \Delta_0 \), then the total number of internal loops necessary for taking the relative residual under a definite tolerance \( \epsilon > 0 \) increases all of a sudden, in such a dramatic way that it can even prevent the process at all from convergence. We try to explain this numerical behavior by inquiring the relationship between the MoM matrix condition number and the number of overall iterations necessary to numerical convergence.

1. INTRODUCTION

It is not easy to give a comprehensive treatment of so many issues involved by analytical and numerical methods in the field of Computational Electromagnetics, and specifically by the method of moments, for solving integral equations relevant to electromagnetic theory. These include a very large amount of mathematical elements and capabilities, ranging from numerical integration and differentiation to the feasibility of converting operator equations to a discrete system of algebraic equations, from the choice of suitable sets of basis and weight functions to a deep knowledge of linear algebra, matrix manipulations, algorithms and actual implementations of iterative and direct solvers and so on going [1]. The lack of analytical solutions provides a further item of difficulty when trying in formal terms to expound mathematical clues intended to be potentially notable from a practical point of view. Thus it is common to consider simple (i.e., typically scalar) problems for preliminary tests and then to extrapolate general ideas to be applied to more complex cases. In fact, this is the approach we follow in the present paper, where we deal with the scalar problem of computing the current density induced on the surface of an ideal metallic strip when struck by a planar wave of a given wavelength \( \lambda \) which is supposed to be small or large with respect to the width \( 2w \) of the strip section. A harmonic time variation \( \exp(i\omega t) \) is assumed and suppressed throughout. The problem serves as an useful and somehow simple example in electromagnetics for tackling basic concepts and discussing practical issues for numerical methods. Specifically, we investigate and try to characterize the behavior of iterative techniques for solving algebraic linear system of equations. Nowadays these are of basic importance, since actual problems give rise to linear systems that are too large to be practically solved by direct methods. There exists much study regarding the numerical aspects of such subjects, but it's not clear at all how the efficiency of iterative solvers are conditioned by the choosing of the basis and weighting functions in the Moment Method (MM) [2, 3]. In this sense, we study as performances of the Generalized Minimum Residual (GMRES) Krylov iterative solver [4] is affected by different choice of these set of functions.

2. A WIDTH-VARYING PULSE-PULSE MOMENT METHOD SCHEME

An implementation of the moment method exploiting width-varying rectangular pulses for both the basis and the weight sets of functions is adopted for solving the EFIE

\[
E_0 e^{i\beta} (x \cos \phi_i + y \sin \phi_i) = \frac{1}{4} \beta \gamma_0 \int_0^{2w} H_0^{(2)} (k|x - \xi|) J(\xi) d\xi \tag{1}
\]
arising from the plane wave TM$_z$ scattering from a planar metallic strip, supposed to have finite width $2w$ and to be indefinitely extended in the direction of the $z$-axis [5]. The medium is the free space, with constitutive parameters ($\varepsilon_0, \mu_0$) and intrinsic impedance $\eta_0$. Equation (1) is indeed a standard Fredholm integral equation of the first kind in the unknown current density $J(\xi)$. Since there exists no chance to express its solution in a simple closed form, we have discretized it through the Method of Moments [6] and solved the subsequent system of linear equations by means of the GMRES. First we spend some words about the descent scheme in order to make clear our plan. As well-known, provided $U = \{u_p(\cdot)\}_{p=1}^\infty$ is some Hilbert complete set of vectors for the unitary space $L^2(a, b)$ on the complex field of all functions being square-summable on the segment $(0, 2w)$ according to Lebesgue (even if in a generalized sense), we can expand the unknown current through

$$J(\xi) = \sum_{p=1}^\infty I_p u_p(\xi), \quad 0 < \xi < 2w$$

and then we form the moment linear system of equations

$$ZI = V,$$

by letting the $(p, q)$-entry in the impedance matrix $Z \in \mathbb{C}^{n \times n}$ be $(Au_q, u_p)$ and arresting the process at a given level $n$ of discretization. Here, $(\cdot, \cdot)$ denotes the usual inner product in $L^2(a, b)$ and $W = \{w_q(\cdot)\}_{q=1}^\infty$ is another Hilbert basis of the same space (eventually, $W = U$). Furthermore, $A : L^2(a, b) \mapsto L^2(a, b)$ is the Hilbert-Schmidt linear operator realizing the mapping

$$u(\cdot) \mapsto \int_0^{2w} H_0^{(2)}(\beta |x - \xi|)u(\xi)d\xi, \quad 0 < x < 2w$$

In fact, in the framework of our numerical experiments, we have set both $u_q(\cdot)$ and $w_p(\cdot)$ to be rectangular pulse functions centered in the points $x_p$ and $x_q$, of width $\Delta_u$ and $\Delta_w$, respectively. Here, $x_1, x_2, \ldots, x_n$ are the middle points of an uniform division of the real segment $(0, 2w)$ into $n$ internally disjoint subintervals of constant width $\Delta = 2w/n$, so that $x_k = \frac{1}{2}(2k - 1)\Delta$ for any $k = 1, 2, \ldots, n$; and $0 < \Delta_u, \Delta_w \leq \Delta$. For completeness, we remark the rectangular pulse of width $T > 0$ centered in the point $t_0$ is the function

$$\mathbb{R} \mapsto \mathbb{R} : t \mapsto \text{rect}\left(\frac{t - t_0}{T}\right) = \left\{\begin{array}{ll} 1 & \text{if } |t - t_0| \leq \frac{1}{2}T \\ 0 & \text{otherwise} \end{array}\right.$$  

3. NUMERICAL RESULTS

Operatively, we have studied the way some numerical properties related to the system (3) are dependent on the actual choice of the basis and weight sets $U_n = \{u_p(\cdot)\}_{p=1}^n$ and $W_n = \{w_q(\cdot)\}_{q=1}^n$, by tuning the widths $\Delta_u$ and $\Delta_w$ of their elements, respectively. In detail, it can be observed (see Fig. 1(a)) there is a critical threshold $\Delta_0 > 0$ such that, whenever $\Delta_u$ or $\Delta_w$ is greater than $\Delta_0$, then the condition number $\kappa$ (as a two-variable function of $\Delta_u$ and $\Delta_w$) presents an abrupt growing, getting even many orders of magnitude larger than in the square $0 < \Delta_u, \Delta_w < \Delta_0$ and describing an irregular surface in the three dimensional space, whereas it maintains a very flat (practically constant) and relatively low profile as far as $0 < \Delta_u, \Delta_w < \Delta_0$. In principle, this wouldn’t be particularly significant by itself, since the pick value attained by $\kappa$ is still too small for resulting into trouble and causing a significant loss of numerical precision. This not standing, it is notable that, for the very same values of $\Delta_u$ and $\Delta_w$ corresponding to the entry into the instability region for the condition number $\kappa$, the total number of iterations necessary for the GMRES to pull down the residual vector 2-norm under a pre-fixed tolerance $\epsilon > 0$ (in our tests, $\epsilon = 1e - 6$) grows dramatically, provided that $\max(\Delta_u, \Delta_w) > \Delta_0$, in such a way that the _tout-courte_ convergence process is indeed very slow (see Fig. 1(b)) and maintains constant not depending on the actual values of the condition number. This seems to be related to the wide spreading of the spectrum through the complex plane, that can be observed entering the instability region, but we feel this is not enough to explain the numerical behavior of the GMRES, suggesting that deeper investigations are necessary for giving a truly satisfactory answer.
4. CONCLUSIONS

Our numerical experiments reveal that, even in the hypothesis of an uniform dissection of the real segment $[0, 2w]$ into $n$ subintervals of amplitude $\Delta$ there exists a critical threshold $\Delta_0$ such that, whenever either the basis or the weight pulses are given with an amplitude greater than $\Delta_0$, then the total number of internal loops necessary for taking the relative residual under a definite tolerance $\epsilon > 0$ increases all of a sudden, in such a dramatic way that it can even prevent the process at all from convergence. Finally, we try to explain this numerical behavior by inquiring into the spectral properties of both the starting integral equation and the impedance matrix $Z$ focusing our attention on its condition number. In conclusion, i) it seems that the condition number solely cannot
be adopted as a reliable quantity to measure the effectiveness of the numerical solutions coming out from the application of the method of moments (3) to the problems relevant with electromagnetics; ii) the choice of the basis and weight sets $U$ and $W$ is critical for ensuring the method can eventually result into a faithful discrete representation of the analytical operator equations.

REFERENCES


