On the Preconditioning of the Algebraic Linear Systems Arising from the Discretization of the EFIE

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Abstract — The rate of convergence of the Generalized Minimum Residual Method (GMRES) applied to the dense linear systems arising from discretization of EFIE integral equation by Method of Moment (MoM) depends heavily by preconditioning. In this work, we evaluate the performances of a simple preconditioner based on the skew hermitian component $S$ of MoM impedance matrix $Z$ for the case of plane wave scattering by CEP bodies.

1. INTRODUCTION

The integral equations involved in electromagnetics are discretized by subdomain or entire domain-type MoM. These eventually result into complex linear systems of equations, whose size depends on both the meshing fineness and the electrical dimensions of the structures at hand. So MoM is ultimately dependent in a critical way on the availability of efficient solvers for algebraic dense linear systems of equations, and this is a topic of active research at present in the computational electromagnetics community [1]. Though at times we can still think of employing direct factorization, as far as the structures are small or the mesh is coarse, this approach is useless in practice, as the size of the discretized problem grows. Then the only concrete alternative is the use of nonstationary iterative methods, i.e., Krylov Subspace methods. In these last years a great interest has been deserved to the Generalized Minimum Residual Method (GMRES), as it has been variously tested to offer higher performances (see [2] and references within) than the standard Conjugate Gradient Method (CGM), especially for indefinite systems. However, in the case of algebraic linear systems produced by discretization of EFIE by MoM, GMRES converges very slowly or not at all. This behavior is caused by unfavorably spectral properties of the impedance matrix $Z$. Therefore, it is crucial to use GMRES in conjunction with an efficient preconditioner. In technical literature, researches relevant preconditioning have been mainly directed toward the design of preconditioners as AINV, SPAI and ILUT (see [1] and references within) based on purely algebraic techniques. In this work, we evaluate the performances of a preconditioner derived by simple considerations on the nature of EFIE. It is based on the skew hermitian component $S$ of the $Z$ impedance matrix. This choice ensures that the spectrum of the preconditioned system will be located around the point $(1,0)$ of the complex plane [4] causing an enhancement of GMRES convergence rate as confirmed by numerical experiments.

2. GMRES BASIC THEORY

In this section we review briefly GMRES method for solving an algebraic linear system of the form

$$Zx = b,$$

where $Z \in \mathbb{C}^{n,n}$ denotes MoM impedance matrix, $b \in \mathbb{C}^{n}$ the righthand side (the excitation vector) and $x \in \mathbb{C}^{n}$ the solution vector. Without loss of generality, let us consider the equivalent system

$$Zy = r_0,$$  \hspace{1cm} where $r_0 = -Zx_0 + b$ and $y = x - x_0.$

The GMRES method is based upon the Arnoldi recursion, which can be simplified as it follows:

1. Given a vector $v_1$ with $\|v_1\| = 1$, compute $v_{k+1} = Zv_k$, for $k = 2, 3, \ldots$

2. For each $k$ and for $i = 1, 2, \ldots, k$, compute

$$h_{i,k} = v_i^H v_{k+1} \text{ and } v_{k+1} = v_{k+1} - h_{i,k}v_i.$$  \hspace{1cm} (3)

3. For each $k$, compute $h_{k+1,k} = \|v_{k+1}\|$ and $v_{k+1} = v_{k+1}/h_{k+1,k}.$
Theoretically, $V_k = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of the Krylov subspace $K_k(Z, v_1)$ and the Hessenberg matrix $H_k = (h_{ij})$ is a matrix representation of the Krylov subspace $K_k(Z, v_1)$ with respect to the $V_k$. In fact, the preceding implementation is a modified Gram-Schmidt orthogonalization. In GMRES, $v_1 = r_0/\|r_0\|$ in the first step of Arnoldi recursion. Theoretically, GMRES select its iterates (say $y_k^G$ and $y_k^F$) from the same Krylov subspaces but subject to different constraints on the corresponding residual vectors, $r_k^G$ and $r_k^F$. At the $k$-th iteration in GMRES, the iterate is selected so that the norm of the corresponding residual vector $r_k^G$ is minimized over the $k$-th Krylov subspace corresponding to $Z$ and $r_0$. For each $k$,

$$\|r_k^G\| = \min_{y \in K_k(Z, r_0)} \|r_0 - Zy\| = \|r_0 - Z y_k^G\|,$$

which Saad proved equivalent to the Petrov-Galerkin condition $r_k^G \perp K_k(Z, r_0)$ [2].

3. PRECONDITIONING

From a theoretical point of view, $Z$ obeys to the Half Plane Condition [2], i.e., its eigenvalues must have real part greater than zero. This is a direct consequence of the Poynting theorem applied on EFIE [3]. The Half Plane Condition is a desirable property because it ensures the convergence of GMRES without preconditioning [2]. However, due to discretization errors, $Z$ always lacks of it and preconditioning becomes mandatory. As well-known, this operation transforms the original algebraic linear system (1) into another one

$$P^{-1}Zx = P^{-1}b$$

named preconditioned system, which has spectral properties able to enhance the convergence rate. In the case of the linear systems arising from EFIE discretization, $P$ can be selected as follows. Let

$$Z = S + H = S(U + S^{-1}H)$$

where $H = \frac{1}{2}(Z + Z^*)$ and $S = \frac{1}{2}(Z - Z^*)$ are the hermitian part and the skew hermitian part of $Z$ and $U = \text{diag}(1, 1, \ldots, 1)$ is the identity matrix of the same size as $Z$, respectively. Since in the near-field zone we have a predominance of the reactive energy on the active one, we can expect that $\|S\| \gg \|H\|$, i.e., the skew-Hermitian component $S$ dominates (in the sense of the matrix norm $\|\cdot\|$) the Hermitian component $H$. If this condition is fulfilled, it follows that the spectrum $\sigma(S^{-1}H)$ of $S^{-1}H$ will be clustered around the origin of the complex plane [4]. Consequently, selecting $P = S$ in (5), we have that the spectrum of the preconditioned coefficient matrix

$$P^{-1}Z = S^{-1}S(U + S^{-1}H) = U + S^{-1}H$$

$\sigma(P^{-1}Z) = \sigma(U + S^{-1}H)$ will be located around the point $(1,0)$ of the complex plane [4] obtaining in this way the desired preconditioning effect [4].

![Figure 1: Spectrum of the system before (in red dots) and after (in blue crosses) the preconditioning (on the left: PEC cube; on the right: PEC sphere).](image-url)
4. NUMERICAL RESULTS AND CONCLUSIONS

The GMRES without restart has been tested on the linear systems (1) coming out from MoM analysis of EFIE for the scattering from i) a PEC cube of 1-meter edges and ii) a PEC sphere of 1-meter radius by an incident \( \theta \)-polarized plane wave \( (f=200\,\text{MHz}) \). In Figure 1 is reported the distribution of the spectrum for unpreconditioned case and preconditioned one, respectively. It can be noticed as the preconditioning causes a clustering of the spectrum. In Figures 2 and 3 the histories of the residual norm are reported.

The preconditioning results in a significant reduction of the overall number of iterations that are necessary to attain convergence within a fixed tolerance \( \epsilon = 10^{-12} \) (starting from an initial guess \( x_0 = 0 \)). All algorithms have been implemented in a MATLAB code on a personal computer equipped with a 2 GHz AMD Athlon processor and 2 GB of DDR2 RAM, with a particular attention devoted to coding time-critical routines using FORTRAN.

REFERENCES