On the irregular Hodge filtration

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Deligne's Hodge theory

- U: smooth quasi-proj. var. $/\mathbb{C}$, $\dim U = n$
- Deligne: $H^k(U,\mathbb{Q})$ endowed with a canonical mixed Hodge struct.
- How to make explicit the Hodge and weight filtr.?
- Choose an embedding $j : U \hookrightarrow X$, X smooth proj., $D := X - U$ a ***ncd***
- $\mathscr{O}_X(*D)$ equipped with the $\searrow$ filtr. by the order of the pole $F^p\mathscr{O}_X(*D)$: locally, $D = \{\prod_{i=1}^{\ell} x_i = 0\}$,

$$F^1\mathscr{O}_X(*D) = 0, \quad F^0\mathscr{O}_X(*D) = \mathscr{O}_X \cdot 1/\prod_{i=1}^{\ell} x_i$$

$$F^p\mathscr{O}_X(*D) = \sum_{|\alpha|\leqslant -p} \partial_x^\alpha F^0\mathscr{O}_X(*D), \quad p \leqslant -1$$

Deligne's Hodge theory

- $F^p \operatorname{DR} \mathscr{O}_X(*D)$:

$$\Big\{0 \to F^p\mathscr{O}_X(*D) \to \cdots \to F^{p-n}\Omega_X^n(*D) \to 0\Big\}$$

- THEOREM (Deligne). $F^pH^k(X, \operatorname{DR} \mathscr{O}_X(*D))$:

image$\Big[H^k(X, F^p \operatorname{DR} \mathscr{O}_X(*D)) \to H^k(X, \operatorname{DR} \mathscr{O}_X(*D))\Big]$

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- $H^k(X, (\Omega_X^\bullet(\log D), \mathrm{d})) \xrightarrow{\sim} H^k(X, \operatorname{DR} \mathscr{O}_X(*D))$
- $H^k(X, F^p(\Omega_X^\bullet(\log D), \mathrm{d})) \hookrightarrow H^k(X, (\Omega_X^\bullet(\log D), \mathrm{d}))$
- $H^k(X, (\Omega_X^\bullet(\log D), \mathrm{d})) \simeq H^k(X, (\Omega_X^\bullet(\log D), 0))$

The Deligne bundles

- Glue $H^k(X, (\Omega^\bullet_X(\log D)[u], u\mathrm{d}))$ with $H^k(X, (\Omega^\bullet_X(\log D)[v], \mathrm{d}))$ $\quad$ ($v = u^{-1}$):

$$u^{-\ell} : \Omega^\ell_X(\log D)[u, u^{-1}] \xrightarrow{\sim} \Omega^\bullet_X(\log D)[v, v^{-1}]$$

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- Deligne's thm $\Rightarrow$:
  - Filtered bundle $F^\bullet \mathscr{H}^k$ on $\mathbb{P}^1_{(u:v)}$.
  - $\mathscr{H}^k \simeq \bigoplus_p \mathscr{O}_{\mathbb{P}^1}(p)^{h^{p,q}}$,
    $h^{p,q} = \dim H^q(X, \Omega_X^p(\log D))$
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  - $\Rightarrow F^p\mathscr{H}^k =$ Harder-Narasimhan filtr. of $\mathscr{H}^k$,
- Moreover, $\exists$ connection ∇ on $\mathscr{H}^k$: ∂_v or $u\partial_u - \ell$, *i.e.*, no pole at $v = 0$ and log. pole at $u = 0$, $\nabla F^p \subset F^p \otimes \Omega^1_{\mathbb{P}^1}$.

What for a regular function?

- $f : U \to \mathbb{A}^1$ regular function.
- ***Hodge filtration on*** $H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$***?***
- Why? Motivations from
  - Arithmetics (p-adic valuations of Gauss sums): Deligne 1984, Adolphson-Sperber $\sim$1989.
  - Mirror symmetry for Fano mani(orbi)folds: Katzarkov, Kontsevich & Pantev $\sim$2012
  - Wild Hodge theory: T. Mochizuki $\sim$2011

What for a regular function?

- If (Y, ω_Y) Fano mfld, expect the mirror given by a LG model $f : U \to \mathbb{A}^1$.
- $H^k_{\mathrm{dR}}(Y) \longleftrightarrow H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$
- Problem: To find a filtr. $F^\bullet H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$ corresp. to $F^\bullet H^k_{\mathrm{dR}}(Y)$
- $\rightsquigarrow$ "Hodge filtr." of $H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$

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- What properties?
- ⚠ Betti structures $/\mathbb{Q}$ ***do not*** correspond.
- $\Rightarrow$ Do not expect $\overline{F}^\bullet$ opposite to $F^\bullet$, even after grading by some $W_\bullet$
- So what?

What for a regular function?

- Example:
  - $U = \mathbb{A}^1$, $f : t \mapsto -t^2$, $\mathscr{M} = \mathscr{O}_U$.
  - $\dim H^1_{\mathrm{dR}}(U, \mathrm{d} - \mathrm{d}(t^2)) = 1$,
  - period: $\displaystyle\int_{\mathbb{R}} \mathrm{e}^{-t^2} \mathrm{d}t = \pi^{1/2}$,
  - (Deligne): $\Rightarrow F^p_{\mathrm{Del}} E^{-t^2}$ with $p \in 1/2 + \mathbb{Z}$.
- In gen., should expect $F^p_{\mathrm{Del}} H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$ with $p \in A + \mathbb{Z}$, $A \subset \mathbb{Q}$ finite.
  (Mirror sym. to Hodge struct. of Fano orbifolds)
- Motivation for $F^{\bullet}_{\mathrm{Del}}$: estimates for p-adic exp. sums in terms of the ***irregular Hodge polygon***.

Exponentially regular hol. $\mathscr{D}$-modules

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- $U \subset X$ smooth, $X \setminus U = D$: divisor in X
- M: reg. hol. $\mathscr{D}_X$-module ($\Leftrightarrow \mathscr{O}_X$mod. $+$ reg. flat ∇)
- $f : U \to \mathbb{A}^1$ regular fctn on U,
  $E^f := (\mathscr{O}_X(*D), \mathrm{d} + \mathrm{d}f)$
- $M \otimes E^f = (M(*D), \nabla + df)$
- $\pi : X \to Y$ proper. $\pi_+(M \otimes E^f)$ may have very complicated irreg. sing.

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- $\pi : X \to Y$ proper. $\pi_+(M \otimes E^f)$ may have very complicated irreg. sing.
- If $(M, F_\bullet M)$ underlies an object of $\mathrm{MHM}(X)$, how to define a filtr. $F_\bullet^{\mathrm{irr}}$ on $M \otimes E^f$?
- Would like: the spectr. seq.
  $E_1^{p,q} = H^{p+q}(X, F_{\mathrm{irr}}^p \mathrm{DR}(M \otimes E^f))$ degen.
- More gen., the spectr. seq. $H^k \pi_+(M, F_\bullet^{\mathrm{irr}} M)$ degenerates

Exponentially twisted MHM's

- $U \subset X$ smooth, $X \smallsetminus U = D$: divisor in X
- $f : U \to \mathbb{A}^1$ regular fctn on U

THEOREM.
$(M, F_\bullet M) \in \mathrm{MHM}(X) \Rightarrow \exists$ good filtr. $F_\bullet^{\mathrm{irr}}(M \otimes E^f)$ s.t.

- $F_\bullet^{\mathrm{irr}}(M \otimes E^f)_{|U} = F_\bullet M_{|U}$
- Any morphism φ in $\mathrm{MHM}(X)$ induces a ***strict*** morphism w.r.t. $F_\bullet^{\mathrm{irr}}$
- $\pi : X \to Y$ projective $\Rightarrow$ $H^k \pi_+(M \otimes E^f, F_\bullet^{\mathrm{irr}}(M \otimes E^f))$ is ***strict***
- ***If*** $\exists\, h : V \to \mathbb{A}^1$ s.t. $f = h \circ \pi$, then

$$F_p H^k \pi_+(M \otimes E^f) = F_p^{\mathrm{irr}}((H^k \pi_+ M) \otimes E^h)$$

Techniques of proofs

- Deligne (1984): proof by hand in case of a unitary local system on a punctured curve U and $f : U \to \mathbb{A}^1$. He remarks: cannot expect opposedness. What is a Hodge filtration without opposedness good for? $\longrightarrow$ p-adic valuations of Gauss sums.
- C.S. (2010): defines F^{Del} in the case of VPHS on $U \subset \mathbb{P}^1$, $f = \mathrm{Id}_U$. Uses ideas coming from twistor D-modules & M. Saito's results on Brieskorn lattices.
- J.-D. Yu (2012): defines $F_\bullet^{\mathrm{Yu}}(\mathscr{O}_X(*D), \mathrm{d} + \mathrm{d}f)$, $D =$ ncd. Various special cases of strictness for $\pi : X \to \mathrm{pt}$.

Techniques of proofs

- Kontsevich (2012): Construction of $(\Omega_f^\bullet, u\mathrm{d} + v\mathrm{d}f)$ and degeneration at E_1 for each $u, v \in \mathbb{C}$ (assumes $f^*(\infty)$ reduced ncd, uses char. p reduction).
- E-S-Y (2013) Degeneration at E_1 for F^{Yu}. Relation with F^{Del} and with Kontsevich construction $\rightsquigarrow$ new proof of E_1-degen. for $(\Omega_f^\bullet, u\mathrm{d} + v\mathrm{d}f)$ without assuming $f^*(\infty)$ reduced. Proves last point of the thm by using Saito's MHM theory.
- S-Y (2014): proof of the thm by using Mochizuki's MTM theory

The Kontsevich complex

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- $f : U \to \mathbb{A}^1$, $f : X \to \mathbb{P}^1$, $D = X \smallsetminus \mathbb{P}^1$ ***ncd***
- $\Omega_f^k \subset \Omega_X^k(\log D)$: $\{\omega \mid \mathrm{d}f \wedge \omega \in \Omega_X^{k+1}(\log D)\}$
- $\forall\, u, v \in \mathbb{C}$, $(\Omega_f^\bullet, u\mathrm{d} + v\mathrm{d}f)$ is a cplx

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- Refinement:
  - $P := f^*(\infty)$, $P = \sum_i e_i P_i$, $P_{\mathrm{red}} \subset D$
  - $\forall\, \alpha \in [0, 1)$, $[\alpha P] := \sum_i [\alpha e_i] P_i$
  - $\Omega_f^k(\alpha) \subset \Omega_X^k(\log D)([\alpha P])$
  - $\rightsquigarrow \forall\, u, v \in \mathbb{C}$, $(\Omega_f^\bullet(\alpha), u\mathrm{d} + v\mathrm{d}f)$ is a cplx

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THEOREM (Kontsevich). $\forall\, k$, $\forall\, \alpha \in [0,1)$, $\forall\, u, v \in \mathbb{C}$

$$\dim H^k(X, (\Omega_f^\bullet(\alpha), u\mathrm{d} + v\mathrm{d}f)) = \dim H^k(X, (\Omega_f^\bullet(\alpha), 0))$$

(Pf by M.K. if $P = P_{\mathrm{red}}$, pf by E-S-Y in general).

The Kontsevich bundles

- Glue $H^k\big(X, (\sigma^{\geqslant p}\Omega_f^\bullet(\alpha)[u], u\mathrm{d} + \mathrm{d}f)\big)$ with $H^k\big(X, (\sigma^{\geqslant p}\Omega_f^\bullet(\alpha)[v], \mathrm{d} + v\mathrm{d}f)\big)$ through $u = v^{-1}$
- Get a filtered bdle on $\mathbb{P}^1$: $F^p\mathscr{K}_\alpha^k$
- $\mathscr{K}_\alpha^k \simeq \bigoplus_p \mathscr{O}_{\mathbb{P}^1}(p)^{h_\alpha^{p,k-p}(f)}$, $h_\alpha^{p,k-p}(f) := \dim H^{k-p}(X, \Omega_f^p(\alpha))$
- $\Rightarrow F^p\mathscr{K}_\alpha^k =$ Harder-Narasimhan filtr.
- $\exists$? a natural (merom.) connection ∇ on $\mathscr{K}_\alpha^k$?

The Kontsevich bundles

THEOREM.

- $\exists$ a natural (merom.) connection ∇ on $\mathscr{K}_\alpha^k$, log. pole at $v=0$ & pole of order $\leqslant 2$ at $u=0$.
- $F^\bullet\mathscr{K}_\alpha^k$ satisfies Griffiths transv. w.r.t. ∇: $\nabla F^p\mathscr{K}_\alpha^k \subset F^{p-1}\mathscr{K}_\alpha^k \otimes \Omega^1_{\mathbb{P}^1}$.
- Eigenvalues of $\mathrm{Res}_{v=0}\nabla \in (-\alpha-1,-\alpha]$.
- On each general. eigensp. of $\mathrm{Res}_{v=0}\nabla$,

$$\mathrm{N} = \mathrm{Nilp}\,\mathrm{Res}_{v=0}\nabla : \frac{F^p\mathscr{K}_\alpha^k}{F^p\cap v\mathscr{K}_\alpha^k} \longrightarrow \frac{F^{p-1}\mathscr{K}_\alpha^k}{F^{p-1}\cap v\mathscr{K}_\alpha^k}$$ ***strict***

Ideas of proof

- To interpret $\mathscr{K}_\alpha^k$ as a sub-bdle in a Gauss-Manin syst.
- $f : X \to \mathbb{P}^1$, $f(U) \subset \mathbb{A}^1$.
- $vf : U \times \mathbb{A}^1_v \to \mathbb{A}^1$
  - pole div.: $\mathcal{P} := (P \times \mathbb{P}^1) \cup (X \times \{u = 0\}) \subset X \times \mathbb{P}^1$
  - Divisor $\mathcal{D} = \mathcal{P} \cup (H \times \mathbb{P}^1)$
  - Projection $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$
- $\mathscr{E} = E^{vf}(*\mathcal{D}) := \big(\mathscr{O}_{X\times\mathbb{P}^1}(*\mathcal{D}), \mathrm{d} + \mathrm{d}(vf)\big)$
- Push-frwd $H^k := \mathscr{H}^{k-\dim X} q_+\mathscr{E}$: ***hol. mod.*** on $\mathbb{P}^1$
- $H^k = H^k(*\{u = 0\})$ (can consider H^k_v as a module on $\mathbb{C}[v]\langle\partial_v\rangle$)

Ideas of proof

- PROPOSITION. H^k has sing. at $v = 0$ and $u = 0$ only, and $v = 0$ is a reg. sing.
- PROOF. H^k_v is the ***Laplace transf.*** of the G-M system $H^{k-\dim X} f_+(\mathscr{O}_X(*D), \mathrm{d})$.
- $\Rightarrow \mathbb{C}[v, v^{-1}] \otimes_{\mathbb{C}[v]} H^k_v$ free $\mathbb{C}[v, v^{-1}]$-mod. of finite rk with a connection ∇, ***reg. sing.*** at $v = 0$, ***irreg. sing.*** at $u = 0$.
- $\alpha \in [0, 1)$ fixed, $\rightsquigarrow V_\alpha H^k_v$: Deligne ext. with log. poles and eigenval. of residue in $(-\alpha - 1, -\alpha]$.
- $V_\alpha H^k_v$: free $\mathbb{C}[v]$-mod. with connection ∇, log. pole at $v = 0$.

Ideas of proof

- Chart $\mathbb{A}^1_u$: extend $H^k_u := H^k[u, u^{-1}]$ as a $\mathbb{C}[u]$-module using the Hodge filtration of $(\mathscr{O}_X(*D), \mathrm{d}) \rightsquigarrow$ Brieskorn lattice $G^0 H^k_u$.
- ∇ has a pole of order $\leqslant 2$ on $G^0 H^k_u$ ($\rightsquigarrow$ n.c. Hodge structure in the sense of KKP)
- $\rightsquigarrow$ Bundle $\mathscr{H}^k_\alpha \subset H^k$ obtained by gluing $V_\alpha H^k_v$ and $G_0 H^k_u$.
- ∇: log pole at $v = 0$ and pole of order $\leqslant 2$ at $u = 0$.
- $\mathscr{K}^k_\alpha \xrightarrow{\sim} \mathscr{H}^k_\alpha$
- This isom. is strict w.r.t. the Harder-Narasimhan filtr.

Ideas of proof

- $\rightsquigarrow$ What about the Harder-Narasimhan filtr. of $\mathscr{H}_{\alpha}^{k}$?
- PROPOSITION. The irreg. Hodge filtr. $F_{\alpha}^{\mathrm{irr},\bullet}H^{k}$ ($=$ push-forward by q of the irreg. Hodge filtr. $F_{\alpha}^{\mathrm{irr},\bullet}\mathscr{E}$) induces on $\mathscr{H}_{\alpha}^{k}$ the H-N filtr.
- LEMMA. Let $\mathscr{H}$ be a v.b. on $\mathbb{P}^{1}$ obtained by gluing a free $\mathbb{C}[v]$-mod. V and a free $\mathbb{C}[u]$-mod. G^{0}, with

$$\mathbb{C}[v,v^{-1}]\otimes_{\mathbb{C}[v]}V=:G:=\mathbb{C}[u,u^{-1}]\otimes_{\mathbb{C}[u]}G^{0}.$$

Then

$$\mathscr{H}^{\geqslant p}{}_{|\,\mathrm{Spec}\,\mathbb{C}[v]}=\mathbb{C}[v]\cdot(u^{p}G^{0}\cap V),$$

$$\mathscr{H}^{\geqslant p}{}_{|\,\mathrm{Spec}\,\mathbb{C}[u]}=\mathbb{C}[u]\cdot(G_{0}\cap v^{p}V).$$