Hodge $\mathscr{D}$-modules 3. On curves: global properties

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$$\boxed{H^k(X, \mathscr{H}) = H^k\big(X, \mathrm{DR}(V, \nabla)\big) = H^k\big(\Gamma(X, (\mathscr{E}_X^\bullet \otimes H, D))\big)}$$

$$H^{p,q}_{d''}(X, H) = H^q\big(X, \Omega_X^p \otimes V\big)$$

$$X \text{ Kähler} \Longrightarrow \Delta_D = 2\Delta_{D'} = 2\Delta_{d''}$$

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THEOREM. $H^{q,p}_{d''}(X, H) = \overline{H^{p,q}_{d''}(X, H^\vee)}$ &

$$\boxed{H^k\big(X, \mathrm{DR}(V, \nabla)\big) = \bigoplus_{p+q=k} H^{p,q}_{d''}(X, H)} \quad \text{(canonical)}$$

Hodge-Deligne thm for a VHS

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- HODGE-DELIGNE THEOREM.
  $(\mathscr{H}_{\mathbb{Q}}, (V, \nabla), F^{\bullet}V, \mathscr{Q}, \alpha) \in \mathrm{VHS}(X, w)$

$$\Rightarrow \boxed{H^k_{\mathbb{Q}} := H^k(X, \mathscr{H}_{\mathbb{Q}}) \in \mathrm{HS}(w+k)^{(\mathrm{p})}}$$

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- ***but*** D neither compatible with h nor with $\bigoplus$

$$D'H^{p,q} \subset \Omega^1_X \otimes (H^{p,q} \oplus H^{p-1,q+1}),$$
$$D''H^{p,q} \subset \overline{\Omega^1_X} \otimes (H^{p,q} \oplus H^{p+1,q-1})$$

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- $D = D' + D'' = \mathcal{D}' + \mathcal{D}'' = (D'_E + \theta''_E) + (D''_E + \theta'_E)$
- $\Rightarrow \boxed{\Delta_D = 2\Delta_{\mathcal{D}'} = 2\Delta_{\mathcal{D}''}}$
- Use the total type of sections.

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Problem: To relate L^2-cohom. with Betti cohom.

Hodge-Zucker thm for a VHS

- $X^* \overset{j}{\hookrightarrow} X$ punctured compact Riemann surface
- $($

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- ${}^h\widetilde{V}$ is a $\mathscr{O}_\Delta[1/t]$-mod., ${}^h\widetilde{V}_{|\Delta^*} = V$.
- ***Parabolic filtration:*** $a \in \mathbb{R}$, ${}^h\widetilde{V}^a$: $\forall\, \varepsilon$, $\exists\, C(\varepsilon) > 0$, $\|v\|_h \leqslant C(\varepsilon)|t|^{a-\varepsilon}$.
- ${}^h\widetilde{V}^a$ is an $\mathscr{O}_\Delta$-submodule of ${}^h\widetilde{V}$ and ${}^h\widetilde{V} = \bigcup_a {}^h\widetilde{V}^a, \qquad \& \qquad \forall\, k \in \mathbb{Z}, \quad t^k \cdot {}^h\widetilde{V}^\bullet = {}^h\widetilde{V}^{\bullet+k}$.
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DEFINITION. h ***moderate*** if $\forall\, (\exists\,)\, a$, ${}^h\widetilde{V}^a$ is $\mathscr{O}_\Delta$ coh.

Schmid's theorems

$\big((V, F^\bullet V), \mathscr{H}_{\mathbb{Q}}, \alpha, \mathscr{Q}\big)$ a $\mathrm{VHS}(w)^{(\mathrm{p})}$ on Δ^*,
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THEOREM (Schmid).

- h is moderate and ${}^h\widetilde{V} = \widetilde{V}$ (Deligne merom. extension)
- the parabolic filtr. ${}^h\widetilde{V}^\bullet$ coincides with Deligne's extensions $\widetilde{V}^\bullet$ (*i.e.*, the Kashiwara-Malgrange filtr. of the $\mathscr{D}_\Delta$-mod. $\widetilde{V}$)
- $v \in \mathrm{M}_\ell \widetilde{V}^a$ induces $[v] \neq 0$ in $\mathrm{gr}_\ell^{\mathrm{M}} \psi_t^a \widetilde{V}$ iff $\|v\|_h \underset{t\to 0}{\sim} |t|^a \mathrm{L}(t)^{\ell/2} \quad (\mathrm{L}(t) = \big|\log(|t|^2)\big|)$

The holomorphic L^2 de Rham complex

Goal 1: $$\boxed{H^k(X, j_*\mathscr{H}) = H^k\big(X^*, \mathscr{L}^\bullet_{(2)}(H, D, h, d\,\mathrm{vol})\big)}$$

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- Consider the ***holomorphic L^2 de Rham cplx***

$$(\mathrm{DR}\,\widetilde{V})_{(2)} = \{0 \longrightarrow \widetilde{V}_{(2)} \xrightarrow{\nabla} (\Omega^1_\Delta \otimes \widetilde{V})_{(2)} \longrightarrow 0\},$$

  - $(\Omega^1_\Delta \otimes \widetilde{V})_{(2)} := (\Omega^1_\Delta \otimes \widetilde{V}) \cap \mathscr{L}^1_{(2)}(H, D, h, d\,\mathrm{vol})$
  - $\widetilde{V}_{(2)} := \widetilde{V} \cap \mathscr{L}^0_{(2)}(H, D, h, d\,\mathrm{vol})$.

The L^2 condition in the Poincaré metric

- Poincaré metric on Δ^*: $d\,\mathrm{vol} = \dfrac{dx^2 + dy^2}{|t|^2 \mathrm{L}(t)^2}$

  $x = \mathrm{Re}\,t$, $y = \mathrm{Im}\,t$.
- $\|v\|_{L^2}^2 = \int \|v\|_h^2 \, d\,\mathrm{vol}$
- L^2 condition in the Poincaré metric, polar coord.

(0) $f \in L^2(d\,\mathrm{vol}) \Leftrightarrow |\log r|^{-1} f \in L^2(d\theta\, dr/r)$;

(1) $f\, dr/r + g\, d\theta \in L^2(d\,\mathrm{vol}) \Leftrightarrow f$ and $g \in L^2(d\theta\, dr/r)$;

(2) $h\, d\theta\, dr/r \in L^2(d\,\mathrm{vol}) \Leftrightarrow |\log r| h \in L^2(d\theta\, dr/r)$.

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- Schmid's thm $\Rightarrow$

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L^2 POINCARÉ LEMMA (Zucker).

$$\boxed{(\mathrm{DR}\,\widetilde{V})_{(2)} \xhookrightarrow{\sim} \mathscr{L}^{\bullet}_{(2)}(H, D, h, d\,\mathrm{vol})}$$

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COROLLARY. $\dim H^k\big(X^*, \mathscr{L}^\bullet_{(2)}(H, D, h, d\,\mathrm{vol})\big) < \infty$

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- $(V,\nabla,F^\bullet V,\dots) \in \mathrm{VHS}(X,w)^{(\mathrm{p})}$, $H = \mathscr{C}_X^\infty \otimes V$, $D = D' + D''$.

$$F^p\,\mathrm{DR}(V,\nabla) = \Big\{0 \to F^pV \xrightarrow{\nabla} \cdots F^{p-\dim X}V \otimes \Omega_X^{\dim X} \to 0\Big\}$$

$F^p\mathscr{E}^\bullet \otimes H$ defined similarly: $\mathscr{E}_X^{i,j} \otimes H^{k,\ell}$ of type $(p,q) = (i+k, j+\ell)$, differential: $\mathcal{D}''$ on $\mathrm{gr}_F^p\mathscr{E}^\bullet(H)$.

The L^2 Dolbeault lemma

Goal 2: What about the Hodge filtration?

- X cpct cplx mfld, classical Dolbeault Lemma:

$$H^q(X,\Omega_X^p) \simeq H_{d''}^{p,q}(X) = H^q\big(\Gamma(X,\mathscr{E}_X^{p,\bullet}),d''\big)$$

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$$\boxed{H^q\big(X,\mathrm{gr}_F^p\,\mathrm{DR}(V,\nabla)\big) = H^q\big(\Gamma(X,(\mathrm{gr}_F^p\mathscr{E}_X^\bullet(V),\mathcal{D}''))\big)}$$

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- $F^p\widetilde{V}_{\min}$ already defined, $\rightsquigarrow$

$$F^p\,\mathrm{DR}\,\widetilde{V}_{\min} := \Big\{0 \to F^p\widetilde{V}_{\min} \to \Omega^1 \otimes F^{p-1}\widetilde{V}_{\min} \to 0\Big\}$$

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- ***Is*** $(\mathrm{DR}\,\widetilde{V})_{(2)} \xrightarrow{\sim} \mathrm{DR}\,\widetilde{V}_{\min}$ ***a filtered isom?***
- ***yes***: $\mathrm{N} : (\widetilde{V}^0/\widetilde{V}^{>0}, \mathrm{M}_\bullet) \to (\widetilde{V}^0/\widetilde{V}^{>0}, \mathrm{M}_{\bullet-2})$ is a morphism of MHS, hence is ***strict*** w.r.t. $F^\bullet$.

The L^2 Dolbeault lemma

- L^2 Dolbeault complex on X^*:

$$\mathrm{gr}_F^p \mathscr{L}_{(2)}^\bullet(H, \mathcal{D}'') := \left\{ 0 \to H^{p,q} \xrightarrow{\mathcal{D}''} (\mathscr{E}^{\bullet,\bullet} \otimes H^{\bullet,\bullet})^{(p,q+1)} \to 0 \right\}$$

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THEOREM (Zucker).

$$\boxed{H^q\big(X, \mathrm{gr}_F^p (\mathrm{DR}\, \widetilde{V})_{(2)}\big) \xrightarrow{\sim} H^q\big(X, \mathrm{gr}_F^p \mathscr{L}_{(2)}^\bullet(H, \mathcal{D}'')\big)}$$

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REMARK. All previous statements where ***local*** on X.
This one is ***only global***, *i.e.*, at the cohomology level.

Hodge-Saito thm for a Hodge module

- $\Sigma = X \smallsetminus X^*$
- An object of $\mathrm{MH}(X, w)^{(\mathrm{p})}$:

$$\big((\widetilde{V}_{\min}, F^\bullet), j_*\mathscr{H}_{\mathbb{Q}}, j_*\alpha, j_*\mathscr{Q}\big) \oplus \mathrm{HS}(w)^{(\mathrm{p})}$$

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$$F^p H^k(X, j_*\mathscr{H}_{\mathbb{C}}) = F^p H^k(X, \mathrm{DR}\,\widetilde{V}_{\min}) = H^k(X, F^p\,\mathrm{DR}\,\widetilde{V}_{\min})$$

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This is essentially Zucker's thm, but expressed hol. on X, not L^2 on X^*.

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- Functors: triples $(G, F_{\mathbb{Q}}, H)$

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- Functor $H^k f_*$:

$$ (M, \mathscr{F}_{\mathbb{Q}}, \alpha) \longmapsto (\mathscr{H}^k f_+M, {}^{\mathrm{p}}R^k f_*\mathscr{F}_{\mathbb{Q}}, \beta_f^{-1} \circ {}^{\mathrm{p}}R^k f_*\alpha) $$

Push-forward functor

- $(M, F_\bullet M) \rightsquigarrow$ can define $F_\bullet \mathscr{H}^k f_+ M$.
- ***But*** does not have good properties (e.g., w.r.t. exact seq., or to $g \circ f$)
- ***IF*** we can ensure strictness $\Rightarrow$ ***OK***.

Nearby cycle functor

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- THEOREM (K-M). M hol. and regular along $f^{-1}(0)$
  $\exists$ functorial iso

$$\gamma_f : ({}^{\mathrm{p}}\mathrm{DR}\psi_f^a M, e^{-2\pi i \mathrm{N}}) \xrightarrow{\sim} \left({}^{\mathrm{p}}\Psi_f^{e^{-2\pi i a}}\mathscr{F}_{\mathbb{C}}, (T - e^{-2\pi i a}\,\mathrm{Id})\right)$$

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- Functor (ψ_f, T):

$$(M, \mathscr{F}_{\mathbb{Q}}, \alpha) \longmapsto ((\psi_f M, T), ({}^{\mathrm{p}}\Psi_f \mathscr{F}_{\mathbb{Q}}, T), \Psi_f\alpha \circ \gamma_f^{-1})$$

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Definition of $\mathrm{MH}_{\leqslant d}(X, w)$ ($\dim \mathrm{Supp} \leqslant d$): induction on d.

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  - with $((H, F^\bullet H), H_\mathbb{Q}, \alpha) \in \mathrm{HS}(w)$.
- $\mathrm{MH}_Z(X, w)$ (strict supp. Z closed irred. in X, $\dim Z = d$):
  - $\forall\, f$ (hol. germ on X), $\psi_f\big((M, F_\bullet M), \mathscr{F}_\mathbb{Q}, \alpha\big)$ exists, T quasi-unip. (eigenval. $\in \mu_N$ for some N)
  - if $f \not\equiv 0$ on Z, each $\mathrm{gr}_\ell^{\mathrm{M}} \psi_f^\lambda\big((M, F_\bullet M), \mathscr{F}_\mathbb{Q}, \alpha\big) \in \mathrm{MH}_{\leqslant d-1}(X, w - 1 + \ell)$

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- $\mathrm{MH}_{\leqslant d}(X, w) = \bigoplus_{\dim Z \leqslant d} \mathrm{MH}_Z(X, w)$.

The Hodge-Saito theorem

THEOREM (Saito).

- $f : X \to Y$ projective morphism, X, Y smooth cplx mflds.
- $((M, F_\bullet M), \mathscr{F}_{\mathbb{Q}}, \alpha) \in \mathrm{MH}(X, w)^{(\mathrm{p})}$
- L relatively ample line bdle on X.

$\Rightarrow$

- $\mathscr{H}^k f_*((M, F_\bullet M), \mathscr{F}_{\mathbb{Q}}, \alpha)$ well-defined and $\in \mathrm{MH}(Y, w+k)^{(\mathrm{p})}$,
- (Hard Lefschetz)

$$\mathscr{H}^{-k} f_*((M, F_\bullet M), \mathscr{F}_{\mathbb{Q}}, \alpha) \xrightarrow[\sim]{c_1(L)^k} \mathscr{H}^k f_*((M, F_\bullet M), \mathscr{F}_{\mathbb{Q}}, \alpha)$$