Hodge $\mathscr{D}$-modules 2.
On curves: local properties

Claude Sabbah

Centre de Mathématiques Laurent Schwartz

UMR 7640 du CNRS

École polytechnique, Palaiseau, France

Programme SISYPH ANR-13-IS01-0001-01

Hodge thm on a Riemann surface

$X = \mathrm{pt}$	X = Riemann surf.
$H_{\mathbb{Q}}$: $\mathbb{Q}$-vect. space	$\mathscr{H}_{\mathbb{Q}}$: $\mathbb{Q}$-loc. syst.
$H_{\mathbb{C}}$	(V, ∇): hol. v.b. with conn. $\alpha : \mathscr{H}_{\mathbb{C}} \xrightarrow{\sim} V^{\nabla}$
$F^{\bullet} H_{\mathbb{C}}$	$F^{\bullet} V$ hol. sub-bdle s.t. $\boxed{\nabla F^p V \subset F^{p-1} V \otimes \Omega^1_X}$
$Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ nondeg.	$\mathscr{Q} : \mathscr{H}_{\mathbb{Q}} \otimes \mathscr{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}_X$ nondeg.
$\mathrm{HS}(w)^{(\mathrm{p})}$	VHS(w): $\forall\, x \in X$, $[\mathscr{H}_{\mathbb{Q},x}, F^{\bullet} V_x, \mathscr{Q}_x] \in \mathrm{HS}(w)^{(\mathrm{p})}$

Hodge thm on a Riemann surface

HODGE-DELIGNE THEOREM.
X is cpct & $(\mathscr{H}_{\mathbb{Q}}, (V, \nabla), F^{\bullet}V, \mathscr{Q}, \alpha) \in \mathrm{VHS}(X, w)$

$$\Rightarrow \boxed{H^k_{\mathbb{Q}} := H^k(X, \mathscr{H}_{\mathbb{Q}}) \in \mathrm{HS}(w+k)^{(\mathrm{p})}}$$

Hodge thm on a Riemann surface

HODGE-DELIGNE THEOREM.
X is cpct & $(\mathscr{H}_{\mathbb{Q}}, (V, \nabla), F^{\bullet}V, \mathscr{Q}, \alpha) \in \mathrm{VHS}(X, w)$

$$\Rightarrow \boxed{H^k_{\mathbb{Q}} := H^k(X, \mathscr{H}_{\mathbb{Q}}) \in \mathrm{HS}(w+k)^{(\mathrm{p})}}$$

Question: What if X not compact?

Hodge thm on a Riemann surface

HODGE-DELIGNE THEOREM.
X is cpct & $(\mathscr{H}_{\mathbb{Q}}, (V, \nabla), F^{\bullet}V, \mathscr{Q}, \alpha) \in \mathrm{VHS}(X, w)$

$$\Rightarrow \boxed{H^k_{\mathbb{Q}} := H^k(X, \mathscr{H}_{\mathbb{Q}}) \in \mathrm{HS}(w+k)^{(\mathrm{p})}}$$

Question: What if X not compact?

Possible answers if $X = Y \setminus \{x_1, \dots, x_r\} \xhookrightarrow{j} Y$ cpct

- (Zucker): ***idem*** for $H^k(Y, j_*\mathscr{H}_{\mathbb{Q}})$,
- (Deligne): $H^k(X, \mathscr{H}_{\mathbb{Q}})$ is ***mixed***

Hodge thm on a Riemann surface

HODGE-DELIGNE THEOREM.
X is cpct & $(\mathscr{H}_{\mathbb{Q}}, (V, \nabla), F^{\bullet}V, \mathscr{Q}, \alpha) \in \mathrm{VHS}(X, w)$

$$\Rightarrow \boxed{H^k_{\mathbb{Q}} := H^k(X, \mathscr{H}_{\mathbb{Q}}) \in \mathrm{HS}(w+k)^{(\mathrm{p})}}$$

Question: What if X not compact?

Possible answers if $X = Y \smallsetminus \{x_1, \dots, x_r\} \xhookrightarrow{j} Y$ cpct

- (Zucker): ***idem*** for $H^k(Y, j_*\mathscr{H}_{\mathbb{Q}})$,
- (Deligne): $H^k(X, \mathscr{H}_{\mathbb{Q}})$ is ***mixed***

Goal of Lect. 2 and 3:

Zucker's thm & relation with Hodge modules.

Extensions through the puncture

- $X = \Delta^* \overset{j}{\hookrightarrow} Y = \Delta$.
- (V, ∇) hol. v.b. with connexion on Δ^*

Extensions through the puncture

- $X = \Delta^* \xhookrightarrow{j} Y = \Delta$.
- (V, ∇) hol. v.b. with connexion on Δ^*
- ***Extension of*** V (Deligne canonical extension)
  $\widetilde{V}^0 \subset j_* V$: sections of $j_* V$ s.t. on any sector, the coef. in some horiz. frame have log. growth at most.
  - $\widetilde{V}^0$ is a vect. bdle, $\widetilde{\nabla}$ has a log. pole on $\widetilde{V}^0$
  - eigenval. of $\mathrm{Res}\,\widetilde{\nabla}$ on $\widetilde{V}^0 / t\widetilde{V}^0$: real part in $[0, 1)$.

Extensions through the puncture

- $X = \Delta^* \xrightarrow{j} Y = \Delta$.
- (V, ∇) hol. v.b. with connexion on Δ^*
- ***Extension of*** V (Deligne canonical extension)
  $\widetilde{V}^0 \subset j_* V$: sections of $j_* V$ s.t. on any sector, the coef. in some horiz. frame have log. growth at most.
  - $\widetilde{V}^0$ is a vect. bdle, $\widetilde{\nabla}$ has a log. pole on $\widetilde{V}^0$
  - eigenval. of $\mathrm{Res}\,\widetilde{\nabla}$ on $\widetilde{V}^0/t\widetilde{V}^0$: real part in $[0, 1)$.
- $\widetilde{V} := \mathscr{O}(*0) \otimes_{\mathscr{O}} \widetilde{V}^0$ is hol. $\mathscr{D}_\Delta$-mod.

Extensions through the puncture

- $X = \Delta^* \overset{j}{\hookrightarrow} Y = \Delta$.
- (V, ∇) hol. v.b. with connexion on Δ^*
- ***Extension of*** V (Deligne canonical extension)
  $\widetilde{V}^0 \subset j_* V$: sections of $j_* V$ s.t. on any sector, the coef. in some horiz. frame have log. growth at most.
  - $\widetilde{V}^0$ is a vect. bdle, $\widetilde{\nabla}$ has a log. pole on $\widetilde{V}^0$
  - eigenval. of $\operatorname{Res} \widetilde{\nabla}$ on $\widetilde{V}^0 / t\widetilde{V}^0$: real part in $[0, 1)$.
- $\widetilde{V} := \mathscr{O}(*0) \otimes_{\mathscr{O}} \widetilde{V}^0$ is hol. $\mathscr{D}_\Delta$-mod.
- $\widetilde{V}^a \subset \widetilde{V}$: eigenval. of $\operatorname{Res} \widetilde{\nabla}$ on $\widetilde{V}^a / t\widetilde{V}^a$ with real part in $[a, a+1)$.
- $\widetilde{V}^{>a} \subset \widetilde{V}$: eigenval. of $\operatorname{Res} \widetilde{\nabla}$ on $\widetilde{V}^{>a} / t\widetilde{V}^{>a}$ with real part in $(a, a+1]$.

Extensions through the puncture

- $\boxed{\widetilde{V}_{\min} := \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}} \subset \widetilde{V}$ suited for Zucker:

Extensions through the puncture

- $\boxed{\widetilde{V}_{\min} := \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}} \subset \widetilde{V}$ suited for Zucker:
- $\widetilde{V}_{\min}$ is a holon. $\mathscr{D}_{\Delta}$-module & $\mathrm{DR}\,\widetilde{V}_{\min} = j_* V^{\nabla}$:

$$0 \longrightarrow j_* V^{\nabla} \longrightarrow \widetilde{V}_{\min} \xhookrightarrow{\widetilde{\nabla}_{\partial_t}} \widetilde{V}_{\min} \longrightarrow 0 \quad \textit{exact}$$

Extensions through the puncture

- $\boxed{\widetilde{V}_{\mathbf{min}} := \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}} \subset \widetilde{V}$ suited for Zucker:
- $\widetilde{V}_{\mathbf{min}}$ is a holon. $\mathscr{D}_{\Delta}$-module & $\mathrm{DR}\,\widetilde{V}_{\mathbf{min}} = j_*V^{\nabla}$:

  $$0 \longrightarrow j_*V^{\nabla} \longrightarrow \widetilde{V}_{\mathbf{min}} \xhookrightarrow{\widetilde{\nabla}_{\partial_t}} \widetilde{V}_{\mathbf{min}} \longrightarrow 0 \quad \textit{exact}$$

  Question: How to extend $F^\bullet V$?

Extensions through the puncture

- $\boxed{\widetilde{V}_{\min} := \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}} \subset \widetilde{V}$ suited for Zucker:
- $\widetilde{V}_{\min}$ is a holon. $\mathscr{D}_{\Delta}$-module & $\mathrm{DR}\,\widetilde{V}_{\min} = j_*V^{\nabla}$:

$$0 \longrightarrow j_*V^{\nabla} \longrightarrow \widetilde{V}_{\min} \xhookrightarrow{\widetilde{\nabla}_{\partial_t}} \widetilde{V}_{\min} \longrightarrow 0 \quad \textit{exact}$$

  Question: How to extend $F^{\bullet}V$?
- Try $F^p\widetilde{V} = j_*F^pV \cap \widetilde{V}$. But for $p \ll 0$, $F^pV = V$, hence $j_*F^pV \cap \widetilde{V} = \widetilde{V}$ ***not*** $\mathscr{O}_{\Delta}$-coh.$\rightsquigarrow$ bad guess!

Extensions through the puncture

- $\boxed{\widetilde{V}_{\min} := \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}} \subset \widetilde{V}$ suited for Zucker:

- $\widetilde{V}_{\min}$ is a holon. $\mathscr{D}_\Delta$-module & $\mathrm{DR}\,\widetilde{V}_{\min} = j_* V^\nabla$:

$$0 \longrightarrow j_* V^\nabla \longrightarrow \widetilde{V}_{\min} \xrightarrow{\widetilde{\nabla}_{\partial_t}} \widetilde{V}_{\min} \longrightarrow 0 \quad \textit{exact}$$

Question: How to extend $F^\bullet V$?

- Try $F^p\widetilde{V} = j_* F^p V \cap \widetilde{V}$. But for $p \ll 0$, $F^p V = V$, hence $j_* F^p V \cap \widetilde{V} = \widetilde{V}$ ***not*** $\mathscr{O}_\Delta$-coh.$\rightsquigarrow$ bad guess!

- Start with $F^p\widetilde{V}^{>-1} = j_* F^p V \cap \widetilde{V}^{>-1}$. ***IF*** $\mathscr{O}_\Delta$-coh.,

$$\boxed{F^p\widetilde{V}_{\min} = \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j F^{p+j}\widetilde{V}^{>-1}}$$

$\rightsquigarrow \widetilde{\nabla}_{\partial_t} F^p\widetilde{V}_{\min} \subset F^{p-1}\widetilde{V}_{\min}$.

Extensions through the puncture

THEOREM (Schmid 1973): If $\big((V, \nabla), F^\bullet V\big)$ underlies a VPHS, then $\forall\, a > -1$,

1. $F^p\widetilde{V}^a := j_*F^pV \cap \widetilde{V}^a$ is $\mathscr{O}_\Delta$-coh.,
2. $F^p\widetilde{V}^a / F^p\widetilde{V}^{>a}$ underlies a MHS whose weight filtr. is determined by the nilpotent part of $\operatorname{Res}\widetilde{\nabla}$.

Extensions through the puncture

THEOREM (Schmid 1973): If $\big((V,\nabla),F^\bullet V\big)$ underlies a VPHS, then $\forall\, a>-1$,

1. $F^p\widetilde{V}^a := j_*F^pV \cap \widetilde{V}^a$ is $\mathscr{O}_\Delta$-coh.,
2. $F^p\widetilde{V}^a/F^p\widetilde{V}^{>a}$ underlies a MHS whose weight filtr. is determined by the nilpotent part of $\operatorname{Res}\widetilde{\nabla}$.

Idea for Hodge modules: Take these properties as a definition.

Basics on $\mathscr{D}$-modules

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$, $F_k\mathscr{D} = \sum_{j\leqslant k} \mathbb{C}\{t\}\partial_t^j$

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$, $F_k\mathscr{D} = \sum_{j\leqslant k}\mathbb{C}\{t\}\partial_t^j$
- M: finitely gen. (left) $\mathscr{D}$-module

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle \partial_t \rangle$, $F_k\mathscr{D} = \sum_{j \leqslant k} \mathbb{C}\{t\}\partial_t^j$
- M: finitely gen. (left) $\mathscr{D}$-module
- $F_\bullet M$: $F\mathscr{D}$-filtration,*i.e.*, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell} M$

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$, $F_k\mathscr{D} = \sum_{j\leqslant k} \mathbb{C}\{t\}\partial_t^j$
- M: finitely gen. (left) $\mathscr{D}$-module
- $F_\bullet M$: $F\mathscr{D}$-filtration,*i.e.*, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$
- $F_\bullet M$ is ***good*** if:
  - $F_k M = 0$ for $k \ll 0$,
  - each $F_k M$ is finitely gener. over $\mathscr{O} := \mathbb{C}\{t\}$,
  - for any $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$,
  - $\exists\, \ell_0 \in \mathbb{Z}$ s.t., $\forall\, k \geqslant 0$ and $\forall\, \ell \geqslant \ell_0$, $F_k\mathscr{D} \cdot F_\ell M = F_{k+\ell}M$.

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$, $F_k\mathscr{D} = \sum_{j\leqslant k} \mathbb{C}\{t\}\partial_t^j$
- M: finitely gen. (left) $\mathscr{D}$-module
- $F_\bullet M$: $F\mathscr{D}$-filtration,*i.e.*, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$
- $F_\bullet M$ is ***good*** if:
  - $F_k M = 0$ for $k \ll 0$,
  - each $F_k M$ is finitely gener. over $\mathscr{O} := \mathbb{C}\{t\}$,
  - for any $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$,
  - $\exists\, \ell_0 \in \mathbb{Z}$ s.t., $\forall\, k \geqslant 0$ and $\forall\, \ell \geqslant \ell_0$,
    $F_k\mathscr{D} \cdot F_\ell M = F_{k+\ell}M$.

M finitely gener. over $\mathscr{D} \Rightarrow \exists$ a good filtr.

Basics on $\mathscr{D}$-modules

- $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$, $F_k\mathscr{D} = \sum_{j\leqslant k} \mathbb{C}\{t\}\partial_t^j$
- M: finitely gen. (left) $\mathscr{D}$-module
- $F_\bullet M$: $F\mathscr{D}$-filtration,*i.e.*, $F_k\mathscr{D}\cdot F_\ell M \subset F_{k+\ell}M$
- $F_\bullet M$ is ***good*** if:
  - $F_kM = 0$ for $k \ll 0$,
  - each F_kM is finitely gener. over $\mathscr{O} := \mathbb{C}\{t\}$,
  - for any $k,\ell \in \mathbb{Z}$, $F_k\mathscr{D}\cdot F_\ell M \subset F_{k+\ell}M$,
  - $\exists\,\ell_0 \in \mathbb{Z}$ s.t., $\forall\,k \geqslant 0$ and $\forall\,\ell \geqslant \ell_0$, $F_k\mathscr{D}\cdot F_\ell M = F_{k+\ell}M$.

  M finitely gener. over $\mathscr{D} \Rightarrow \exists$ a good filtr.
- M ***holonomic***: M finitely gener. over $\mathscr{D}$ and

$$\forall\, m \in M,\ \exists\, P \in \mathscr{D} \smallsetminus \{0\}, \quad Pm = 0.$$

Basics on $\mathscr{D}$-modules

- $V^\ell \mathscr{D} = \sum_{j-k\geqslant \ell} t^j \mathbb{C}\{t\}\partial_t^k$, $\searrow$ fitlr. of $\mathscr{D}$

Basics on $\mathscr{D}$-modules

- $V^\ell \mathscr{D} = \sum_{j-k\geqslant \ell} t^j \mathbb{C}\{t\}\partial_t^k$, $\searrow$ fitlr. of $\mathscr{D}$
- Immediate properties:
  - $V^0\mathscr{D} = \mathbb{C}\{t\}\langle t\partial_t\rangle$,
  - $V^\ell\mathscr{D} = t^\ell V^0\mathscr{D}$, $V^{-\ell}\mathscr{D} = \sum_{k=0}^{\ell} V^0\mathscr{D}\cdot\partial_t^k$ ($\ell \geqslant 0$)
  - $\mathrm{gr}_V^0\mathscr{D} = \mathbb{C}[E]$, $E =$ class of $t\partial_t$.

Basics on $\mathscr{D}$-modules

- $V^\ell\mathscr{D} = \sum_{j-k\geqslant\ell} t^j\mathbb{C}\{t\}\partial_t^k$, $\searrow$ fitlr. of $\mathscr{D}$
- Immediate properties:
  - $V^0\mathscr{D} = \mathbb{C}\{t\}\langle t\partial_t\rangle$,
  - $V^\ell\mathscr{D} = t^\ell V^0\mathscr{D}$, $V^{-\ell}\mathscr{D} = \sum_{k=0}^{\ell} V^0\mathscr{D}\cdot\partial_t^k$ $(\ell\geqslant 0)$
  - $\mathrm{gr}_V^0\mathscr{D} = \mathbb{C}[E]$, $E =$ class of $t\partial_t$.
- $U^\bullet M$: $V\mathscr{D}$-filtration,*i.e.*, $V^k\mathscr{D}\cdot U^\ell M \subset U^{k+\ell}M$.
  good: if U^kM is finitely gener. over $V^0\mathscr{D}$, and

$$\exists\,\ell_0\geqslant 0,\quad V^k\mathscr{D}\cdot V^\ell M = V^{k+\ell}M \quad \begin{cases}\forall\, k\geqslant 0\ \&\ \forall\,\ell\geqslant\ell_0\\ \forall\, k\leqslant 0\ \&\ \forall\,\ell\leqslant -\ell_0\end{cases}$$

Basics on $\mathscr{D}$-modules

- $V^\ell\mathscr{D} = \sum_{j-k\geqslant\ell} t^j\mathbb{C}\{t\}\partial_t^k$, $\searrow$ fitlr. of $\mathscr{D}$
- Immediate properties:
  - $V^0\mathscr{D} = \mathbb{C}\{t\}\langle t\partial_t\rangle$,
  - $V^\ell\mathscr{D} = t^\ell V^0\mathscr{D}$, $V^{-\ell}\mathscr{D} = \sum_{k=0}^{\ell} V^0\mathscr{D}\cdot\partial_t^k$ $(\ell\geqslant 0)$
  - $\mathrm{gr}_V^0\mathscr{D} = \mathbb{C}[E]$, $E =$ class of $t\partial_t$.
- $U^\bullet M$: $V\mathscr{D}$-filtration,*i.e.*, $V^k\mathscr{D}\cdot U^\ell M \subset U^{k+\ell}M$.
  good: if $U^k M$ is finitely gener. over $V^0\mathscr{D}$, and

$$\exists\,\ell_0\geqslant 0,\quad V^k\mathscr{D}\cdot V^\ell M = V^{k+\ell}M \quad \begin{cases}\forall\,k\geqslant 0\ \&\ \forall\,\ell\geqslant\ell_0\\ \forall\,k\leqslant 0\ \&\ \forall\,\ell\leqslant -\ell_0\end{cases}$$

M finitely gener. over $\mathscr{D}\Rightarrow\exists$ a good V-filtr.

Basics on $\mathscr{D}$-modules

THEOREM:

- M holonomic $\Leftrightarrow \forall$ good V-filtr. $U^\bullet M$, $\exists\, b_U \in \mathbb{C}[s]$ s.t. $b(E-k)=0$ on $\mathrm{gr}_U^k M\ \forall\, k$.

Basics on $\mathscr{D}$-modules

THEOREM:

- M holonomic $\Leftrightarrow \forall$ good V-filtr. $U^\bullet M$, $\exists\, b_U \in \mathbb{C}[s]$ s.t. $b(E-k)=0$ on $\mathrm{gr}_U^k M\ \forall\, k$.
- $\exists\,!$ good V-filtr. $V^\bullet M$ s.t. $\mathrm{Re}\big(b_V^{-1}(0)\big) \subset [0,1)$. (Kashiwara-Malgrange filtr.)

Basics on $\mathscr{D}$-modules

THEOREM:

- M holonomic $\Leftrightarrow \forall$ good V-filtr. $U^\bullet M$, $\exists\, b_U \in \mathbb{C}[s]$ s.t. $b(E-k)=0$ on $\mathrm{gr}_U^k M\ \forall k$.
- $\exists\,!$ good V-filtr. $V^\bullet M$ s.t. $\mathrm{Re}\big(b_V^{-1}(0)\big) \subset [0,1)$. (Kashiwara-Malgrange filtr.)

CONCLUSION:

- $\widetilde{V}$ hol. $\mathscr{D}_\Delta$-mod. & $\widetilde{V}^k =$ K-M filtr. of $\widetilde{V}$
- $\widetilde{V}_{\min}$ hol. $\mathscr{D}_\Delta$-mod. & $\widetilde{V}_{\min}^{>k} := \sum_{j\leqslant k+1}(\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^{>-1}$ is the K-M filtr. of $\widetilde{V}$
- ***IF*** $F^p\widetilde{V}^{>-1}$ $\mathscr{O}_\Delta$-coh. then $F^p\widetilde{V}_{\min}$ ***good*** filtr. of $\widetilde{V}_{\min}$.

Intrinsic properties of $F^\bullet\widetilde{V}_{\min}$

- V-filtration of $\widetilde{V}_{\min}$ ($a \in (-1, 0]$):

$$V^{a+k}(\widetilde{V}_{\min}) = \begin{cases} \widetilde{V}^{a+k} = t^k \widetilde{V}^a & \text{if } k \geqslant 0, \\ \sum_{j=0}^{-k} (\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^a & \text{if } k < 0 \end{cases}$$

Intrinsic properties of $F^\bullet \widetilde{V}_{\min}$

- V-filtration of $\widetilde{V}_{\min}$ ($a \in (-1, 0]$):

$$V^{a+k}(\widetilde{V}_{\min}) = \begin{cases} \widetilde{V}^{a+k} = t^k \widetilde{V}^a & \text{if } k \geqslant 0, \\ \sum_{j=0}^{-k} (\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^a & \text{if } k < 0 \end{cases}$$

- $$\boxed{\begin{aligned} F^p \widetilde{V}^{>-1} &= j_* F^p V \cap \widetilde{V}^{>-1}, \\ F^p \widetilde{V}_{\min} &:= \sum_{j \geqslant 0} (\widetilde{\nabla}_{\partial_t})^j F^{p+j} \widetilde{V}^{>-1} \end{aligned}}$$

Intrinsic properties of $F^\bullet \widetilde{V}_{\min}$

- V-filtration of $\widetilde{V}_{\min}$ ($a \in (-1,0]$):

$$V^{a+k}(\widetilde{V}_{\min}) = \begin{cases} \widetilde{V}^{a+k} = t^k \widetilde{V}^a & \text{if } k \geqslant 0, \\ \sum_{j=0}^{-k} (\widetilde{\nabla}_{\partial_t})^j \widetilde{V}^a & \text{if } k < 0 \end{cases}$$

- $$\boxed{\begin{aligned} &F^p \widetilde{V}^{>-1} = j_* F^p V \cap \widetilde{V}^{>-1}, \\ &F^p \widetilde{V}_{\min} := \sum_{j \geqslant 0} (\widetilde{\nabla}_{\partial_t})^j F^{p+j} \widetilde{V}^{>-1} \end{aligned}}$$

THEOREM ($\Leftarrow$ Schmid). $\forall\, k \geqslant 0$, $\forall\, a \in (-1,0]$

- $t^k : F^p \widetilde{V}_{\min} \cap V^a(\widetilde{V}_{\min}) \xrightarrow{\sim} F^p \widetilde{V}_{\min} \cap V^{a+k}(\widetilde{V}_{\min})$,
- $(\widetilde{\nabla}_{\partial_t})^k : F^p \mathrm{gr}_V^a \widetilde{V}_{\min} \xrightarrow{\sim} F^{p-k} \mathrm{gr}_V^{a-k} \widetilde{V}_{\min}$.

Nearby cycles

Nearby cycles

- $\rho : \widetilde{\Delta}^* \longrightarrow \Delta^*$: universal cover.

Nearby cycles

- $\rho : \widetilde{\Delta}^* \to \Delta^*$: universal cover.
- $\Psi_t \mathscr{H}_{\mathbb{Q}} = \widetilde{H}_{\mathbb{Q}} = \Gamma(\widetilde{\Delta}^*, \rho^{-1}\mathscr{H}_{\mathbb{Q}})$ with an autom. T.

Nearby cycles

- $\rho : \widetilde{\Delta}^* \to \Delta^*$: universal cover.
- $\Psi_t \mathscr{H}_{\mathbb{Q}} = \widetilde{H}_{\mathbb{Q}} = \Gamma(\widetilde{\Delta}^*, \rho^{-1}\mathscr{H}_{\mathbb{Q}})$ with an autom. T.
- $\lambda \in \mathbb{C}^*$, $\Psi_t^\lambda \mathscr{H}_{\mathbb{C}} := \mathrm{Ker}(T - \lambda\,\mathrm{Id})^N$ ($N \gg 0$),

$$\Psi_t \mathscr{H}_{\mathbb{C}} = \bigoplus_\lambda \Psi_t^\lambda \mathscr{H}_{\mathbb{C}}, \quad \Psi_t \mathscr{H}_{\mathbb{Q}} = \Psi_t \mathscr{H}_{\mathbb{Q}}^1 \oplus \Psi_t \mathscr{H}_{\mathbb{Q}}^{\neq 1}.$$

Nearby cycles

- $\rho : \widetilde{\Delta}^* \to \Delta^*$: universal cover.
- $\Psi_t \mathscr{H}_{\mathbb{Q}} = \widetilde{H}_{\mathbb{Q}} = \Gamma(\widetilde{\Delta}^*, \rho^{-1}\mathscr{H}_{\mathbb{Q}})$ with an autom. T.
- $\lambda \in \mathbb{C}^*$, $\Psi_t^{\lambda} \mathscr{H}_{\mathbb{C}} := \mathrm{Ker}(T - \lambda \,\mathrm{Id})^N$ ($N \gg 0$),

$$\Psi_t \mathscr{H}_{\mathbb{C}} = \bigoplus_{\lambda} \Psi_t^{\lambda} \mathscr{H}_{\mathbb{C}}, \quad \Psi_t \mathscr{H}_{\mathbb{Q}} = \Psi_t \mathscr{H}_{\mathbb{Q}}^{1} \oplus \Psi_t \mathscr{H}_{\mathbb{Q}}^{\neq 1}.$$

Monodromy filtration: $\mathrm{N} :=$ nilp. part of T,
i.e., $\mathrm{N} = (T - \lambda \,\mathrm{Id})$ on $\Psi_t^{\lambda} \mathscr{H}_{\mathbb{C}}$.

Nearby cycles

- $\rho : \widetilde{\Delta}^* \to \Delta^*$: universal cover.
- $\Psi_t \mathscr{H}_{\mathbb{Q}} = \widetilde{H_{\mathbb{Q}}} = \Gamma(\widetilde{\Delta}^*, \rho^{-1}\mathscr{H}_{\mathbb{Q}})$ with an autom. T.
- $\lambda \in \mathbb{C}^*$, $\Psi_t^\lambda \mathscr{H}_{\mathbb{C}} := \mathrm{Ker}(T - \lambda\,\mathrm{Id})^N$ ($N \gg 0$),

$$\Psi_t \mathscr{H}_{\mathbb{C}} = \bigoplus_\lambda \Psi_t^\lambda \mathscr{H}_{\mathbb{C}}, \quad \Psi_t \mathscr{H}_{\mathbb{Q}} = \Psi_t \mathscr{H}_{\mathbb{Q}}^1 \oplus \Psi_t \mathscr{H}_{\mathbb{Q}}^{\neq 1}.$$

Monodromy filtration: $\mathrm{N} :=$ nilp. part of T,
i.e., $\mathrm{N} = (T - \lambda\,\mathrm{Id})$ on $\Psi_t^\lambda \mathscr{H}_{\mathbb{C}}$.

THEOREM (Jacobson-Morosov). $\exists\,!\nearrow$ filtr. $\mathrm{M}_\bullet \Psi_t \mathscr{H}_{\mathbb{Q}}$ ($\mathrm{M}_\bullet \Psi_t \mathscr{H}_{\mathbb{C}} = \bigoplus_\lambda \mathrm{M}_\bullet \Psi_t^\lambda \mathscr{H}_{\mathbb{C}}$) s.t.

- $\mathrm{N}\big(\mathrm{M}_\ell \Psi_t \mathscr{H}_{\mathbb{Q}}\big) \subset \mathrm{M}_{\ell-2} \Psi_t \mathscr{H}_{\mathbb{Q}}$,
- $\forall\, \ell \geqslant 1$, $\mathrm{N}^\ell : \mathrm{gr}_\ell^{\mathrm{M}} \Psi_t \mathscr{H}_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{gr}_{-\ell}^{\mathrm{M}} \Psi_t \mathscr{H}_{\mathbb{Q}}$

Nearby cycles

- Given $\mathscr{Q} : \mathscr{H}_{\mathbb{Q}} \otimes \mathscr{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ nondeg. $\rightsquigarrow \Psi_t \mathscr{Q}$ s.t.

$$\Psi_t \mathscr{Q}(T\bullet, T\bullet) = \Psi_t \mathscr{Q}(\bullet, \bullet)$$

$\Rightarrow \Psi_t \mathscr{Q}(\Psi_t^{\lambda} \mathscr{H}_{\mathbb{C}}, \Psi_t^{\mu} \mathscr{H}_{\mathbb{C}}) = 0$ unless $\mu = \lambda^{-1}$.

Nearby cycles

- Given $\mathscr{Q} : \mathscr{H}_{\mathbb{Q}} \otimes \mathscr{H}_{\mathbb{Q}} \to \mathbb{Q}$ nondeg. $\rightsquigarrow \Psi_t \mathscr{Q}$ s.t.

$$\Psi_t \mathscr{Q}(T\bullet, T\bullet) = \Psi_t \mathscr{Q}(\bullet, \bullet)$$

$$\Rightarrow \Psi_t \mathscr{Q}(\Psi_t^{\lambda} \mathscr{H}_{\mathbb{C}}, \Psi_t^{\mu} \mathscr{H}_{\mathbb{C}}) = 0 \text{ unless } \mu = \lambda^{-1}.$$

THEOREM. For each $a \in (-1, 0]$, there is a canonical isomorphism ($\lambda = \exp(-2\pi i a)$):

$$\left(\psi_t^a \widetilde{V} := \widetilde{V}^a / \widetilde{V}^{>a}, \exp(-2\pi i t \widetilde{\nabla}_{\partial_t})\right) \simeq \left(\Psi_t^{\lambda} V^{\nabla}, T\right).$$

Nearby cycles

- Given $\mathscr{Q} : \mathscr{H}_{\mathbb{Q}} \otimes \mathscr{H}_{\mathbb{Q}} \to \mathbb{Q}$ nondeg. $\rightsquigarrow \Psi_t \mathscr{Q}$ s.t.

$$\Psi_t \mathscr{Q}(T\bullet, T\bullet) = \Psi_t \mathscr{Q}(\bullet, \bullet)$$

$$\Rightarrow \Psi_t \mathscr{Q}(\Psi_t^\lambda \mathscr{H}_{\mathbb{C}}, \Psi_t^\mu \mathscr{H}_{\mathbb{C}}) = 0 \text{ unless } \mu = \lambda^{-1}.$$

THEOREM. For each $a \in (-1, 0]$, there is a canonical isomorphism ($\lambda = \exp(-2\pi i a)$):

$$(\psi_t^a \widetilde{V} := \widetilde{V}^a / \widetilde{V}^{>a}, \exp(-2\pi i t \widetilde{\nabla}_{\partial_t})) \simeq (\Psi_t^\lambda V^\nabla, T).$$

Schmid's thm is more precise:

Set $\mathrm{M}_\ell \widetilde{V}^a =$ pull-back of $\mathrm{M}_\ell(\widetilde{V}^a / \widetilde{V}^{>a})$ by the proj. $\widetilde{V}^a \to (\widetilde{V}^a / \widetilde{V}^{>a})$. Then (usefull later)

$$\forall\, a > -1,\ \forall\, \ell \in \mathbb{Z}, \quad \boxed{j_* F^p V \cap \mathrm{M}_\ell \widetilde{V}^a \text{ is } \mathscr{O}_\Delta\text{-coh.}}$$

Pure Hodge modules on a Riemann surf.

Pure Hodge modules on a Riemann surf.

DEFINITION of $\mathrm{MH}_{\boldsymbol{\Delta}}((\boldsymbol{\Delta}, 0), w)$

Pure Hodge modules on a Riemann surf.

DEFINITION of $\mathrm{MH}_{\Delta}((\Delta,0),w)$

- Objects: $((\widetilde{V}_{\min}, F^{\bullet}\widetilde{V}_{\min}), j_*\mathscr{H}_{\mathbb{Q}}, \alpha)$ s.t.
  - $F^{\bullet}\widetilde{V}_{\min}$: good filtr. of $\widetilde{V}_{\min}$,
  - monodromy of $\mathscr{H}_{\mathbb{Q}}$ ***quasi-unipotent*** (*i.e.*, eigenval. $=$ roots of unity).
  - $\alpha : j_*\mathscr{H}_{\mathbb{C}} \simeq \mathrm{DR}\,\widetilde{V}_{\min}$
  - On Δ^*, belong to $\mathrm{VHS}(\Delta^*, w)$,
  - $(\widetilde{V}_{\min}, F^{\bullet}\widetilde{V}_{\min})$ strict. specializable at $t=0$
  - $((\psi_t\widetilde{V}_{\min}, F^{\bullet}\psi_t\widetilde{V}_{\min}), \Psi_t j_*\mathscr{H}_{\mathbb{Q}}, \psi_t\alpha)$ is a MHS with weight filtr. $W_k = \mathrm{M}_{w+k}$.

Pure Hodge modules on a Riemann surf.

DEFINITION of $\mathrm{MH}_{\Delta}((\Delta, 0), w)^{(\mathrm{p})}$

- Objects: $((\widetilde{V}_{\min}, F^{\bullet}\widetilde{V}_{\min}), j_*\mathscr{H}_{\mathbb{Q}}, j_*\mathscr{Q}, \alpha)$ s.t.
  - $F^{\bullet}\widetilde{V}_{\min}$: good filtr. of $\widetilde{V}_{\min}$,
  - monodromy of $\mathscr{H}_{\mathbb{Q}}$ ***quasi-unipotent*** (*i.e.*, eigenval. $=$ roots of unity).
  - $\alpha : j_*\mathscr{H}_{\mathbb{C}} \simeq \mathrm{DR}\,\widetilde{V}_{\min}$
  - On Δ^*, belong to $\mathrm{VHS}(\Delta^*, w)^{(\mathrm{p})}$,
  - $(\widetilde{V}_{\min}, F^{\bullet}\widetilde{V}_{\min})$ strict. specializable at $t = 0$
  - $((\psi_t\widetilde{V}_{\min}, F^{\bullet}\psi_t\widetilde{V}_{\min}), \Psi_t j_*\mathscr{H}_{\mathbb{Q}}, \Psi_t j_*\mathscr{Q}, \psi_t\alpha)$ is a $\mathrm{MHS}^{(\mathrm{p})}$ with weight filtr. $W_k = \mathrm{M}_{w+k}$.

Pure Hodge modules on a Riemann surf.

DEFINITION of $\mathrm{MH}_0(\Delta, w)^{(\mathrm{p})}$

- Start with $((H, F^\bullet H), H_{\mathbb{Q}}, Q, \alpha : H \simeq H_{\mathbb{Q}} \otimes \mathbb{C})$ in $\mathrm{MH}(w)^{(\mathrm{p})}$
- $i : \{0\} \hookrightarrow \Delta$
- $i_+(H, F^\bullet H) := \big(H \otimes \mathbb{C}[\partial_t], F^\bullet(H \otimes \mathbb{C}[\partial_t])\big)$

$$F^p(H \otimes \mathbb{C}[\partial_t]) = \bigoplus_{k \geqslant 0} F^{p+k} H \partial_t^k$$

- $\mathrm{DR}\, i_+ H = Ri_* H[-1]$
- $Ri_*\alpha : \mathrm{DR}(i_+ H) \simeq Ri_* H_{\mathbb{Q}}[-1] \otimes \mathbb{C}$
- Objects:

$$\big(i_+(H, F^\bullet H), Ri_* H_{\mathbb{Q}}, Ri_* Q, Ri_*\alpha\big)$$

Pure Hodge modules on a Riemann surf.

DEFINITION of $\mathbf{MH}((\Delta, 0), w)^{(\mathbf{p})}$

$$\boxed{\mathbf{MH}_{\Delta}((\Delta, 0), w)^{(\mathbf{p})} \oplus \mathbf{MH}_{0}(\Delta, w)^{(\mathbf{p})}}$$

THEOREM.

- $\mathbf{MH}_{\Delta}((\Delta, 0), w)^{(\mathbf{p})} \sim \mathbf{VHS}(\Delta^*, w)^{(\mathbf{p})}$
- $\mathbf{MH}_{0}(\Delta, w)^{(\mathbf{p})} \sim \mathbf{HS}(w)^{(\mathbf{p})}$

$\mathscr{D}_X$-modules in higher dim.

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)
- $\mathscr{D}_X$ sheaf of differential oper. with coefs in $\mathscr{O}_X$.
- In local analytic (or étale) coord.:
  $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial_x^\alpha$.

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)
- $\mathscr{D}_X$ sheaf of differential oper. with coefs in $\mathscr{O}_X$.
- In local analytic (or étale) coord.:
  $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial_x^\alpha$.
- $\nearrow$ Filtration $F_\bullet\mathscr{D}_X$: $P \in F_k \Leftrightarrow a_\alpha = 0 \ \forall\, |\alpha| > k$.

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)
- $\mathscr{D}_X$ sheaf of differential oper. with coefs in $\mathscr{O}_X$.
- In local analytic (or étale) coord.: $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial_x^\alpha$.
- $\nearrow$ Filtration $F_\bullet \mathscr{D}_X$: $P \in F_k \Leftrightarrow a_\alpha = 0 \; \forall \, |\alpha| > k$.
- Left $\mathscr{D}_X$-mod. $\iff$ $\mathscr{O}_X$-mod. with an ***integrable*** connection

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)
- $\mathscr{D}_X$ sheaf of differential oper. with coefs in $\mathscr{O}_X$.
- In local analytic (or étale) coord.: $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial_x^\alpha$.
- $\nearrow$ Filtration $F_\bullet\mathscr{D}_X$: $P \in F_k \Leftrightarrow a_\alpha = 0\ \forall\, |\alpha| > k$.
- Left $\mathscr{D}_X$-mod. $\iff$ $\mathscr{O}_X$-mod. with an ***integrable*** connection
- $\mathrm{DR}\, M = \Big\{0 \to \underset{\bullet}{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^{\dim X} \otimes M \to 0\Big\}$

$\mathscr{D}_X$-modules in higher dim.

- X a cplx mfld or smooth cplx alg. var. (any dim.)
- $\mathscr{D}_X$ sheaf of differential oper. with coefs in $\mathscr{O}_X$.
- In local analytic (or étale) coord.: $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial_x^\alpha$.
- $\nearrow$ Filtration $F_\bullet \mathscr{D}_X$: $P \in F_k \Leftrightarrow a_\alpha = 0 \ \forall \, |\alpha| > k$.
- Left $\mathscr{D}_X$-mod. $\iff$ $\mathscr{O}_X$-mod. with an ***integrable*** connection
- $\mathrm{DR}\, M = \Big\{ 0 \to \underset{\bullet}{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^{\dim X} \otimes M \to 0 \Big\}$
- ${}^{\mathrm{p}}\mathrm{DR} M = \Big\{ 0 \to M \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \underset{\bullet}{\Omega_X^{\dim X}} \otimes M \to 0 \Big\}$

$${}^{\mathrm{p}}\mathrm{DR} M = R\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{O}_X, M)$$

$\mathscr{D}_X$-modules in higher dim.

- ***Good filtration*** $F_\bullet M$:
  - $\nearrow$ filtr. s.t. $F_k\mathscr{D}_X \cdot F_\ell M \subset F_{k+\ell}M$
  - $F_\ell M$ $\mathscr{O}_X$-coh. and 0 for $\ell \ll 0$,
  - $\subset$ is an equality for $\ell \geqslant \ell_o$ and all k.

$\mathscr{D}_X$-modules in higher dim.

- ***Good filtration*** $F_\bullet M$:
  - $\nearrow$ filtr. s.t. $F_k\mathscr{D}_X \cdot F_\ell M \subset F_{k+\ell}M$
  - $F_\ell M$ $\mathscr{O}_X$-coh. and $\mathbf{0}$ for $\ell \ll 0$,
  - $\subset$ is an equality for $\ell \geqslant \ell_o$ and all k.
  - $F^pM := F_{-p}M$

$\mathscr{D}_X$-modules in higher dim.

- ***Good filtration*** $F_\bullet M$:
  - $\nearrow$ filtr. s.t. $F_k\mathscr{D}_X \cdot F_\ell M \subset F_{k+\ell}M$
  - $F_\ell M$ $\mathscr{O}_X$-coh. and 0 for $\ell \ll 0$,
  - $\subset$ is an equality for $\ell \geqslant \ell_o$ and all k.
  - $F^pM := F_{-p}M$
- $\rightsquigarrow$ filtered DR cplx: $F^p({}^{\mathrm{p}}\mathrm{DR}M)$ defined by

$$\left\{0 \to F^{p+\dim X}M \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^{\dim X} \underset{\bullet}{\otimes} F^pM \to 0\right\}$$

$\mathscr{D}_X$-modules in higher dim.

- ***Good filtration*** $F_\bullet M$:
  - $\nearrow$ filtr. s.t. $F_k\mathscr{D}_X \cdot F_\ell M \subset F_{k+\ell}M$
  - $F_\ell M$ $\mathscr{O}_X$-coh. and 0 for $\ell \ll 0$,
  - $\subset$ is an equality for $\ell \geqslant \ell_o$ and all k.
  - $F^pM := F_{-p}M$
- $\rightsquigarrow$ filtered DR cplx: $F^p({}^{\mathrm{p}}\mathrm{DR}M)$ defined by

$$\left\{0 \to F^{p+\dim X}M \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^{\dim X} \underset{\bullet}{\otimes} F^pM \to 0\right\}$$

- EXAMPLE. $M = \mathscr{O}_X$, $F^k\mathscr{O}_X = \begin{cases} \mathscr{O}_X & k \leqslant \dim X \\ 0 & k > \dim X \end{cases}$

$$F^p({}^{\mathrm{p}}\mathrm{DR}\mathscr{O}_X) = \left\{\cdots \to 0 \to \Omega_X^p \xrightarrow{\nabla} \cdots \to \underset{\bullet}{\Omega_X^{\dim X}} \to 0\right\}$$

The nearby cycle functor for $\mathscr{D}$-mod.

THEOREM (Kashiwara). $Y \subset X$ a closed submfld $\Rightarrow$

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y) \simeq \mathrm{Mod}_{\mathrm{coh},Y}(\mathscr{D}_X)$$

The nearby cycle functor for $\mathscr{D}$-mod.

THEOREM (Kashiwara). $Y \subset X$ a closed submfld $\Rightarrow$

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y) \simeq \mathrm{Mod}_{\mathrm{coh},Y}(\mathscr{D}_X)$$

$\Rightarrow$ For $Y \subset X$ closed alg. (or anal.) subset, ***define*** $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ as $\mathrm{Mod}_{\mathrm{coh},Y}(\mathscr{D}_X)$ (but do not use $\mathscr{D}_Y$).

The nearby cycle functor for $\mathscr{D}$-mod.

THEOREM (Kashiwara). $Y \subset X$ a closed submfld $\Rightarrow$

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y) \simeq \mathrm{Mod}_{\mathrm{coh},Y}(\mathscr{D}_X)$$

$\Rightarrow$ For $Y \subset X$ closed alg. (or anal.) subset, ***define*** $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ as $\mathrm{Mod}_{\mathrm{coh},Y}(\mathscr{D}_X)$ (but do not use $\mathscr{D}_Y$).

- $f : X \to \mathbb{C}$ alg. (or hol.) fctn, $M \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.
- $i_f : X \hookrightarrow X \times \mathbb{C}, \quad x \mapsto (x, f(x))$
- $M \longleftrightarrow i_{f,+}M \in \mathrm{Mod}_{\mathrm{coh},i_f(X)}(\mathscr{D}_{X\times\mathbb{C}})$

$\Rightarrow$ may assume $f =$ projection $X \times \mathbb{C} \to \mathbb{C}$

The nearby cycle functor for $\mathscr{D}$-mod.

The nearby cycle functor for $\mathscr{D}$-mod.

- $$V^k\mathscr{D}_{X\times\mathbb{C}} = \begin{cases} t^k\mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle & k \geqslant 0 \\ \sum_{j=0}^{|k|}\mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle\partial_t^j & k < 0 \end{cases}$$

The nearby cycle functor for $\mathscr{D}$-mod.

- $$V^k \mathscr{D}_{X\times\mathbb{C}} = \begin{cases} t^k \mathscr{O}_{X\times\mathbb{C}}\langle \partial_x, t\partial_t\rangle & k \geqslant 0 \\ \sum_{j=0}^{|k|} \mathscr{O}_{X\times\mathbb{C}}\langle \partial_x, t\partial_t\rangle \partial_t^j & k < 0 \end{cases}$$
- M a coh. $\mathscr{D}_{X\times\mathbb{C}}$-mod., notion of good V-filtr.

The nearby cycle functor for $\mathscr{D}$-mod.

- $$V^k\mathscr{D}_{X\times\mathbb{C}} = \begin{cases} t^k \mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle & k \geqslant 0 \\ \sum_{j=0}^{|k|} \mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle\partial_t^j & k < 0 \end{cases}$$
- M a coh. $\mathscr{D}_{X\times\mathbb{C}}$-mod., notion of good V-filtr.
- ***Kashiwara***: M holon. $\Rightarrow \exists\,!$ good V-filtr. of M s.t.

  α eigenval. of E on $V^0M/tV^0M \Rightarrow \mathrm{Re}(\alpha) \in [0,1)$

  $\rightsquigarrow$ Kashiwara-Malgrange filtr. $V^\bullet M$.

The nearby cycle functor for $\mathscr{D}$-mod.

- $$V^k\mathscr{D}_{X\times\mathbb{C}} = \begin{cases} t^k\mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle & k \geqslant 0 \\ \sum_{j=0}^{|k|}\mathscr{O}_{X\times\mathbb{C}}\langle\partial_x, t\partial_t\rangle\partial_t^j & k < 0 \end{cases}$$
- M a coh. $\mathscr{D}_{X\times\mathbb{C}}$-mod., notion of good V-filtr.
- ***Kashiwara***: M holon. $\Rightarrow \exists\,!$ good V-filtr. of M s.t.

  α eigenval. of E on $V^0M/tV^0M \Rightarrow \mathrm{Re}(\alpha) \in [0, 1)$

  $\rightsquigarrow$ Kashiwara-Malgrange filtr. $V^\bullet M$.

DEFINITION. $\psi_t^a M =$ gener. eigenmod. w.r.t. a of E on V^0M/tV^0M, $\mathrm{N} =$ corresponding nilp. end.

- $\psi_t^a M$ is a hol. $\mathscr{D}_X$-mod. (Kashiwara)

The case of filtered $\mathscr{D}$-mod.

- In general, Kashiwara's equiv. ***not true*** for filtered $\mathscr{D}$-mod.
- Can define $i_{f,+}(M, F_\bullet M)$
- In general, the induced filtration $F^\bullet \psi_t^a M$ does not have good properties.
- $\rightsquigarrow$ notion of ***strictly specializable*** filtered $\mathscr{D}$-mod. modelled on the dim. 1 case (Schmid).

The case of filtered $\mathscr{D}$-mod.

- In general, Kashiwara's equiv. ***not true*** for filtered $\mathscr{D}$-mod.
- Can define $i_{f,+}(M, F_{\bullet}M)$
- In general, the induced filtration $F^{\bullet}\psi_t^a M$ does not have good properties.
- $\rightsquigarrow$ notion of ***strictly specializable*** filtered $\mathscr{D}$-mod. modelled on the dim. 1 case (Schmid).
- In the following, assume M hol. and ***quasi-unipotent***, *i.e.*, $\psi_t^a M \neq 0$ only for $a \in \mathbb{Q}$ (automatic in Schmid's case according to a lemma of Borel). $\Rightarrow$ can consider $V^{\bullet}M$ indexed by $\mathbb{Q}$, and $\psi_t^a M = V^a M / V^{>a} M$.

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

- $(M, F^\bullet M)$ ***strict. special.*** along $t = 0$ if
  - $\forall\, a > -1$, $t : F^p M \cap V^a M \xrightarrow{\sim} F^p M \cap V^{a+1} M$
  - $\forall\, a < 0$, $\quad \partial_t : F^p \psi_t^a M \xrightarrow{\sim} F^{p-1} \psi_t^{a-1} M$

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

- $(M, F^\bullet M)$ ***strict. special.*** along $t = 0$ if
  - $\forall\, a > -1$, $t : F^pM \cap V^aM \xrightarrow{\sim} F^pM \cap V^{a+1}M$
  - $\forall\, a < 0$, $\quad \partial_t : F^p\psi_t^aM \xrightarrow{\sim} F^{p-1}\psi_t^{a-1}M$
- $(M, F^\bullet M)$ ***regular*** along $t = 0$ if
  - $\forall\, a \in [-1, 0]$, $F^\bullet\psi_t^aM$ is a ***good*** filtr.

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

- $(M, F^\bullet M)$ ***strict. special.*** along $t=0$ if
  - $\forall\, a > -1$, $t : F^pM \cap V^aM \xrightarrow{\sim} F^pM \cap V^{a+1}M$
  - $\forall\, a < 0$, $\quad \partial_t : F^p\psi_t^a M \xrightarrow{\sim} F^{p-1}\psi_t^{a-1}M$
- $(M, F^\bullet M)$ ***regular*** along $t=0$ if
  - $\forall\, a \in [-1, 0]$, $F^\bullet\psi_t^a M$ is a ***good*** filtr.

$(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_X$-mod.

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

- $(M, F^\bullet M)$ ***strict. special.*** along $t = 0$ if
  - $\forall\, a > -1$, $t : F^p M \cap V^a M \xrightarrow{\sim} F^p M \cap V^{a+1} M$
  - $\forall\, a < 0$, $\quad \partial_t : F^p \psi_t^a M \xrightarrow{\sim} F^{p-1} \psi_t^{a-1} M$
- $(M, F^\bullet M)$ ***regular*** along $t = 0$ if
  - $\forall\, a \in [-1, 0]$, $F^\bullet \psi_t^a M$ is a ***good*** filtr.

$(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_X$-mod.

- $(M, F^\bullet M)$ is ***strict. special.*** resp. ***regular*** if ***for any*** hol. fctn $f : X \supset U \to \mathbb{C}$, $i_{f,+}(M, F^\bullet M)$ is strict. special. resp. regular along $t = 0$.

The case of filtered $\mathscr{D}$-mod.

DEFINITION. $(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_{X\times\mathbb{C}}$-mod.

- $(M, F^\bullet M)$ ***strict. special.*** along $t = 0$ if
  - $\forall\, a > -1$, $t : F^pM \cap V^aM \xrightarrow{\sim} F^pM \cap V^{a+1}M$
  - $\forall\, a < 0$, $\quad \partial_t : F^p\psi_t^a M \xrightarrow{\sim} F^{p-1}\psi_t^{a-1}M$
- $(M, F^\bullet M)$ ***regular*** along $t = 0$ if
  - $\forall\, a \in [-1, 0]$, $F^\bullet\psi_t^a M$ is a ***good*** filtr.

$(M, F^\bullet M)$ a coh. filtered $\mathscr{D}_X$-mod.

- $(M, F^\bullet M)$ is ***strict. special.*** resp. ***regular*** if ***for any*** hol. fctn $f : X \supset U \to \mathbb{C}$, $i_{f,+}(M, F^\bullet M)$ is strict. special. resp. regular along $t = 0$.
- Axiom for $\mathrm{MHM}(X) \Rightarrow (M, F^\bullet M)$ strict. special. and regular. E.g., $(\mathscr{O}_X, F^\bullet\mathscr{O}_X)$.