Hodge $\mathscr{D}$-modules 1. Overview

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Review of mixed Hodge theory

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- finite dim. $\mathbb{Q}$-vect. space $H_{\mathbb{Q}}$,
- a decomp. $H := H_{\mathbb{C}} = \bigoplus_p H^p$ s.t.

$$\forall\, p, \quad \overline{H^p} = H^{w-p}.$$

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EQUIV. DEFINITION: Pure $\mathbb{Q}$-Hodge struct. of weight w:

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- a $\searrow$ filtr. F^pH s.t.

$$\forall\, p, \quad \boxed{F^pH \cap \overline{F^{w-p+1}H} = 0}$$

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Category of Hodge structures HS:

- Objects $=$ Pure $\mathbb{Q}$-Hodge structures (of some weight).
- Morphisms: $\varphi : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ s.t. $\varphi(F^p H) \subset F^p H'$ $\forall\, p$.

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PROPOSITION:

- $\mathrm{HS}(w)$: Abelian category.
- Any morphism is ***strict***: $\varphi(F^pH) = F^pH' \cap \varphi(H)$
- $\varphi : (H_{\mathbb{Q}}, F^\bullet H, w) \to (H'_{\mathbb{Q}}, F^\bullet H', w')$. Then

$$\boxed{w > w' \Longrightarrow \varphi = 0}$$

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Polarization of $(H_{\mathbb{Q}}, F^{\bullet}H) \in \mathrm{HS}(w)$:

- $Q : H_{\mathbb{Q}} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ ***nondeg. bilin. pairing*** s.t.
  - Q is $(-1)^w$-symm.
  - $Q(H^{p,w-p}, H^{p',w-p'}) = 0$ if $p' \neq w-p$, $\rightsquigarrow$ $k_{p,w-p} : H^{p,w-p} \otimes \overline{H^{p,w-p}} \longrightarrow \mathbb{C}$ Hermitian form.
  - (positivity): on $H^{p,w-p}$,

    $$\boxed{h_{p,w-p} := (-1)^p i^{-w} k_{p,w-p} \text{ is } > 0}$$

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PROPOSITION: $(H_{\mathbb{Q}}, F^{\bullet}H, Q, w)$ ***pol.*** HS & $(H'_{\mathbb{Q}}, F^{\bullet}H', w)$ sub-HS $\Rightarrow$ $(H'_{\mathbb{Q}}, F^{\bullet}H', w)$ ***pol.***
$\rightsquigarrow$ $\mathrm{HS}^{(\mathrm{p})}(w)$ ***semi-simple***.

Review of mixed Hodge theory

- X: cpct Riemann surf., genus g, connected.
- ***Poincaré duality***: skew-symm. nondeg. bilin. form

$$Q = \langle \bullet, \bullet \rangle : H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X, \mathbb{Z}) \xrightarrow{\bullet \cup \bullet} H^2(X, \mathbb{Z}) \xrightarrow{\int_{[X]}} \mathbb{Z}$$

- $\dim H^1(X, \mathscr{O}_X) = g$ & $H^1(X, \mathscr{O}_X) \simeq H^0(X, \Omega^1_X)^\vee$
- $\rightsquigarrow H^1(X, \mathbb{C}) \simeq H^0(X, \Omega^1_X) \oplus H^1(X, \mathscr{O}_X)$
- $\langle H^{1,0}, H^{1,0} \rangle = 0$ & $\langle H^{0,1}, H^{0,1} \rangle = 0$
- ***Hodge-Riemann bilin. rel.*** $\Rightarrow h = ik = (-1)^1 i^{-1} k$ is > 0 on $H^{1,0}$.

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- a $\searrow$ filtr. $F^{\bullet}H_{\mathbb{C}}$,

s.t. $\forall\, w$, the following data in $\mathrm{HS}(w)$:

$$\left(\mathrm{gr}_w^W H_{\mathbb{Q}} := \frac{W_w H_{\mathbb{Q}}}{W_{w-1} H_{\mathbb{Q}}}, \quad F^p \mathrm{gr}_w^W H_{\mathbb{C}} := \frac{F^p H_{\mathbb{C}} \cap W_w H_{\mathbb{C}}}{F^p H_{\mathbb{C}} \cap W_{w-1} H_{\mathbb{C}}}\right)$$

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From now on, all MHS's are assumed ***gr.-pol.***

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THEOREM (Deligne):
X: quasi-proj. var. $/\mathbb{C}$ (or separ. scheme $/\mathbb{C}$).
Then $H^q(X, \mathbb{C})$ and $H^q_c(X, \mathbb{C})$ have canonical MHS's.

Grothendieck RS

"Of course, it was part of my big working program around motives, of which Deligne was informed directly and every day, to make explicit a notion of "***Hodge coefficients***" on a scheme of finite type over $\mathbb{C}$, in such a way that, to any motive on X there corresponds a "Hodge realization", and that, for pure smooth motives on X (for example those coming from a smooth and proper scheme over X by taking its "motivic cohomology in dimension i"), one recovers the notion (more or less known already) of "families of Hodge structures" (studied in particular by Griffiths in the sixties). Moreover, for a varying X, these categories of "Hodge coefficients" should satisfy a ***formalism of the six operations***, reflecting the same formalism at the level of motives. The contribution of Deligne represents a first step to the fulfillment of this program, namely (essentially) the description of the category $\mathbf{Hdg}(X)$ for X reduced to a point, and that of the "realization" functor *i.e.*, essentially, the construction of a cohomology theory on separated $\mathbb{C}$-scheme of finite type, with values in this category of Hodge-Deligne."

Formalism of 6 operations

Coefficients: bounded cplxes of $\mathbb{Q}$-vect. spaces with constr. cohom.: $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X) \rightsquigarrow H^\bullet(X, \mathscr{F})$, e.g., $\mathscr{F} = \mathbb{Q}_X$.

- Bi-functors $R\mathscr{H}om_{\mathbb{Q}_X}(\mathscr{F}, \mathscr{G})$ and $\mathscr{F} \otimes \mathscr{G}$
- Given $f : X \to Y$, functors

$$f_*, f_! : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_Y), \quad f^*, f^! : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_Y) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X)$$

- Duality D (Poincaré-Verdier):
  $D\mathscr{F} = R\mathscr{H}om(\mathscr{F}, a_X^!\mathbb{Q}), \quad a_X =: X \to \mathrm{pt}$

Formalism of 6 operations

Relations between the functors

- Adjunction for pairs (f^*, f_*) and $(f_!, f^!)$
- $D \circ D \simeq \mathrm{Id}$
- D exchanges (f_*, f^*) and $(f_!, f^!)$
- $\ldots$

Formalism of 6 operations

Cohomology computations

- $H^\bullet(X,\mathbb{Q}) = H^\bullet(a_{X,*}a_X^*\mathbb{Q})$,
- $H_c^\bullet(X,\mathbb{Q}) = H^\bullet(a_{X,!}a_X^*\mathbb{Q})$,
- $H_\bullet(X,\mathbb{Q}) = H^\bullet(a_{X,!}a_X^!\mathbb{Q})$,
- $H_\bullet^{\mathrm{BM}}(X,\mathbb{Q}) = H^\bullet(a_{X,*}a_X^!\mathbb{Q})$,
- for $i : Z \hookrightarrow X$ a closed immersion,
  $H_Z^\bullet(X,\mathbb{Q}) = H^\bullet(a_{X,*}i_*i^!a_X^*\mathbb{Q})$,
- etc.

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Main theorems by M. Saito

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THEOREM 1. Given X, $\exists$ an abelian cat. $\mathrm{MHM}(X)$ with a functor $\mathrm{rat} : \mathsf{D}^{\mathrm{b}}(\mathrm{MHM}(X)) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X)$. The six operations and the duality functor can be naturally lifted to $\mathsf{D}^{\mathrm{b}}(\mathrm{MHM}(X))$, and satisfy the same relations.

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THEOREM 2. The category $\mathrm{MHM}(\mathrm{pt})$ is the category of (graded polarizable) mixed Hodge structures.

$\Longrightarrow \exists\,!\ \mathbb{Q}_X^{\mathrm{H}} \in \mathbf{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ s.t. $\mathrm{rat}(\mathbb{Q}_X^{\mathrm{H}}) = \mathbb{Q}_X$.

$\Longrightarrow \exists$ canon. MHS on $H^\bullet(X, \mathbb{Q}) = H^\bullet(a^{\mathrm{H}}_{X,*} a_X^{\mathrm{H},*} \mathbb{Q}^{\mathrm{H}}_{\mathrm{pt}})$.

The weight filtration

PROPOSITION. Each $\mathscr{M}$ in $\mathrm{MHM}(X)$ carries a funct. $\nearrow$ filtr. $W_\bullet\mathscr{M}$, called the ***weight filtration of*** $\mathscr{M}$, such that $\mathscr{M} \mapsto W_k\mathscr{M}$ and $\mathscr{M} \mapsto \mathrm{gr}_k^W\mathscr{M}$ are ***exact*** functors.

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DEFINITION. $\mathscr{M}$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{MHM}(X))$. Then $\mathscr{M}$ has

- ***weights*** $\leqslant w$ if $\mathrm{gr}_i^W H^j\mathscr{M} = 0$, $\forall\, i, j$ s.t. $i > j + w$,
- ***weights*** $\geqslant w$ if $\mathrm{gr}_i^W H^j\mathscr{M} = 0$, $\forall\, i, j$ s.t. $i < j + w$,
- ***pure weight*** w if $\mathrm{gr}_i^W H^j\mathscr{M} = 0$, $\forall\, i, j$ s.t. $i \neq j + w$, *i.e.*, $H^j\mathscr{M}$ has pure weight $j + w$.

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$\Longrightarrow$ full sub-cat. $\mathbf{D}^{\mathrm{b}}_{\leqslant w}(\mathrm{MHM}(X))$ and $\mathbf{D}^{\mathrm{b}}_{\geqslant w}(\mathrm{MHM}(X))$ of $\mathbf{D}^{\mathrm{b}}(\mathrm{MHM}(X))$.

Weight filtration and functors

THEOREM.

- $f_!, f^*$ preserve $\mathsf{D}^{\mathrm{b}}_{\leqslant w}(\mathrm{MHM}(X))$,
- $f^!, f_*$ preserve $\mathsf{D}^{\mathrm{b}}_{\geqslant w}(\mathrm{MHM}(X))$,
- $\otimes$ sends $\mathsf{D}^{\mathrm{b}}_{\leqslant w}(\mathrm{MHM}(X)) \times \mathsf{D}^{\mathrm{b}}_{\leqslant w'}(\mathrm{MHM}(X))$ in $\mathsf{D}^{\mathrm{b}}_{\leqslant w+w'}(\mathrm{MHM}(X))$,
- $\mathscr{H}om$ sends $\mathsf{D}^{\mathrm{b}}_{\leqslant w}(\mathrm{MHM}(X)) \times \mathsf{D}^{\mathrm{b}}_{\geqslant w'}(\mathrm{MHM}(X))$ in $\mathsf{D}^{\mathrm{b}}_{\geqslant -w+w'}(\mathrm{MHM}(X))$,
- $D : \mathsf{D}^{\mathrm{b}}_{\leqslant w}(\mathrm{MHM}(X)) \longleftrightarrow \mathsf{D}^{\mathrm{b}}_{\geqslant -w}(\mathrm{MHM}(X))$.

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- COROLLARY.
  L: loc. syst. on X, underlying an adm. var. of MHS

  $\Rightarrow$ $H^\bullet(X, L)$, $H^\bullet_{\mathrm{c}}(X, L)$ carry a natural MHS

Pure objects

THEOREM.
$\mathscr{M}$ ***pure*** in $\mathsf{D}^{\mathrm{b}}(\mathrm{MHM}(X)) \Rightarrow \mathscr{M} \simeq \bigoplus_k H^k\mathscr{M}[-k]$ (non-canon.).

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THEOREM.

- $\mathscr{M}$ pure in $\mathrm{MHM}(X)$. $\Rightarrow \mathscr{M}$ is ***semi-simple*** in $\mathrm{MHM}(X)$.
- Conversely, $\mathscr{M}$ ***simple*** in $\mathrm{MHM}(X) \Rightarrow \mathscr{M}$ ***pure***. Moreover, if X is ***projective***, $\mathrm{rat}(\mathscr{M})$ is a semi-simple $\mathbb{Q}$-perverse sheaf.

Pure objects

- $\mathscr{M} \in \mathrm{MH}(X, w)$
  $\Rightarrow \mathscr{M} = \bigoplus_Z \mathscr{M}_Z$ (decompos. by the ***strict support***):
  - $Z \subset X$ ***closed irred.***
  - $\operatorname{Supp} \mathscr{M}_Z = Z$,
  - $\nexists\, \mathscr{N} \neq 0, \mathscr{M}_Z$ in $\mathrm{MHM}(X)$, $\mathscr{N} \subset \mathscr{M}_Z$, s.t.

$$\operatorname{Supp} \mathscr{N} \text{ or } \operatorname{Supp} \mathscr{M}_Z/\mathscr{N} \subsetneq Z$$

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  - M: regular hol. $\mathscr{D}_X$-mod.
  - $F_\bullet M$: good $F_\bullet \mathscr{D}_X$-filtr. subject to some properties.
  - $\mathscr{F}_{\mathbb{Q}}$: $\mathbb{Q}$-perverse sheaf on X
  - $W_\bullet \mathscr{F}_{\mathbb{Q}}$: $\nearrow$ filtr. by $\mathbb{Q}$-perverse subsheaves
  - $\alpha : \mathscr{F}_{\mathbb{C}} \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR} M$.

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- If X ***not smooth***
  - Use local embed. $(X, x) \hookrightarrow (Y, x)$, Y smooth
  - Set $\mathrm{MHM}(X, x) = \mathrm{MHM}_X(Y, x)$
  - Independence of the embedding $\Rightarrow$ can glue the categories.