A REMARK ON THE MEROMORPHIC EXTENSION OF HORIZONTAL SECTIONS

by

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Abstract. We give a criterion for horizontal sections of a meromorphic connection on $(\mathbb{C}^2, 0)$ with poles along the coordinate axes to extend as meromorphic sections. An application is given to morphisms between wild twistor $\mathscr{D}$-modules on the disc.

Let M be the germ of a meromorphic bundle with a flat connection on the germ $X = (\mathbb{C}^2, 0)$ equipped with coordinates x_1, x_2, with poles contained in the (germ of) divisor $D = \{x_1x_2 = 0\}$. In other words, M is a free $\mathbb{C}\{x_1, x_2\}[(x_1x_2)^{-1}]$-module of finite rank equipped with a flat connection $\nabla : M \to \Omega^1_X \otimes M$. If ∇ has regular singularities along D, it is well-known that any ∇-horizontal section of M on $X^* := X \smallsetminus D$ is meromorphic along D, because it has moderate growth (cf. [**Del70**]). In particular, given two such meromorphic bundles M' and M'', any morphism of bundles with connection $(M', \nabla)_{|X^*} \to (M'', \nabla)_{|X^*}$ can be extended as a morphism $(M', \nabla) \to (M'', \nabla)$.

Without the assumption of regular singularity, the previous statement is evidently not true in general: it suffices to consider the free module $M = \mathbb{C}\{x_1, x_2\}[(x_1x_2)^{-1}]$ of rank one and the connection ∇ such that $\nabla 1 = \frac{1}{x_1x_2}(dx_1/x_1 + dx_2/x_2)$. Then $e^{1/x_1x_2} \cdot 1$ is a horizontal section on X^*, but is not meromorphic.

We wish to give a sufficient condition so that the following extension property is satisfied:

(P) *Any ∇-horizontal section of M on X^*, which is meromorphic along $D_2 := \{x_2 = 0\}$, is meromorphic along D.*

By Hartogs' theorem, if (M, ∇) has regular singularity generically along $D_1 := \{x_1 = 0\}$, Property (P) holds. We will introduce a less restrictive condition.

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We will say that M has a *good formal decomposition* at the origin if, setting $\widehat{M} = \mathbb{C}\llbracket x_1, x_2\rrbracket \otimes_{\mathbb{C}\{x_1,x_2\}} M$, there is an isomorphism

$$\widehat{M} \simeq \bigoplus_{\varphi\in\Phi} (\mathscr{E}^{\varphi} \otimes \widehat{\mathscr{R}}_{\varphi}), \tag{DEC^}$$

where (cf. [**Sab00**, § I.2.1.4, p. 10])

(1) the φ's vary in a finite subset Φ of $\mathbb{C}\{x_1, x_2\}[1/x_1x_2]/\mathbb{C}\{x_1, x_2\}$, are pairwise distinct and, for any $\varphi, \psi \in \Phi$, the divisor of φ and of $\varphi - \psi$ is $\leqslant 0$;

(2) $\mathscr{E}^{\varphi}$ is the meromorphic bundle with flat connection of rank one having a basis in which the matrix of ∇ is $d\varphi$; $\mathscr{R}_{\varphi}$ has regular singularities along D.

Proposition. *If M has a good formal decomposition at the origin and all the $\varphi \in \Phi\smallsetminus\{0\}$ have a pole along $D_2 = \{x_2 = 0\}$, then M satisfies Property* (P)*.*

The condition on the polar locus of the φ's will prevent us from the example $\varphi = e^{1/x_1}$, for which Property (P) is clearly not satisfied.

Proof. Let us denote by $e : \widetilde{X} \to X$ the real blow-up of X along both components of D. Then $\widetilde{X}$ is a real analytic space isomorphic to the product $([0, \varepsilon) \times S^1)^2$, equipped with polar coordinates $(\rho_1, \theta_1; \rho_2, \theta_2)$. Let $\mathscr{A}_{\widetilde{X}}$ be the sheaf of functions which are C^∞ on $\widetilde{X}$ and holomorphic on X^*. After [**Sab00**, Th. II.2.1.1], the formal decomposition can be locally lifted to $\widetilde{X}$ with coefficients in $\mathscr{A}_{\widetilde{X}}$, and gives rise to an analogous decomposition of $M^{\mathscr{A}} := \mathscr{A}_{\widetilde{X}} \otimes_{e^{-1}\mathscr{O}_X} e^{-1}M$.

Let us now work in the neighbourhood of some point $\theta^o = (\theta_1^o, \theta_2^o)$ of the torus $(S^1)^2 = e^{-1}(0)$.

Lemma. *Under the assumption of the proposition, let m be a horizontal section of M on the intersection with X^* of a neighbourhood of θ^o in $\widetilde{X}$ (in other words, an open bi-sector of bi-direction θ^o). If, in some (or any) $\mathscr{A}_{\widetilde{X},\theta^o}$-basis of $M^{\mathscr{A}}_{\theta^o}$, the entries of the section m have moderate growth along $D_2 = \{x_2 = 0\}$, then they also have moderate growth along $D_1 = \{x_1 = 0\}$.*

Proof. As the choice of the local $\mathscr{A}$-basis is irrelevant, we can assume that the basis is adapted to the $\mathscr{A}$-decomposition into elementary connections, hence we can assume that $M^{\mathscr{A}}_{\theta^o} = (\mathscr{E}^{\varphi} \otimes \mathscr{R}_{\varphi})^{\mathscr{A}}_{\theta^o}$. In a suitable basis of $\mathscr{R}_{\varphi}$, the entries of $m^{\mathscr{A}} := 1 \otimes m$ take the form $e^{-\varphi} x_1^{a_1} x_2^{a_2} (\log x_1)^{k_1} (\log x_2)^{k_2}$ and, by assumption, if $\varphi \neq 0$, then φ has a pole along D_2. Then, such an entry has moderate growth along D_2 in the neighbourhood of θ^o if and only if one of the following conditions is satisfied:

- $\varphi = 0$,
- $\operatorname{Re} \varphi > 0$ in some neighbourhood of θ_0.

If $\varphi = 0$ or if φ has no pole along D_1, then the corresponding entry has moderate growth along D_1. If φ has a pole along D_1, it also has a pole along D_2 and the corresponding entry has rapid decay along $\{x_1x_2 = 0\}$ (all this understood in some neighbourhood of θ^o). □

We can now end the proof of the proposition. If m is a horizontal section of M on X^*, the entries of which in some $\mathscr{O}_X[1/x_1x_2]$-basis of M are meromorphic along D_2, then the entries of $m^{\mathscr{A}}$ have moderate growth along D_2 in the neighbourhood of any $\theta^o \in (S^1)^2$ (in any $\mathscr{A}_{\widetilde{X},\theta^o}$-basis of $M^{\mathscr{A}}_{\theta^o}$). After the lemma, its entries (in some local $\mathscr{A}$-basis of $M^{\mathscr{A}}$ or some $\mathscr{O}_X$-basis de M) have moderate growth along D_1 in a small sector of bi-direction θ^o for any $\theta^o \in (S^1)^2$, hence on a neighbourhood of 0 in X^*. In other words, m is a (meromorphic) section of M in the neighbourhood of 0. It is then meromorphic, according to Hartogs, on its domain of definition. □

Let us now consider the situation where the decomposition (DEC^) exists but is maybe not good, i.e., does not satisfy (1). We associate a Newton polygon $N(\varphi) \subset \mathbb{R}^2$ to any exponent $\varphi \in \mathbb{C}\{x_1, x_2\}[1/x_1x_2]/\mathbb{C}\{x_1, x_2\}$: this is the convex hull of the union of subsets $(k_1, k_2) + \mathbb{N}^2$, where (k_1, k_2) is the exponent of some monomial in φ.

Corollary 1. *If M has a formal decomposition* (DEC^) *at the origin (but maybe not good) and if, for any $\varphi \in \Phi \smallsetminus \{0\}$, the polygon $N(\varphi)$ has no vertex (k_1, k_2) with $k_1 < 0$ and $k_2 \geqslant 0$, then M satisfies Property* (P).

Proof. We perform a sequence π of toric blowing-up above the origin of $\mathbb{C}^2$ in order to reduce to the case where, in any crossing point of the pull-back divisor of D, the pulled-back connection has a good formal decomposition (this is easy, see for example [**Sab00**, lemme III.1.2.4, p. 83]). The source space of π is covered by a finite number of charts. Typically, each chart has coordinates y_1, y_2 and the map π is given by formulas like

$$x_1 \circ \pi = y_1^a y_2^b$$
$$x_2 \circ \pi = y_1^c y_2^d,$$

with $ad - bc = 1$. By assumption, for any $\varphi, \psi \in \Phi$ the functions $\varphi \circ \pi$ and $\varphi \circ \pi - \psi \circ \pi$ are holomorphic or have a non positive divisor.

In the source space of π, the dual graph of the pulled-back divisor of D is a tree of the form • — • — • ··········· • — • where the extremal vertices correspond to the strict transforms by π of D_1 and D_2. A chart of this space corresponds to a subgraph • — • and, in this chart, one can distinguish a divisor "on the side of D_1" and a divisor "on the side of D_2". In the coordinates given above, where we assume $ad - bc = 1$, the divisor $\{y_1 = 0\}$ is on the side of D_1 and $\{y_2 = 0\}$ on the side of D_2.

The proof is done by induction on the length of the tree, starting from the vertex corresponding to the strict transform of D_2. It is a straightforward application of the proposition, once we have proved the following property:

At any crossing point of the divisor $\pi^{-1}(D)$, and for any $\varphi \in \Phi$, if $\varphi \circ \pi$ has a pole along the divisor on the side of D_1, then φ also has a pole along the divisor on the side of D_2.

By assumption, if $\varphi \neq 0$, it is a minimal finite sum of terms of the form $x_1^{k_1} x_2^{k_2} u(x_1, x_2)$, where u is a holomorphic unit. Moreover, by assumption, if $k_1 < 0$, then $k_2 < 0$. In any chart as above, $ak_1 + ck_2$ and $bk_1 + dk_2$ have the same sign, because of the assumption of good formal decomposition. It is then a matter of checking that, if $bk_1 + dk_2 = 0$, we cannot have $ak_1 + ck_2 < 0$. Let us recall that a, b, c, d are non negative integers such that $ad - bc = 1$. This implies $d > 0$. We then have $k_2 = -bk_1/d$, hence $ak_1 + ck_2 = (ad - bc)k_1/d = k_1/d$; but in such a situation we cannot have $k_1 < 0$, otherwise we would also have $k_2 < 0$ and $bk_1 + dk_2 < 0$, a contradiction. □

Corollary 2. *If M satisfies the property of Corollary* 1 *after a cyclic ramification around D_1 and/or D_2, then M satisfies Property* (P)*.*

Proof. Easy. □

Corollary 3. *If M', M'' both satisfy the assumptions of Corollary* 2*, any morphism of meromorphic bundles $(M', \nabla)_{|X \smallsetminus D_1} \to (M'', \nabla)_{|X \smallsetminus D_1}$ compatible with the connections can be extended as a morphism $(M', \nabla) \to (M'', \nabla)$.*

Proof. Such a morphism is a horizontal section of $M' \otimes M''^{\vee}$ on $X \smallsetminus D_1$, hence a horizontal section of $M' \otimes M''^{\vee}$ on X^* with moderate growth along D_2. One can immediately check that the hypotheses of Corollary 2 are satisfied by $M' \otimes M''^{\vee}$ if they are satisfied by M' and M''. This is enough to conclude. □

Example. Here is an example related to wild twistor $\mathscr{D}$-modules (cf. [**Sab09**]). Let M be a meromorphic bundle with flat connection on X. Let us assume that, after a ramification along D_1 (that we forget in the following), there exists a finite family Φ consisting of pairwise distinct $\varphi \in x_1^{-1}\mathbb{C}[x_1^{-1}]$ such that, denoting by $M_{\widehat{X|D_1}}$ the formalized bundle of M along D_1, we have a decomposition

$$M_{\widehat{X|D_1}} \simeq \oplus_{\varphi \in \Phi} (\mathscr{E}^{\varphi/x_2} \otimes N_\varphi),$$

where N_φ is a free $\mathscr{O}_{\widehat{X|D_1}}[1/x_1x_2]$-module having a basis in which the matrix of $x_1 \nabla_{\partial_{x_1}}$ has no pole. Then, formalizing once more with respect to x_2, any N_φ can be decomposed as the direct sum of terms $\mathscr{E}^\psi \otimes \widehat{R}_{\varphi,\psi}$, with pairwise distinct $\psi \in x_2^{-1}\mathbb{C}[x_2^{-1}]$ and $\widehat{R}_{\varphi,\psi}$ with regular singularity (cf. [**Sab00**, prop. III.2.1.1(2), p. 89]). The Newton polygon of any nonzero exponent $\psi(x_2) + \varphi(x_1)/x_2$ has no vertex (k_1, k_2) with $k_1 < 0$ and $k_2 \geqslant 0$. We are thus in the situation of Corollary 1. We deduce that M satisfies Property (P).

We now use notation and definitions of [**Sab09**, §4.5], where X is $(\mathbb{C}, 0)$ with coordinate t and $\mathscr{X} = X \times \mathbb{C}$ with coordinates (t, z). We set $\mathscr{X}^* = \mathscr{X} \smallsetminus \{t = 0\}$.

Corollary 4. *Let* $\widetilde{\mathscr{M}}', \widetilde{\mathscr{M}}''$ *be two free* $\mathscr{O}_{\mathscr{X}}[t^{-1}]$*-modules of finite rank, equipped with a compatible action of* $\mathscr{R}_{\mathscr{X}}$*. Let us also assume that they are integrable* $\mathscr{R}_{\mathscr{X}}$*-modules, that is, are equipped with a flat meromorphic connection with a pole of Poincaré rank one along* $\{z=0\}$*, extending the* z*-connection coming from the* $\mathscr{R}_{\mathscr{X}}$*-structure. Let us assume that* $\widetilde{\mathscr{M}}', \widetilde{\mathscr{M}}''$ *are strictly specializable with ramification and exponential twist along* $\{t=0\}$*.*

Then, any morphism $\widetilde{\mathscr{M}}'_{|\mathscr{X}^*} \to \widetilde{\mathscr{M}}''_{|\mathscr{X}^*}$ *which is compatible with the connections can be extended to a morphism* $\widetilde{\mathscr{M}}' \to \widetilde{\mathscr{M}}''$*.*

Proof. In the previous notation, we set $x_1 = t$, $x_2 = z$. According to [**Sab09**, Prop. 4.5.4], we can apply the argument above. □

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